Volume 27, Number 1 July 15, 2019
ISSN:1521-1398 PRINT,1572-9206 ONLINE

Journal of Computational Analysis and Applications

EUDOXUS PRESS,LLC

Journal of Computational Analysis and Applications
ISSNno.'s:1521-1398 PRINT,1572-9206 ONLINE

SCOPE OF THE JOURNAL

An international publication of Eudoxus Press, LLC
(fifteen times annually)

Editor in Chief: George Anastassiou
Department of Mathematical Sciences,
University of Memphis, Memphis, TN 38152-3240, U.S.A
ganastss@memphis.edu
http://www.msci.memphis.edu/~ganastss/jocaaa

The main purpose of "J.Computational Analysis and Applications" is to publish high quality research articles from all subareas of Computational Mathematical Analysis and its many potential applications and connections to other areas of Mathematical Sciences. Any paper whose approach and proofs are computational,using methods from Mathematical Analysis in the broadest sense is suitable and welcome for consideration in our journal, except from Applied Numerical Analysis articles. Also plain word articles without formulas and proofs are excluded. The list of possibly connected mathematical areas with this publication includes, but is not restricted to: Applied Analysis, Applied Functional Analysis, Approximation Theory, Asymptotic Analysis, Difference Equations, Differential Equations, Partial Differential Equations, Fourier Analysis, Fractals, Fuzzy Sets, Harmonic Analysis, Inequalities, Integral Equations, Measure Theory, Moment Theory, Neural Networks, Numerical Functional Analysis, Potential Theory, Probability Theory, Real and Complex Analysis, Signal Analysis, Special Functions, Splines, Stochastic Analysis, Stochastic Processes, Summability, Tomography, Wavelets, any combination of the above, e.t.c.

"J.Computational Analysis and Applications" is a peer-reviewed Journal. See the instructions for preparation and submission of articles to JoCAAA. Assistant to the Editor: Dr.Razvan Mezei,mezei_razvan@yahoo.com, Madison,WI,USA.

Journal of Computational Analysis and Applications(JoCAAA) is published by **EUDOXUS PRESS,LLC**,1424 Beaver Trail Drive,Cordova,TN38016,USA,anastassioug@yahoo.com http://www.eudoxuspress.com. **Annual Subscription Prices**:For USA and Canada,Institutional:Print $800, Electronic OPEN ACCESS. Individual:Print $400. For any other part of the world add $160 more(handling and postages) to the above prices for Print. No credit card payments.

JoCAAA is reviewed and abstracted by AMS Mathematical Reviews,MATHSCI,and Zentralblaat MATH.

Dartmouth College
8000 Cummings Hall,
Hanover, NH 03755-8000
603-646-3843 (X 3546 Secr.)
e-mail:george.cybenko@dartmouth.edu
Approximation Theory and Neural Networks

Sever S. Dragomir
School of Computer Science and Mathematics, Victoria University,
PO Box 14428,
Melbourne City,
MC 8001, AUSTRALIA
Tel. +61 3 9688 4437
Fax +61 3 9688 4050
sever.dragomir@vu.edu.au
Inequalities, Functional Analysis, Numerical Analysis, Approximations, Information Theory, Stochastics.

Oktay Duman
TOBB University of Economics and Technology,
Department of Mathematics, TR-06530,
Ankara, Turkey,
oduman@etu.edu.tr
Classical Approximation Theory, Summability Theory, Statistical Convergence and its Applications

Saber N. Elaydi
Department Of Mathematics
Trinity University
715 Stadium Dr.
San Antonio, TX 78212-7200
210-736-8246
e-mail: selaydi@trinity.edu
Ordinary Differential Equations, Difference Equations

J .A. Goldstein
Department of Mathematical Sciences
The University of Memphis
Memphis, TN 38152
901-678-3130
jgoldste@memphis.edu
Partial Differential Equations, Semigroups of Operators

H. H. Gonska
Department of Mathematics
University of Duisburg
Duisburg, D-47048
Germany
011-49-203-379-3542
e-mail: heiner.gonska@uni-due.de
Approximation Theory, Computer Aided Geometric Design

John R. Graef
Department of Mathematics
University of Tennessee at Chattanooga
Chattanooga, TN 37304 USA
John-Graef@utc.edu
Ordinary and functional differential equations, difference equations, impulsive systems, differential inclusions, dynamic equations on time scales, control theory and their applications

Weimin Han
Department of Mathematics
University of Iowa
Iowa City, IA 52242-1419
319-335-0770
e-mail: whan@math.uiowa.edu
Numerical analysis, Finite element method, Numerical PDE, Variational inequalities, Computational mechanics

Tian-Xiao He
Department of Mathematics and Computer Science
P.O. Box 2900, Illinois Wesleyan University
Bloomington, IL 61702-2900, USA
Tel (309)556-3089
Fax (309)556-3864
the@iwu.edu
Approximations, Wavelet, Integration Theory, Numerical Analysis, Analytic Combinatorics

Margareta Heilmann
Faculty of Mathematics and Natural Sciences, University of Wuppertal
Gaußstraße 20
D-42119 Wuppertal, Germany,
heilmann@math.uni-wuppertal.de
Approximation Theory (Positive Linear Operators)

Xing-Biao Hu
Institute of Computational Mathematics
AMSS, Chinese Academy of Sciences
Beijing, 100190, CHINA
hxb@lsec.cc.ac.cn

Instructions to Contributors
Journal of Computational Analysis and Applications

An international publication of Eudoxus Press, LLC, of TN.

Editor in Chief: George Anastassiou

Department of Mathematical Sciences
University of Memphis
Memphis, TN 38152-3240, U.S.A.

1. Manuscripts files in Latex and PDF and in English, should be submitted via email to the Editor-in-Chief:

Prof.George A. Anastassiou
Department of Mathematical Sciences
The University of Memphis
Memphis,TN 38152, USA.
Tel. 901.678.3144
e-mail: ganastss@memphis.edu

Authors may want to recommend an associate editor the most related to the submission to possibly handle it.

Also authors may want to submit a list of six possible referees, to be used in case we cannot find related referees by ourselves.

2. Manuscripts should be typed using any of TEX,LaTEX,AMS-TEX,or AMS-LaTEX and according to EUDOXUS PRESS, LLC. LATEX STYLE FILE. (Click HERE to save a copy of the style file.)They should be carefully prepared in all respects. Submitted articles should be brightly typed (not dot-matrix), double spaced, in ten point type size and in 8(1/2)x11 inch area per page. Manuscripts should have generous margins on all sides and should not exceed 24 pages.

3. Submission is a representation that the manuscript has not been published previously in this or any other similar form and is not currently under consideration for publication elsewhere. A statement transferring from the authors(or their employers,if they hold the copyright) to Eudoxus Press, LLC, will be required before the manuscript can be accepted for publication.The Editor-in-Chief will supply the necessary forms for this transfer.Such a written transfer of copyright,which previously was assumed to be implicit in the act of submitting a manuscript,is necessary under the U.S.Copyright Law in order for the publisher to carry through the dissemination of research results and reviews as widely and effective as possible.

4. The paper starts with the title of the article, author's name(s) (no titles or degrees), author's affiliation(s) and e-mail addresses. The affiliation should comprise the department, institution (usually university or company), city, state (and/or nation) and mail code.

The following items, 5 and 6, should be on page no. 1 of the paper.

5. An abstract is to be provided, preferably no longer than 150 words.

6. A list of 5 key words is to be provided directly below the abstract. Key words should express the precise content of the manuscript, as they are used for indexing purposes.

The main body of the paper should begin on page no. 1, if possible.

7. All sections should be numbered with Arabic numerals (such as: 1. INTRODUCTION) .
Subsections should be identified with section and subsection numbers (such as 6.1. Second-Value Subheading).
If applicable, an independent single-number system (one for each category) should be used to label all theorems, lemmas, propositions, corollaries, definitions, remarks, examples, etc. The label (such as Lemma 7) should be typed with paragraph indentation, followed by a period and the lemma itself.

8. Mathematical notation must be typeset. Equations should be numbered consecutively with Arabic numerals in parentheses placed flush right, and should be thusly referred to in the text [such as Eqs.(2) and (5)]. The running title must be placed at the top of even numbered pages and the first author's name, et al., must be placed at the top of the odd numbed pages.

9. Illustrations (photographs, drawings, diagrams, and charts) are to be numbered in one consecutive series of Arabic numerals. The captions for illustrations should be typed double space. All illustrations, charts, tables, etc., must be embedded in the body of the manuscript in proper, final, print position. In particular, manuscript, source, and PDF file version must be at camera ready stage for publication or they cannot be considered.

Tables are to be numbered (with Roman numerals) and referred to by number in the text. Center the title above the table, and type explanatory footnotes (indicated by superscript lowercase letters) below the table.

10. List references alphabetically at the end of the paper and number them consecutively. Each must be cited in the text by the appropriate Arabic numeral in square brackets on the baseline.
References should include (in the following order):
initials of first and middle name, last name of author(s)
title of article,

name of publication, volume number, inclusive pages, and year of publication.

Authors should follow these examples:

Journal Article

1. H.H.Gonska,Degree of simultaneous approximation of bivariate functions by Gordon operators, (journal name in italics) *J. Approx. Theory*, 62,170-191(1990).

Book

2. G.G.Lorentz, (title of book in italics) *Bernstein Polynomials* (2nd ed.), Chelsea,New York,1986.

Contribution to a Book

3. M.K.Khan, Approximation properties of beta operators,in(title of book in italics) *Progress in Approximation Theory* (P.Nevai and A.Pinkus,eds.), Academic Press, New York,1991,pp.483-495.

11. All acknowledgements (including those for a grant and financial support) should occur in one paragraph that directly precedes the References section.

12. Footnotes should be avoided. When their use is absolutely necessary, footnotes should be numbered consecutively using Arabic numerals and should be typed at the bottom of the page to which they refer. Place a line above the footnote, so that it is set off from the text. Use the appropriate superscript numeral for citation in the text.

13. After each revision is made please again submit via email Latex and PDF files of the revised manuscript, including the final one.

14. Effective 1 Nov. 2009 for current journal page charges, contact the Editor in Chief. Upon acceptance of the paper an invoice will be sent to the contact author. The fee payment will be due one month from the invoice date. The article will proceed to publication only after the fee is paid. The charges are to be sent, by money order or certified check, in US dollars, payable to Eudoxus Press, LLC, to the address shown on the Eudoxus homepage.

No galleys will be sent and the contact author will receive one (1) electronic copy of the journal issue in which the article appears.

15. This journal will consider for publication only papers that contain proofs for their listed results.

On common fixed point theorems of weakly compatible mappings in fuzzy metric spaces

Afshan Batool[1], Tayyab Kamran[2], Dong Yun Shin[3] and Choonkil Park[4]

[1,2]Department of Mathematics, Quaid-i-Azam University, Islamabad, Pakistan
[3]Department of Mathematics, University of Seoul, Seoul 02504, Korea
[4]Research Institute for Natural Sciences, Hanyang University, Seoul 04763, Korea
afshan.batoolqau@gmail.com, tayyabkamran@gmail.com, dyshin@uos.ac.kr, baak@hanyang.ac.kr

Abstract: The purpose of this paper is to obtain common fixed point theorem involving two pair of weakly compatible mappings in complete fuzzy metric spaces. Some related results and illustrative examples are also discussed.

Keywords: common fixed point; weakly compatible mapping; complete fuzzy metric space; coincidence point; point of coincidence

2010 MSC: 47H10, 54E50, 54E40, 46S50.

1. Introduction and preliminaries

Let (X, d) be a metric space. A mapping $T : X \to X$ is said to be *contraction* if there exists $\alpha \in (0, 1)$ such that for all $x, y \in X$,

$$d(Tx, Ty) \leq \alpha d(x, y). \tag{1}$$

If the metric space (X, d) is complete, then the mapping satisfying (1) has a unique fixed point.
Rhoades [11] assumed a weakly contractive mapping $f : X \to X$ which satisfies the condition

$$d(fx, fy) \leq d(x, y) - \varphi(d(x, y)), \tag{2}$$

where $x, y \in X$ and $\varphi : [0, \infty) \to [0, \infty)$ is a continuous and nondecreasing function such that $\varphi(t) = 0$ if and only if $t = 0$. Rhoades [11] obtained the following extension.

Theorem 1.1. ([11]) *Let $T : X \to X$ be a weakly contractive mapping, where (X, d) is a complete metric space. Then T has a unique fixed point.*

Dutta and Choudhury [7] introduced a new generalization of contraction principle in the following theorem.

Theorem 1.2. ([7]) *Let (X, d) be a complete metric space and let $T : X \to X$ be a self-mapping satisfying the inequality*

$$\psi(d(fx, fy)) \leq \psi(d(x, y)) - \varphi(d(x, y)) \tag{3}$$

for all $x, y \in X$, where $\phi, \varphi : [0, \infty) \to [0, \infty)$ are both continuous and monotone nondecreasing functions with $\psi(t) = \varphi(t) = 0$ if and only if $t = 0$. Then T has a unique fixed point.

[0]Corresponding authors: dyshin@uos.ac.kr (Dong Yun Shin), baak@hanyang.ac.kr (Choonkil Park)

Several researchers have studied the existence of fixed points and common fixed points of mappings (see [1, 2, 3, 4, 5, 6, 8, 9, 10, 12]).

In this article, we give a fixed point theorem for contraction maps in complete fuzzy metric space, which improves and generalizes the above-mentioned result of Dutta and Choudhury.

We recall some definitions before giving the main result of this article.

Definition 1.3. *A binary operation* $* : [0,1]^2 \to [0,1]$ *is called a continuous* t*-norm if* $([0,1], *)$ *is an Abelian topological monoid, i.e.,*

(1) $*$ *is associative and commutative;*
(2) $*$ *is continuous;*
(3) $a * 1 = a$ *for all* $a \in [0,1]$;
(4) $a * b \leq c * d$ *whenever* $a \leq c$ *and* $b \leq d$ *for all* $a, b, c, d \in [0,1]$.

Definition 1.4. *A 3-tuple* $(X, M, *)$ *is called a fuzzy metric space if* X *is an arbitrary set,* $*$ *is a continuous* t*-norm and* M *is a fuzzy set on* $X^2 \times (0,\infty)$ *satisfying the following conditions:*

(1) $M(x, y, t) > 0$,
(2) $M(x, y, t) = 1$ *if and only if* $x = y$,
(3) $M(x, y, t) = M(y, x, t)$,
(4) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$,
(5) $M(x, y, .) : (0,\infty) \to [0,1]$ *is continuous,*
for all $x, y, z \in X$ *and* $t, s > 0$.

Definition 1.5. *Let* f *and* g *be self-maps on a set* X*. If* $w = fx = gx$ *for some* $x \in X$*, then* x *is called coincidence point of* f *and* g*, and* w *is called a point of coincidence of* f *and* g*.*

Definition 1.6. *Let* f *and* g *be two self-maps on a set* X*. Then* f *and* g *are said to be weakly compatible if they commute at every coincidence point.*

2. Main results

Theorem 2.1. *Let* (X, M, t) *be a complete fuzzy metric space, and let* E *be a nonempty closed subset of* X*. Let* $S, T : E \to E$ *and* $I, J : E \to X$ *be mappings satisfying* $T(E) \subset I(E)$ *and* $S(E) \subset J(E)$ *and for every* $x, y \in X$*,*

$$\psi(M(Sx, Ty, t)) \leq \psi(M_{I,J}(x, y)) - \varphi(M_{I,J}(x, y)), \tag{4}$$

where $\psi : [0,\infty) \to [0,\infty)$ *is a continuous and nondecreasing function such that* $\psi(t) = 0$ *if and only if* $t = 0$*.* $\varphi : [0,\infty) \to [0,\infty)$ *is a lower semi-continuous function such that* $\varphi(t) = 0$ *if and only if* $t = 0$*, and*

$$\begin{aligned} M_{I,J}(x, y) \;=\; \max\Big\{ & M(Ix, Jy, t), M(Ix, Sx, t), M(Jy, Ty, t), \\ & \frac{1}{2}\Big(M(Ix, Ty, t) + M(Jy, Sx, t)\Big)\Big\}. \end{aligned} \tag{5}$$

If one of $S(E)$, $T(E)$, $I(E)$, JE is a closed subset of X, then $\{S, I\}$ and $\{T, J\}$ have a unique point of coincidence in X. Moreover, if $\{S, I\}$ and $\{T, J\}$ are weakly compatible, then S, T, I and J have a unique common fixed point in X.

Proof. Let x_0 be an arbitrary point in X. Since $T(E) \subset I(E)$ and $S(E) \subset J(E)$, we can define the sequences $\{x_n\}$ and $\{y_n\}$ in X by

$$y_{2n-1} = Sx_{2n-2} = Jx_{2n-1}, \ \ y_{2n} = Tx_{2n-1} = Ix_{2n}, \qquad n = 1, 2, \cdots .$$

Suppose that $y_{n_0} = y_{n_0+1}$ for some n_0. Then the sequence $\{y_n\}$ is constant for $n \geq n_0$. Indeed, let $n_0 = 2k$. Then $y_{2k} = y_{2k+1}$ and it follows from (4) that

$$\begin{aligned} \psi(M(y_{2k+1}, y_{2k+2}, t)) &= \psi(M(Sx_{2k}, Tx_{2k+1}, t)) \\ &\leq \psi(M_{I,J}(x_{2k}, x_{2k+1})) - \varphi(M_{I,J}(x_{2k}, x_{2k+1})), \end{aligned} \tag{6}$$

where

$$\begin{aligned} &M_{I,J}(x_{2k}, x_{2k+1}) \\ &= \max\Big\{M(y_{2k}, y_{2k+1}, t), M(y_{2k}, Sx_{2k}, t), M(y_{2k+1}, Tx_{2k+1}, t), \\ &\qquad \frac{1}{2}\Big(M(y_{2k}, Tx_{2k+1}, t) + M(y_{2k+1}, Sx_{2k}, t)\Big)\Big\} \\ &= \max\Big\{0, 0, M(y_{2k+1}, y_{2k+2}, t), \frac{1}{2}\Big(M(y_{2k}, y_{2k+2}, t) + 0\Big)\Big\} \\ &= \max\Big\{M(y_{2k+1}, y_{2k+2}, t), \frac{1}{2}M(y_{2k}, y_{2k+2}, t)\Big\} \\ &= M(y_{2k+1}, y_{2k+2}, t). \end{aligned}$$

By (6), we get

$$\psi(M(y_{2k+1}, y_{2k+2}, t)) \leq \psi(M(y_{2k+1}, y_{2k+2}, t)) - \varphi(M(y_{2k+1}, y_{2k+2}, t)),$$

and so $\varphi(M(y_{2k+1}, y_{2k+2}, t)) \leq 0$ and $y_{2k+1} = y_{2k+2}$.

Similarly, if $n_0 = 2k+1$, then one easily obtains that $y_{2k+2} = y_{2k+3}$ and the sequence $\{y_n\}$ is constant (starting from some n_0). Therefore, $\{S, I\}$ and $\{T, J\}$ have a point of coincidence in X.

Now, suppose that $M(y_n, y_{n+1}, t) > 0$ for each n. We shall show that for each $n = 0, 1, \cdots$,

$$M(y_{n+1}, y_{n+2}, t) \leq M_{I,J}(x_n, x_{n+1}) = M(y_n, y_{n+1}, t). \tag{7}$$

Using (4), we obtain that

$$\begin{aligned} \psi(M(y_{2n+1}, y_{2n+2}, t)) &= \psi(M(Sx_{2n}, Tx_{2n+1}, t)) \\ &\leq \psi(M_{I,J}(x_{2n}, x_{2n+1})) - \varphi(M_{I,J}(x_{2n}, x_{2n+1})) \\ &< \psi(M_{I,J}(x_{2n}, x_{2n+1})). \end{aligned} \tag{8}$$

On the other hand, the control function ψ is nondecreasing. Then

$$M(y_{2n+1}, y_{2n+2}, t) \leq M_{I,J}(x_{2n}, x_{2n+1}). \tag{9}$$

Moreover, we have

$$\begin{aligned}
&M_{I,J}(x_{2n}, x_{2n+1})\\
&= \max\Big\{M(y_{2n}, y_{2n+1}, t), M(y_{2n}, Sx_{2n}, t), M(y_{2n+1}, Tx_{2n+1}, t),\\
&\qquad \frac{1}{2}\Big(M(y_{2n}, Tx_{2n+1}, t) + M(y_{2n+1}, Sx_{2n}, t)\Big)\Big\}\\
&= \max\Big\{M(y_{2n}, y_{2n+1}, t), M(y_{2n}, y_{2n+1}, t),\\
&\qquad M(y_{2n+1}, y_{2n+2}, t), \frac{1}{2}M(y_{2n}, y_{2n+2}, t)\Big\}\\
&\le \max\Big\{M(y_{2n}, y_{2n+1}, t), M(y_{2n+1}, y_{2n+2}, t),\\
&\qquad \frac{1}{2}\Big(M(y_{2n}, y_{2n+1}, t) + M(y_{2n+1}, y_{2n+2})\Big)\Big\}\\
&\le \max\Big\{M(y_{2n}, y_{2n+1}, t), M(y_{2n+1}, y_{2n+2}, t)\Big\}.
\end{aligned}$$

If $M(y_{2n+1}, y_{2n+2}, t) \ge M(y_{2n}, y_{2n+1}, t)$, then by using the last inequality and (9), we have $M_{I,J}(x_{2n}, x_{2n+1}) = M(y_{2n+1}, y_{2n+2}, t)$ and (8) implies that

$$\begin{aligned}
\psi(M(y_{2n+1}, y_{2n+2}, t)) &= \psi(M(Sx_{2n}, Tx_{2n+1}, t))\\
&\le \psi(M(y_{2n+1}, y_{2n+2}, t)) - \varphi(M(y_{2n+1}, y_{2n+2}, t)),
\end{aligned}$$

which is only possible when $M(y_{2n+1}, y_{2n+2}, t) = 0$. It is a contradiction. Hence $M(y_{2n+1}, y_{2n+2}, t) \le M(y_{2n}, y_{2n+1}, t)$ and $M_{I,J}(x_{2n}, x_{2n+1}) \le M(y_{2n}, y_{2n+1}, t)$. By definition, $M_{I,J}(x_{2n}, x_{2n+1}) \ge M(y_{2n}, y_{2n+1}, t)$, and so (7) is proved for $\{M(y_{2n+1}, y_{2n+2}, t)\}$.

In a similar way, one can obtain that

$$M(y_{2n+3}, y_{2n+2}, t) \le M_{I,J}(x_{2n+2}, x_{2n+1}) = M(y_{2n+2}, y_{2n+1}, t).$$

So (7) holds for each $n \in \mathbb{N}$.
It follows that the sequence $\{M(y_n, y_{n+1}, t)\}$ is nondecreasing and the limit

$$\lim_{n\to\infty} M(y_n, y_{n+1}, t) = \lim_{n\to\infty} M_{I,J}(x_n, x_{n+1})$$

exists. We denote this limit by d^*. We have $d^* \ge 0$.
Suppose that $d^* > 0$. Then

$$\psi(M(y_{n+1}, y_{n+2}, t)) \le \psi(M_{I,J}(x_n, x_{n+1})) - \varphi(M_{I,J}(x_n, x_{n+1})).$$

Passing to the (upper) limit when $n \to \infty$, we get

$$\psi(d^*) \le \psi(d^*) - \liminf_{n\to\infty} \varphi(M_{I,J}(x_n, x_{n+1})) \le \psi(d^*) - \varphi(d^*),$$

i.e., $\varphi(d^*) \le 0$. Using the properties of control functions, we get that $d^* = 0$, which is a contradiction. Hence we have $\lim_{n\to\infty} M(y_n, y_{n+1}, t) = 0$.
Now we show that $\{y_n\}$ is a Cauchy sequence in X.
It is enough to prove that $\{y_{2n}\}$ is a Cauchy sequence. Suppose the contrary. Then,

for some $\epsilon > 0$, there exist subsequences $\{y_{2n(k)}\}$ and $\{y_{2m(k)}\}$ of $\{y_{2n}\}$ such that $n(k)$ is the smallest index satisfying

$$n(k) > m(k) \text{ and } M(y_{n(k)}, y_{m(k)}, t) \geq \epsilon.$$

In particular, $M(y_{n(k)-2}, y_{m(k)}, t) < \epsilon$. Using the triangle inequality and the known relation $|d(x,z) - d(x,y)| \leq d(x,z)$, we obtain that

$$\begin{aligned}\lim_{k\to\infty} M(y_{2n(k)}, y_{2m(k)}, t) &= \lim_{k\to\infty} M(y_{2n(k)}, y_{2m(k)-1}, t) = \lim_{k\to\infty} M(y_{2n(k)+1}, y_{2m(k)}, t)\\ &= \lim_{k\to\infty} M(y_{2n(k)+1}, y_{2m(k)-1}, t) = \epsilon.\end{aligned}$$

By the definition of $M(x,y,t)$ and by using the previous limits, we get that

$$\lim_{k\to\infty} M_{I,J}(x_{2n(k)}, x_{2m(k)-1}) = \epsilon.$$

Indeed,

$$\begin{aligned}&M_{I,J}(x_{2n(k)}, x_{2m(k)-1})\\ &= \max\Big\{M(y_{2n(k)}, y_{2m(k)-1}, t), M(y_{2n(k)}, y_{2n(k)+1}, t), M(y_{2m(k)-1}, y_{2m(k)}, t),\\ &\quad \frac{1}{2}\Big(M(y_{2n(k)}, y_{2m(k)}, t) + M(y_{2n(k)+1}, y_{2m(k)-1}, t)\Big)\Big\}\\ &\quad \to \max\Big\{\epsilon, 0, 0, \frac{1}{2}(\epsilon+\epsilon)\Big\} = \epsilon.\end{aligned}$$

Applying (4), we obtain

$$\begin{aligned}\psi(M(y_{2n(k)+1}, y_{2m(k)}, t)) &= \psi(M(Sx_{2n(k)}, Tx_{2m(k)-1}, t))\\ &\leq \psi(M_{I,J}(x_{2n(k)}, x_{2m(k)-1})) - \varphi(M_{I,J}(x_{2n(k)}, x_{2m(k)-1})).\end{aligned}$$

Passing to the limit $k \to \infty$, we obtain that $\psi(\epsilon) \leq \psi(\epsilon) - \varphi(\epsilon)$, which is a contradiction. Therefore, $\{y_n\}$ is a Cauchy sequence in the complete metric (X,d). So there exists $u \in X$ such that $\lim_{n\to\infty} y_n = u$.

On the other hand, E is closed and $\{y_n\} \subset E$. Then $u \in E$. Suppose that $I(E)$ is closed. Then there exists $v \in E$ such that

$$u = Iv. \tag{10}$$

We claim that $Sv = u$. Using (4) and (10), we have

$$\psi(M(Sv, y_{2n}, t)) = \psi(M(Sv, Tx_{2n-1}, t)) \leq \psi(M_{I,J}(v, x_{2n-1})) - \varphi(M_{I,J}(v, x_{2n-1})), \tag{11}$$

where

$$\begin{aligned}M_{I,J}(v, x_{2n-1}) &= \max\Big\{M(y_{2n-1}, u, t), M(u, Sv, t), M(y_{2n-1}, Tx_{2n-1}, t),\\ &\qquad \frac{1}{2}\Big(M(y_{2n-1}, Sv, t) + M(u, Tx_{2n-1}, t)\Big)\Big\}\\ &\to \max\Big\{0, M(u, Sv, t), 0, \frac{1}{2}M(u, Sv, t)\Big\} = M(u, Sv, t).\end{aligned}$$

Passing to the limit when $n \to \infty$ in (11), we get

$$\psi(M(u,Sv,t)) \leq \psi(M(u,Sv,t)) - \varphi(M(u,Sv,t)).$$

It follows that

$$u = Sv. \tag{12}$$

Since $u = Sv \in SE \subset JE$, there exists $w \in E$ such that

$$u = Jw. \tag{13}$$

We claim that $Tw = u$. By (4), we get

$$\psi(M(u,Tw,t)) = \psi(M(Sv,Tw,t)) \leq \psi(M_{I,J}(v,w)) - \varphi(M_{I,J}(v,w)),$$

where

$$\begin{aligned} M_{I,J}(v,w) = \max\Big\{ & M(u,u,t), M(Iv,Sv,t), M(Jw,Tw,t), \\ & \frac{1}{2}\Big(M(Jw,Sv,t) + M(Iv,Tw,t)\Big)\Big\} \\ = \max\Big\{ & 0, 0, M(u,Tw,t), \frac{1}{2}M(u,Tw,t)\Big\} = M(u,Tw,t). \end{aligned}$$

Hence (2) implies that

$$\psi(M(u,Tw,t)) \leq \psi(M(u,Tw,t)) - \varphi(M(u,Tw,t)).$$

It follows that

$$u = Tw. \tag{14}$$

Combining (10) and (12) yields

$$u = Iv = Sv, \tag{15}$$

that is, u is a point of coincidence of I and S. Combining (13) and (14) yields

$$u = Jw = Tw, \tag{16}$$

that is, u is a point of coincidence of J and T.

To prove the uniqueness property of u, suppose that $u^{'}$ is another point of coincidence of I and S, that is,

$$u^{'} = Iv^{'} = Sv^{'}$$

for some $v^{'} \in E$. By (4), we have

$$\psi(M(u^{'},u,t)) = \psi(M(Sv^{'},Tw,t)) \leq \psi(M_{I,J}(v^{'},w)) - \varphi(M_{I,J}(v^{'},w)),$$

where

$$\begin{aligned} M_{I,J}(v^{'},w) &= \max\Big\{M(u^{'},u,t), 0, 0, \frac{1}{2}\Big(M(u^{'},u,t) + M(u^{'},u,t)\Big)\Big\} \\ &= M(u^{'},u,t). \end{aligned}$$

It follows from (2) that $u^{'} = u$.
Now, suppose that $\overline{u}$ is another point of coincidence of J and T, that is,

$$\overline{u} = jw^{'} = Tw^{'}$$

for some $w^{'} \in E$. Using (4), we obtain

$$\psi(M(\overline{u}, u, t)) = \psi(M(Sv, Tw, t^{'})) \leq \psi(M_{I,J}(v, w^{'})) - \varphi(M_{I,J}(v, w^{'})),$$

where

$$\begin{aligned} M_{I,J}(v, w^{'}) &= \max\Big\{M(\overline{u}, u, t), 0, 0, \frac{1}{2}\Big(M(\overline{u}, u, t) + M(\overline{u}, u, t)\Big)\Big\} \\ &= M(\overline{u}, u, t). \end{aligned}$$

It follows from (2) that $\overline{u} = u$.
Therefore, u is the unique point of coincidence of $\{S, I\}$ and $\{T, J\}$.

Now, if $\{S, I\}$ and $\{T, J\}$ are weakly compatible, then by (15) and (16), we have $Su = S(Iv) = I(Sv) = Iu = w_1$ and $Tu = T(Jw) = J(Tw) = Ju = w_2$. By (4), we have

$$\psi(M(w_1, w_2, t)) = \psi(M(Su, Tu, t)) \leq \psi(M_{I,J}(u, u)) - \varphi(M_{I,J}(u, u)),$$

where

$$\begin{aligned} M_{I,J}(u, u) &= \max\Big\{M(w_1, w_2, t), 0, 0, \frac{1}{2}\Big(M(w_1, w_2, t) + M(w_1, w_2, t)\Big)\Big\} \\ &= M(w_1, w_2, t). \end{aligned}$$

It follows that $w_1 = w_2$, that is,

$$Su = Iu = Tu = Ju. \tag{17}$$

By (4) and (17), we have

$$\psi(M(Sv, Tu, t)) \leq \psi(M_{I,J}(v, u) - \varphi(M_{I,J}(v, u)),$$

where

$$\begin{aligned} M_{I,J}(v, u) &= \max\Big\{M(Iv, Ju, t), M(Iv, Sv, t), M(Ju, Tu, t), \\ &\qquad \frac{1}{2}\Big(M(Iv, Tu, t) + M(Sv, Tu, t)\Big)\Big\} \\ &= \max\Big\{M(Sv, Tu, t), 0, 0, \frac{1}{2}\Big(M(Sv, Tu, t) + M(Sv, Tu, t)\Big)\Big\} \\ &= M(Sv, Tu, t). \end{aligned}$$

Therefore, we deduce that $Sv = Tu$, that is, $u = Tu$. It follows from (17) that

$$u = Su = Iu = Tu = Ju.$$

Then u is the unique common fixed point of S, I, J and T.

The rest of the proof is similar to the above case and so the rest will be omitted. □

Example 2.2. *Let $X = [0, 1]$ be equipped with the natural metric $d(x, y) = |x - y|$. Now for $t \in [0, \infty)$ define*

$$M(x, y, t) = \begin{cases} 0 \ \textit{if}\ t = 0 \ \textit{and}\ x, y \ \in X \\ \frac{t}{t+|x-y|} \ \textit{if}\ t \neq 0 \ \textit{and}\ x, y \ \in X. \end{cases}$$

Clearly, $(X, M, *)$ *is a fuzzy metric on* X*, where* $*$ *is defined as* $a * b = ab$*. This fuzzy metric space is complete.*

Let $E = \{0, \frac{1}{2}, 1\}$ *and we define* $T, S : E \to E$ *as*
$T0 = T1 = 0$ *and* $T\frac{1}{2} = 1$*,* $Sx = 0$*.*
We also define $I, J : E \to X$ *as*
$I0 = I1 = 0$ *and* $I\frac{1}{2} = 1$*,* $J0 = J1 = 0$ *and* $J\frac{1}{2} = 1$*.*
The functions $\psi : \varphi : [0, \infty) \to [0, \infty)$ *are defined as*
$\psi(t) = t$ *and* $\varphi(t) = \frac{t}{4}$*.*
Then

$$\psi(M(Sx, Ty, t)) \leq \psi(M_{I,J}(x, y)) - \varphi(M_{I,J}(x, y)).$$

References

[1] I. Beg, M. Abbas, Coincidence point and invariant approximation for mappings satisfying generalized weak contractive condition, Fixed Point Theory Appl. **2006**, Article ID 74503 (2006).

[2] G.A. Anastassiou, I.K. Argyros, Approximating fixed points with applications in fractional calculus, J. Comput. Anal. Appl. **21** (2016), 1225–1242.

[3] S. Banach, Sur les oprations dans les ensembles abstraits et leur application aux quations intgrales, Fund. Math. **3** (1922), 133–181.

[4] A. Batool, T. Kamran, S. Jang, C. Park, Generalized φ-weak contractive fuzzy mappings and related fixed point results on complete metric space, J. Comput. Anal. Appl. **21** (2016), 729–737.

[5] V. Berinde, Approximating fixed points of weak φ-contractions, Fixed Point Theory **4** (2003), 131–142.

[6] C. E. Chidume, H. Zegeye, S. J. Aneke, Approximation of fixed points of weakly contractive nonself maps in Banach spaces, J. Math. Anal. Appl. **270** (2002), 189–199.

[7] P. N. Dutta, B. S. Choudhury, A generalization of contraction principle in metric spaces, Fixed Point Theory Appl. **2008**, Article ID 406368 (2008).

[8] G. Jungck, B. E. Rhoades, Fixed points for set valued functions without continuity, Indian J. Pure Appl. Math. **29** (1998), 227–238.

[9] J. H. Mai, X. H. Liu, Fixed points of weakly contractive maps and boundedness of orbits, Fixed Point Theory Appl. **2007**, Article ID 20962 (2007).

[10] S. Moradi, Z. Fathi, E. Analouee, The common fixed point of single-valued generalized φ_f-weakly contractive mappings, Appl. Math. Lett. **24** (2011), 771–776.

[11] B.E. Rhoades, Some theorems on weakly contractive maps, Nonlinear Anal. **47** (2001), 2683–2693.

[12] Q. Zhang, Y. Song, Y Fixed point theory for generalized φ-weak contractions, Appl. Math. Lett. **22** (2009), 75–78.

Analysis of latent CHIKV dynamics model with time delays

Ahmed. M. Elaiw, Taofeek O. Alade and Saud M. Alsulami
Department of Mathematics, Faculty of Science, King Abdulaziz University,
P.O. Box 80203, Jeddah 21589, Saudi Arabia.
Email: a_m_elaiw@yahoo.com (A. Elaiw)

Abstract

This paper proposes a latent Chikungunya viral infection model with saturated incidence rate. To take into account the time lag between the initial viral contacts uninfected monocytes and the production of new active CHIKV particles the model is incorporated by intracellular discrete or distributed time delays. We study the qualitative behavior of the model. Using the method of Lyapunov function, we established the global stability of the steady states of the model. The effect of the time delay on the stability of the steady states has also been shown by numerical simulations.

Keywords: Chikungunya virus infection; Latency; Time delay; Global stability; Lyapunov function.

1 Introduction

Mathematical analysis of viral infection models plays a substantial role in understanding the dynamics of human viruses (such as HIV, HCV, HBV, HTLV and Chikungunya virus). The models have been developed to mainly describe the relation among virus particles, uninfected target cells and infected cells [1]-[15]. The effect of Cytotoxic T Lymphocytes (CTL) immune response or humoral immune response has also been modeled (see e.g. [10]-[15]. Two main classes of mathematical models of viral infection have been proposed in the literature. The first class of models are given by ordinary differential equations. The second class of models is given by delay differential equations which incorporate the time lag between the initial viral contacts a target cell and the production of new active viruses. Modeling the virus dynamics with two types of infected cells, latently infected cells and actively infected cells has been studied by several researchers (see e.g. [2] and [14]). The latent viral infection model has been formulated as [2]:

$$\dot{S}(t) = \mu - aS(t) - bV(t)S(t), \tag{1}$$

$$\dot{L}(t) = (1-\rho)bV(t)S(t) - (\theta+\lambda)L(t), \tag{2}$$

$$\dot{I}(t) = \rho bV(t)S(t) + \lambda L(t) - \epsilon I(t), \tag{3}$$

$$\dot{V}(t) = mI(t) - rV(t), \tag{4}$$

where, S, L, I and V are the concentrations of uninfected cells, latently infected cells, actively infected cells and free virus particles. Parameters a and μ represent the death rate and birth rate constants of the uninfected cells, respectively. The uninfected cells become infected at rate bSV, where b is a constant. The parameters θ, ϵ and r denote the death rate constants of the latently infected cells, actively infected cells and free virus particles, respectively. An actively infected cell produces an average number m of virus particles. The parameter λ is the latent to active transmission rate constant. A fraction $(1-\rho)$ of infected cells is assumed to be latently infected cells and the remaining ρ becomes actively infected cells, where $0 < \rho < 1$.

Chikungunya virus (CHIKV) is an alphavirus and is transmitted to humans by Aedes aegypti and Aedes albopictus mosquitos. In the CHIKV literature, most of the mathematical models have been presented to describe the disease transmission in mosquito and human populations (see e.g. [17]-[22]). However, only few works have devoted for mathematical modeling of the dynamics of the CHIKV within host. In 2017, Wang and Liu [16] have presented a mathematical model for in host CHIKV infection model without considering the latent infection.

The objective of this paper is to propose a CHIKV infection model which improves the model presented in [16] by taking into account (i) two types of infected monocytes, latently infected monocytes and actively infected monocytes, (ii) two types of discrete or distributed time delays (iii) saturated incidence rate which is suitable to model the nonlinear dynamics of the CHIKV especially when its concentration is high. We investigate the nonnegativity and boundedness of the solutions of the CHIKV dynamics model. We show that the CHIKV dynamics is governed by one bifurcation parameter (the basic reproduction numbers $\mathcal{R}_0$). We use Lyapunov direct method to establish the global stability of the model's equilibria.

2 CHIKV model with discrete time delays

We consider a within-host CHIKV dynamics model with latently infected monocytes taking into account two discrete time delays.

$$\dot{S}(t) = \mu - aS(t) - \frac{bV(t)S(t)}{1+\pi V(t)}, \tag{5}$$

$$\dot{L}(t) = \frac{(1-\rho)e^{-\delta_1\tau_1}bV(t-\tau_1)S(t-\tau_1)}{1+\pi V(t-\tau_1)} - (\theta+\lambda)L(t), \tag{6}$$

$$\dot{I}(t) = \frac{\rho e^{-\delta_2\tau_2}bV(t-\tau_2)S(t-\tau_2)}{1+\pi V(t-\tau_2)} + \lambda L(t) - \epsilon I(t), \tag{7}$$

$$\dot{V}(t) = mI(t) - rV(t) - qB(t)V(t), \tag{8}$$

$$\dot{B}(t) = \eta + cB(t)V(t) - \delta B(t), \tag{9}$$

where, S, L, I, V, and B are the concentrations of uninfected monocytes, latently infected monocytes, actively infected monocytes CHIKV particles and B cells, respectively. The CHIKV particles are attacked by the B cells at rate qVB. The B cells are produced at constant rate η, proliferated at rate cBV and die at rate δB. τ_1 denotes the time between the CHIKV contacts the uninfected monocytes and latent infection, while τ_2 denotes the time between monocytes infection and the production of active CHIKV particles. The probability of latently and actively infected monocytes surviving to the age of τ_1 and τ_2 are represented by $e^{-\delta_1\tau_1}$ and $e^{-\delta_2\tau_2}$, respectively, where δ_1 and δ_2 are. We consider the following initial conditions:

$$\begin{aligned} &S(\vartheta) = \varphi_1(\vartheta),\ L(\vartheta) = \varphi_2(\vartheta),\ I(\vartheta) = \varphi_3(\vartheta),\ V(\vartheta) = \varphi_4(\vartheta),\ B(\vartheta) = \varphi_5(\vartheta), \\ &\varphi_i(\vartheta) \geq 0, \vartheta \in [-\tau, 0] \text{ and } \varphi_i \in C\left([-\tau,0], \mathbb{R}_{\geq 0}\right), \quad i = 1,2,...,5, \end{aligned} \tag{10}$$

where $\tau = \max\{\tau_1, \tau_2\}$ and C is the Banach space of continuous functions mapping the interval $[-\tau, 0]$ into $\mathbb{R}_{\geq 0}$ with norm $\|\varphi_j\| = \sup_{-\tau\leq\vartheta\leq 0} |\varphi_j(\vartheta)|$. Then the uniqueness of the solution for $t > 0$ is guaranteed [23].

2.1 Preliminaries

In this subsection we show the nonnegativity and boundedness of solutions as well as the existence of the steady states of system (5)-(9).

Lemma 1. The solutions of system (5)-(9) with the initial states (10) are nonnegative and ultimately bounded.

Proof. From Eqs. (5) and (9) we have $\dot{S}\Big|_{S=0} = \mu > 0$ and $\dot{B}\Big|_{B=0} = \eta > 0$. Thus, $S(t) > 0$ and $B(t) > 0$ for all $t \geq 0$. Moreover, for $t \in [0, \tau]$ we have

$$\begin{aligned}
L(t) &= \varphi_2(0)e^{-(\theta+\lambda)t} + \int_0^t \left(\frac{(1-\rho)e^{-\delta_1\tau_1} bS(\omega-\tau_1)V(\omega-\tau_1)}{1+\pi V(\omega-\tau_1)} \right) e^{-(\theta+\lambda)(t-\omega)} d\omega \geq 0, \\
I(t) &= \varphi_3(0)e^{-\epsilon t} + \int_0^t \left(\frac{\rho e^{-\delta_2\tau_2} bS(\omega-\tau_2)V(\omega-\tau_2)}{1+\pi V(\omega-\tau_2)} + \lambda L(\omega) \right) e^{-\epsilon(t-\omega)} d\omega \geq 0, \\
V(t) &= \varphi_4(0)e^{-\int_0^t (c+qB(u))u} + \int_0^t mI(\omega)e^{-\int_\omega^t (c+qB(u))du} d\omega \geq 0.
\end{aligned}$$

By recursive argument, we get $L(t) \geq 0, I(t) \geq 0$ and $V(t) \geq 0$ for all $t \geq 0$.

Next, we establish the boundedness of the model's solutions. The nonnegativity of the model's solution implies that $\frac{dS(t)}{dt} \leq \mu - aS(t)$, which yields $\lim\limits_{t\to\infty} \sup S(t) \leq \frac{\mu}{a}$. Let us define

$$X_1(t) = (1-\rho)e^{-\delta_1\tau_1} S(t-\tau_1) + L(t),$$

then

$$\begin{aligned}
\dot{X}_1(t) &= (1-\rho)e^{-\delta_1\tau_1} \left(\mu - aS(t-\tau_1) - \frac{bV(t-\tau_1)S(t-\tau_1)}{1+\pi V(t-\tau_1)} \right) + (1-\rho)e^{-\delta_1\tau_1} \frac{bV(t-\tau_1)S(t-\tau_1)}{1+\pi V(t-\tau_1)} - (\theta+\lambda)L(t) \\
&\leq \mu(1-\rho)e^{-\delta_1\tau_1} - \sigma_1 \left((1-\rho)e^{-\delta_1\tau_1} S(t-\tau_1) + L(t) \right) \leq \mu(1-\rho) - \sigma_1 X_1(t),
\end{aligned}$$

where $\sigma_1 = \min\{a, \theta+\lambda\}$. Then, $\lim\limits_{t\to\infty} \sup X_1(t) \leq M_1$, and $\lim\limits_{t\to\infty} \sup L(t) \leq M_1$, where $M_1 = \frac{\mu(1-\rho)}{\sigma_1}$. Let

$$X_2(t) = \rho e^{-\delta_2\tau_2} S(t-\tau_2) + I(t) + \frac{\epsilon}{2m} V(t) + \frac{\epsilon q}{2mc} B(t),$$

then we get

$$\begin{aligned}
\dot{X}_2(t) &= \rho e^{-\delta_2\tau_2} \left(\mu - aS(t-\tau_2) - \frac{bV(t-\tau_2)S(t-\tau_2)}{1+\pi V(t-\tau_2)} \right) + \rho e^{-\delta_2\tau_2} \frac{bV(t-\tau_2)S(t-\tau_2)}{1+\pi V(t-\tau_2)} + \lambda L(t) - \epsilon I(t) \\
&\quad + \frac{\epsilon}{2m} \left(mI(t) - rV(t) - qV(t)B(t) \right) + \frac{\epsilon q}{2mc} \left(\eta + cB(t)V(t) - \delta B(t) \right) \\
&= \rho\mu e^{-\delta_2\tau_2} - \rho e^{-\delta_2\tau_2} aS(t-\tau_2) + \lambda L(t) - \frac{\epsilon}{2} I(t) + \frac{\epsilon q\eta}{2mc} - \frac{\epsilon r}{2m} V(t) - \frac{\epsilon q\delta}{2mc} B(t) \\
&\leq \rho\mu + \lambda M_1 + \frac{\epsilon q\eta}{2mc} - \sigma_2 \left(\rho e^{-\delta_2\tau_2} S(t-\tau_2) + I(t) + \frac{\epsilon}{2m} V(t) + \frac{\epsilon q}{2mc} B(t) \right) \\
&= \rho\mu + \lambda M_1 + \frac{\epsilon q\eta}{2mc} - \sigma_2 X_2(t),
\end{aligned}$$

where $\sigma_2 = \min\{a, \frac{\epsilon}{2}, r, \delta\}$. It follows that $\lim\limits_{t\to\infty} \sup I(t) \leq M_2$, $\lim\limits_{t\to\infty} \sup V(t) \leq M_3$ and $\lim\limits_{t\to\infty} \sup B(t) \leq M_4$, where $M_2 = \frac{\rho\mu+\lambda M_1}{\sigma_2} + \frac{\epsilon q\eta}{2mc\sigma_2}$, $M_3 = \frac{2m}{\epsilon}$ and $M_4 = \frac{2mc}{\epsilon q}$. This shows the ultimate boundedness of $S(t), L(t), I(t), V(t)$ and $B(t)$. $\square$

Lemma 2. For system (5)-(9) there exists a threshold parameter $\mathcal{R}_0 > 0$, such that

(i) if $\mathcal{R}_0 \leq 1$, then there exists only one positive steady state, virus-free steady state Q_0.

(i) if $\mathcal{R}_0 > 1$, then in addition to Q_0, there exists an endemic steady state Q_1

Proof.

To calculate the steady states we let the R.H.S of system (5)-(9) be equal zero

$$0=\mu-aS-\frac{bVS}{1+\pi V}, \tag{11}$$
$$0=(1-\rho)\frac{e^{-\delta_1\tau_1}bVS}{1+\pi V}-(\theta+\lambda)L, \tag{12}$$
$$0=\frac{\rho e^{-\delta_2\tau_2}bVS}{1+\pi V}+\lambda L-\epsilon I, \tag{13}$$
$$0=mI-rV-qVB, \tag{14}$$
$$0=\eta+cBV-\delta B. \tag{15}$$

From Eqs. (11)-(15) we obtain

$$S=\frac{\mu\left(1+\pi V\right)}{bV+a\left(1+\pi V\right)},\ L=\frac{(1-\rho)e^{-\delta_1\tau_1}bVS}{\left(1+\pi V\right)\left(\theta+\lambda\right)},\ I=\frac{b\beta VS}{\epsilon\left(1+\pi V\right)\left(\theta+\lambda\right)},\ B=\frac{\eta}{\delta-cV}. \tag{16}$$

where $\beta=\lambda(1-\rho)e^{-\delta_1\tau_1}+\rho e^{-\delta_2\tau_2}(\theta+\lambda)$. Substituting Eq. (16) into Eq. (14) we have

$$\left[\frac{m\mu b\beta}{\epsilon\left(\theta+\lambda\right)\left(bV+a\left(1+\pi V\right)\right)}-r-\frac{q\eta}{\delta-cV}\right]V=0. \tag{17}$$

Equation (17) has two possibilities:
(i) $V=0$ which gives the virus-free steady state $Q_0=(S_0,L_0,I_0,V_0,B_0)=(\frac{\mu}{a},0,0,0,\frac{\eta}{\delta})$,
(ii) $V\neq 0$ which gives

$$\frac{m\mu b\beta}{\epsilon\left(\theta+\lambda\right)\left(bV+a\left(1+\pi V\right)\right)}-r-\frac{q\eta}{\delta-cV}=0. \tag{18}$$

Equation (18) takes the form $P_1V^2-P_2V+P_3=0$, where

$$\begin{aligned}
P_1&=r\epsilon c(\theta+\lambda)(b+\pi a),\\
P_2&=-r\epsilon ca(\theta+\lambda)+m\mu bc(\beta)+\epsilon\left(r\delta+q\eta\right)\left(\theta+\lambda\right)(b+\pi a),\\
P_3&=m\rho b\mu\delta(\theta+\lambda)e^{-\delta_2\tau_2}+m\mu b\lambda\delta(1-\rho)e^{-\delta_1\tau_1}-\epsilon a\left(r\delta+q\eta\right)\left(\theta+\lambda\right).
\end{aligned}$$

The constants P_1, P_2 and P_3 can be rewritten as

$$\begin{aligned}
P_1&=r\epsilon c(\theta+\lambda)(b+\pi a),\\
P_2&=\frac{\epsilon ca\left(r\delta+q\eta\right)\left(\theta+\lambda\right)}{\delta}(\mathcal{R}_0-1)+\epsilon\left(r\delta+q\eta\right)\left(\theta+\lambda\right)(b+\pi a)+\frac{caq\epsilon\eta(\theta+\lambda)}{\delta},\\
P_3&=\epsilon a\left(r\delta+q\eta\right)\left(\theta+\lambda\right)(\mathcal{R}_0-1),
\end{aligned}$$

where

$$\mathcal{R}_0=\frac{m\mu b\delta\beta}{\epsilon a(r\delta+q\eta)(\theta+\lambda)}.$$

Let

$$\Theta_1(V)=P_1V^2-P_2V+P_3=0. \tag{19}$$

If $\mathcal{R}_0>1$, then $P_2>0$ and $P_3>0$. We have $\Theta_1(0)=P_3>0$, $\Theta_1\left(\frac{\delta}{c}\right)=-\frac{q\epsilon\eta(\theta+\lambda)(ca+\delta(b+\pi a))}{c}<0$, and $\Theta_1'(0)=-P_2<0$. Then, Eq. (19) has two positive roots

$$V_1=\frac{P_2-\sqrt{P_2^2-4P_1P_3}}{2P_1}<\frac{\delta}{c}\quad\text{and}\quad V_2=\frac{P_2+\sqrt{P_2^2-4P_1P_3}}{2P_1}>\frac{\delta}{c}.$$

If $V = V_2$, then from Eq. (16) we get $B_2 = \frac{\eta}{\delta - cV_2} < 0$. Thus, when $\mathcal{R}_0 > 1$, a positive endemic steady state $Q_1 = (S_1, L_1, I_1, V_1, B_1)$ will appear, where

$$S_1 = \frac{\mu\,(1+\pi V_1)}{bV_1 + a\,(1+\pi V_1)},\ L_1 = \frac{b\mu V_1(1-\rho)e^{-\delta_1\tau_1}}{(\theta+\lambda)(bV_1 + a\,(1+\pi V_1))},\ I_1 = \frac{b\mu\beta V_1}{\epsilon(\theta+\lambda)(bV_1 + a\,(1+\pi V_1))},$$
$$V_1 = \frac{\rho_2 - \sqrt{\rho_2^2 - 4\rho_1\rho_3}}{2\rho_1},\ B_1 = \frac{\eta}{\delta - cV_1}.$$

The parameter $\mathcal{R}_0$ represents the basic reproduction number. □

2.2 Global stability

We define $H(x) = x - \ln x - 1$. Clearly, $H(1) = 0$ and $H(u) \geq 0$ for $u > 0$. Denote $(S, L, I, V, B) = (S(t), L(t), I(t), V(t), B(t))$.

Theorem 1. Suppose that $\mathcal{R}_0 \leq 1$, then Q_0 is globally asymptotically stable (GAS).

Proof. We define a Lyapunov functional Y_0 as:

$$\begin{aligned} Y_0(S,L,I,V,B) &= \left(\frac{\beta}{\theta+\lambda}\right) S_0 H\left(\frac{S}{S_0}\right) + \frac{\lambda}{\theta+\lambda} L + I + \frac{\epsilon}{m} V + \frac{\epsilon q}{mc} B_0 H\left(\frac{B}{B_0}\right) \\ &+ \frac{\lambda(1-\rho)e^{-\delta_1\tau_1}}{\theta+\lambda} \int_0^{\tau_1} \frac{bV(t-\vartheta)S(t-\vartheta)}{1+\pi V(t-\vartheta)} d\vartheta + \rho e^{-\delta_2\tau_2} \int_0^{\tau_2} \frac{bV(t-\vartheta)S(t-\vartheta)}{1+\pi V(t-\vartheta)} d\vartheta. \end{aligned} \tag{20}$$

Note that, $Y_0(S,L,I,V,B) > 0$ for all $S,L,I,V,B > 0$ and $Y_0(S_0,0,0,0,B_0) = 0$. Calculating $\frac{dY_0}{dt}$ along the trajectories of (5)-(9) we get

$$\begin{aligned} \frac{dY_0}{dt} &= \frac{\beta}{\theta+\lambda}\left(1 - \frac{S_0}{S}\right)\left(\mu - aS - \frac{bVS}{1+\pi V}\right) \\ &+ \frac{\lambda}{\theta+\lambda}\left(\frac{(1-\rho)e^{-\delta_1\tau_1} bV(t-\tau_1)S(t-\tau_1)}{1+\pi V(t-\tau_1)} - (\theta+\lambda)L\right) + \frac{\rho e^{-\delta_2\tau_2} bV(t-\tau_2)S(t-\tau_2)}{1+\pi V(t-\tau_2)} \\ &+ \lambda L - \epsilon I + \frac{\epsilon}{m}(mI - rV - qVB) + \frac{\epsilon q}{mc}\left(1 - \frac{B_0}{B}\right)(\eta + cBV - \delta B) \\ &+ \frac{\lambda(1-\rho)e^{-\delta_1\tau_1}}{\theta+\lambda}\left(\frac{bVS}{1+\pi V} - \frac{bV(t-\tau_1)S(t-\tau_1)}{1+\pi V(t-\tau_1)}\right) + \rho e^{-\delta_2\tau_2}\left(\frac{bVS}{1+\pi V} - \frac{bV(t-\tau_2)S(t-\tau_2)}{1+\pi V(t-\tau_2)}\right) \\ &= -\frac{a\beta}{\theta+\lambda}\frac{(S-S_0)^2}{S} + \frac{\beta}{\theta+\lambda}\frac{bS_0V}{1+\pi V} - \frac{\epsilon rV}{m} - \frac{\epsilon q B_0 V}{m} + \frac{\epsilon q}{mc}\left(1 - \frac{B_0}{B}\right)(\delta B_0 - \delta B) \\ &= -\frac{a\beta}{\theta+\lambda}\frac{(S-S_0)^2}{S} - \frac{\epsilon q\delta}{mc}\frac{(B-B_0)^2}{B} + \frac{\epsilon(r\delta+q\eta)}{m\delta}\left(\frac{m\mu b\delta\beta}{\epsilon a(r\delta+q\eta)(\theta+\lambda)(1+\pi V)} - 1\right)V \\ &= -\frac{a\beta}{\theta+\lambda}\frac{(S-S_0)^2}{S} - \frac{\epsilon q\delta}{mc}\frac{(B-B_0)^2}{B} + \frac{\epsilon(r\delta+q\eta)}{m\delta}(\mathcal{R}_0 - 1)V - \frac{\epsilon(r\delta+q\eta)\mathcal{R}_0\pi V^2}{m\delta(1+\pi V)}. \end{aligned} \tag{21}$$

Therefore, $\frac{dY_0}{dt} \leq 0$ holds if $\mathcal{R}_0 \leq 1$. Further, $\frac{dY_0}{dt} = 0$ if and only if $S = S_0$, $B = B_0$ and $V = 0$. By LaSalle's invariance principle, Q_0 is GAS. □

In the next theorem we show the global stability of Q_1.

Theorem 2. Suppose that $\mathcal{R}_0 > 1$, then Q_1 is GAS.

Proof. Consider

$$\begin{aligned}Y_1(S,L,I,V,B) &= \frac{\beta}{\theta+\lambda}S_1 H\left(\frac{S}{S_1}\right)+\frac{\lambda}{\theta+\lambda}L_1 H\left(\frac{L}{L_1}\right)+I_1 H\left(\frac{I}{I_1}\right)+\frac{\epsilon}{m}V_1 H\left(\frac{V}{V_1}\right)+\frac{\epsilon q}{mc}B_1 H\left(\frac{B}{B_1}\right)\\ &+\frac{\lambda(1-\rho)e^{-\delta_1\tau_1}}{\theta+\lambda}\frac{bS_1V_1}{1+\pi V_1}\int_0^{\tau_1} H\left(\frac{V(t-\vartheta)S(t-\vartheta)(1+\pi V_1)}{S_1V_1\left(1+\pi V(t-\vartheta)\right)}\right)d\vartheta\\ &+\rho e^{-\delta_2\tau_2}\frac{bS_1V_1}{1+\pi V_1}\int_0^{\tau_2} H\left(\frac{V(t-\vartheta)S(t-\vartheta)(1+\pi V_1)}{S_1V_1\left(1+\pi V(t-\vartheta)\right)}\right)d\vartheta.\end{aligned}$$

We have $Y_1(S,L,I,V,B)>0$ for all $S,L,I,V,B>0$ and $Y_1(S_1,L_1,I_1,V_1,B_1)=0$. Calculating $\frac{dY_1}{dt}$ along the trajectories of (5)-(9) we get

$$\begin{aligned}\frac{dY_1}{dt} &= \frac{\beta}{\theta+\lambda}\left(1-\frac{S_1}{S}\right)\left(\mu-aS-\frac{bVS}{1+\pi V}\right)\\ &+\frac{\lambda}{\theta+\lambda}\left(1-\frac{L_1}{L}\right)\left(\frac{(1-\rho)e^{-\delta_1\tau_1}bV(t-\tau_1)S(t-\tau_1)}{1+\pi V(t-\tau_1)}-(\theta+\lambda)L\right)\\ &+\left(1-\frac{I_1}{I}\right)\left(\frac{\rho e^{-\delta_2\tau_2}bV(t-\tau_2)S(t-\tau_2)}{1+\pi V(t-\tau_2)}+\lambda L-\epsilon I\right)+\frac{\epsilon}{m}\left(1-\frac{V_1}{V}\right)(mI-rV-qVB)\\ &+\frac{\epsilon q}{mc}\left(1-\frac{B_1}{B}\right)(\eta+cBV-\delta B)+\frac{\lambda(1-\rho)e^{-\delta_1\tau_1}}{\theta+\lambda}\left(\frac{bVS}{1+\pi V}-\frac{bV(t-\tau_1)S(t-\tau_1)}{1+\pi V(t-\tau_1)}\right)\\ &+\frac{\lambda(1-\rho)e^{-\delta_1\tau_1}}{\theta+\lambda}\frac{bS_1V_1}{1+\pi V_1}\ln\left(\frac{V(t-\tau_1)S(t-\tau_1)(1+\pi V)}{VS\left(1+\pi V(t-\tau_1)\right)}\right)+\rho e^{-\delta_2\tau_2}\left(\frac{bVS}{1+\pi V}-\frac{bV(t-\tau_2)S(t-\tau_2)}{1+\pi V(t-\tau_2)}\right)\\ &+\rho e^{-\delta_2\tau_2}\frac{bS_1V_1}{1+\pi V_1}\ln\left(\frac{V(t-\tau_2)S(t-\tau_2)(1+\pi V)}{VS\left(1+\pi V(t-\tau_2)\right)}\right).\end{aligned} \tag{22}$$

Applying

$$\mu=aS_1+\frac{bS_1V_1}{1+\pi V_1},\ \eta=\delta B_1-cB_1V_1,$$

we obtain

$$\begin{aligned}\frac{dY_1}{dt} &= \frac{\beta}{\theta+\lambda}\left(1-\frac{S_1}{S}\right)(aS_1-aS)+\frac{\beta}{\theta+\lambda}\frac{bS_1V_1}{1+\pi V_1}\left(1-\frac{S_1}{S}\right)+\frac{\beta}{\theta+\lambda}\frac{bS_1V}{1+\pi V}\\ &-\frac{\lambda(1-\rho)e^{-\delta_1\tau_1}}{\theta+\lambda}\frac{bV(t-\tau_1)S(t-\tau_1)L_1}{1+\pi V(t-\tau_1)L}+\lambda L_1-\rho e^{-\delta_2\tau_2}\frac{bV(t-\tau_2)S(t-\tau_2)I_1}{1+\pi V(t-\tau_2)I}-\frac{\lambda LI_1}{I}+\epsilon I_1-\frac{\epsilon IV_1}{V}\\ &-\frac{r\epsilon V}{m}+\frac{r\epsilon V_1}{m}+\frac{\epsilon qBV_1}{m}+\frac{\epsilon q}{mc}\left(1-\frac{B_1}{B}\right)(\delta B_1-\delta B)-\frac{\epsilon qB_1V}{m}-\frac{\epsilon qB_1V_1}{m}+\frac{\epsilon qB_1V_1}{m}\left(\frac{B_1}{B}\right)\\ &+\frac{bS_1V_1}{1+\pi V_1}\left[\frac{\lambda(1-\rho)e^{-\delta_1\tau_1}}{\theta+\lambda}\ln\left(\frac{V(t-\tau_1)S(t-\tau_1)(1+\pi V)}{VS\left(1+\pi V(t-\tau_1)\right)}\right)+\rho e^{-\delta_2\tau_2}\ln\left(\frac{V(t-\tau_2)S(t-\tau_2)(1+\pi V)}{VS\left(1+\pi V(t-\tau_2)\right)}\right)\right].\end{aligned}$$

Using the equilibrium conditions for Q_1:

$$(1-\rho)e^{-\delta_1\tau_1}\frac{bS_1V_1}{1+\pi V_1}=(\theta+\lambda)L_1,\ \rho e^{-\delta_1\tau_1}\frac{bS_1V_1}{1+\pi V_1}+\lambda L_1=\epsilon I_1,\ mI_1=rV_1+qB_1V_1,$$

we get

$$\epsilon I_1=\frac{\beta}{\theta+\lambda}\frac{bS_1V_1}{(1+\pi V_1)},\quad \frac{r\epsilon V_1}{m}=\frac{\beta}{\theta+\lambda}\frac{bS_1V_1}{(1+\pi V_1)}-\frac{\epsilon qB_1V_1}{m},$$

and

$$\begin{aligned}
\frac{dY_1}{dt} &= -\frac{a\beta}{\theta+\lambda}\frac{(S-S_1)^2}{S} + \frac{\lambda(1-\rho)e^{-\delta_1\tau_1}}{\theta+\lambda}\frac{bS_1V_1}{(1+\pi V_1)}\left(1-\frac{S_1}{S}\right)\\
&+ \rho e^{-\delta_2\tau_2}\frac{bS_1V_1}{1+\pi V_1}\left(1-\frac{S_1}{S}\right) + \left(\frac{\beta}{\theta+\lambda}\right)\frac{bS_1V_1}{1+\pi V_1}\left(\frac{(1+\pi V_1)V}{(1+\pi V)V_1} - \frac{V}{V_1}\right)\\
&- \frac{\lambda(1-\rho)e^{-\delta_1\tau_1}}{\theta+\lambda}\frac{bS_1V_1}{1+\pi V_1}\frac{V(t-\tau_1)S(t-\tau_1)(1+\pi V_1)L_1}{(1+\pi V(t-\tau_1))S_1V_1L} + \frac{\lambda(1-\rho)e^{-\delta_1\tau_1}}{\theta+\lambda}\frac{bS_1V_1}{(1+\pi V_1)}\\
&- \rho e^{-\delta_2\tau_2}\frac{bS_1V_1}{1+\pi V_1}\frac{V(t-\tau_2)S(t-\tau_2)(1+\pi V_1)I_1}{(1+\pi V(t-\tau_2))S_1V_1I} - \frac{\lambda(1-\rho)e^{-\delta_1\tau_1}}{\theta+\lambda}\frac{bS_1V_1}{1+\pi V_1}\frac{I_1L}{L_1I}\\
&+ \frac{\lambda(1-\rho)}{\theta+\lambda}e^{-\delta_1\tau_1}\frac{bS_1V_1}{(1+\pi V_1)} + \rho e^{-\delta_2\tau_2}\frac{bS_1V_1}{(1+\pi V_1)} - \frac{\lambda(1-\rho)e^{-\delta_1\tau_1}}{\theta+\lambda}\frac{bS_1V_1}{1+\pi V_1}\frac{IV_1}{I_1V}\\
&- \rho e^{-\delta_2\tau_2}\frac{bS_1V_1}{1+\pi V_1}\frac{IV_1}{I_1V} + \frac{\lambda(1-\rho)e^{-\delta_1\tau_1}}{\theta+\lambda}\frac{bS_1V_1}{(1+\pi V_1)} + \rho e^{-\delta_2\tau_2}\frac{bS_1V_1}{(1+\pi V_1)}\\
&- \frac{2\epsilon qB_1V_1}{m} + \frac{\epsilon qBV_1}{m} + \frac{\epsilon qB_1V_1}{m}\left(\frac{B_1}{B}\right) - \frac{\epsilon q\delta}{mc}\frac{(B-B_1)^2}{B}\\
&+ \frac{bS_1V_1}{1+\pi V_1}\left[\frac{\lambda(1-\rho)e^{-\delta_1\tau_1}}{\theta+\lambda}\ln\left(\frac{V(t-\tau_1)S(t-\tau_1)(1+\pi V)}{VS\,(1+\pi V(t-\tau_1))}\right) + \rho e^{-\delta_2\tau_2}\ln\left(\frac{V(t-\tau_2)S(t-\tau_2)(1+\pi V)}{VS\,(1+\pi V(t-\tau_2))}\right)\right].
\end{aligned}$$

Using the following equalities:

$$\begin{aligned}
\ln\left(\frac{V(t-\tau_1)S(t-\tau_1)(1+\pi V)}{VS\,(1+\pi V(t-\tau_1))}\right) &= \ln\left(\frac{S_1}{S}\right) + \ln\left(\frac{IV_1}{I_1V}\right) + \ln\left(\frac{LI_1}{L_1I}\right) + \ln\left(\frac{1+\pi V}{1+\pi V_1}\right)\\
&+ \ln\left(\frac{V(t-\tau_1)S(t-\tau_1)(1+\pi V_1)L_1}{(1+\pi V(t-\tau_1))S_1V_1L}\right),\\
\ln\left(\frac{V(t-\tau_2)S(t-\tau_2)(1+\pi V)}{VS\,(1+\pi V(t-\tau_2))}\right) &= \ln\left(\frac{S_1}{S}\right) + \ln\left(\frac{IV_1}{I_1V}\right) + \ln\left(\frac{1+\pi V}{1+\pi V_1}\right)\\
&+ \ln\left(\frac{V(t-\tau_2)S(t-\tau_2)(1+\pi V_1)I_1}{(1+\pi V(t-\tau_2))S_1V_1I}\right),
\end{aligned}$$

we get

$$\begin{aligned}
\frac{dY_1}{dt} &= -\frac{a\beta}{\theta+\lambda}\frac{(S-S_1)^2}{S} + \frac{\beta}{\theta+\lambda}\frac{bS_1V_1}{1+\pi V_1}\left(-1+\frac{(1+\pi V_1)V}{(1+\pi V)V_1} - \frac{V}{V_1} + \frac{1+\pi V}{1+\pi V_1}\right)\\
&+ \frac{\lambda(1-\rho)e^{-\delta_1\tau_1}}{\theta+\lambda}\frac{bS_1V_1}{(1+\pi V_1)}\left[1-\frac{S_1}{S}+\ln\left(\frac{S_1}{S}\right)\right] + \frac{\lambda(1-\rho)e^{-\delta_1\tau_1}}{\theta+\lambda}\frac{bS_1V_1}{(1+\pi V_1)}\left[1-\frac{IV_1}{I_1V}+\ln\left(\frac{IV_1}{I_1V}\right)\right]\\
&+ \frac{\lambda(1-\rho)e^{-\delta_1\tau_1}}{\theta+\lambda}\frac{bS_1V_1}{(1+\pi V_1)}\left[1-\frac{V(t-\tau_1)S(t-\tau_1)(1+\pi V_1)L_1}{(1+\pi V(t-\tau_1))S_1V_1L}+\ln\left(\frac{V(t-\tau_1)S(t-\tau_1)(1+\pi V_1)L_1}{(1+\pi V(t-\tau_1))S_1V_1L}\right)\right]\\
&+ \frac{\lambda(1-\rho)e^{-\delta_1\tau_1}}{\theta+\lambda}\frac{bS_1V_1}{(1+\pi V_1)}\left[1-\frac{1+\pi V}{1+\pi V_1}+\ln\left(\frac{1+\pi V}{1+\pi V_1}\right)\right]\\
&+ \frac{\lambda(1-\rho)e^{-\delta_1\tau_1}}{\theta+\lambda}\frac{bS_1V_1}{(1+\pi V_1)}\left[1-\frac{LI_1}{L_1I}+\ln\left(\frac{LI_1}{L_1I}\right)\right]\\
&+ \rho e^{-\delta_2\tau_2}\frac{bS_1V_1}{1+\pi V_1}\left[1-\frac{S_1}{S}+\ln\left(\frac{S_1}{S}\right)\right] + \rho e^{-\delta_2\tau_2}\frac{bS_1V_1}{(1+\pi V_1)}\left[1-\frac{IV_1}{I_1V}+\ln\left(\frac{IV_1}{I_1V}\right)\right]\\
&+ \rho e^{-\delta_2\tau_2}\frac{bS_1V_1}{1+\pi V_1}\left[1-\frac{V(t-\tau_2)S(t-\tau_2)(1+\pi V_1)I_1}{(1+\pi V(t-\tau_2))S_1V_1I}+\ln\left(\frac{V(t-\tau_2)S(t-\tau_2)(1+\pi V_1)I_1}{(1+\pi V(t-\tau_2))S_1V_1I}\right)\right]\\
&+ \rho e^{-\delta_2\tau_2}\frac{bS_1V_1}{1+\pi V_1}\left[1-\frac{1+\pi V}{1+\pi V_1}+\ln\left(\frac{1+\pi V}{1+\pi V_1}\right)\right] - \frac{\epsilon q\delta}{mc}\frac{(B-B_1)^2}{B} - \frac{\epsilon qB_1V_1}{m}\left[2-\frac{B}{B_1}-\frac{B_1}{B}\right]
\end{aligned}$$

$$
\begin{aligned}
&= -\frac{a\beta(S-S_1)^2}{(\theta+\lambda)S} - \frac{\epsilon q\eta}{mcB_1}\frac{(B-B_1)^2}{B} - \frac{\beta}{\theta+\lambda}\frac{\pi b S_1(V-V_1)^2}{(1+\pi V)(1+\pi V_1)^2} - \frac{\lambda(1-\rho)e^{-\delta_1\tau_1}}{\theta+\lambda}\frac{bS_1V_1}{(1+\pi V_1)}\left[H\left(\frac{S_1}{S}\right)\right.\\
&\left.+H\left(\frac{IV_1}{I_1V}\right)+H\left(\frac{1+\pi V}{1+\pi V_1}\right)+H\left(\frac{V(t-\tau_1)S(t-\tau_1)(1+\pi V_1)L_1}{(1+\pi V(t-\tau_1))S_1V_1L}\right)+H\left(\frac{LI_1}{L_1I}\right)\right]\\
&-\rho e^{-\delta_2\tau_2}\frac{bS_1V_1}{(1+\pi V_1)}\left[H\left(\frac{S_1}{S}\right)+H\left(\frac{IV_1}{I_1V}\right)+H\left(\frac{1+\pi V}{1+\pi V_1}\right)+H\left(\frac{V(t-\tau_2)S(t-\tau_2)(1+\pi V_1)I_1}{(1+\pi V(t-\tau_2))S_1V_1I}\right)\right]. \quad (23)
\end{aligned}
$$

It can be seen that if $\mathcal{R}_0 > 1$, then $\frac{dY_1}{dt} \le 0$ for all $S, L, I, V, B > 0$ and $\frac{dY_1}{dt} = 0$ if and only if $S = S_1$, $L = L_1$, $I = I_1$, $V = V_1$, and $B = B_1$. It follows from LaSalle's invariance principle that, Q_1 is GAS. □

3 CHIKV model with delay-distributed

We suggest a dynamical model for within-host CHIKV infection with latently infected monocytes taking into account the distributed delays.

$$
\begin{aligned}
\dot{S}(t) &= \mu - aS(t) - \frac{bV(t)S(t)}{1+\pi V(t)}, && (24)\\
\dot{L}(t) &= (1-\rho)b\int_0^{\kappa_1}\xi_1(\tau)e^{-\delta_1\tau}\frac{V(t-\tau)S(t-\tau)}{1+\pi V(t-\tau)}d\tau - (\theta+\lambda)L(t), && (25)\\
\dot{I}(t) &= \rho b\int_0^{\kappa_2}\xi_2(\tau)e^{-\delta_2\tau}\frac{V(t-\tau)S(t-\tau)}{1+\pi V(t-\tau)}d\tau + \lambda L(t) - \epsilon I(t), && (26)\\
\dot{V}(t) &= mI(t) - rV(t) - qV(t)B(t), && (27)\\
\dot{B}(t) &= \eta + cB(t)V(t) - \delta B(t). && (28)
\end{aligned}
$$

where, $\xi_1(\tau)$ and $\xi_2(\tau)$ are probability distribution functions which satisfy $\xi_1(\tau) > 0$ and $\xi_2(\tau) > 0$, and

$$
\int_0^{\kappa_1}\xi_1(\tau)d\tau = \int_0^{\kappa_2}\xi_2(\tau)d\tau = 1, \ \int_0^{\kappa_1}\xi_1(u)e^{nu}du < \infty, \quad \int_0^{\kappa_2}\xi_2(u)e^{nu}du < \infty, \tag{29}
$$

where n is a positive number. Let

$$
E = \int_0^{\kappa_1}\xi_1(\tau)e^{-\delta_1\tau}d\tau \text{ and } \quad K = \int_0^{\kappa_2}\xi_2(\tau)e^{-\delta_2\tau}d\tau
$$

Then $0 < E \le 1$, $0 < K \le 1$. The initial conditions for model (24)-(28) take the form

$$
\begin{aligned}
&S(\varphi) = \psi_1(\varphi), \ L(\varphi) = \psi_2(\varphi), \ I(\varphi) = \psi_3(\varphi),\\
&V(\varphi) = \psi_4(\varphi), \ B(\varphi) = \psi_5(\varphi),\\
&\psi_j(\varphi) \ge 0, \ \varphi \in [-\ell, 0], \ j = 1, ..., 5, && (30)
\end{aligned}
$$

where $\ell = \max\{\kappa_1, \kappa_2\}$, $\psi_j \in C([-\ell, 0], \mathbb{R}_{\ge 0})$. This guarantees the uniqueness of solution of the system [23].

3.1 Preliminaries

Lemma 3. The solutions of system (24)-(28) with the initial states (30) are nonnegative and ultimately bounded.

Proof. From Lemma 1 we have $S(t) > 0$ and $B(t) > 0$ for all $t \geq 0$. Moreover, one can show that for $t \geq 0$

$$\begin{aligned}
L(t) &= e^{-(\theta+\lambda)t}\psi_2(0) + (1-\rho)b\int_0^t e^{-(\theta+\lambda)(t-u)}\int_0^{\kappa_1} e^{-\delta_1\tau}\xi_1(\tau)\frac{S(u-\tau)V(u-\tau)}{1+\pi V(u-\tau)}d\tau du \geq 0,\\
I(t) &= e^{-\epsilon t}\psi_3(0) + \rho b\int_0^t e^{-\epsilon(t-u)}\left(\int_0^{\kappa_2} e^{-\delta_2\tau}\xi_2(\tau)\frac{S(u-\tau)V(u-\tau)}{1+\pi V(u-\tau)}d\tau + \lambda L(u)\right)du \geq 0,\\
V(t) &= e^{-\int\limits_0^t (c+qB(u))u}\psi_4(0) + \int_0^t mI(\omega)e^{-\int\limits_\omega^t (c+qB(u))du}d\omega \geq 0.
\end{aligned}$$

From (24), we have $\lim\limits_{t\to\infty}\sup S(t) \leq \frac{\mu}{a}$. Let $T_1(t) = (1-\rho)\int_0^{\kappa_1}\xi_1(\tau)e^{-\delta_1\tau}S(t-\tau)d\tau + L(t)$, then

$$\begin{aligned}
\dot{T}_1(t) &= (1-\rho)\int_0^{\kappa_1}\xi_1(\tau)e^{-\delta_1\tau}\left(\mu - aS(t-\tau) - \frac{bV(t-\tau)S(t-\tau)}{1+\pi V(t-\tau)}\right)d\tau\\
&+ (1-\rho)b\int_0^{\kappa_1}\xi_1(\tau)e^{-\delta_1\tau}\frac{V(t-\tau)S(t-\tau)}{1+\pi V(t-\tau)}d\tau - (\theta+\lambda)L(t)\\
&\leq \mu(1-\rho)E - \sigma_1\left((1-\rho)\int_0^{\kappa_1}\xi_1(\tau)e^{-\delta_1\tau}S(t-\tau)d\tau + L(t)\right)\\
&\leq \mu(1-\rho) - \sigma_1 T_1(t).
\end{aligned}$$

It follows that, $\lim\limits_{t\to\infty}\sup T_1(t) \leq M_1$. Since $\int_0^{\kappa_1}\xi_1(\tau)e^{-\delta_1\tau}S(t-\tau)d\tau > 0$, then $\lim\limits_{t\to\infty}\sup L(t) \leq M_1$. Let

$$T_2(t) = \rho\int_0^{\kappa_2}\xi_2(\tau)e^{-\delta_2\tau}S(t-\tau)d\tau + I(t) + \frac{\epsilon}{2m}V(t) + \frac{\epsilon q}{2mc}B(t),$$

then we have

$$\begin{aligned}
\dot{T}_2(t) &= \rho\int_0^{\kappa_2}\xi_2(\tau)e^{-\delta_2\tau}\left(\mu - aS(t-\tau) - \frac{bV(t-\tau)S(t-\tau)}{1+\pi V(t-\tau)}\right)d\tau\\
&+ \rho b\int_0^{\kappa_2}\xi_2(\tau)e^{-\delta_2\tau}\frac{V(t-\tau)S(t-\tau)}{1+\pi V(t-\tau)}d\tau + \lambda L(t) - \epsilon I(t)\\
&+ \frac{\epsilon}{2m}\left(mI(t) - rV(t) - qV(t)B(t)\right) + \frac{\epsilon q}{2mc}\left(\eta + cB(t)V(t) - \delta B(t)\right)\\
&\leq \mu\rho K + \lambda L_1 - \sigma_2\left(\rho\int_0^{\kappa_2}\xi_2(\tau)e^{-\delta_2\tau}S(t-\tau)d\tau + I(t) + \frac{\epsilon}{2m}V(t) + \frac{\epsilon q}{2mc}B(t)\right)\\
&\leq \mu\rho + \lambda L_1 - \sigma_2 T_2(t).
\end{aligned}$$

Then $\lim\limits_{t\to\infty}\sup T_2(t) \leq M_2$. It follows that $\lim\limits_{t\to\infty}\sup I(t) \leq M_2$, $\lim\limits_{t\to\infty}\sup V(t) \leq M_3$ and $\lim\limits_{t\to\infty}\sup B(t) \leq M_4$. Therefore $S(t), L(t), I(t), V(t)$, and $B(t)$ are ultimately bounded. □

Lemma 4. For system (24)-(28) there exists a threshold parameter $\mathcal{R}_0^D > 0$, such that

(i) if $\mathcal{R}_0^D \leq 1$, then there exists only one positive steady state, virus-free steady state Q_0.

(i) if $\mathcal{R}_0^D > 1$, then in addition to Q_0, there exists an endemic steady state Q_1

Proof. Similar to the proof of Lemma 2 we can show that if $\mathcal{R}_0^D \leq 1$ then there exists $Q_0 = (S_0, 0, 0, 0, B_0)$, where $S_0 = \frac{\mu}{a}$ and $B_0 = \frac{\eta}{\delta}$, and if $\mathcal{R}_0^D > 1$ then there exists $Q_1 = (S_1, L_1, I_1, V_1, B_1)$, with

$$\begin{aligned}
S_1 &= \frac{\mu\left(1+\pi V_1\right)}{bV_1 + a\left(1+\pi V_1\right)},\ L_1 = \frac{E(1-\rho)b\mu V_1}{(\theta+\lambda)(bV_1 + a\left(1+\pi V_1\right))},\ I_1 = \frac{b\mu V_1\gamma}{\epsilon(\theta+\lambda)(bV_1 + a\left(1+\pi V_1\right))},\\
V_1 &= \frac{P_2^D - \sqrt{\left(P_2^D\right)^2 - 4P_1^D P_3^D}}{2P_1^D} < \frac{\delta}{c},\ B_1 = \frac{\eta}{\delta - cV_1},
\end{aligned}$$

where

$$\begin{aligned}
\gamma &= E\lambda(1-\rho) + K\rho(\theta+\lambda), \quad P_1^D = r\epsilon c(\theta+\lambda)(b+\pi a), \\
P_2^D &= \frac{\epsilon ca\,(r\delta+q\eta)\,(\theta+\lambda)}{\delta}(\mathcal{R}_0^D-1) + \epsilon\,(r\delta+q\eta)\,(\theta+\lambda)(b+\pi a) + \frac{caq\epsilon\eta(\theta+\lambda)}{\delta}, \\
P_3^D &= \epsilon a\,(r\delta+q\eta)\,(\theta+\lambda)(\mathcal{R}_0^D-1).
\end{aligned}$$

The basic reproduction number for system (24)-(28) is defined as

$$\mathcal{R}_0^D = \frac{m\mu b\delta\gamma}{\epsilon a(r\delta+q\eta)(\theta+\lambda)}. \ \square$$

3.2 Global stability

In this section we construct suitable Lyapunov functions to prove that the steady states Q_0 and Q_1 of system (24)-(28) are GAS.

Theorem 3. Suppose that $\mathcal{R}_0^D \le 1$, then Q_0 is GAS.

Proof. Let us define $Y_0^D(S,L,I,V,B)$ as:

$$\begin{aligned}
Y_0^D &= \frac{\gamma}{\theta+\lambda}S_0 H\left(\frac{S}{S_0}\right) + \frac{\lambda}{\theta+\lambda}L + I + \frac{\epsilon}{m}V + \frac{\epsilon q}{mc}B_0 H\left(\frac{B}{B_0}\right) \\
&+ \frac{\lambda(1-\rho)b}{\theta+\lambda}\int_0^{\kappa_1}\xi_1(\tau)e^{-\delta_1\tau}\int_0^{\tau}\frac{V(t-\vartheta)S(t-\vartheta)}{1+\pi V(t-\vartheta)}d\vartheta d\tau + \rho b\int_0^{\kappa_2}\xi_2(\tau)e^{-\delta_2\tau}\int_0^{\tau}\frac{V(t-\vartheta)S(t-\vartheta)}{1+\pi V(t-\vartheta)}d\vartheta d\tau. \quad (31)
\end{aligned}$$

Note that, $Y_0^D(S,L,I,V,B) > 0$ for all $S,L,I,V,B > 0$ and $Y_0^D(S_0,0,0,0,B_0) = 0$. Calculating $\frac{dY_0^D}{dt}$ along the trajectories of (24)-(28) we get

$$\begin{aligned}
\frac{dY_0^D}{dt} &= \frac{\gamma}{\theta+\lambda}\left(1-\frac{S_0}{S}\right)\left(\mu - aS - \frac{bVS}{1+\pi V}\right) \\
&+ \frac{\lambda}{\theta+\lambda}\left((1-\rho)b\int_0^{\kappa_1}\xi_1(\tau)e^{-\delta_1\tau}\frac{V(t-\tau)S(t-\tau)}{1+\pi V(t-\tau)}d\tau - (\theta+\lambda)L\right) \\
&+ \rho b\int_0^{\kappa_2}\xi_2(\tau)e^{-\delta_2\tau}\frac{V(t-\tau)S(t-\tau)}{1+\pi V(t-\tau)}d\tau + \lambda L - \epsilon I + \frac{\epsilon}{m}(mI - rV - qVB) \\
&+ \frac{\epsilon q}{mc}\left(1-\frac{B_0}{B}\right)(\eta + cBV - \delta B) + \frac{\lambda(1-\rho)}{\theta+\lambda}\int_0^{\kappa_1}\xi_1(\tau)e^{-\delta_1\tau}\left(\frac{bVS}{1+\pi V} - \frac{bV(t-\tau)S(t-\tau)}{1+\pi V(t-\tau)}\right)d\tau \\
&+ \rho b\int_0^{\kappa_2}\xi_2(\tau)e^{-\delta_2\tau}\left(\frac{bVS}{1+\pi V} - \frac{bV(t-\tau)S(t-\tau)}{1+\pi V(t-\tau)}\right)d\tau \\
&= -\frac{a\gamma}{\theta+\lambda}\frac{(S-S_0)^2}{S} + \left(\frac{\gamma}{(\theta+\lambda)}\right)\frac{bS_0V}{1+\pi V} - \frac{\epsilon rV}{m} - \frac{\epsilon qB_0V}{m} + \frac{\epsilon q}{mc}\left(1-\frac{B_0}{B}\right)(\delta B_0 - \delta B) \\
&= -\frac{a\gamma}{\theta+\lambda}\frac{(S-S_0)^2}{S} - \frac{\epsilon q\delta}{mc}\frac{(B-B_0)^2}{B} + \frac{\epsilon(r\delta+q\eta)}{m\delta}(\mathcal{R}_0^D-1)V - \frac{\epsilon(r\delta+q\eta)\mathcal{R}_0^D\pi V^2}{m\delta(1+\pi V)}. \quad (32)
\end{aligned}$$

Therefore, $\frac{dY_0^D}{dt} \le 0$ holds if $\mathcal{R}_0^D \le 1$. Further, $\frac{dY_0^D}{dt} = 0$ if and only if $S = S_0$, $B = B_0$, $V = 0$. Applying LaSalle's invariance principle, we get that Q_0 is GAS . $\square$

Theorem 4. Suppose that $\mathcal{R}_0^D > 1$, then Q_1 is GAS.

Proof. Consider

$$\begin{aligned}
Y_1^D(S,L,I,V,B) &= \frac{\gamma}{\theta+\lambda}S_1 H\left(\frac{S}{S_1}\right) + \frac{\lambda}{\theta+\lambda}L_1 H\left(\frac{L}{L_1}\right) + I_1 H\left(\frac{I}{I_1}\right) + \frac{\epsilon}{m}V_1 H\left(\frac{V}{V_1}\right) \\
&+ \frac{\epsilon q}{mc}B_1 H\left(\frac{B}{B_1}\right) + \frac{\lambda(1-\rho)}{\theta+\lambda}\frac{bS_1V_1}{1+\pi V_1}\int_0^{\kappa_1}\xi_1(\tau)e^{-\delta_1\tau}\int_0^{\tau}H\left(\frac{V(t-\vartheta)S(t-\vartheta)(1+\pi V_1)}{S_1V_1\,(1+\pi V(t-\vartheta))}\right)d\vartheta d\tau \\
&+ \frac{\rho bS_1V_1}{1+\pi V_1}\int_0^{\kappa_2}\xi_2(\tau)e^{-\delta_2\tau}\int_0^{\tau}H\left(\frac{V(t-\vartheta)S(t-\vartheta)(1+\pi V_1)}{S_1V_1\,(1+\pi V(t-\vartheta))}\right)d\vartheta d\tau.
\end{aligned}$$

We have $Y_1^D(S,L,I,V,B) > 0$ for all $S,L,I,V,B > 0$ and $Y_1^D(S_1,L_1,I_1,V_1,B_1) = 0$. Calculating $\dfrac{dY_1^D}{dt}$ along the trajectories of (24)-(28) we get

$$\begin{aligned}
\frac{dY_1^D}{dt} &= \frac{\gamma}{\theta+\lambda}\left(1-\frac{S_1}{S}\right)\left(\mu - aS - \frac{bVS}{1+\pi V}\right) \\
&+ \frac{\lambda}{\theta+\lambda}\left(1-\frac{L_1}{L}\right)\left((1-\rho)b\int_0^{\kappa_1}\xi_1(\tau)e^{-\delta_1\tau}\frac{V(t-\tau)S(t-\tau)}{1+\pi V(t-\tau)}d\tau - (\theta+\lambda)L\right) \\
&+ \left(1-\frac{I_1}{I}\right)\left(\rho b\int_0^{\kappa_2}\xi_2(\tau)e^{-\delta_2\tau}\frac{V(t-\tau)S(t-\tau)}{1+\pi V(t-\tau)}d\tau + \lambda L - \epsilon I\right) + \frac{\epsilon}{m}\left(1-\frac{V_1}{V}\right)(mI - rV - qVB) \\
&+ \frac{\epsilon q}{mc}\left(1-\frac{B_1}{B}\right)(\eta + cBV - \delta B) + \frac{\lambda(1-\rho)}{(\theta+\lambda)}\int_0^{\kappa_1}\xi_1(\tau)e^{-\delta_1\tau}\left(\frac{bVS}{1+\pi V} - \frac{bV(t-\tau)S(t-\tau)}{1+\pi V(t-\tau)}\right)d\tau \\
&+ \frac{\lambda(1-\rho)}{(\theta+\lambda)}\frac{bS_1V_1}{1+\pi V_1}\int_0^{\kappa_1}\xi_1(\tau)e^{-\delta_1\tau}\ln\left(\frac{V(t-\tau)S(t-\tau)(1+\pi V)}{VS\,(1+\pi V(t-\tau))}\right)d\tau \\
&+ \rho\int_0^{\kappa_2}\xi_2(\tau)e^{-\delta_2\tau}\left(\frac{bVS}{1+\pi V} - \frac{bV(t-\tau)S(t-\tau)}{1+\pi V(t-\tau)}\right)d\tau \\
&+ \frac{\rho bS_1V_1}{1+\pi V_1}\int_0^{\kappa_2}\xi_2(\tau)e^{-\delta_2\tau}\ln\left(\frac{V(t-\tau)S(t-\tau)(1+\pi V)}{VS\,(1+\pi V(t-\tau))}\right)d\tau. \qquad (33)
\end{aligned}$$

Applying

$$\mu = aS_1 + \frac{bS_1V_1}{1+\pi V_1},\ \eta = \delta B_1 - cB_1V_1,$$

we obtain

$$\begin{aligned}
\frac{dY_1^D}{dt} &= \frac{\gamma}{\theta+\lambda}\left(1-\frac{S_1}{S}\right)(aS_1 - aS) + \left(\frac{\gamma}{\theta+\lambda}\right)\frac{bS_1V_1}{1+\pi V_1}\left(1-\frac{S_1}{S}\right) \\
&+ \frac{\gamma}{\theta+\lambda}\frac{bS_1V}{1+\pi V} - \frac{\lambda(1-\rho)b}{\theta+\lambda}\int_0^{\kappa_1}\xi_1(\tau)e^{-\delta_1\tau}\frac{V(t-\tau)S(t-\tau)L_1}{(1+\pi V(t-\tau))\,L}d\tau + \lambda L_1 \\
&- \rho b\int_0^{\kappa_2}\xi_2(\tau)e^{-\delta_2\tau}\frac{V(t-\tau)S(t-\tau)I_1}{(1+\pi V(t-\tau))\,I}d\tau - \frac{\lambda L I_1}{I} + \epsilon I_1 - \frac{\epsilon I V_1}{V} - \frac{r\epsilon V}{m} + \frac{r\epsilon V_1}{m} + \frac{\epsilon qBV_1}{m} \\
&+ \frac{\epsilon q}{mc}\left(1-\frac{B_1}{B}\right)(\delta B_1 - \delta B) - \frac{\epsilon qB_1V}{m} - \frac{\epsilon qB_1V_1}{m} + \frac{\epsilon qB_1V_1}{m}\left(\frac{B_1}{B}\right) \\
&+ \frac{\lambda(1-\rho)}{\theta+\lambda}\frac{bS_1V_1}{1+\pi V_1}\int_0^{\kappa_1}\xi_1(\tau)e^{-\delta_1\tau}\ln\left(\frac{V(t-\tau)S(t-\tau)(1+\pi V)}{VS\,(1+\pi V(t-\tau))}\right)d\tau \\
&+ \frac{\rho bS_1V_1}{1+\pi V_1}\int_0^{\kappa_2}\xi_2(\tau)e^{-\delta_2\tau}\ln\left(\frac{V(t-\tau)S(t-\tau)(1+\pi V)}{VS\,(1+\pi V(t-\tau))}\right)d\tau.
\end{aligned}$$

The components of the steady state Q_1 satisfy

$$E(1-\rho)\frac{bS_1V_1}{1+\pi V_1} = (\theta+\lambda)L_1,\ K\rho\frac{bS_1V_1}{1+\pi V_1} + \lambda L_1 = \epsilon I_1,\ mI_1 = rV_1 + qB_1V_1,$$

then

$$\epsilon I_1 = \frac{\gamma}{\theta+\lambda}\frac{bS_1V_1}{(1+\pi V_1)}, \qquad \frac{r\epsilon V_1}{m} = \frac{\gamma}{\theta+\lambda}\frac{bS_1V_1}{(1+\pi V_1)} - \frac{\epsilon qB_1V_1}{m},$$

and

$$\begin{aligned}
\frac{dY_1^D}{dt} &= -\frac{a\gamma}{\theta+\lambda}\frac{(S-S_1)^2}{S} + \frac{E\lambda(1-\rho)}{\theta+\lambda}\frac{bS_1V_1}{1+\pi V_1}\left(1-\frac{S_1}{S}\right) \\
&+ K\rho\frac{bS_1V_1}{(1+\pi V_1)}\left(1-\frac{S_1}{S}\right) + \left(\frac{\gamma}{\theta+\lambda}\right)\frac{bS_1V_1}{1+\pi V_1}\left(\frac{(1+\pi V_1)V}{(1+\pi V)V_1}-\frac{V}{V_1}\right) \\
&- \frac{\lambda(1-\rho)}{\theta+\lambda}\frac{bS_1V_1}{1+\pi V_1}\int_0^{\kappa_1}\xi_1(\tau)e^{-\delta_1\tau}\frac{V(t-\tau)S(t-\tau)(1+\pi V_1)L_1}{(1+\pi V(t-\tau))S_1V_1L}d\tau + \frac{E\lambda(1-\rho)}{\theta+\lambda}\frac{bS_1V_1}{(1+\pi V_1)} \\
&- \frac{\rho bS_1V_1}{1+\pi V_1}\int_0^{\kappa_2}\xi_2(\tau)e^{-\delta_2\tau}\frac{V(t-\tau)S(t-\tau)(1+\pi V_1)I_1}{(1+\pi V(t-\tau))S_1V_1I}d\tau - \frac{E\lambda(1-\rho)}{\theta+\lambda}\frac{bS_1V_1}{1+\pi V_1}\frac{I_1L}{L_1I} \\
&+ \frac{E\lambda(1-\rho)}{\theta+\lambda}\frac{bS_1V_1}{(1+\pi V_1)} + K\rho\frac{bS_1V_1}{(1+\pi V_1)} - \frac{E\lambda(1-\rho)}{\theta+\lambda}\frac{bS_1V_1}{1+\pi V_1}\frac{IV_1}{I_1V} - K\rho\frac{bS_1V_1}{1+\pi V_1}\frac{IV_1}{I_1V} \\
&+ \frac{E\lambda(1-\rho)}{\theta+\lambda}\frac{bS_1V_1}{(1+\pi V_1)} + K\rho\frac{bS_1V_1}{(1+\pi V_1)} - \frac{2\epsilon qB_1V_1}{m} + \frac{\epsilon qBV_1}{m} + \frac{\epsilon qB_1V_1}{m}\left(\frac{B_1}{B}\right) \\
&- \frac{\epsilon q\delta}{mc}\frac{(B-B_1)^2}{B} + \frac{\lambda(1-\rho)}{(\theta+\lambda)}\frac{bS_1V_1}{1+\pi V_1}\int_0^{\kappa_1}\xi_1(\tau)e^{-\delta_1\tau}\ln\left(\frac{V(t-\tau)S(t-\tau)(1+\pi V)}{VS\left(1+\pi V(t-\tau)\right)}\right)d\tau \\
&+ \frac{\rho bS_1V_1}{1+\pi V_1}\int_0^{\kappa_2}\xi_2(\tau)e^{-\delta_2\tau}\ln\left(\frac{V(t-\tau)S(t-\tau)(1+\pi V)}{VS\left(1+\pi V(t-\tau)\right)}\right)d\tau.
\end{aligned}$$

Utilizing the following equalities

$$\begin{aligned}
\ln\left(\frac{V(t-\tau)S(t-\tau)(1+\pi V)}{VS\left(1+\pi V(t-\tau)\right)}\right) &= \ln\left(\frac{S_1}{S}\right) + \ln\left(\frac{IV_1}{I_1V}\right) + \ln\left(\frac{1+\pi V}{1+\pi V_1}\right) + \ln\left(\frac{LI_1}{L_1I}\right) \\
&+ \ln\left(\frac{V(t-\tau)S(t-\tau)(1+\pi V_1)L_1}{(1+\pi V(t-\tau))S_1V_1L}\right), \\
\ln\left(\frac{V(t-\tau)S(t-\tau)(1+\pi V)}{VS\left(1+\pi V(t-\tau)\right)}\right) &= \ln\left(\frac{S_1}{S}\right) + \ln\left(\frac{IV_1}{I_1V}\right) + \ln\left(\frac{1+\pi V}{1+\pi V_1}\right) \\
&+ \ln\left(\frac{V(t-\tau)S(t-\tau)(1+\pi V_1)I_1}{(1+\pi V(t-\tau))S_1V_1I}\right),
\end{aligned}$$

we have

$$\begin{aligned}
\frac{dY_1^D}{dt} &= -\frac{a\gamma}{\theta+\lambda}\frac{(S-S_1)^2}{S} + \frac{\gamma}{\theta+\lambda}\frac{bS_1V_1}{1+\pi V_1}\left(-1+\frac{(1+\pi V_1)V}{(1+\pi V)V_1}-\frac{V}{V_1}+\frac{1+\pi V}{1+\pi V_1}\right) \\
&+ \frac{E\lambda(1-\rho)}{(\theta+\lambda)}\frac{bS_1V_1}{(1+\pi V_1)}\left[1-\frac{S_1}{S}+\ln\left(\frac{S_1}{S}\right)\right] + \frac{E\lambda(1-\rho)}{(\theta+\lambda)}\frac{bS_1V_1}{(1+\pi V_1)}\left[1-\frac{IV_1}{I_1V}+\ln\left(\frac{IV_1}{I_1V}\right)\right] \\
&+ \frac{E\lambda(1-\rho)}{\theta+\lambda}\frac{bS_1V_1}{1+\pi V_1}\frac{1}{E}\int_0^{\kappa_1}\xi_1(\tau)e^{-\delta_1\tau}\left[1-\frac{V(t-\tau)S(t-\tau)(1+\pi V_1)L_1}{(1+\pi V(t-\tau))S_1V_1L}\right. \\
&\left.+\ln\left(\frac{V(t-\tau)S(t-\tau)(1+\pi V_1)L_1}{(1+\pi V(t-\tau))S_1V_1L}\right)\right]d\tau \\
&+ \frac{E\lambda(1-\rho)}{\theta+\lambda}\frac{bS_1V_1}{1+\pi V_1}\left[1-\frac{1+\pi V}{1+\pi V_1}+\ln\left(\frac{1+\pi V}{1+\pi V_1}\right)\right] + \frac{E\lambda(1-\rho)}{\theta+\lambda}\frac{bS_1V_1}{(1+\pi V_1)}\left[1-\frac{LI_1}{L_1I}+\ln\left(\frac{LI_1}{L_1I}\right)\right] \\
&+ K\rho\frac{bS_1V_1}{(1+\pi V_1)}\left[1-\frac{S_1}{S}+\ln\left(\frac{S_1}{S}\right)\right] + K\rho\frac{bS_1V_1}{1+\pi V_1}\left[1-\frac{IV_1}{I_1V}+\ln\left(\frac{IV_1}{I_1V}\right)\right] \\
&+ K\rho\frac{bS_1V_1}{1+\pi V_1}\frac{1}{K}\int_0^{\kappa_2}\xi_2(\tau)e^{-\delta_2\tau}\left[1-\frac{V(t-\tau)S(t-\tau)(1+\pi V_1)I_1}{(1+\pi V(t-\tau))S_1V_1I}\right. \\
&\left.+\ln\left(\frac{V(t-\tau)S(t-\tau)(1+\pi V_1)I_1}{(1+\pi V(t-\tau))S_1V_1I}\right)\right]d\tau \\
&+ K\rho\frac{bS_1V_1}{(1+\pi V_1)}\left[1-\frac{1+\pi V}{1+\pi V_1}+\ln\left(\frac{1+\pi V}{1+\pi V_1}\right)\right] - \frac{\epsilon q\delta}{mc}\frac{(B-B_1)^2}{B} - \frac{\epsilon qB_1V_1}{m}\left[2-\frac{B}{B_1}-\frac{B_1}{B}\right]
\end{aligned}$$

$$
\begin{aligned}
&= -a\left(\frac{\gamma}{\theta+\lambda}\right)\frac{(S-S_1)^2}{S} - \left(\frac{\gamma}{\theta+\lambda}\right)\frac{\pi b S_1 (V-V_1)^2}{(1+\pi V)(1+\pi V_1)^2} \\
&- \frac{\epsilon q \eta}{mcB_1}\frac{(B-B_1)^2}{B} - \frac{E\lambda(1-\rho)}{\theta+\lambda}\frac{bS_1V_1}{1+\pi V_1}\left[H\left(\frac{S_1}{S}\right) + H\left(\frac{IV_1}{I_1V}\right) + H\left(\frac{1+\pi V}{1+\pi V_1}\right) + H\left(\frac{LI_1}{L_1 I}\right)\right. \\
&\left. + \frac{1}{E}\int_0^{\kappa_1} \xi_1(\tau) e^{-\delta_1 \tau} H\left(\frac{V(t-\tau)S(t-\tau)(1+\pi V_1)L_1}{(1+\pi V(t-\tau))S_1V_1L}\right) d\tau\right] \\
&- K\rho\frac{bS_1V_1}{(1+\pi V_1)}\left[H\left(\frac{S_1}{S}\right) + H\left(\frac{IV_1}{I_1V}\right) + H\left(\frac{1+\pi V}{1+\pi V_1}\right)\right. \\
&\left. + \frac{1}{K}\int_0^{\kappa_2} \xi_2(\tau) e^{-\delta_2 \tau} H\left(\frac{V(t-\tau)S(t-\tau)(1+\pi V_1)I_1}{(1+\pi V(t-\tau))S_1V_1I}\right) d\tau\right].
\end{aligned}
$$

It can be seen that if $\mathcal{R}_0^D > 1$, then $S_1, L_1, I_1, V_1, B_1 > 0$ and $\frac{dY_1^D}{dt} \leq 0$ for all $S, L, I, V, B > 0$. We have $\frac{dY_1^D}{dt} = 0$ if and only if $S = S_1$, $L = L_1$, $I = I_1$, $V = V_1, B = B_1$ and $H = 0$. Then using from LaSalle's invariance principle, we show that Q_1 is GAS. □

4 Numerical simulations

Next we conduct numerical simulations for system (5)-(9). The values of the parameters are listed in Table 1. We let $\tau_i = \tau_1 = \tau_2$. The following initial conditions are used:

$$\varphi_1(\vartheta) = 1.7, \varphi_2(\vartheta) = 0.4, \varphi_3(\vartheta) = 0.6, \varphi_4(\vartheta) = 0.6, \varphi_5(\vartheta) = 1.6, \quad \vartheta \in [-\tau_i, 0]$$

In Figures 1-5, we show the evolution of the five states of the system S, L, I, V and B with respect to the time. The effect of τ_i on the stability of Q_0 and Q_1 is also shown. We can see that, for smaller values of τ_i e.g. $\tau_i = 0.0, 0.5, 1.0$ and 2.0, the corresponding values of $\mathcal{R}_0$ satisfy $\mathcal{R}_0 > 1$, and the trajectory of the system converges to the steady states Q_1. This confirm the results of Theorem 2 that Q_1 is GAS. On the the other hand, when τ_i become larger e.g. $\tau_i = 3.0$ and 5.0, then $\mathcal{R}_0 < 1$, and the system has one steady state Q_0. and according to Theorem 1 it is GAS. For this case, the concentrations of the uninfected monocytes and B cells return to their values $S_0 = \frac{\mu}{a} = 2.2885$ and $B_0 = \frac{\eta}{\delta} = 1.1207$, respectively, while the CHIKV particles are cleared from the body.

Let τ^{cr} be the critical value of the parameter τ_i, such that

$$\mathcal{R}_0 = \frac{bm\delta\mu(\lambda(1-\rho)e^{-\delta_1\tau^{cr}} + \rho(\theta+\lambda)e^{-\delta_1\tau^{cr}})}{\epsilon a(r\delta + q\eta)(\theta+\lambda)} = 1.$$

Using the data given in Table 1 we obtain $\tau^{cr} = 2.01206$. The value of $\mathcal{R}_0$ for different values of τ_i are listed in Table 2. We can observed that as τ_i is increased then $\mathcal{R}_0$ is decreased. Moreover, we have the following cases:

(i) if $0 \leq \tau_i < 2.01206$, then Q_1 exists and it is GAS,

(ii) if $\tau_i \geq 2.01206$, then Q_0 is GAS. It is clearly seen that, an increasing in time delay will stabilize the system around Q_0. Biologicaly, the time delay has a similar effect as the antiviral treatment which can be used to eliminate the CHIKV. We observe that, when the delay period is sufficiently long the CHIKV replication will be cleared.

Table 1: The values of the parameters of model (5)-(9).

Parameter	Value	Parameter	Value
μ	1.826	m	2.02
π	0.1	q	0.5964
c	1.2129	r	0.4418
a	0.7979	η	1.402
θ	0.5	δ_1	0.5
λ	0.1	τ_1	*varied*
ϵ	0.4441	τ_2	*varied*
δ	1.251	b	0.5
ρ	0.5		

Table 2: The values of steady states, $\mathcal{R}_0$ for model (5)-(9) with different values of τ_i.

τ_i	Steady states	R_0	
0.0	$Q_1 = (1.6788, 0.4054, 0.6390, 0.6152, 2.7772)$	2.7347	
0.5	$Q_1 = (1.7636, 0.2718, 0.4284, 0.4986, 2.1694)$	2.1298	
1.0	$Q_1 = (1.8827, 0.1637, 0.2580, 0.3562, 1.7120)$	1.6587	
1.5	$Q_1 = (2.0497, 0.0750, 0.1182, 0.1895, 1.3729)$	1.2918	
2.0	$Q_1 = (2.2819, 0.0016, 0.0025, 0.0046, 1.1257)$	1.0060	
2.01206	$Q_0 = (2.2885, 0, 0, 0, 1.1207)$	1.0000	
2.5	$Q_0 = (2.2885, 0, 0, 0, 1.1207)$	0.7835	
3.0	$Q_0 = (2.2885, 0, 0, 0, 1.1207)$	0.6102	
3.5	$Q_0 = (2.2885, 0, 0, 0, 1.1207)$	0.4752	
4.0	$Q_0 = (2.2885, 0, 0, 0, 1.1207)$	0.3701	
4.5	$Q_0 = (2.2885, 0, 0, 0, 1.1207)$	0.2882	
5.0	$Q_0 = (2.2885, 0, 0, 0, 1.1207)$	0.2245	

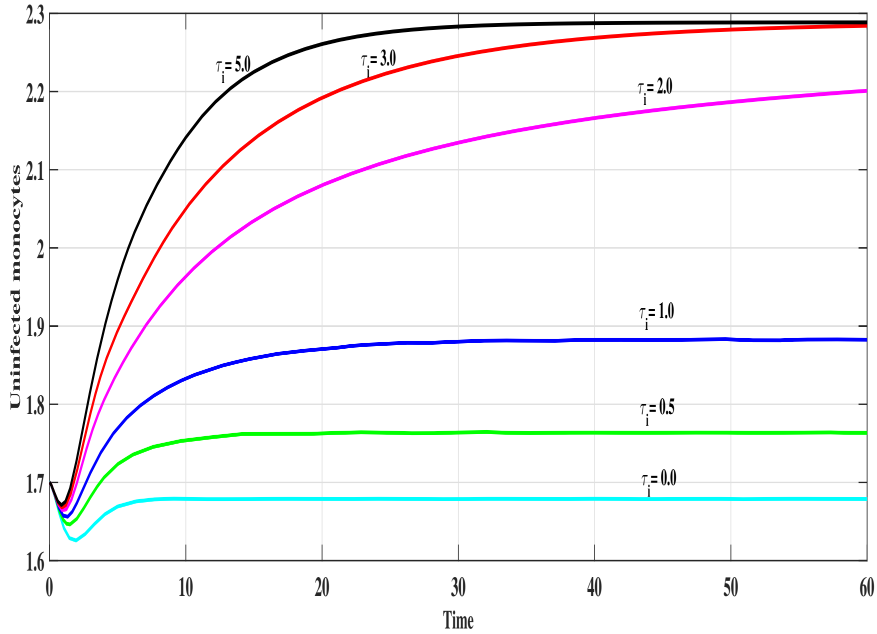


Figure 1: The evolution of uninfected monocytes.

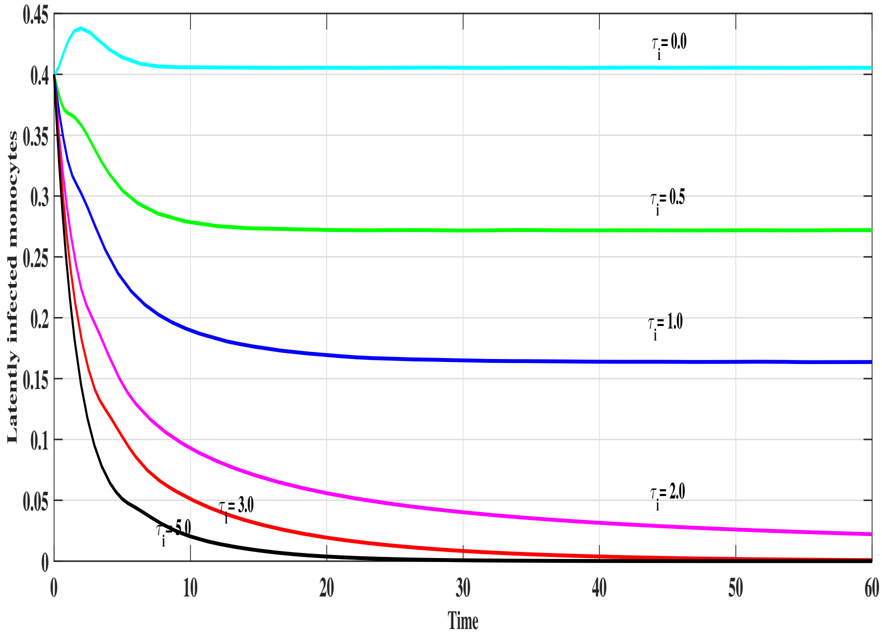


Figure 2: The evolution of latently infected monocytes.

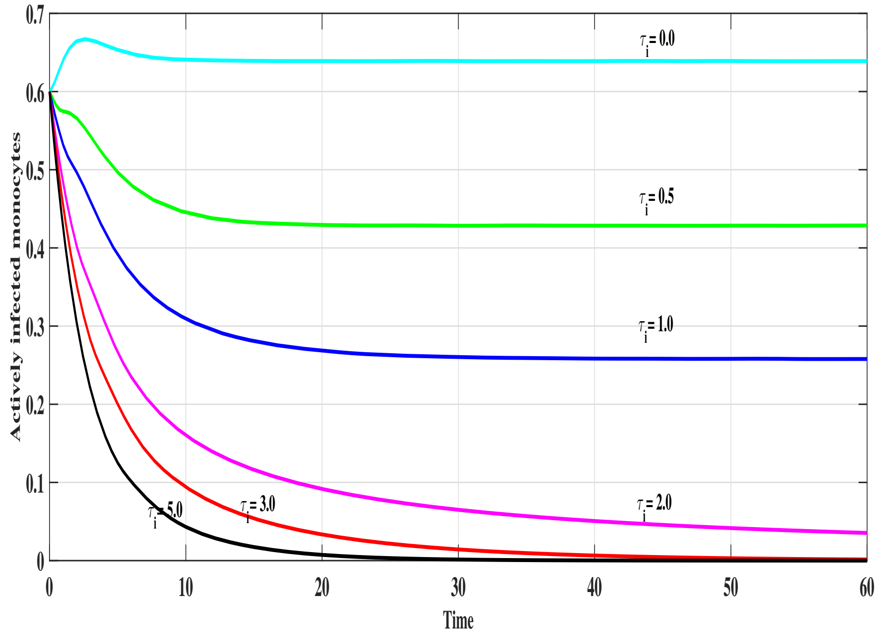


Figure 3: The evolution of actively infected monocytes.

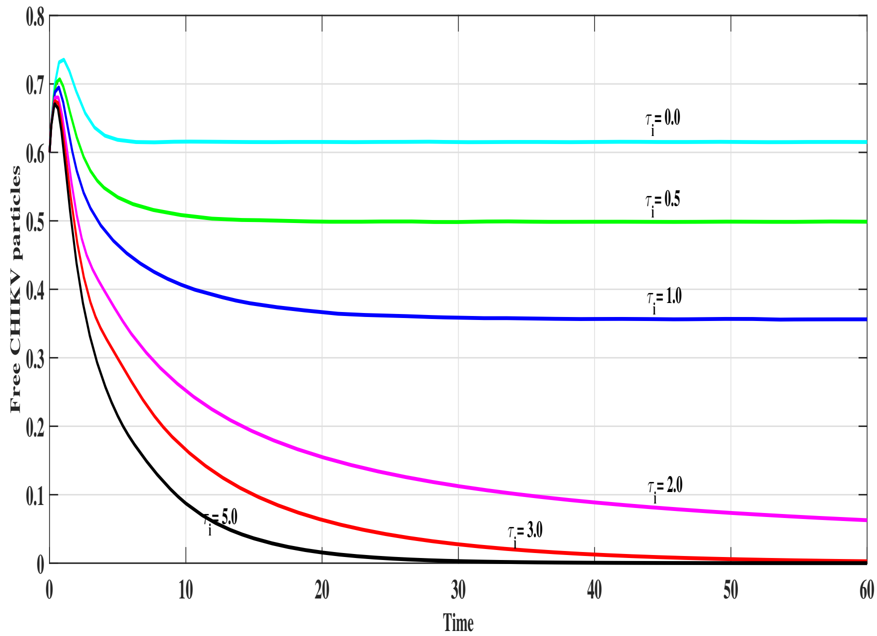


Figure 4: The evolution of free CHIKV particles.

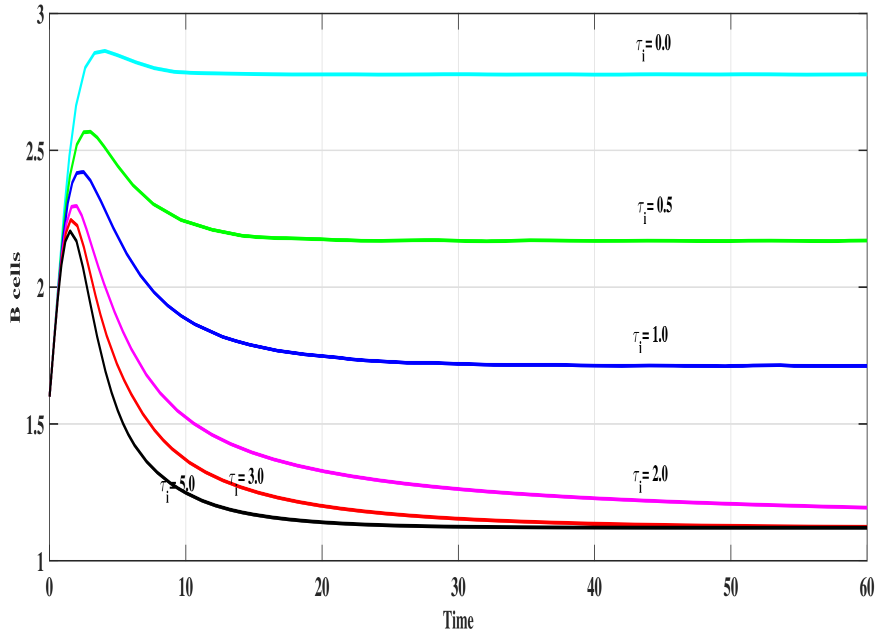


Figure 5: The evolution of B cells.

5 Acknowledgment

This article was funded by the Deanship of Scientific Research (DSR) at King Abdulaziz University, Jeddah. The authors, therefore, acknowledge with thanks DSR for technical and financial support.

References

[1] M. A. Nowak and C. R. M. Bangham, *Population dynamics of immune responses to persistent viruses*, Science, **272** (1996) 74-79.

[2] D.S. Callaway, and A.S. Perelson, *HIV-1 infection and low steady state viral loads*, Bull. Math. Biol., **64** (2002), 29-64.

[3] C. Connell McCluskey, Y. Yang, *Global stability of a diffusive virus dynamics model with general incidence function and time delay*, Nonlinear Analysis: Real World Applications, **25** (2015), 64-78.

[4] A. M. Elaiw and S.A. Azoz, *Global properties of a class of HIV infection models with Beddington-DeAngelis functional response,* Mathematical Methods in the Applied Sciences, **36** (2013), 383-394.

[5] A. M. Elaiw, *Global properties of a class of HIV models*, Nonlinear Analysis: Real World Applications, **11** (2010), 2253-2263.

[6] G. Huang, Y. Takeuchi and W. Ma, *Lyapunov functionals for delay differential equations model of viral infections*, SIAM J. Appl. Math., **70**(7) (2010), 2693-2708.

[7] D. Huang, X. Zhang, Y. Guo, and H. Wang, *Analysis of an HIV infection model with treatments and delayed immune response*, Applied Mathematical Modelling, **40**(4) (2016), 3081-3089.

[8] K. Wang, A. Fan, and A. Torres, *Global properties of an improved hepatitis B virus model*, Nonlinear Analysis: Real World Applications, **11** (2010), 3131-3138.

[9] A. U. Neumann, N. P. Lam, H. Dahari, D. R. Gretch, T. E. Wiley, T. J, Layden, and A. S. Perelson, *Hepatitis C viral dynamics in vivo and the antiviral efficacy of interferon-alpha therapy*, Science, **282** (1998), 103-107.

[10] A. M. Elaiw, A. A. Raezah and A. S. Alofi, *Stability of delay-distributed virus dynamics model with cell-to-cell transmission and CTL immune response*, Journal of Computational Analysis and Applications, **25(**8) (2018), 1518-1531.

[11] X. Shi, X. Zhou, and X. Son, *Dynamical behavior of a delay virus dynamics model with CTL immune response*, Nonlinear Analysis: Real World Applications, **11** (2010), 1795-1809.

[12] H. Shu, L. Wang and J. Watmough, *Global stability of a nonlinear viral infection model with infinitely distributed intracellular delays and CTL imune responses*, SIAM Journal of Applied Mathematics, **73**(3) (2013), 1280-1302.

[13] A. M. Elaiw, A. M. Althiabi, M. A. Alghamdi and N. Bellomo, *Dynamical behavior of a general HIV-1 infection model with HAART and cellular reservoirs*, Journal of Computational Analysis and Applications, **24**(4) (2018), 728-743.

[14] A. M. Elaiw and N. H. AlShamrani, *Global stability of humoral immunity virus dynamics models with nonlinear infection rate and removal*, Nonlinear Analysis: Real World Applications, **26**, (2015), 161-190.

[15] A. M. Elaiw and N. H. AlShamrani, *Stability of a general delay-distributed virus dynamics model with multi-staged infected progression and immune response*, Mathematical Methods in the Applied Sciences, **40(3)** (2017), 699-719.

[16] Y. Wang, X. Liu, *Stability and Hopf bifurcation of a within-host chikungunya virus infection model with two delays,* Mathematics and Computers in Simulation, **138** (2017), 31-48.

[17] Y. Dumont, F. Chiroleu, *Vector control for the chikungunya disease*, Mathematical Biosciences and Engineering, **7** (2010), 313-345.

[18] Y. Dumont, J. M. Tchuenche, *Mathematical studies on the sterile insect technique for the chikungunya disease and aedes albopictus*, Journal of Mathematical Biology **65**(5) (2012), 809-854.

[19] D. Moulay, M. Aziz-Alaoui, M.Cadivel, *The chikungunya disease: modeling, vector and transmission global dynamics,* Mathematical Biosciences, **229** (2011) 50-63.

[20] C. A. Manore, K. S. Hickmann, S. Xu, H. J. Wearing, J. M. Hyman, *Comparing dengue and chikungunya emergence and endemic transmission in A. aegypti and A. albopictus*, Journal of Theoretical Biology **356** (2014), 174-191.

[21] L. Yakob, A.C. Clements, *A mathematical model of chikungunya dynamics and control: the major epidemic on Reunion Island*, PLoS One, **8** (2013), e57448.

[22] X. Liu, and P. Stechlinski, *Application of control strategies to a seasonal model of chikungunya disease*, Applied Mathematical Modelling, **39** (2015), 3194-3220.

[23] J. K. Hale and S. M. Verduyn Lunel, *Introduction to Functional Differential Equations*, Springer Verlag, New York, 1993.

Dynamical behavior of MERS-CoV model with discrete delays

H. Batarfi, A. Elaiw and A. Alshareef
Department of Mathematics, Faculty of Science, King Abdulaziz University,
P.O. Box 80203, Jeddah 21589, Saudi Arabia
Emails: hbatarfi@kau.edu.sa (H. Batarfi),
a_m_elaiw@yahoo.com (A. Elaiw),
3beer23sh@gmail.com (A. Alshareef)

Abstract

A nonlinear mathematical model for Middle East Respiratory Syndrome Corona Virus (MERS-CoV) with two discrete time delays is proposed and analyzed. We show that the solutions of the model are nonnegative and bounded. We derive the basic reproduction number for the MERS-CoV model, R_0. we prove that if $R_0 \leq 1$ then there exists a disease-free equilibrium P_0 and $R_0 > 1$ then in addition to P_0 the model has an endemic equilibrium P^*. Utilizing Lyapunov method, the global asymptotic stability of disease-free equilibrium of the proposed model is obtained. The dynamical behaviour of the model is also shown by numerical simulations.

Keywords: Infectious diseases; global stability; Lyapunov functional.

1 Introduction

Mathematical of infectious diseases have received the attention of several researchers during the past decides. Some of the models are given by a set of ODEs (see e.g. [1]-[12]). For some disease such as influenza, on adequate contact with an infective, a susceptible individual becomes exposed, that is, infected but not infective. This individual stays in exposed class for a certain latent period before becoming infective. This period can been described as delays on the spread of infectious diseases, and thus, delays should be incorporated into infection term in the system. As a result, the models are given by DDEs (see e.g. [13]-[19]). There are two types of time delays: (i) discrete delay, where the time delay is assumed to be constant (see e.g. [13]-[15]), (ii) distributed delays, where the time delay is assumed to be random parameter taken from probability distributed function (see e.g. [16]-[19]). Recently, Chowell et al. [20] have studied the spread of a Middle East Respiratory Syndrome

Corona virus (MERS-CoV) by using a SEIR-type compartmental transmission model as:

$$\frac{dS}{dt} = -\frac{\beta\, S\,(I_i + I_s + \ell\, H)}{N} - \alpha, \tag{1}$$
$$\frac{dE_i}{dt} = \alpha - k\, E_i, \tag{2}$$
$$\frac{dE_s}{dt} = \frac{\beta\, S\,(I_i + I_s + \ell\, H)}{N} - k\, E_s, \tag{3}$$
$$\frac{dI_i}{dt} = k\,\rho_{c,i}\, E_i - \gamma_a\, I_i - \gamma_{I,i}\, I_i \tag{4}$$
$$\frac{dA_i}{dt} = k\,(1 - \rho_{c,i})\, E_i, \tag{5}$$
$$\frac{dI_s}{dt} = k\,\rho_{c,s}\, E_s - \gamma_a\, I_s - \gamma_{I,s}\, I_s, \tag{6}$$
$$\frac{dA_s}{dt} = k\,(1 - \rho_{c,s})\, E_s, \tag{7}$$
$$\frac{dH}{dt} = \gamma_a\,(I_i + I_s) - \gamma_r\, H, \tag{8}$$
$$\frac{dR}{dt} = \gamma_r H + \gamma_{I,i}\, I_i + \gamma_{I,s}\, I_s. \tag{9}$$

In model (1)-(9), the populations divided into 9 compartment: susceptible individuals S, individuals exposed to the zoonotic reservoir E_i or to infectious humans E_s, infectious and symptomatic individuals arising from reservoir I_i, or from human-to-human transmission I_s, asymptomatic and non-infectious individuals arising from environmental/animal exposure A_i or arising from human-to-human transmission A_s, hospitalized individuals H, and removed individuals after recovery or disease-induced death R [20]. Susceptible individuals are infected uniformly at random from the zoonotic reservoir at rate α. The parameter β is the mean human-to-human transmission rate per day, ℓ is relative transmissibility of hospitalized cases, $\frac{1}{k}$ mean latent period (days), $\rho_{c,i}$ is proportion of symptomatic and infectious cases among index cases, $\rho_{s,i}$ denote to proportion of symptomatic and infectious cases among secondary cases, $\rho_{h,i}$ proportion of hospitalized individuals among symptomatic and infectious index cases, $\rho_{h,s}$ is proportion of hospitalized individuals among symptomatic and infectious secondary cases, $\frac{1}{\gamma_{I,i}}$ represent the mean infectious period among primary cases (days), $\frac{1}{\gamma_{I,s}}$ is the mean infectious period among secondary cases (days), $\frac{1}{\gamma_a}$ is the mean time from symptom onset to hospital admission (days) and $\frac{1}{\gamma_r}$ denote to mean length of hospital stay (days). Chowell et al., assume that the asymptomatic individuals do not contribute to the transmission process. Moreover, the basic properties of model (1)-(9) are not well studied. Therefore, the aim of this paper is to study the effect of asymptomatic individuals on the transmission of MERS-CoV. Our proposed model is a modification of model (1)-(9) by incorporate the asymptomatic individuals as a carrier individuals. We assume that the first scenario describes only the carrier cases and the second one describes the infected cases which demonstrate symptoms. We introduce two types of discrete time delays into the MERS-CoV model. We study the basic properties of the model such as nonnegativity and boundedness of the solutions, stability analysis of the equilibria. At the end we perform some numerical simulations.

2 The MERS-CoV model

In this section, we propose a MERS-CoV model with two discrete delays . Let us define

$$\Upsilon(t) = S(t)(\beta I_c(t) + \gamma I_m(t) + \ell H(t)).$$

Then we propose the following model:

$$\dot{S}(t) = b - \Upsilon(t) - d_1 S(t), \tag{10}$$

$$\dot{E}_c(t) = p\, e^{-\mu_1 \tau_1}\Upsilon(t-\tau_1) - (k\rho_1 + d_2)E_c(t), \tag{11}$$

$$\dot{E}_m(t) = (1-p)e^{-\mu_2\tau_2}\Upsilon(t-\tau_2) - (k\rho_2 + d_3)E_m(t), \tag{12}$$

$$\dot{I}_c(t) = k\,\rho_1\, E_c(t) - \gamma_a I_c(t) - qI_c(t) - \gamma_1 I_c(t) - d_4 I_c(t), \tag{13}$$

$$\dot{I}_m(t) = k\,\rho_2 E_m(t) - \gamma_a I_m(t) - \gamma_2 I_m(t) + qI_c(t) - d_5 I_m(t), \tag{14}$$

$$\dot{H}(t) = \gamma_a(I_c(t) + I_m(t)) - \gamma_r H(t) - d_6 H(t), \tag{15}$$

$$\dot{R}(t) = \gamma_1 I_c(t) + \gamma_2 I_m(t) + \gamma_r H(t) - d_7 R(t), \tag{16}$$

where S is susceptible individuals, E_c exposed individuals to carrier, E_m exposed individuals to infected, I_c carrier individuals, I_m infected individuals, H hospitalized infected and R recovered individuals. The parameters $\tau_1 \geq 0$ and $\tau_2 \geq 0$ represents for the time between contact the susceptible individuals with exposed to carrier E_c and exposed to infected E_m, respectively. The factors $e^{-\mu_1\tau_1}$ and $e^{-\mu_2\tau_2}$ are the probability that an individuals survives during the delay periods $[0, \tau_1]$ and $[0, \tau_2]$, respectively. The other parameters are defined in section 6. The initial conditions of system (10)-(16) are given by

$$\begin{aligned} &S(\theta) = \varphi_1(\theta),\ E_c(\theta) = \varphi_2(\theta),\ E_m(\theta) = \varphi_3(\theta), \\ &I_c(\theta) = \varphi_4(\theta),\ I_m(\theta) = \varphi_5(\theta),\ H(\theta) = \varphi_6(\theta),\ R(\theta) = \varphi_7(\theta), \\ &\varphi_i(\theta) \geq 0,\ \theta \in [-\varrho, 0],\ i = 1, ..., 7, \end{aligned} \tag{17}$$

where, $\varrho = \max\{\tau_1, \tau_2]$ and $(\varphi_1(\theta), \varphi_2(\theta), ..., \varphi_7(\theta)) \in C([-\varrho, 0], \mathbf{R}^7_{\geq 0})$ where C is the Banach space of continuous functions mapping the interval $[-\varrho, 0]$ into $\mathbf{R}^7_{\geq 0}$. By the fundamental theory of functional differential equations [21], system (10)-(16) has a unique solution satisfying the initial conditions.

3 Nonnegativity and boundedness

In this section, we will study the nonnegativity and boundedness of the model's solutions.

Theorem 1. The solutions of system (10)-(16) are nonnegative and there exist a positive number Q such that the compact set:
$\Gamma = \{(S,\, E_c,\, E_m,\, I_c,\, I_m,\, H,\, R) \in \mathbf{R}^7_{\geq 0} : 0 \leq\ S,\, E_c,\, E_m,\, I_c,\, I_m,\, H,\, R\ \leq Q\}$
is positively invariant.

Proof First, we show the nonnegativity solutions, we will write the system in the matrix form as $\dot{Y} = \phi(Y)$, where $Y = (S,\, E_c,\, E_m,\, I_c,\, I_m,\, H,\, R)^T$ and $\phi = (\phi_1, \phi_2, \phi_3,\, \phi_4,\, \phi_5,\, \phi_6,\, \phi_7)^T$. Then,

$$\begin{pmatrix} \phi_1(Y) \\ \phi_2(Y) \\ \phi_3(Y) \\ \phi_4(Y) \\ \phi_5(Y) \\ \phi_6(Y) \\ \phi_7(Y) \end{pmatrix} = \begin{pmatrix} b - \Upsilon(t) - d_1\, S(t) \\ p\, e^{-\mu_1\,\tau_1}\,\Upsilon(t-\tau_1) - k\,\rho_1\, E_c(t) - d_2\, E_c(t) \\ (1-p)\, e^{-\mu_2\,\tau_2}\,\Upsilon(t-\tau_2) - k\,\rho_2\, E_m(t) - d_3\, E_m(t) \\ k\,\rho_1\, E_c(t) - \gamma_a\, I_c(t) - q\, I_c(t) - \gamma_1\, I_c(t) - d_4\, I_c(t) \\ k\,\rho_2\, E_m(t) - \gamma_a\, I_m(t) - \gamma_2\, I_m(t) + q\, I_c(t) - d_5\, I_m(t) \\ \gamma_a\,(I_c(t) + I_m(t)) - \gamma_r\, H(t) - d_6\, H(t) \\ \gamma_1\, I_c(t) + \gamma_2\, I_m(t) + \gamma_r\, H(t) - d_7\, R(t) \end{pmatrix}.$$

It is easy to see that functions ϕ_i satisfies the following condition

$$\phi_i(Y(t))|_{Y_i(t)=0, Y(t)\in\mathbf{R}^7_{\geq 0}} \geq 0\,.$$

Due to lemma 2 in [22], any solution of (10)-(16) with initial (17) is such that $Y(t) \in \mathbf{R}^7_{\geq 0}$ for all $t \geq 0$. Next, we prove the ultimate bound of the solutions of system (10)-(16). Let us define
$L(t) = p\, e^{-\mu_1\tau_1} S(t-\tau_1) + (1-p)\, e^{-\mu_2\tau_2} S(t-\tau_2) + E_c(t) + E_m(t) + I_c(t) + I_m(t) + H(t) + R(t)$.
Then,

$$
\begin{aligned}
\dot{L}(t) = &\, p\, e^{-\mu_1\tau_1} \left(b - \Upsilon(t-\tau_1) - d_1 S(t-\tau_1)\right) \\
&+ (1-p)\, e^{-\mu_2\tau_2} \left(b - \Upsilon(t-\tau_2) - d_1\, S(t-\tau_2)\right) \\
&+ p\, e^{-\mu_1\tau_1}\, \Upsilon(t-\tau_1) - k\,\rho_1\, E_c(t) - d_2\, E_c(t) \\
&+ (1-p) e^{-\mu_2\tau_2}\, \Upsilon(t-\tau_2) - k\rho_2 E_m(t) - d_3 E_m(t) \\
&+ k\,\rho_1\, E_c(t) - \gamma_a\, I_c(t) - q\, I_c(t) - \gamma_1\, I_c(t) - d_4\, I_c(t) \\
&+ k\,\rho_2\, E_m(t) - \gamma_a\, I_m(t) - \gamma_2\, I_m(t) + q\, I_c(t) - d_5\, I_m(t) + \gamma_a\, (I_c(t) + I_m(t)) \\
&- \gamma_r\, H(t) - d_6\, H(t) + \gamma_1\, I_c(t) + \gamma_2\, I_m(t) + \gamma_r\, H(t) - d_7\, R(t), \\
= &\, (p\, e^{-\mu_1\tau_1} + (1-p)\, e^{-\mu_2\tau_2})\, b - p\, e^{-\mu_1\tau_1}\, d_1\, S(t-\tau_1) - (1-p)\, e^{-\mu_2\tau_2} d_1\, S(t-\tau_2) \\
&- d_2 E_c(t) - d_3\, E_m(t) - d_4 I_c(t) - d_5 I_m(t) - d_6 H(t) - d_7 R(t) \\
\leq &\, b - \overline{d}\, L(t),
\end{aligned}
$$

where $\overline{d} = min\{d_i\},\ \ i = 1, .., 7$. It follows that, $\limsup_{t\to\infty} L(t) \leq Q$, where $Q = \frac{b}{\overline{d}}$. Then, $\limsup_{t\to\infty} S(t) \leq Q, \limsup_{t\to\infty} E_c(t) \leq Q, \limsup_{t\to\infty} E_m(t) \leq Q, \limsup_{t\to\infty} I_c(t) \leq Q, \limsup_{t\to\infty} I_m(t) \leq Q, \limsup_{t\to\infty} H(t) \leq Q$, and $\limsup_{t\to\infty} R(t) \leq Q$. $\square$

4 Equilibria and biological thresholds

To calculated the equilibria of model (10)-(16), we put the R.H.S of Eqs. (10)-(16) equals zero, we get

$$
b - S\,(d_1 + \beta\, I_c + \gamma\, I_m + \ell\, H) = 0, \tag{18}
$$

$$
p\, e^{-\mu_1\tau_1} S\,(\beta\, I_c + \gamma\, I_m + \ell\, H) - a_1\, E_c = 0, \tag{19}
$$

$$
(1-p)\, e^{-\mu_2\tau_2} S(\,(\beta\, I_c + \gamma\, I_m + \ell\, H) - a_2\, E_m, = 0, \tag{20}
$$

$$
\lambda_1\, E_c - a_3\, I_c = 0, \tag{21}
$$

$$
\lambda_2\, E_m - a_4\, I_m + q\, I_c = 0, \tag{22}
$$

$$
\gamma_a\, (I_c + I_m) - a_5\, H = 0, \tag{23}
$$

$$
\gamma_1\, I_c + \gamma_2\, I_m + \gamma_r\, H - d_7\, R = 0, \tag{24}
$$

where

$$
\begin{aligned}
a_1 &= k\,\rho_1\, + d_2, & a_2 &= k\,\rho_2\, + d_3 \\
a_3 &= \gamma_a + \gamma_1 + q + d_4, & a_4 &= \gamma_a + \gamma_2 + d_5, \\
a_5 &= \gamma_r + d_6, & \lambda_1 &= k\rho_1,\ \ \lambda_2 = k\rho_2.
\end{aligned}
$$

Solving system (18)-(24), we find that the system has two equilibria

- The disease-free equilibrium

$$
P_0\, = (S_0,\, 0,\, 0,\, 0,\, 0,\, 0,\, 0) = \left(\frac{b}{d_1},\, 0,\, 0,\, 0,\, 0,\, 0,\, 0\right). \tag{25}
$$

- The endemic equilibrium

$$P^* = (S^*, E_c^*, E_m^*, I_c^*, I_m^*, H^*, R^*), \tag{26}$$

where

$$S^* = \frac{a_0}{A_1}, \ E_c^* = \frac{p(A_2 - a_0\, d_1\, e^{\mu_2\, \tau_2})}{a_1\, A_3}, \ E_m^* = \frac{(1-p)\,(A_4 - a_0\, d_1\, e^{\mu_1 \tau_1})}{a_2\, A_3},$$
$$I_c^* = \frac{\lambda_1\, p\,(A_2 - a_0\, d_1\, e^{\mu_2 \tau_2})}{a_1\, a_3\, A_3}, \ I_m^* = \frac{A_5 + a_2\,(A_6 - 2\, a_1\, a_3\, A_7)}{a_1\, a_2\, a_3\, a_4\, A_3},$$
$$H^* = \frac{\gamma_a\,(2\, a_1\, a_3\,(a_2\, A_{10} + A_9) - A_8)}{a_0\, A_3}, \ R^* = \frac{A_{11} - a_1\, a_3\,(A_{12} + a_2\, A_{13})}{d_1\, a_0\, A_3},$$

and

$$\begin{aligned}
a_0 &= a_1\, a_2\, a_3\, a_4\, a_5,\\
A_1 &= ((a_4\beta + \gamma\, q)a_5 + \gamma_a \ell(q + a_4))a_2\lambda_1\, p\, e^{(-\mu_1\tau_1)} + (1-p)\lambda_2\, a_3\, a_1\,(a_5\gamma + \ell\gamma_a)\, e^{(-\mu_2\tau_2)},\\
A_2 &= ((a_4\beta + \gamma\, q)a_5 + \gamma_a \ell(q + a_4))a_2\lambda_1\, p\, b\, e^{(-\mu_1\tau_1 + \mu_2\tau_2)} + (1-p)\lambda_2\, b\, a_3\, a_1\,(a_5\gamma + \ell\gamma_a),\\
A_3 &= ((a_4\beta + \gamma\, q)a_5 + \gamma_a \ell(q + a_4))a_2\lambda_1\, p\, e^{(\mu_2\tau_2)} + (1-p)\lambda_2\, a_3\, a_1\,(a_5\gamma + \ell\gamma_a)\, e^{(\mu_1\tau_1)},\\
A_4 &= ((a_4\beta + \gamma\, q)a_5 + \gamma_a \ell(q + a_4))a_2\lambda_1\, p\, b + (1-p)\lambda_2\, a_3\, a_1\, b\,(a_5\gamma + \ell\gamma_a)\, e^{(\mu_1\tau_1 - \mu_2\tau_2)},\\
A_5 &= b\,\lambda_2^2\, a_1^2\, a_3^2\,(p-1)^2\,(a_5\gamma + \ell\gamma_a)\, e^{\mu_1\tau_1 - \mu_2\tau_2},\\
A_6 &= q\, b\,\lambda_1^2\,((a_4\beta + \gamma\, q)\, a_5 + \ell\,\gamma_a\,(q + a_4))\, p^2 a_2\, e^{-\mu_1\tau_1 + \mu_2\tau_2},\\
A_7 &= -\frac{1}{2}\, d_1\lambda_2\, a_1\, a_3\, a_4\, a_5\,(p-1)\, e^{\mu_1\tau_1} + \lambda_1\, \left(\frac{1}{2}\, d_1\, e^{\mu_2\tau_2}\, q\, a_2\, a_4\, a_5\right.\\
&\quad \left. + b\,\left(\left(\frac{1}{2}\, a_4\beta + \gamma\, q\right)\, a_5 + \ell\gamma_a\,\left(q + \frac{1}{2} a_4\right)\right)\,(p-1)\lambda_2\right)\, p,\\
A_8 &= b\,\lambda_1^2\,((a_4\beta + \gamma\, q)\, a_5 + \ell\,\gamma_a\,(q + a_4))\,(q + a_4)\, p^2 a_2^2\, e^{-\mu_1\tau_1 + \mu_2\tau_2},\\
A_9 &= -\frac{1}{2}(b\lambda_2^2\, a_1\, a_3\,(p-1)^2\,(a_5\gamma + \ell\gamma_a)\, e^{\mu_1\tau_1 - \mu_2\tau_2}),\\
A_{10} &= -\frac{1}{2}\, d_1\lambda_2\, a_1\, a_3\, a_4 a_5\,(p-1)\, e^{\mu_1\tau_1} + \lambda_1\, \frac{1}{2}\, d_1\, a_2\, a_4\, a_5\,(q + a_4)\, e^{\mu_2\tau_2}\\
&\quad + b\,(p-1)\,\left(\left(\left(\frac{1}{2}\,\beta + \frac{1}{2}\gamma\right)\, a_4 + \gamma\right)\, a_5 + \ell\gamma_a\,(q + a_4)\right)\lambda_2\, p,\\
A_{11} &= b\,\lambda_1^2\,((a_4\gamma_1 + \gamma_2\, q)\, a_5 + \gamma_r\gamma_a\,(q + a_4))\, p^2\, a_2^2\,((a_4\beta + \gamma\, q)\, a_5 + \ell\gamma_a\,(q + a_4))\, e^{-\mu_1\tau_1 + \mu_2\tau_2},\\
A_{12} &= -b\lambda_2^2\, a_1\, a_3\,(p-1)^2\,(a_5\gamma_2 + \gamma_a\gamma_r)\,(a_5\gamma + \ell\,\gamma_a)\, e^{\mu_1\tau_1 - \mu_2\tau_2},\\
A_{13} &= -d_1\lambda_2\, a_1\, a_3\, a_4\, a_5\,(p-1)\,(a_5\,\gamma_2 + \gamma_a\gamma_r)\, e^{\mu_1\tau_1} + \lambda_1\,(((a_4\gamma_1 + \gamma_2\, q)\, a_5\\
&\quad + \gamma_r\gamma_a\,(q + a_4))\, a_4\, a_2\, a_5\, d_1\, e^{\mu_2\tau_2} + \lambda_2\, b\,(((\beta\gamma_2 + \gamma\,\gamma_1)\, a_4 + 2\gamma\,\gamma_2\, q)\, a_5^2\\
&\quad + \gamma_a\,(((\beta + \gamma)\gamma_r + \ell(\gamma_1 + \gamma_2))\, a_4 + 2\, q\,(\gamma\,\gamma_r + \gamma_2\ell))\, a_5 + 2\ell\gamma_a^2\gamma_r(q + a_4))\,(p-1))\, p.
\end{aligned}$$

4.1 Calculating the basic reproduction number

We will apply the next generation method [23] to determine the basic reproduction number R_0 for system (10)-(16). We follow the following steps

(i) We evaluated the matrix F at P_0 as:

$$F = \begin{pmatrix}
0 & 0 & p\, e^{-\mu_1\tau_1}\,\beta\,\frac{b}{d_1} & p\, e^{-\mu_1\tau_1}\,\gamma\,\frac{b}{d_1} & p\, e^{-\mu_1\tau_1}\,\ell\,\frac{b}{d_1}\\
0 & 0 & (1-p)\, e^{-\mu_2\tau_2}\,\beta\,\frac{b}{d_1} & (1-p)\, e^{-\mu_2\tau_2}\,\gamma\,\frac{b}{d_1} & (1-p)\, e^{-\mu_2\tau_2}\,\ell\,\frac{b}{d_1}\\
0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0
\end{pmatrix}.$$

(ii) We get the matrix V at P_0 as:

$$V = \begin{pmatrix} a_1 & 0 & 0 & 0 & 0 \\ 0 & a_2 & 0 & 0 & 0 \\ -\lambda_1 & 0 & a_3 & 0 & 0 \\ 0 & -\lambda_2 & -q & a_4 & 0 \\ 0 & 0 & -\gamma_a & -\gamma_a & a_5 \end{pmatrix}.$$

(iii) Finally, the basic reproduction number is given by

$$R_0 = \rho(FV^{-1}) = \tfrac{A_1\, S_0}{a_0}.$$

4.2 Existence of equilibria

Theorem 2. For system (10)-(16), we have
(i) If $R_0 \le 1$, then there exists only one positive equilibria P_0.
(ii) If $R_0 > 1$, then there exist two positive equilibria P_0 and P^*.

Proof We have

$$\begin{aligned} S^* &= \frac{a_0}{A_1} = \frac{S_0}{R_0}, \\ E_c^* &= \frac{p\,(A_2 - a_0\, d_1\, e^{\mu_2\, \tau_2})}{a_1\, A_3} = \frac{p}{a_1 A_3}\left(\frac{A_2}{e^{\mu_2 \tau_2}} - a_0\, d_1\right) = \frac{p}{a_1 A_3}(b\, A_1 - a_0\, d_1) = \frac{p}{a_1 A_3}(R_0 - 1), \\ E_m^* &= \frac{(1-p)(A_4 - a_0\, d_1\, e^{\mu_1 \tau_1})}{a_2\, A_3} = \frac{(1-p)}{a_2\, A_3}\left(\frac{A_4}{e^{\mu_1 \tau_1}} - a_0\, d_1\right), \\ &= \frac{(1-p)}{a_2\, A_3}(b\, A_1 - a_0\, d_1) = \frac{(1-p)}{a_2\, A_3}(R_0 - 1). \end{aligned}$$

From Eq. (13)-(16), we have

$$I_c^* = \frac{\lambda_1}{a_3} E_c^* = \frac{\lambda_1}{a_3}\, \frac{p}{a_1\, A_3}(R_0 - 1) = \frac{\lambda_1\, p}{a_1\, a_3\, A_3}(R_0 - 1), \tag{27}$$

$$I_m^* = \frac{1}{a_4}(\lambda_2\, E_m^* + q\, I_c^*) = C_1(R_0 - 1), \tag{28}$$

$$H^* = \frac{1}{a_5}(\gamma_a(I_c^* + I_m^*)) = C_2\,(R_0 - 1)), \tag{29}$$

$$R^* = \frac{1}{d_7}(\gamma_1 I_c^* + \gamma_2 I_m^* + \gamma_r H^*) = C_3(R_0 - 1), \tag{30}$$

where,

$$\begin{aligned} C_1 &= \frac{1}{a_4}(\lambda_2\, \frac{(1-p)}{a_2\, A_3} + q\, \frac{\lambda_1\, p}{a_1\, a_3\, A_3}), \\ C_2 &= \frac{\gamma_a}{a_5}(\frac{\lambda_1\, p}{a_1\, a_3\, A_3} + \frac{1}{a_4}(\lambda_2\, \frac{(1-p)}{a_2\, A_3} + q\, \frac{\lambda_1\, p}{a_1\, a_3\, A_3})), \\ C_3 &= \frac{1}{d_7}(\gamma_1\, \frac{\lambda_1\, p}{a_1\, a_3\, A_3} + \gamma_2\, C_1 + \gamma_r\, C_2\,). \end{aligned} \tag{31}$$

$\square$

5 Global stability analysis of P_0

In this section, we use Lyapunov function and LaSalle's invariance principle to establish the global stability of P_0.

Theorem 3. For system (10)-(16), if $R_0 \leq 1$, then P_0 is GAS.

Proof We define the following Lyapunov functional

$$\begin{aligned}
W_0 = S_0\left(\frac{S}{S_0} - 1 - \ln\frac{S}{S_0}\right) &+ \varepsilon_1\, E_c + \varepsilon_2\, E_m + \varepsilon_3 I_c + \varepsilon_4 I_m + \varepsilon_5 H \\
&+ \varepsilon_6 \int_0^{\tau_1} S(t-s)(\beta I_c(t-s) + \gamma I_m(t-s) + \ell H(t-s))\, ds \\
&+ \varepsilon_7 \int_0^{\tau_2} S(t-s)(\beta I_c(t-s) + \gamma I_m(t-s) + \ell H(t-s))\, ds.
\end{aligned}$$

The time derivative of W_0 along the trajectory of system (10)-(16) is given by

$$\begin{aligned}
\frac{dW_0}{dt} = &\left(1 - \frac{S_0}{S(t)}\right)(b - S(t)\,(\beta\, I_c(t) + \gamma\, I_m(t) + \ell\, H(t)) - d_1\, S(t)) \\
&+ \varepsilon_1\,(p\, e^{-\mu_1\tau_1}\, S(t-\tau_1)\,(\beta\, I_c(t-\tau_1) + \gamma\, I_m(t-\tau_1) + \ell\, H(t-\tau_1)) - a_1\, E_c(t)) \\
&+ \varepsilon_2\,((1-p)\, e^{-\mu_2\tau_2}\, S(t-\tau_2)\,(\beta\, I_c(t-\tau_2) + \gamma\, I_m(t-\tau_2) + \ell\, H(t-\tau_2)) - a_2\, E_m(t)) \\
&+ \varepsilon_3\,(\lambda_1 E_c(t) - a_3 I_c(t)) + \varepsilon_4\,(\lambda_2 E_m(t) - a_4 I_m(t) + q\, I_c(t)) + \varepsilon_5\,(\gamma_a(I_c(t) + I_m) - a_5 H(t)) \\
&+ \varepsilon_6\,\{(S(t)\,(\beta\, I_c(t) + \gamma\, I_m(t) + \ell\, H(t)))\; - \;(S(t-\tau_1)\,(\beta\, I_c(t-\tau_1) + \gamma\, I_m(t-\tau_1) + \ell H(t-\tau_1)))\} \\
&+ \varepsilon_7\,\{(S(t)\,(\beta\, I_c(t) + \gamma\, I_m(t) + \ell\, H(t)))\; - \;(S(t-\tau_2)\,(\beta\, I_c(t-\tau_2) + \gamma\, I_m(t-\tau_2) + \ell H(t-\tau_2)))\},
\end{aligned} \tag{32}$$

The parameters $\varepsilon_i,\ i = 1, ..., 7$ are chosen such that

$$\varepsilon_6 + \varepsilon_7 = 1, \tag{33}$$

$$p\,\varepsilon_1\, e^{-\mu_1\tau_1} - \varepsilon_6 = 0, \tag{34}$$

$$(1-p)\,\varepsilon_2\, e^{-\mu_2\tau_2} - \varepsilon_7 = 0, \tag{35}$$

$$-\varepsilon_1\, a_1 + \lambda_1\varepsilon_3 = 0, \tag{36}$$

$$-\varepsilon_2\, a_2 + \lambda_2\varepsilon_4 = 0, \tag{37}$$

$$-a_3\,\varepsilon_3 + q\,\varepsilon_4 + \gamma_a\,\varepsilon_5 + \beta S_0 = 0, \tag{38}$$

$$-a_4\,\varepsilon_4 + \gamma_a\,\varepsilon_5 + \gamma S_0 = 0. \tag{39}$$

Solving Eqs. (33)-(39), we get

$$\varepsilon_5 = \frac{G(1 - R_0) + \ell S_0}{a_5},$$

where

$$G = \frac{a_1 a_2 a_3 a_4 a_5}{\gamma_a(\lambda_1 p a_2\,(a_4 + q)\, e^{-\mu_1\tau_1} + \lambda_2\, e^{-\mu_2\tau_2}\, a_1\, a_3\,(1-p))}.$$

We can see that $\varepsilon_5 > 0$ if $R_0 \leq 1$.

From Eqs. (34)-(38) we get

$$\begin{aligned}
\varepsilon_4 &= \frac{1}{a_4}\left(\gamma_a\,\varepsilon_5 + \gamma S_0\right) \,>\, 0.\\
\varepsilon_3 &= \frac{1}{a_3}\left(q\varepsilon_4 + \gamma_a\varepsilon_5 + \beta S_0\right) \,>\, 0.\\
\varepsilon_2 &= \frac{\lambda_2\varepsilon_4}{a_2} \,>\, 0.\\
\varepsilon_1 &= \frac{\lambda_1\,\varepsilon_3}{a_1} \,>\, 0.\\
\varepsilon_7 &= (1-p)\,\varepsilon_2\,e^{-\mu_2\tau_2} > 0.\\
\varepsilon_6 &= p\,\varepsilon_1\,e^{-\mu_1\tau_1} \,>\, 0,
\end{aligned}$$

Thus, Eq. (32) becomes

$$\frac{dW_0}{dt} = -b\frac{(S-S_0)^2}{S} + (\ell S_0 - a_5\varepsilon_5)\,H, \tag{40}$$

we have

$$\ell S_0 - a_5\varepsilon_5 = G(R_0 - 1).$$

. Then

$$\frac{dW_0}{dt} = -b\frac{(S-S_0)^2}{S} + \frac{G}{a_5}\,(R_0-1)\,H, \tag{41}$$

From Eq (41), $\frac{dW_0}{dt} \le 0$ if $R_0 \le 1$. Then, $\frac{dW_0}{dt}$ equal to zero if $S = S_0$ and $H = 0$. Let $\Omega = \{(S, E_c, E_m, I_c, I_m, H, R) : S = S_0, H = 0\}$. From system (10)-(16), if $H = 0$, then $\dot{H} = 0$ and $0 = \gamma_a(I_c + I_m)$. Since, $I_c \ge 0$, $I_m \ge 0$ then $I_c = 0, I_m = 0, \Rightarrow \dot{I_c} = \dot{I_m} = 0$. From system (10)-(16), we have $0 = \dot{I_c} = \lambda_1 E_c \Rightarrow E_c = 0$. Similarly, we have $0 = \dot{I_m} = \lambda_2 E_m \Rightarrow E_m = 0$. Finally, $\dot{R}(t) = -d_7 R$ it follows that $R \to 0$ as $t \to \infty$. From LaSalle's invariance principle, P_0 is GAS in Γ. □

6 Numerical simulations and discussions

In this section, we introduce the numerical results of system (10)-(16). We consider the following initial conditions
IC : $S(\theta) = 600,\ E_c(\theta) = 30,\ E_m(\theta) = 80,\ I_c(\theta) = 3,\ I_m(\theta) = 12,\ H(\theta) = 8,\ R(\theta) = 40$, $\theta \in [-\max\{\tau_1, \tau_2\}, 0]$. we use the values of the parameters in Table 1. In addition we choose $\mu_1 = \mu_2 = 1$.

We study the following cases:

6.1 Effect of parameters β, γ and ℓ on the stability of equilibria:

In this case, we fix the values $\tau_1 = \tau_2 = 0.01$. Figure 1 shows the evaluation of system states for two scenarios:

i) $R_0 \le 1$. We choose $\beta = 0.002$, $\gamma = 0.0001$, and $\ell = 0.0001$ then we compute $R_0 = 0.23$. We can see from the figure that the states of the system approach $P_0 = (1000, 0, 0, 0, 0, 0, 0)$. This means that according to Theorem 3 P_0 is GAS.

ii) $R_0 > 1$. We choose $\beta = 0.02$, $\gamma = 0.001$, and $\ell = 0.001$ then we compute $R_0 = 2.37$ and $P^* = (421.6, 55.4, 129.2, 5.08, 20.9, 14.5, 72.4)$. Then P^* exists and this confirm the results of Theorem 2. Figure 1 shows that the states of the system converge to P^*.

Table 1: The parameters values of MERS-CoV model

Symbol	Parameter	Value
b	Rate of generation of new susceptible individuals	100
β	Rate constant of transmission for carriers	Varied
γ	Rate constant of symptomatically infected individuals	Varied
ℓ	Relative transmissibility of hospitalized cases	Varied
γ_a	Mean time from carrier and infected to hospital admission (days)	0.3
d_1	Death rate of susceptible individuals	0.1
d_2	Death rate of exposed to carrier	0.2
d_3	Death rate of exposed to infected individuals	0.2
d_4	Death rate of carrier individuals	0.2
d_5	Death rate of infected individuals	0.3
d_6	Death rate of hospitalized individuals	0.4
d_7	Death rate of recovered individuals	0.1
k	Mean latent period	0.19
$\rho_1 = \rho_2$	Proportion of carrier and infected cases	0.58
$\gamma_1 = \gamma_2$	Mean infectious period	0.2
γ_r	Mean length of hospital stay	0.14
p	Rate of infected individual who becomes carrier	0.3
q	Rate of carrier individual who becomes infected	0.5

6.2 Effect of the time delays on the asymptotic behaviour of the equilibria:

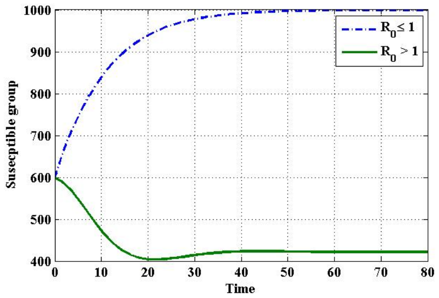
In this case, we take the values $\beta = 0.02$, $\gamma = 0.001$, and $\ell = 0.001$. Let us consider the case $\tau_1 = \tau_2 = \tau$. In Table 2, we present the values of R_0 and the equilibria of system (10)-(16) with different values of τ.From

Table 2: Values of R_0 and steady states of system (10)-(16) with different values of τ

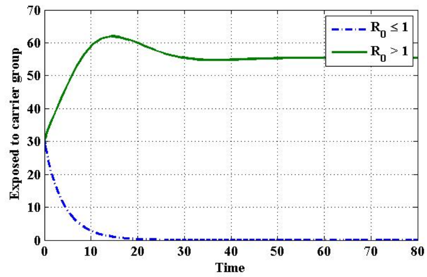
τ	R_0	Steady states
0.067	2.24	$P^* = (446.37, 50.07, 116.83, 4.6, 18.97, 13.09, 65.46)$
0.082	2.21	$P^* = (453.12, 48.73, 113.69, 4.47, 18.97, 12.74, 63.70)$
0.67	1.23	$P^* = (815.79, 9.12, 21.27, 0.84, 3.45, 2.38, 11.92)$
0.8735928143	1.00	$P_0 = (1000, 0, 0, 0, 0, 0, 0)$
1.2	0.72	$P_0 = (1000, 0, 0, 0, 0, 0, 0)$
1.5	0.5	$P_0 = (1000, 0, 0, 0, 0, 0, 0)$
2.5	0.19	$P_0 = (1000, 0, 0, 0, 0, 0, 0)$
3.1	0.11	$P_0 = (1000, 0, 0, 0, 0, 0, 0)$
3.5	0.07	$P_0 = (1000, 0, 0, 0, 0, 0, 0)$

Table 2, we can observe that the value of R_0 is decreased as τ is increased. Moreover, for small values of τ, P^* exists and for large values of τ the system moved from P^* to P_0 with is GAS. Figures 2 shows the effect of the parameter τ on the evaluation of the states of the system. We can see that as the time delay parameter is increased, the number of susceptible individuals are increased and tend to its normal number, while the number

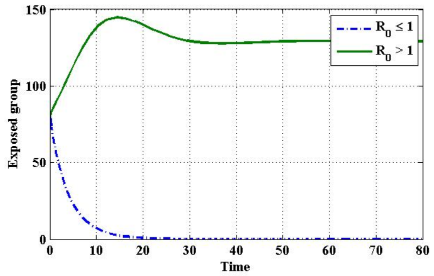
of individuals in other groups is reduced and tends to zero. It means that, the time delay play the role of controlling the disease transmission.



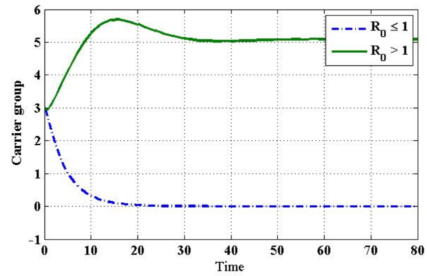
(a) Evaluation of $S(t)$.



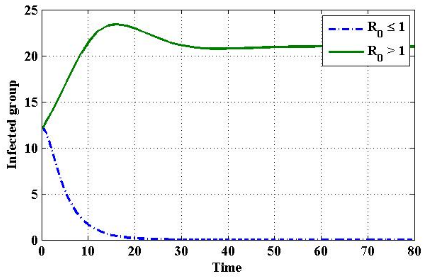
(b) Evaluation of $E_c(t)$.



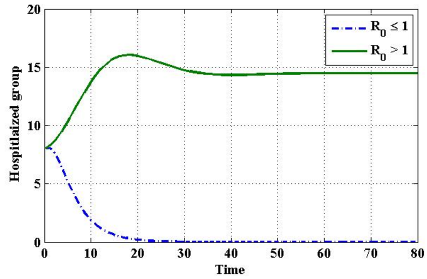
(c) Evaluation of $E_m(t)$.



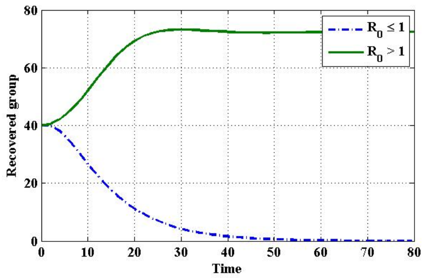
(d) Evaluation of $I_c(t)$.



(e) Evaluation of carrier $I_m(t)$.



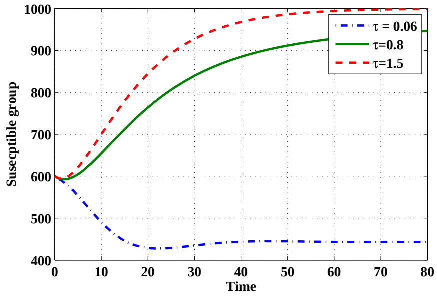
(f) Evaluation of $H(t)$.



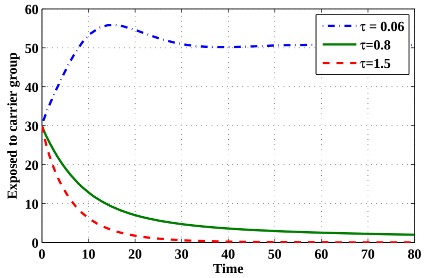
(g) Evaluation of $R(t)$.

Figure 1: The evaluations of the system states (10)-(16) with two delays $\tau_1 = \tau_2 = 0.01$.

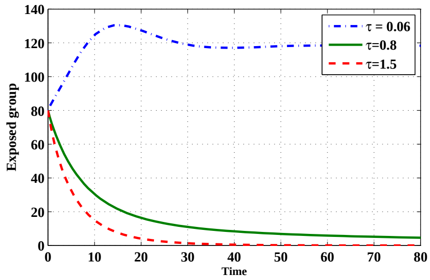
.



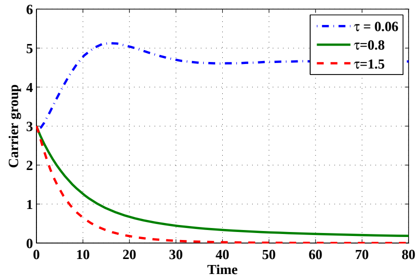
(a) Evaluation of $S(t)$.



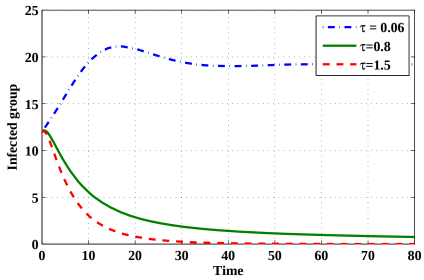
(b) Evaluation of $E_c(t)$.



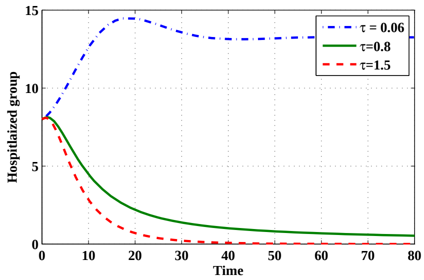
(c) Evaluation of $E_m(t)$.



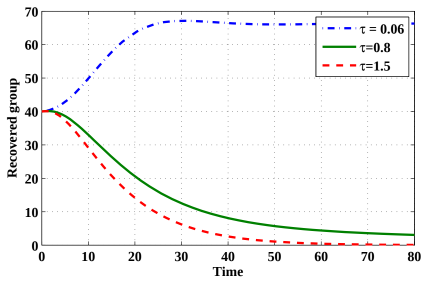
(d) Evaluation of $I_c(t)$.



(e) Evaluation of carrier $I_m(t)$.



(f) Evaluation of $H(t)$.



(g) Evaluation of $R(t)$.

Figure 2: The evaluations of system (10) -(16) with different values of τ.

7 Conclusion

We have proposed a MERS-CoV model with two times delay. We have obtained the biological threshold, the basic reproduction number R_0. The existence of the model's equilibria has been proven. The global asymptotic stability of the disease free equilibria P_0 has been investigated by constructing Lyapunov functional and using LaSalle's invariance principle. To support our theoretical results, we have presented the numerical simulations.

References

[1] Rachsh, A., and Torres, D., (2015), *Mathematical modelling, simulation, and optimal control of 2014 Ebola outbreak in West Africa*, Discrete Dynamics in Nature and Society, Vol. 2015, 1-9.

[2] Elaiw, A., (2012), *Global properties of a class of virus infection models with multi target cells*, Nonlinear Dynamics , Vol.69, 423-435.

[3] Wester, T., (2015), *Analysis and simulation of a mathematical model of Ebola virus dynamics in vivo*, Society for Industrial and Applied Mathematics, Vol. 8, 236-256.

[4] Ullah, R., Zaman, G., and Islam, S., (2013), *Stability analysis of a general SIR epidemic model*, VFAST Transactions on Mathematics, Vol. 1, 16-20.

[5] Ma, X., Zhou, Y., and Cao, H., (2013), *Global stability of the endemic equilibrium of a discrete SIR epidemic model*, Advances in Difference Equations, Vol. 2013, 1-19.

[6] Korobeinikov, A., (2004), *Lyapunov functions and global properties for SEIR and SEIS epidemic models*, Mathematical Medicine and Biology, Vol. 21, 75-83.

[7] Grigorieva, E., Khailov, E., and Korobeinikov A., (2016), *Optimal control for a SIR epidemic model with nonlinear incident rate*, Mathematical Modelling of Natural Phenomena, Vol. 11, 89-104.

[8] Ledzewicz, U., and Schattler, H., (2011), *On optimal Singular control for a general SIR-model with vaccination and treatment*, Discrete and Continuous Dynamical System, Vol. 2011, 981-990.

[9] Bakare1, E., Nwagwo, A., and Danso-Addo, E., (2014), *Optimal control analysis of an SIR epidemic model with constant recruitment*, International Journal of Applied Mathematical Research, Vol. 3, 273-285.

[10] Kara, T., and Batabyalb, A., (2011), *Stability analysis and optimal control of an SIR epidemic model with vaccination*, BioSystems, Vol. 104, 127-135.

[11] Grigorieva, E., Khailov, E., and Korobeinikov, A., (2015), *Optimal control for an epidemic in populations of varying size*, American Institute of Mathematical Sciences, Vol. 2015, 549-561.

[12] Pinho, M., and Nogueira, F., (2017), *On application of optimal control to SEIR normalized models: pros and cons*, Mathematical Biosciences and Engineering, Vol. 14, 111-126.

[13] Rui Xu, (2013), *Global dynamics of a delayed epidemic model with latency and relapse*, Nonlinear Analysis: Modelling and Control, Vol. 18, 250-263.

[14] Gau, S., Chen, L., and Teng, Z., (2008), *Pulse vaccination of an SEIR epidemic model with time delay*, Nonlinear Analysis: Real world Applications, Vol. 9, 599-607.

[15] Gao, S., Teng, Z., and Xie, D., (2008), *The effects of pulse vaccination on SEIR model with two time delays*, Applied Mathematics and Computation, Vol. 201, 282-292.

[16] Connell McCluskey, C., (2010), *Complete global stability for an SIR epidemic model with delay distributed or discrete*, Nonlinear Analysis: Real World Applications, Vol. 11, 55-59.

[17] Enatsu, Y., and Nakata, Y., (2010), *Global stability for a class of discrete SIR epidemic models*, Mathematical Bioscience and Engineering, Vol. 7, 347-361.

[18] Beretta, E., and Takeuchi, Y., (1997), *Convergence results in SIR epidemic models with varying population size*, Nonlinear Analysis: Theory, Methods and Application, Vol. 28, 1909-1921.

[19] Enatsu, Y., Yukihiko and Muroya, Y., (2012), *Global stability of SIRS epidemic models with a class of nonlinear incidence rate and distributed delayes*, Acta Mathematical Scientia, Vol. 32, 851-865.

[20] Chowell, G., Blumberg, S., Simonsen, L., Miller, M., and Viboud, C., (2014), *Synthesizing data and models for the spread of MERS-CoV, 2013:Key role of index cases and hospital transmission*, Epidemics, Vol. 9, 40-51.

[21] Hale, J., and Lunel, S., (1993), Introduction to functional differential equations, *Science and Business Media.*

[22] X. Yang, L.S. Chen and J.F. Chen, (1996), *Permanence and positive periodic solution for the single-species nonautonomous delay diffusive models*, Computers and Mathematics with Applications, Vol. 32, 109-116.

[23] Heffernan, J., smith, R., and Wahl, L., (2005), *Perspectives on the basic reproduction ratio*, Journal of the Royal Society Interface, Vol. 2, 281-293.

CONVEXITY AND HYPERCONVEXITY IN FUZZY METRIC SPACE

EBRU YIĞIT AND HAKAN EFE

Abstract. In this paper, firstly we give the definion of fuzzy convex metric, in a different way. Then we introduce the concept of hyperconvexity in fuzzy metric space and prove that every fuzzy hyperconvex metric space is complete. Also it is proved that for $m-$seperable fuzzy metric spaces, fuzzy $m-$hyperconvexity is equivalent to fuzzy hyperconvexity.

1. INTRODUCTION

The concept of convex metric space has been studied by many authors, in some different ways [7, 9, 11, 14, 15]. After that, some authors examined this concept for fuzzy metric space by using the definition of fuzzy metric which is introduced by George and Veeremani [1], for example; Thanithamil [4] introduced the convex structure in fuzzy metric spaces and Vosoughi and Hosseni [8] gave the definion of metrically convex fuzzy metric space $(X, M, *)$. The other common concept for metric space is hyperconvexity which was introduced by Aronszajn and Panitchpakdi [10] in 1956. Since then many interesting works have been appeared for hyperconvex spaces [5, 11, 13].

In this paper, we give the notion of fuzzy convex metric space by using the closed balls, in a different way. Also, we introduce a new notion for fuzzy metric space which is called fuzzy hyperconvex metric space. One of the main result of this paper is that every fuzzy hyperconvex metric space is complete. Also, the fuzzy $m-$hyperconvexity is introduced for any cardinal $m \geq 3$, which is a weaker property than fuzzy hyperconvexity. The definition $m-$seperability for fuzzy metric space is used, so the other result for this paper is that for any $m-$seperable fuzzy metric spaces, fuzzy $m-$hyperconvexity is equivalent to fuzzy hyperconvexity.

2. PRELIMINARIES

Definition 1. [6] *A binary operation* $* : [0,1] \times [0,1] \to [0,1]$ *is continuous t-norm if* $*$ *satisfies the following conditions:*

(i) $*$ *is commutative and associative;*

(ii) $*$ *is continuous;*

(iii) $a * 1 = a$ *for all* $a \in [0,1]$;

(iv) $a * b \leq c * d$ *whenever* $a \leq c$ *and* $b \leq d$, *for* $a, b, c, d \in [0,1]$.

Remark 1. [1] *(i) For any* $r_1 \in (0,1)$ *with* $r_1 > r_2$, *there exist* $r_3 \in (0,1)$ *such that* $r_1 * r_3 \geq r_2$.

Date: July 8, 2017.

1991 *Mathematics Subject Classification.* 46S40, 54A40 .

Key words and phrases. convexity, hyperconvexity, fuzzy metric space.

(ii) For any $r_4 \in (0,1)$, *there exist* $r_5 \in (0,1)$ *such that* $r_5 * r_5 \geq r_4$.

Definition 2. [1] *The 3-tuple* $(X, M, *)$ *is said to be a fuzzy metric space if* X *is an arbitrary set,* $*$ *is a continuous t-norm and* M *is a fuzzy set on* $X^2 \times (0, \infty)$ *satisfying the following conditions, for all* $x, y, z \in X$ *and* $s, t > 0$:

(FM-1) $M(x, y, t) > 0$,
(FM-2) $M(x, y, t) = 1$ if and only if $x = y$,
(FM-3) $M(x, y, t) = M(y, x, t)$,
(FM-4) $M(x, z, t + s) \geq M(x, y, t) * M(y, z, s)$,
(FM-5) $M(x, y, .) : (0, \infty) \to [0, 1]$ is continuous.

Example 1. [1] *(Induced fuzzy metric). Let* (X, d) *be a metric space. Define* $a * b = \min\{a, b\}$ *for all* $\forall a, b \in [0, 1]$ *and let* M *be fuzzy set on* $X \times X \times (0, \infty)$ *as follows:*

$$M(x, y, t) = \frac{kt^n}{kt^n + md(x, y)},\ k, m, n \in \mathbb{R}^+.$$

Then $(X, M, *)$ *is a fuzzy metric space. In this example by taking* $k = m = n = 1$, *we get*

$$M(x, y, t) = \frac{t}{t + d(x, y)}.$$

We call this fuzzy metric induced by a metric d *the standard fuzzy metric.*

Definition 3. [1] *Let* $(X, M, *)$ *be a fuzzy metric space and let* $r \in (0, 1)$, $t > 0$ *and* $x \in X$. *The open ball and the closed ball with center* x *and radius* r *with respect to* t *are defined as follows, respectively*

$$B_M(x, r, t) = \{y \in X : M(x, y, t) > 1 - r\}$$

$$\bar{B}_M(x, r, t) = \{y \in X : M(x, y, t) \geq 1 - r\}.$$

Remark 2. [1] *Every open ball is an open set and every closed ball is a closed set in a fuzzy metric space* $(X, M, *)$.

Theorem 1. [1] *Let* $(X, M, *)$ *be a fuzzy metric space. Define*

$$\tau_M = \{A \subset X : \forall\ x \in A,\ \exists r \in (0, 1) \text{ and } t > 0\ \ni\ B_M(x, r, t) \subset A\}.$$

Then τ_M *is a topology on* X.

Definition 4. [1] *Let* $(X, M, *)$ *be a fuzzy metric space. Then*

(a) A sequence $\{x_n\}$ *in* X *is said to be Cauchy sequence if for each* $\varepsilon > 0$ *and each* $t > 0$, *there exists* $n_0 \in \mathbb{N}$ *such that* $M(x_n, x_m, t) > 1 - \varepsilon$, *for all* $n, m \geq n_0$.

(b) $(X, M, *)$ *is called complete if every Cauchy sequence is convergent with respect to* τ_M.

Definition 5. [3] *Let* $(X, M, *)$ *be a fuzzy metric space. A collection of sets* $\{F_n\}_{n \in \mathbb{N}}$ *is said to have fuzzy diameter zero if and only if for each pair* $r, t > 0$, *(*$r \in (0, 1)$ *and* $t > 0$*), there exists* $n \in \mathbb{N}$ *such that* $M(x, y, t) > 1 - r$ *for all* $x, y \in F_n$.

Remark 3. [3] *A non-empty subset* F *of a fuzzy metric space* X *has fuzzy diameter zero if and only if* F *is a singleton set.*

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 27, NO.1, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC

Theorem 2. [3] *A necessary and sufficient condition that a fuzzy metric space* $(X, M, *)$ *be complete is that every nested sequence of non-empty closed sets* $\{F_n\}_{n=1}^{\infty}$ *with fuzzy diameter zero have non-empty intersection. And the element* $x \in \bigcap_{n\in\mathbb{N}} F_n$ *is unique.*

Definition 6. [12] *Let* $(X, M, *)$ *be a fuzzy metric space. Let the mappings* $\delta_A(t) : (0, \infty) \longrightarrow [0, 1]$ *be defined as*

$$\delta_A(t) = \inf_{x,y\in A} \sup_{\varepsilon<t} M(x, y, \varepsilon).$$

The constant $\delta_A = \sup_{t>0} \delta_A(t)$ *will be called fuzzy diameter of set A. If* $\delta_A = 1$ *the set A will be called F-strongly bounded.*

Definition 7. [11] *Let* (X, d) *be a metric space. We say that X is metrically convex if for any points* $x_1, x_2 \in X$ *and positive numbers* α *and* β *such that* $d(x_1, x_2) \leq \alpha + \beta$*, there exists* $z \in X$ *such that* $d(x_1, z) \leq \alpha$ *and* $d(x_2, z) \leq \beta$*, or equivalently* $z \in \bar{B}(x_1, \alpha) \cap \bar{B}(x_2, \beta)$.

Definition 8. [11] *Let* (X, d) *be a metric space and* Γ *be an index set. The metric space X is said to has the ball intersection property (BIP in short) if* $\bigcap_{\alpha\in\Gamma} \bar{B}_\alpha \neq \varnothing$ *for any collection of closed balls* $(\bar{B}_\alpha)_{\alpha\in\Gamma}$ *such that* $\bigcap_{\alpha\in\Gamma_f} \bar{B}_\alpha \neq \varnothing$*, for any finite subset* $\Gamma_f \subset \Gamma$.

Definition 9. [11] *Let* (X, d) *be a metric space and* Γ *be an index set. The metric space X is said to be hyperconvex if* $\bigcap_{\alpha\in\Gamma} \bar{B}(x_\alpha, r_\alpha) \neq \varnothing$ *for any collection of points* $\{x_\alpha\}_{\alpha\in\Gamma}$ *in X and positive numbers* $\{r_\alpha\}_{\alpha\in\Gamma}$ *such that* $d(x_\alpha, x_\beta) \leq r_\alpha + r_\beta$ *for any* α *and* β *in* Γ.

Example 2. [11] *The real line* $\mathbb{R}$ *is hyperconvex with the usual metric d.*

Example 3. [11] *The infinite dimensional Banach space* l_∞ *is hyperconvex.*

Definition 10. [10] *A metric space* (X, d) *is called* $m-$*seperable if it contains a dense subset of cardinal* $< m$.

3. MAIN RESULTS

Before we give the definition of fuzzy metrically convexity, we give the following Lemma for the definition to be clear.

Lemma 1. *Let* $(X, M, *)$ *be a fuzzy metric space,* $x_1, x_2 \in X$*,* $r_1, r_2 \in (0, 1)$ *and* $t_1, t_2 \in (0, \infty)$*. If* $\bar{B}_M(x_1, r_1, t_1) \cap \bar{B}_M(x_2, r_2, t_2) \neq \varnothing$ *then* $M(x_1, x_2, t_1 + t_2) \geq (1 - r_1) * (1 - r_2)$ *for any* $x_1, x_2 \in X$ *and each pair of* $r_1, t_1 > 0$ *and* $r_2, t_2 > 0$.

Proof. Let $\bar{B}_M(x_1, r_1, t_1) \cap \bar{B}_M(x_2, r_2, t_2) \neq \varnothing$. Then there exists $z \in X$ such that

$$\begin{aligned} z &\in \bar{B}_M(x_1, r_1, t_1) \cap \bar{B}_M(x_2, r_2, t_2) \\ &\Longrightarrow z \in \bar{B}_M(x_1, r_1, t_1) \text{ and } z \in \bar{B}_M(x_2, r_2, t_2) \\ &\Longrightarrow M(x_1, z, t_1) \geq (1 - r_1) \text{ and } M(x_2, z, t_2) \geq (1 - r_2). \end{aligned}$$

By the Definition 1-(vi) we have $M(x_1, z, t_1) * M(x_2, z, t_2) \geq (1-r_1)*(1-r_2)$ and by the condition (FM-4) of fuzzy metric we get $M(x_1, x_2, t_1+t_2) \geq (1-r_1)*(1-r_2)$. □

The converse of Lemma 1 may not be true. Example 4 explain this situation.

Example 4. *Let $X = \mathbb{N}$. Define $a * b = a.b$ for all $\forall a, b \in [0,1]$ and let M be fuzzy set on $\mathbb{N} \times \mathbb{N} \times (0,\infty)$ as follows:*

$$M(x,y,t) = \frac{\min\{x,y\} + t}{\max\{x,y\} + t}.$$

In this case we know that M is a fuzzy metric on $\mathbb{N}$. If we choose $t_1 = 1$, $t_2 = 1$, $r_1 = 0.3$, $r_2 = 0.5$, $x_1 = 3$ and $x_2 = 10$ then the inequality $M(x_1, x_2, t_1 + t_2) \geq (1-r_1)*(1-r_2)$ is satisfied but $\bar{B}_M(3, 0.3, 1) \cap \bar{B}_M(10, 0.5, 1) = \varnothing$ and so we can not find any point $z \in X$ such that $M(x_1, z, t_1) \geq (1 - r_1)$ and $M(x_2, z, t_2) \geq (1 - r_2)$.

Consequently, when the converse of Lemma 1 also be true, we give Definition 11.

Definition 11. *Let $(X, M, *)$ be a fuzzy metric space. We say that X is fuzzy metrically convex if for any points $x_1, x_2 \in X$ and for each pair $r_1, t_1 > 0$ and $r_2, t_2 > 0$ ($r_1, r_2 \in (0,1)$ and $t_1, t_2 \in (0,\infty)$) such that $M(x_1, x_2, t_1 + t_2) \geq (1 - r_1)*(1-r_2)$, there exists $z \in X$ such that $M(x_1, z, t_1) \geq (1-r_1)$ and $M(x_2, z, t_2) \geq (1 - r_2)$ or equivalently $z \in \bar{B}_M(x_1, r_1, t_1) \cap \bar{B}_M(x_2, r_2, t_2)$.*

Example 5. *Let the metric space (X, d) be metrically convex. Define continuous t-norm as $a * b = a.b$ for all $\forall a, b \in [0,1]$ and let M be fuzzy set on $X \times X \times (0,\infty)$ as follows:*

$$M(x,y,t) = e^{\frac{-d(x,y)}{t}}.$$

Then the 3-tuple $(X, M, *)$ is a fuzzy metric space and under these conditions $(X, M, *)$ is fuzzy metrically convex. Indeed, let (X, d) be metrically convex then for any points $x_1, x_2 \in X$ and positive numbers α and β such that $d(x_1, x_2) \leq \alpha + \beta$, there exists $z \in X$ such that $d(x_1, z) \leq \alpha$ and $d(x_2, z) \leq \beta$, or equivalently $z \in B(x_1, \alpha) \cap B(x_2, \beta)$. Take $\alpha = -t_1 \ln(1 - r_1)$ and $\beta = -t_2 \ln(1 - r_2)$. By the choices of α, β, the inequality $M(x_1, x_2, t_1 + t_2) \geq (1 - r_1) * (1 - r_2)$ is satisfied and also r_1, $r_2 \in (0,1)$. By using the metrically convexity of (X, d);

$$\begin{aligned}
d(x_1, z) &\leq -t_1 \ln(1 - r_1) \text{ and } d(x_2, z) \leq -t_2 \ln(1 - r_2) \\
&\Longrightarrow -d(x_1, z) \geq t_1 \ln(1 - r_1) \text{ and } -d(x_2, z) \geq t_2 \ln(1 - r_2) \\
&\Longrightarrow e^{-d(x_1,z)} \geq e^{t_1 \ln(1-r_1)} \text{ and } e^{-d(x_2,z)} \geq e^{t_2 \ln(1-r_2)} \\
&\Longrightarrow e^{\frac{-d(x_1,z)}{t_1}} \geq (1 - r_1) \text{ and } e^{\frac{-d(x_2,z)}{t_2}} \geq (1 - r_2) \\
&\Longrightarrow M(x_1, z, t_1) \geq (1 - r_1) \text{ and } M(x_2, z, t_2) \geq (1 - r_2).
\end{aligned}$$

This implies that $z \in \bar{B}_M(x_1, r_1, t_1) \cap \bar{B}_M(x_2, r_2, t_2)$, then the fuzzy metric space $(X, M, *)$ is fuzzy metrically convex.

Definition 12. *Let $(X, M, *)$ be a metric space, Γ be an index set, $r_i \in (0,1)$ and $t_i \in (0,\infty)$ for all $i \in \Gamma$. The fuzzy metric space X is said to has the ball intersection property (BIP in short) if $\bigcap_{i \in \Gamma} \bar{B}_M(x_i, r_i, t_i) \neq \varnothing$ for any collection of closed balls $(\bar{B}_M(x_i, r_i, t_i))_{i \in \Gamma}$ such that $\bigcap_{i \in \Gamma_f} \bar{B}_M(x_i, r_i, t_i) \neq \varnothing$ for any finite subset $\Gamma_f \subset \Gamma$.*

Definition 13. *Let $(X, M, *)$ be a metric space, Γ be an index set, $r_i \in (0,1)$ and $t_i \in (0,\infty)$ for all $i \in \Gamma$. The fuzzy metric space X is said to be fuzzy hyperconvex*

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 27, NO.1, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC

if for any indexed class of closed balls $\bar{B}_M(x_i, r_i, t_i)$ in X, satifying the condition that

$$M(x_i, x_j, t_i + t_j) \geq (1 - r_i) * (1 - r_j)$$

for all $i, j \in \Gamma$, the intersection $\bigcap_{i \in \Gamma} \bar{B}_M(x_i, r_i, t_i) \neq \varnothing$.

Theorem 3. *Let $(X = \mathbb{R}, d)$ be the usual metric space. Consider the standard fuzzy metric M where $M(x, y, t) = \frac{t}{t + d(x,y)}$ with $a * b = \min\{a, b\}$ for all $a, b \in (0, 1)$. Then $(X, M, *)$ is fuzzy metrically hyperconvex (or fuzzy hyperconvex).*

Proof. Since $(\mathbb{R}, d)$ is hyperconvex, for any collection of closed balls $\bar{B}(x_\alpha, r_\alpha)$ satisfying the condition that $d(x_\alpha, x_\beta) \leq r_\alpha + r_\beta$ for any α and β in Γ, the intersection $\bigcap_{\alpha \in \Gamma} \bar{B}(x_\alpha, r_\alpha) \neq \varnothing$. Now choose $R_\alpha = \frac{r_\alpha}{t_\alpha + r_\alpha}$ ve $R_\beta = \frac{r_\beta}{t_\beta + r_\beta}$. It is clear that R_α, $R_\beta \in (0, 1)$ and by these choices and the minimum t-norm, $M(x_\alpha, x_\beta, t_\alpha + t_\beta) \geq (1 - R_\alpha) * (1 - R_\beta) = \min\{(1 - R_\alpha), (1 - R_\beta)\} = (1 - R_\alpha)$ (without lost generality we can take $R_\alpha \geq R_\beta$) is satisfied. By the hyperconvexity of $(\mathbb{R}, d)$;

$$\begin{aligned}
\bigcap_{\alpha \in \Gamma} \bar{B}(x_\alpha, r_\alpha) &\neq \varnothing, \text{ for all } \alpha \in \Gamma \text{ , then there exsists } z \in X \text{ such that} \\
z &\in \bigcap_{\alpha \in \Gamma} \bar{B}(x_\alpha, r_\alpha) \\
&\Longrightarrow d(x_\alpha, z) \leq r_\alpha \\
&\Longrightarrow t_\alpha + d(x_\alpha, z) \leq t_\alpha + r_\alpha \\
&\Longrightarrow \frac{t_\alpha}{t_\alpha + d(x_\alpha, z)} \geq \frac{t_\alpha}{t_\alpha + r_\alpha} \\
&\Longrightarrow M(x_\alpha, z, t_\alpha) \geq 1 - R_\alpha \\
&\Longrightarrow z \in \bar{B}_M(x_\alpha, r_\alpha, t_\alpha), \text{ for all } \alpha \in \Gamma \\
&\Longrightarrow z \in \bigcap_{i \in \Gamma} \bar{B}_M(x_\alpha, r_\alpha, t_\alpha).
\end{aligned}$$

So $(\mathbb{R}, M, *)$ is fuzzy hyperconvex. □

Example 6. *In particular if we take $t = 1$ in the Theorem 3 M becomes $M(x, y) = \frac{1}{1 + d(x,y)}$. M is stationary fuzzy metric on $\mathbb{R}$ with the continuous minimum t-norm and $(\mathbb{R}, M, *)$ is fuzzy hyperconvex.*

Proposition 1. *If the space $(X, M, *)$ is fuzzy hyperconvex then it has the ball intersection property.*

Proof. Let $(X, M, *)$ be fuzzy hyperconvex and be $\bigcap_{i \in \Gamma_f} \bar{B}_M(x_i, r_i, t_i) \neq \varnothing$ for any finite subset $\Gamma_f \subset \Gamma$. Then it follows that

$$\begin{aligned}
\bigcap_{i \in \Gamma_f} \bar{B}_M(x_i, r_i, t_i) &\neq \varnothing, \text{ then there exists } z \in X \text{ such that} \\
z &\in \bigcap_{i \in \Gamma_f} \bar{B}_M(x_i, r_i, t_i) \text{ for } i = \{1, 2, \ ..., \ n\} \\
&\Longrightarrow z \in \bar{B}_M(x_1, r_1, t_1) \cap \bar{B}_M(x_2, r_2, t_2) \cap ... \cap \bar{B}_M(x_n, r_n, t_n) \\
&\Longrightarrow M(x_1, z, t_1) \geq (1 - r_1), M(x_2, z, t_2) \geq (1 - r_2), ..., M(x_n, z, t_n) \geq (1 - r_n)
\end{aligned}$$

so $M(x_i, z, t_i) \geq (1 - r_i)$ and $M(x_j, z, t_j) \geq (1 - r_j)$ for arbitrary $i, j \in \Gamma_f$. By the condition (FM-4)

$$M(x_i, x_j, t_i + t_j) \geq M(x_i, z, t_i) * M(x_j, z, t_j) \geq (1 - r_i) * (1 - r_j). \tag{3.1}$$

Since $(X, M, *)$ is fuzzy hyperconvex and the inequality (3.1) is satisfied for all $i, j \in \Gamma$, then $\bigcap_{i \in \Gamma} \bar{B}_M(x_i, r_i, t_i) \neq \varnothing$ for all $i \in \Gamma$ and so $(X, M, *)$ has the ball intersection property. □

Theorem 4. *Any fuzzy metric space $(X, M, *)$ which has the ball intersection property is complete. In particular any fuzzy hyperconvex metric space is complete.*

Proof. Let $(X, M, *)$ be a fuzzy metric space which has ball intersection property and let $\{x_n\}$ be a Cauchy sequence in X. For any $n \geq 1$, take the set

$$r_n = \sup_{t_n > 0} \left\{ \inf_{m \geq n} \left\{ \sup_{s < t_n} \{M(x_n, x_m, s)\} \right\} \right\}.$$

Consider the collection of closed balls $\left(\bar{B}_M(x_n, r_n, t_n)\right)_{n \geq 1}$. Since $\{x_n\}$ is Cauchy and by the choice of r_n, for $m \geq n$ we have $M(x_n, x_m, t_n) \geq 1 - r_n$ i.e $\{r_n\}$ has fuzzy diameter zero. Now we examine this situation for any finite index $n_1 < n_2 < ... < n_k$. For $n_1 < n_2 < ... < n_k$, we have

$$M(x_{n_1}, x_{n_k}, t_{n_1}) \geq 1 - r_{n_1},\ M(x_{n_2}, x_{n_k}, t_{n_2}) \geq 1 - r_{n_2},\ ...,\ M(x_{n_k}, x_{n_k}, t_{n_k}) \geq 1 - r_{n_k}$$

which means that

$$\begin{array}{rcl} x_{n_1}, x_{n_2}, ..., x_{n_k} & \in & \bar{B}_M(x_{n_1}, r_{n_1}, t_{n_1}) \\ x_{n_2}, ..., x_{n_k} & \in & \bar{B}_M(x_{n_2}, r_{n_2}, t_{n_2}) \\ & & ... \\ x_{n_k} & \in & \bar{B}_M(x_{n_k}, r_{n_k}, t_{n_k}) \end{array}$$

therefore

$$x_{n_k} \in \bar{B}_M(x_{n_1}, r_{n_1}, t_{n_1}) \cap \bar{B}_M(x_{n_2}, r_{n_2}, t_{n_2}) \cap ... \cap \bar{B}_M(x_{n_k}, r_{n_k}, t_{n_k}).$$

Since X has the ball intersection property, then we may conclude that $\bigcap_{n \geq 1} \bar{B}_M(x_n, r_n, t_n) \neq \varnothing$ for any $n \in \mathbb{N}$. Since $\{x_n\}$ is a Cauchy sequence and $\{r_n\}$ has fuzzy diameter zero, the intersection $\bigcap_{n \geq 1} \bar{B}_M(x_n, r_n, t_n)$ is reduced to one point z which is the limit of the sequence $\{x_n\}$. So indeed, the point $z \in \bigcap_{n \geq 1} \bar{B}_M(x_n, r_n, t_n)$ then for each pair of r_n, $t_n > 0$ there exists $n_1 \in \mathbb{N}$ such that $M(x_n, z, t_n) > 1 - r_n$ for all $n \geq n_1$. Therefore, $M(x_n, z, t_n)$ converges to 1 when $n \longrightarrow \infty$, for each $t_n > 0$ and $(X, M, *)$ is complete. □

Proposition 2. *Fuzzy hyperconvexity is equivalent to the ball intersection property and fuzzy metrically convexity.*

Proof. If $(X, M, *)$ is fuzzy hyperconvex, by Proposition 1 X satisfies the ball intersection property and it is easy to see that X is fuzzy convex metric space. Conversely, if two closed balls $\bar{B}_M(x_i, r_i, t_i)$ and $\bar{B}_M(x_j, r_j, t_j)$ satisfy the relation $M(x_i, x_j, t_i + t_j) \geq (1 - r_i) * (1 - r_j)$, they must intersect since X has ball intersection property. □

Now we give the definition of fuzzy $m-$hyperconvexity. Note that fuzzy m-hyperconvexity is a weaker property than fuzzy hyperconvexity. The definitions of fuzzy hyperconvexity and fuzzy $m-$hyperconvexity can be considered structurally similar.

Definition 14. *Let $(X, M, *)$ be a metric space, Γ be an index set such that card(Γ)$< m$, $r_i \in (0,1)$ and $t_i \in (0,\infty)$ for all $i \in \Gamma$. The fuzzy metric space X is said to be fuzzy $m-$hyperconvex if for any indexed class of closed balls $\bar{B}_M(x_i, r_i, t_i)$ in X, satifying the condition that*

$$M(x_i, x_j, t_i + t_j) \geq (1 - r_i) * (1 - r_j)$$

for all $i, j \in \Gamma$, the intersection $\bigcap_{i\in\Gamma} \bar{B}_M(x_i, r_i, t_i) \neq \varnothing$.

Proposition 3. *(i) It is clear that fuzzy hyperconvexity is stronger than fuzzy $m-$hyperconvexity, which is stronger than fuzzy $n-$hyperconvexity if $n < m$.*

(ii) It is easy to see that every fuzzy metric space $(X, M, *)$ is fuzzy $1-$hyperconvex.

Theorem 5. *For $m = 3$, fuzzy $3-$hyperconvexity is equivalent to fuzzy metrically convexity.*

Proof. Let $(X, M, *)$ be fuzzy $3-$hyperconvex. Since card(Γ)$< m = 3$, the index set Γ is $\Gamma = \{1, 2\}$. It follows that for any points $x_1, x_2 \in X$ and for each pair $r_1, t_1 > 0$ and $r_2, t_2 > 0$ ($r_1, r_2 \in (0, 1)$ and $t_1, t_2 \in (0,\infty)$) such that $M(x_1, x_2, t_1 + t_2) \geq (1 - r_1) * (1 - r_2)$, there exists $z \in X$ such that $M(x_1, z, t_1) \geq (1 - r_1)$ and $M(x_2, z, t_2) \geq (1 - r_2)$. This means that $(X, M, *)$ is fuzzy metrically convex.

Conversely, Let $(X, M, *)$ be fuzzy metrically convex. Then for $\Gamma = \{1, 2\}$, we have $\bar{B}_M(x_1, r_1, t_1) \cap \bar{B}_M(x_2, r_2, t_2) \neq \varnothing$. So for any indexed class of closed balls $\bar{B}_M(x_i, r_i, t_i)$ in X, satifying the condition that

$$M(x_i, x_j, t_i + t_j) \geq (1 - r_i) * (1 - r_j)$$

for all $i, j \in \Gamma = \{1, 2\}$, the intersection $\bigcap_{i=1}^{2} \bar{B}_M(x_i, r_i, t_i) \neq \varnothing$. So $(X, M, *)$ is fuzzy $3-$hyperconvex. □

Definition 15. *A fuzzy metric space $(X, M, *)$ is called $m-$seperable if it contains a dense subset of cardinal $(K) < m$ where $K \subset \Gamma$, Γ is index set. (This definition is the same with Definition 10 except for the spaces.)*

Note that when $n < m$, $m-$seperability is weaker than $n-$seperability for any fuzzy metric space $(X, M, *)$. $m-$seperability for a finite cardinal m means that the fuzzy metric space $(X, M, *)$ is a finite set, and at the same time it contains at most $m - 1$ points.

Theorem 6. *If the fuzzy metric space $(X, M, *)$ is fuzzy $m-$hyperconvex and at the same time $m-$seperable, then it is fuzzy hyperconvex.*

Proof. Consider an arbitrary indexed family of closed balls $\bar{B}_M(x_i, r_i, t_i)$ satisfying the condition that $M(x_i, x_j, t_i + t_j) \geq (1 - r_i) * (1 - r_j)$, for all $i, j \in \Gamma$. Let X be fuzzy m-hyperconvex and let $\{p_k\}$, $k \in K$ with card$(K)$$< m$, be an indexed set of points, which is dense in X. Take the pair of $r_k^{'}$, $t_k^{'} > 0$ as follows, respectively

(3.2) $\quad r_k^{'}, t_k^{'} = {}''$the infimum of all $r \in (0, 1)$ and the infimum of all $t > 0$ such that $\exists\, i \in \Gamma$ with $\bar{B}_M(x_i, r_i, t_i) \subset \bar{B}_M(p_k, r, t)''$.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 27, NO.1, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC

Now we claim that the class of closed balls $\bar{B}_M(p_k, r_k^{'}, t_k^{'})$, $k \in K$, satisfies the requirement of fuzzy $m-$hyperconvexity. So indeed, take any indices $k, l \in K$ and arbitrary $\varepsilon \in (0,1)$ and arbitrary $\varepsilon^{'} > 0$. By (3.2) there exist i, $j \in \Gamma$ such that

$$\bar{B}_M(x_i, r_i, t_i) \subset \bar{B}_M(p_k, r_k^{'} + \varepsilon, t_k^{'} + \varepsilon^{'}) \tag{3.3}$$

and

$$\bar{B}_M(x_j, r_j, t_j) \subset \bar{B}_M(p_l, r_l^{'} + \varepsilon, t_l^{'} + \varepsilon'). \tag{3.4}$$

Since X is fuzzy $m-$hyperconvex, there exist a point q in $\bar{B}_M(x_i, r_i, t_i) \cap \bar{B}_M(x_j, r_j, t_j)$, at the same time by (3.3), (3.4) q is in $\bar{B}_M(p_k, r_k^{'} + \varepsilon, t_k^{'} + \varepsilon^{'}) \cap \bar{B}_M(p_l, r_l^{'} + \varepsilon, t_l^{'} + \varepsilon')$. Then,

$$\begin{aligned} q &\in \bar{B}_M(p_k, r_k^{'} + \varepsilon, t_k^{'} + \varepsilon^{'}) \cap \bar{B}_M(p_l, r_l^{'} + \varepsilon, t_l^{'} + \varepsilon') \\ &\Longrightarrow M(p_k, q, t_k^{'} + \varepsilon^{'}) \geq 1 - (r_k^{'} + \varepsilon) \text{ and } M(p_l, q, t_l^{'} + \varepsilon^{'}) \geq 1 - (r_l^{'} + \varepsilon) \\ &\Longrightarrow M(p_k, p_l, t_k^{'} + t_l^{'} + 2\varepsilon^{'}) \geq \Big[1 - (r_k^{'} + \varepsilon)\Big] * \Big[1 - (r_l^{'} + \varepsilon)\Big]. \end{aligned} \tag{3.5}$$

Since $\varepsilon \in (0,1)$ and $\varepsilon' > 0$ are arbitrary, by (3.5) we find the requirement for $m-$hyperconvexity for the collection of closed balls $\bar{B}_M(p_k, r_k^{'}, t_k^{'})$, $k \in K$. So, there is a point x in $\bigcap_{k \in K} \bar{B}_M(p_k, r_k^{'}, t_k^{'})$.

Now we need to show that $x \in \bar{B}_M(x_i, r_i, t_i)$, for all $i \in \Gamma$, i.e. $M(x, x_i, t_i) \geq 1 - r_i$ to see the fuzzy hyperconvexity of X. For this, take an arbitrary $\varepsilon \in (0,1)$ and arbitrary $\varepsilon' > 0$. Since the set $\{p_k\}$, $k \in K$ is dense in X, there exists a point p_k for each $x_i \in X$ such that

$$M(x_i, p_k, \varepsilon') > 1 - \varepsilon. \tag{3.6}$$

Therefore

$$\bar{B}_M(x_i, r_i, t_i) \subset \bar{B}_M(p_k, r_i + \varepsilon, t_i + \varepsilon').$$

Due to the choices of $r_k^{'}$ and $t_k^{'}$, it follows that $\bar{B}_M(p_k, r_k^{'}, t_k^{'}) \subset \bar{B}_M(p_k, r_i + \varepsilon, t_i + \varepsilon')$ and so, we get that $r_k^{'} \leq r_i + \varepsilon$ and $t_k^{'} \leq t_i + \varepsilon'$.Therefore, by the triangle inequality for fuzzy metric (i.e. the condition (FM-4)),

$$\begin{aligned} M(x, x_i, t_i + 2\varepsilon^{'}) &\geq M(x, p_k, t_i + \varepsilon^{'}) * M(p_k, x_i, \varepsilon') \\ &\geq M(x, p_k, t_k^{'}) * M(p_k, x_i, \varepsilon') \\ &> (1 - r_k^{'}) * (1 - \varepsilon) \\ &\geq [1 - (r_i + \varepsilon)] * (1 - \varepsilon). \end{aligned}$$

Since ε and ε' is arbitrary, $M(x, x_i, t_i) \geq 1 - r_i$. This means that $x \in \bigcap_{i \in \Gamma} \bar{B}_M(x_i, r_i, t_i)$ and so X is fuzzy hyperconvex. $\square$

Remark 4. *It is clear that if the fuzzy metric space $(X, M, *)$ is $m-$seperable and the space X has finite number of points, then we can not mention fuzzy $m-$hyperconvexity. So indeed, since fuzzy $m-$hyperconvexity ($m \geq 3$) implies the fuzzy metrically convexity (Proposition 2), X can not be a finite set except when the set is reduced to a single point.*

Consequently, Teorem 6 indicate this situation i.e. if the fuzzy metric space $(X, M, *)$ is fuzzy $m-$ hyperconvex and $m-$seperable then $(X, M, *)$ is fuzzy hyperconvex.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 27, NO.1, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC

References

[1] A.George and P.Veeramani, On some results in fuzzy metric spaces, *Fuzzy Sets and Systems*, 64(3), 395-399 (1994).

[2] A.George and P.Veeramani, Some theorems in fuzzy metric spaces, *Journal of Fuzzy Mathematics*, 3, 933-940 (1995).

[3] A.George and P.Veeramani, On some results of analysis for fuzzy metric spaces, *Fuzzy Sets and Systems*, 90(3), 365-368 (1997).

[4] A.Thanithamil and P.Thirunavukarasu, Some Results in Fuzzy Metric Spaces with Convex Structure, *International Journal of Mathematical Analysis*, 8(57), 2827-2836 (2014).

[5] B.Miesch, M.Pavón, Ball intersection properties in metric Spaces, *arXiv preprint arXiv:1610.03307* (2016).

[6] B.Schweizer and A.Sklar, Statistical metric spaces, *Pacific J. Math*, 10(3), 313-334 (1960).

[7] B.K.Sharma and C.L.Dewangan, Fixed point theorem in convex metric space, *Novi Sad Journal of Mathematics*, 25(1), 9-18 (1995).

[8] H.Vosoughi and S.J.Hosseini Ghoncheh, Extension of fuzzy contraction mappings, *Iranian Journal of Fuzzy Systems*, 9(5), 1-6 (2014).

[9] K.Menger, Untersuchungen über allegemeine Metrik, *Math. Ann.*, 100, 75–63 (1928).

[10] N.Aronszajn and P.Panitchpakdi, Extension of uniformly continuous transformations and hyperconvex metric spaces, *Pacific J. Math*, 6, 405–439. MR 18:917c (1956).

[11] R.Esp´ınola and M.A.Khamsi, Introduction to hyperconvex spaces, in *Handbook of Metric Fixed Point Theory* (W. A. Kirk and B. Sims, eds.), Kluwer Acad. Publ., Dordrecht,2001,pp.391-435.

[12] S.N.Ješić, N.A.Babačev and R.M.Nikolić, A common fixed point theorem in fuzzy metric spaces with nonlinear contractive type condition defined using Φ-function, In *Abstract and Applied Analysis* (Vol. 2013). Hindawi Publishing Corporation (2013).

[13] S.Park, Fixed point theorems in hyperconvex metric spaces, *Nonlinear Analysis: Theory, Methods & Applications*, 37(4), 467-472 (1999).

[14] T.Shimizu, Fixed Points of Multivalued Nonexpansive Mappings in Certain Convex Metric Spaces, *Nonlinear Analysıs and Convex Analysıs*, (1998).

[15] W.Takahashi, A convexity in metric space and nonexpansive mappings, I. In *Kodai Mathematical Seminar Reports* (Vol. 22, No. 2, pp. 142-149). Department of Mathematics, Tokyo Institute of Technology (1970).

Graduate School of Natural and Applied Science Gazi University Teknikokullar, Ankara, 06500, TURKEY

E-mail address: yigittebru@gmail.com

Department of Mathematics Faculty of Science Gazi University Teknikokullar, Ankara, 06500, TURKEY

E-mail address: hakanefe@gazi.edu.tr

ON GENERALIZATIONS OF A REVERSE HARDY-HILBERT'S TYPE INEQUALITY

ZHENGPING ZHANG, GAOWEN XI

ABSTRACT. By introducing a parameter α and using the expression of the β function establishing the inequality of the weight coefficient, a generalizations of the reverse Hardy-Hilbert's type inequality is proved. As applications, some equivalent form and a number of particular cases are obtained.

1. INTRODUCTION

Let $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $a_n \geq 0$, $b_n \geq 0$, and $0 < \sum_{n=0}^{\infty} a_n^p < \infty$, $0 < \sum_{n=0}^{\infty} b_n^q < \infty$, then

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{a_m b_n}{m+n+1} < \frac{\pi}{\sin(\pi/p)} \left(\sum_{n=0}^{\infty} a_n^p\right)^{1/p} \left(\sum_{n=0}^{\infty} b_n^q\right)^{1/q}, \tag{1.1}$$

and

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{a_m b_n}{\max\{m,n\}} < pq \left\{\sum_{n=1}^{\infty} a_n^p\right\}^{\frac{1}{p}} \left\{\sum_{n=1}^{\infty} b_n^q\right\}^{\frac{1}{q}}, \tag{1.2}$$

where the constant $\frac{\pi}{\sin\frac{\pi}{p}}$ and pq is best possible for each inequality respectively. Inequality (1.1) is Hardy-Hilbert's inequality. Inequality (1.2) is a Hilbert's type inequality [1].

For (1.1), Yang et al. [7], [8], [9], [10] and [11] gave some strengthened versions and extensions as follows:

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{a_m b_n}{m+n+1} < \left\{\sum_{n=0}^{\infty} \left[\pi - \frac{7}{5(\sqrt{n}+3)}\right] a_n^2 \sum_{n=0}^{\infty} \left[\pi - \frac{7}{5(\sqrt{n}+3)}\right] b_n^2\right\}^{1/2}; \tag{1.3}$$

$$\begin{aligned}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{a_m b_n}{m+n+1} \quad &< \quad \left\{\sum_{n=0}^{\infty} \left[\frac{\pi}{\sin(\pi/p)} - \frac{ln2 - C}{(2n+1)^{1+1/p}}\right] a_n^p\right\}^{1/p} \\ &\times \left\{\sum_{n=0}^{\infty} \left[\frac{\pi}{\sin(\pi/p)} - \frac{\ln 2 - C}{(2n+1)^{1+1/q}}\right] b_n^q\right\}^{1/q},\end{aligned} \tag{1.4}$$

2010 Mathematics Subject Classification: 26D15.
Key Words: Hardy-Hilbert's inequality; weight coefficient; β function; reverse; generalizations .

where $\ln 2 - C = 0.1159315^{+}$ (C is the Euler constant).

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{a_m b_n}{(m+n+1)^{\lambda}} < B\left(\frac{p+\lambda-2}{p}, \frac{q+\lambda-2}{q}\right)\left\{\sum_{n=0}^{\infty}(n+\frac{1}{2})^{1-\lambda}a_n^p\right\}^{1/p} \times\left\{\sum_{n=0}^{\infty}(n+\frac{1}{2})^{1-\lambda}b_n^q\right\}^{1/q}, \tag{1.5}$$

where the constant $B\left(\frac{p+\lambda-2}{p}, \frac{q+\lambda-2}{q}\right)$ is the best possible ($2-\min\{p,\ q\} < \lambda \le 2$).

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{a_m b_n}{(m+n+1)^{\lambda}} < B\left(\frac{\lambda}{p},\ \frac{\lambda}{q}\right)\left\{\sum_{n=0}^{\infty}(n+\frac{1}{2})^{p-1-\lambda}a_n^p\right\}^{1/p} \times\left\{\sum_{n=0}^{\infty}(n+\frac{1}{2})^{q-1-\lambda}b_n^q\right\}^{1/q}. \tag{1.6}$$

where the constant $B(\frac{\lambda}{p},\ \frac{\lambda}{q})$ is the best possible ($0 < \lambda \le \min\{p,\ q\}$).

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{a_m b_n}{(m+n+1)^{\lambda}} < B\left(\frac{(r-2)t+\lambda}{r}, \frac{(s-2)t+\lambda}{s}\right)\left\{\sum_{n=0}^{\infty}(n+\frac{1}{2})^{p(1-t+\frac{2t-\lambda}{r})-1}a_n^p\right\}^{1/p} \times\left\{\sum_{n=0}^{\infty}(n+\frac{1}{2})^{q(1-t+\frac{2t-\lambda}{s})-1}b_n^q\right\}^{1/q}, \tag{1.7}$$

where the constant $B(\frac{(r-2)t+\lambda}{r}, \frac{(s-2)t+\lambda}{s})$ is the best possible ($r>1,\ \frac{1}{r}+\frac{1}{s}=1,\ \ t\in[0,\ 1]$, $2-\min\{r,\ s\}t < \lambda \le 2-\min\{r,\ s\}t+\min\{r,\ s\}$).

In [5] and [6], Xi gave a generalizations and reinforcements of inequalities (1.2:

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{\max(m^{\lambda}, n^{\lambda})} < \left\{\sum_{n=1}^{\infty}\left[\kappa(\lambda)-\frac{1}{3qn^{\frac{q+\lambda-2}{q}}}\right]n^{1-\lambda}a_n^p\right\}^{\frac{1}{p}} \times\left\{\sum_{n=1}^{\infty}\left[\kappa(\lambda)-\frac{1}{3pn^{\frac{p+\lambda-2}{p}}}\right]n^{1-\lambda}b_n^q\right\}^{\frac{1}{q}}, \tag{1.8}$$

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{\max\{m^{\lambda}+A, n^{\lambda}+B\}} < \left\{\sum_{n=1}^{\infty}\left[\kappa(\lambda)-\frac{1}{n^{\frac{q+\lambda-2}{q}}}\left(\frac{1}{3q}-\frac{B}{1+B}\right)\right]n^{1-\lambda}a_n^p\right\}^{\frac{1}{p}} \times\left\{\sum_{n=1}^{\infty}\left[\kappa(\lambda)-\frac{1}{n^{\frac{p+\lambda-2}{p}}}\left(\frac{1}{3p}-\frac{A}{1+A}\right)\right]n^{1-\lambda}b_n^q\right\}^{\frac{1}{q}}, \tag{1.9}$$

where $\kappa(\lambda)=\frac{pq\lambda}{(p+\lambda-2)(q+\lambda-2)} > 0$, $2-\min\{p,q\} < \lambda \le 2$, $0 \le A \le B \le \min\{\frac{1}{3p-1}, \frac{1}{3q-1}\}$.

For the reverse Hardy-Hilbert's inequality, recently, Yang [12] gave a reverse form of inequalities (1.5), (1.6) and (1.7) for $\lambda = 2$. In [4], Xi gave an extension of the above Yang's work for $1.5 \le \lambda < 3$:

$$\begin{aligned}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{a_m b_n}{(m+n+1)^{\lambda}} &> \frac{2}{\lambda-1}\left\{\sum_{n=0}^{\infty}\frac{(n+1)^{2-\lambda}}{2n+3-\lambda}\left[1-\frac{(\lambda-1)^2}{4(n+1)^2}\right]a_n^p\right\}^{1/p}\\ &\quad\times\left\{\sum_{n=0}^{\infty}\frac{(n+1)^{2-\lambda}}{2n+3-\lambda}b_n^q\right\}^{1/q}, \end{aligned}\tag{1.10}$$

where $0 < p < 1$, $\frac{1}{p}+\frac{1}{q}=1$, $1.5 \le \lambda < 3$ and $a_n \ge 0, b_n > 0$, such that $0 < \sum_{n=0}^{\infty}\frac{(n+1)^{2-\lambda}a_n^p}{2n+3-\lambda} < \infty$, $0 < \sum_{n=0}^{\infty}\frac{(n+1)^{2-\lambda}b_n^q}{2n+3-\lambda} < \infty$.

In this paper, by introducing a parameter α and using the expression of the β function establishing the inequality of the weight coefficient. The purpose of this paper is to give a generalization of inequality (1.10).

For this, we need the following expression of the β function $B(p,\ q)$ (see [3])

$$B(p,\ q)=B(q,\ p)=\int_0^{\infty}\frac{1}{(1+u)^{p+q}}u^{p-1}du \quad (p,\quad q>0), \tag{1.11}$$

and the following inequality [8]:

$$\int_0^{\infty}f(x)dx+\frac{1}{2}f(0)<\sum_{m=0}^{\infty}f(m)<\int_0^{\infty}f(x)dx+\frac{1}{2}f(0)-\frac{1}{12}f'(0) \tag{1.12}$$

where $f(x)\in C^3[0,\quad\infty)$, and $\int_0^{\infty}f(x)dx<\infty$, $(-1)^n f^{(n)}(x)>0, f^{(n)}(\infty)=0(n=0,\ 1,\ 2,\ 3)$.

2. Main Results

Lemma 2.1. *Let N_0 be the set of non-negative integers, N be the set of positive integers and R be the set of real numbers. The weight coefficient $\omega_\lambda(n,\alpha)$ is defined by*

$$\omega_\lambda(n,\alpha)=\sum_{m=0}^{\infty}\frac{1}{(m+n+\alpha)^{\lambda}}, \qquad n\in N_0,\quad 1.5\le\lambda<3, \alpha\ge 1.$$

Then we have

$$\begin{aligned}\frac{2(n+\alpha)^{2-\lambda}}{(\lambda-1)(2n+2\alpha-\lambda+1)}\left[1-\frac{(\lambda-1)^2}{4(n+\alpha)^2}\right] &< \omega_\lambda(n,\alpha)\\ &< \frac{2(n+\alpha)^{2-\lambda}}{(\lambda-1)(2n+2\alpha-\lambda+1)}.\end{aligned}\tag{2.1}$$

Proof If $n\in N_0$, let $f(x)=\frac{1}{(x+n+\alpha)^{\lambda}}$, $\quad x\in[0,\quad\infty)$. By (1.12), we obtain

$$\omega_\lambda(n,\alpha)>\int_0^{\infty}\frac{dx}{(x+n+\alpha)^{\lambda}}+\frac{1}{2(n+\alpha)^{\lambda}}=\frac{1}{(\lambda-1)(n+\alpha)^{\lambda-1}}+\frac{1}{2(n+\alpha)^{\lambda}}.$$

$$\begin{aligned}\omega_\lambda(n,\alpha) &< \int_0^{\infty}\frac{1}{(x+n+\alpha)^{\lambda}}dx+\frac{1}{2(n+\alpha)^{\lambda}}+\frac{\lambda}{12(n+\alpha)^{\lambda+1}}\\ &= \frac{1}{(\lambda-1)(n+\alpha)^{\lambda-1}}+\frac{1}{2(n+\alpha)^{\lambda}}+\frac{\lambda}{12(n+\alpha)^{\lambda+1}}.\end{aligned}$$

Since we find

$$\left[\frac{1}{(\lambda-1)(n+\alpha)^{\lambda-1}}+\frac{1}{2(n+\alpha)^{\lambda}}\right]\left[2(n+\alpha)^{\lambda-1}-(\lambda-1)(n+\alpha)^{\lambda-2}\right]=\frac{2}{\lambda-1}-\frac{\lambda-1}{2(n+\alpha)^2},$$

$$\left[\frac{1}{(\lambda-1)(n+\alpha)^{\lambda-1}}+\frac{1}{2(n+\alpha)^{\lambda}}+\frac{1}{12(n+\alpha)^{\lambda+1}}\right]\left[2(n+\alpha)^{\lambda-1}-(\lambda-1)(n+\alpha)^{\lambda-2}\right]$$
$$=\frac{2}{\lambda-1}-\frac{2\lambda-3}{6(n+\alpha)^2}-\frac{\lambda(\lambda-1)}{12(n+\alpha)^3}.$$

Then we obtain

$$\frac{1}{(\lambda-1)(n+\alpha)^{\lambda-1}}+\frac{1}{2(n+\alpha)^{\lambda}}=\frac{2(n+\alpha)^{2-\lambda}}{(\lambda-1)(2n+2\alpha-\lambda+1)}\left[1-\frac{(\lambda-1)^2}{4(n+\alpha)^2}\right],$$
$$\frac{1}{(\lambda-1)(n+\alpha)^{\lambda-1}}+\frac{1}{2(n+\alpha)^{\lambda}}+\frac{1}{12(n+\alpha)^{\lambda+1}}=\frac{2(n+\alpha)^{2-\lambda}}{(\lambda-1)(2n+2\alpha-\lambda+1)}$$
$$\times\left[1-\frac{(2\lambda-3)(\lambda-1)}{12(n+\alpha)^2}-\frac{\lambda(\lambda-1)^2}{24(n+\alpha)^3}\right].$$

Sine $1.5\le\lambda<3$, $\alpha\ge 1$, so $\frac{2(n+\alpha)^{2-\lambda}}{(\lambda-1)(2n+2\alpha-\lambda+1)}>0, \frac{(2\lambda-3)(\lambda-1)}{12(n+\alpha)^2}\ge 0$, $\frac{\lambda(\lambda-1)^2}{24(n+\alpha)^3}>0$. Then we have (2.1). The lemma is proved.

Theorem 2.2. *Let* $0<p<1$, $\frac{1}{p}+\frac{1}{q}=1$, $1.5\le\lambda<3$, $\alpha\ge 1$, *and* $a_n\ge 0, b_n>0$, *such that* $0<\sum_{n=0}^{\infty}\frac{(n+\alpha)^{2-\lambda}}{2n+2\alpha-\lambda+1}\left[1-\frac{(\lambda-1)^2}{4(n+\alpha)^2}\right]a_n^p<\infty$, $0<\sum_{n=0}^{\infty}\frac{(n+\alpha)^{2-\lambda}}{2n+2\alpha-\lambda+1}b_n^q<\infty$. *Then we have*

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{a_m b_n}{(m+n+\alpha)^{\lambda}} > \frac{2}{\lambda-1}\left\{\sum_{n=0}^{\infty}\frac{(n+\alpha)^{2-\lambda}}{2n+2\alpha-\lambda+1}\left[1-\frac{(\lambda-1)^2}{4(n+\alpha)^2}\right]a_n^p\right\}^{1/p}$$
$$\times\left\{\sum_{n=0}^{\infty}\frac{(n+\alpha)^{2-\lambda}}{2n+2\alpha-\lambda+1}b_n^q\right\}^{1/q}. \tag{2.2}$$

Proof By the reverse Hölder 's inequality [2], we have

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{a_m b_n}{(m+n+\alpha)^{\lambda}} = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{a_m}{(m+n+\alpha)^{\frac{\lambda}{p}}}\cdot\frac{b_n}{(m+n+\alpha)^{\frac{\lambda}{q}}}$$
$$\ge \left\{\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{a_m^p}{(m+n+\alpha)^{\lambda}}\right\}^{\frac{1}{p}}\cdot\left\{\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{b_n^q}{(m+n+\alpha)^{\lambda}}\right\}^{\frac{1}{q}}$$
$$= \left\{\sum_{m=0}^{\infty}\omega_\lambda(m,\alpha)a_m^p\right\}^{\frac{1}{p}}\cdot\left\{\sum_{n=0}^{\infty}\omega_\lambda(n,\alpha)b_n^q\right\}^{\frac{1}{q}}.$$

Since $0<p<1$ and $q<0$, then by (2.1), we obtain (2.2). The theorem is proved.

In Theorem 2.2, for $\alpha=1$ we have

Corollary 2.3. *Let* $0 < p < 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $1.5 \leq \lambda < 3$ *and* $a_n \geq 0, b_n > 0$, *such that* $0 < \sum_{n=0}^{\infty} \frac{(n+1)^{2-\lambda} a_n^p}{2n+3-\lambda} < \infty$, $0 < \sum_{n=0}^{\infty} \frac{(n+1)^{2-\lambda} b_n^q}{2n+3-\lambda} < \infty$. *Then we have*

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{a_m b_n}{(m+n+1)^{\lambda}} > \frac{2}{\lambda-1} \left\{ \sum_{n=0}^{\infty} \frac{(n+1)^{2-\lambda}}{2n+3-\lambda} \left[1 - \frac{(\lambda-1)^2}{4(n+1)^2} \right] a_n^p \right\}^{1/p} \times \left\{ \sum_{n=0}^{\infty} \frac{(n+1)^{2-\lambda}}{2n+3-\lambda} b_n^q \right\}^{1/q}. \tag{2.3}$$

Remark. Inequality (2.3) is inequality (1.10). Hence, inequality (2.2) is an extension inequality (1.10)

Theorem 2.4. *Let* $0 < p < 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $1.5 \leq \lambda < 3, \alpha \geq 1$, *and* $a_n \geq 0$, *such that* $0 < \sum_{n=0}^{\infty} \frac{(n+\alpha)^{2-\lambda} a_n^p}{2n+2\alpha-\lambda+1} < \infty$. *Then we have*

$$\sum_{n=0}^{\infty} \left[\frac{(n+\alpha)^{2-\lambda}}{2n+2\alpha-\lambda+1} \right]^{1-p} \left[\sum_{m=0}^{\infty} \frac{a_m}{(m+n+\alpha)^{\lambda}} \right]^p > \left(\frac{2}{\lambda-1} \right)^p \sum_{n=0}^{\infty} \frac{(n+\alpha)^{2-\lambda}}{2n+2\alpha-\lambda+1} \left[1 - \frac{(\lambda-1)^2}{4(n+\alpha)^2} \right] a_n^p. \tag{2.4}$$

Inequalities (2.4) *and* (2.2) *are equivalent.*

Proof Let

$$b_n = \left[\frac{(n+1)^{2-\lambda}}{2n+2\alpha-\lambda+1} \right]^{1-p} \left[\sum_{m=0}^{\infty} \frac{a_m}{(m+n+\alpha)^{\lambda}} \right]^{p-1}, \qquad n \in N_0.$$

By (2.2), we have

$$\begin{aligned} \left\{ \sum_{n=0}^{\infty} \frac{(n+1)^{2-\lambda} b_n^q}{2n+2\alpha-\lambda+1} \right\}^p &= \left\{ \sum_{n=0}^{\infty} \left[\frac{(n+1)^{2-\lambda}}{2n+2\alpha-\lambda+1} \right]^{1-p} \left[\sum_{m=0}^{\infty} \frac{a_m}{(m+n+\alpha)^{\lambda}} \right]^p \right\}^p \\ &= \left\{ \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{a_m b_n}{(m+n+\alpha)^{\lambda}} \right\}^p \\ &\geq \left(\frac{2}{\lambda-1} \right)^p \sum_{n=0}^{\infty} \frac{(n+\alpha)^{2-\lambda}}{2n+2\alpha-\lambda+1} \left[1 - \frac{(\lambda-1)^2}{4(n+\alpha)^2} \right] a_n^p \\ &\quad \times \left\{ \sum_{n=0}^{\infty} \frac{(n+\alpha)^{2-\lambda} b_n^q}{2n+2\alpha-\lambda+1} \right\}^{p-1}. \end{aligned}$$

Then we obtain

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{(n+\alpha)^{2-\lambda} b_n^q}{2n+2\alpha-\lambda+1} &= \sum_{n=0}^{\infty} \left[\frac{(n+\alpha)^{2-\lambda}}{2n+2\alpha-\lambda+1} \right]^{1-p} \left[\sum_{m=0}^{\infty} \frac{a_m}{(m+n+\alpha)^{\lambda}} \right]^p \\ &\geq \left(\frac{2}{\lambda-1} \right)^p \sum_{n=0}^{\infty} \frac{(n+\alpha)^{2-\lambda}}{2n+2\alpha-\lambda+1} \left[1 - \frac{(\lambda-1)^2}{4(n+\alpha)^2} \right] a_n^p. \end{aligned} \tag{2.5}$$

If $\sum_{n=0}^{\infty} \frac{(n+\alpha)^{2-\lambda} b_n^q}{2n+2\alpha-\lambda+1} = \infty$, then in view of

$$0 < \sum_{n=0}^{\infty} \frac{(n+\alpha)^{2-\lambda} a_n^p}{2n+2\alpha-\lambda+1}\left[1-\frac{(\lambda-1)^2}{4(n+\alpha)^2}\right] \leq \sum_{n=0}^{\infty} \frac{(n+\alpha)^{2-\lambda} a_n^p}{2n+2\alpha-\lambda+1} < \infty$$

and (2.5), we have

$$\sum_{n=0}^{\infty}\left[\frac{(n+\alpha)^{2-\lambda}}{2n+2\alpha-\lambda+1}\right]^{1-p}\left[\sum_{m=0}^{\infty}\frac{a_m}{(m+n+\alpha)^{\lambda}}\right]^p > \left(\frac{2}{\lambda-1}\right)^p \times \sum_{n=0}^{\infty}\frac{(n+\alpha)^{\lambda-2}}{2n+2\alpha-\lambda+1}\left[1-\frac{(\lambda-1)^2}{4(n+\alpha)^2}\right] a_n^p;$$

if $0 < \sum_{n=0}^{\infty} \frac{(n+\alpha)^{\lambda-2} b_n^q}{2n+2\alpha-\lambda+1} < \infty$, then by (2.2), we find

$$\sum_{n=0}^{\infty}\left[\frac{(n+\alpha)^{2-\lambda}}{2n+2\alpha-\lambda+1}\right]^{1-p}\left[\sum_{m=0}^{\infty}\frac{a_m}{(m+n+\alpha)^{\lambda}}\right]^p > \left(\frac{2}{\lambda-1}\right)^p \times \sum_{n=0}^{\infty}\frac{(n+\alpha)^{2-\lambda}}{2n+2\alpha-\lambda+1}\left[1-\frac{(\lambda-1)^2}{4(n+\alpha)^2}\right] a_n^p.$$

Hence we obtain (2.4).

On the other-hand, by the reverse Hölder 's inequality [2], we have

$$\begin{aligned}
\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{a_m b_n}{(m+n+\alpha)^{\lambda}} &= \left[\sum_{n=0}^{\infty}(n+\alpha)^{\frac{\lambda-2}{q}}(2n+2\alpha-\lambda+1)^{\frac{1}{q}}\sum_{m=0}^{\infty}\frac{a_m}{(m+n+\alpha)^{\lambda}}\right] \\
&\quad \times \left[\frac{b_n}{(n+\alpha)^{\frac{\lambda-2}{q}}(2n+2\alpha-\lambda+1)^{\frac{1}{q}}}\right] \\
&\geq \left\{\sum_{n=0}^{\infty}\left[\frac{(n+\alpha)^{2-\lambda}}{2n+2\alpha-\lambda+1}\right]^{1-p}\left[\sum_{m=0}^{\infty}\frac{a_m}{(m+n+\alpha)^{\lambda}}\right]^p\right\}^{\frac{1}{p}} \\
&\quad \times \left\{\sum_{n=0}^{\infty}\frac{(n+\alpha)^{2-\lambda} b_n^q}{2n+2\alpha-\lambda+1}\right\}^{\frac{1}{q}}.
\end{aligned}$$

Hence by (2.4), it follows

$$\begin{aligned}
\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{a_m b_n}{(m+n+\alpha)^{\lambda}} &> \frac{2}{\lambda-1}\left\{\sum_{n=0}^{\infty}\frac{(n+\alpha)^{2-\lambda} a_n^p}{2n+2\alpha-\lambda+1}\left[1-\frac{(\lambda-1)^2}{4(n+\alpha)^2}\right]\right\}^{\frac{1}{p}} \\
&\quad \times \left\{\sum_{n=0}^{\infty}\frac{(n+\alpha)^{2-\lambda} b_n^q}{2n+2\alpha-\lambda+1}\right\}^{\frac{1}{q}}.
\end{aligned}$$

Then, (2.4) and (2.2) are equivalent. The theorem is proved.

In (2.4), for $\alpha = 1$, we have

Corollary 2.5. *Let* $1.5 \leq \lambda < 3$, $0 < p < 1$, $\frac{1}{p}+\frac{1}{q} = 1$, $a_n \geq 0$, $0 < \sum_{n=0}^{\infty} \frac{(n+1)^{2-\lambda} a_n^p}{2n-\lambda+3} < \infty$, *Then we have*

$$\sum_{n=0}^{\infty} \left[\frac{(n+1)^{2-\lambda}}{2n-\lambda+3}\right]^{1-p} \left[\sum_{m=0}^{\infty} \frac{a_m}{(m+n+1)^{\lambda}}\right]^p > \left(\frac{2}{\lambda-1}\right)^p \sum_{n=0}^{\infty} \frac{(n+1)^{2-\lambda}}{2n-\lambda+3} \left[1 - \frac{(\lambda-1)^2}{4(n+1)^2}\right] a_n^p. \tag{2.6}$$

Inequalities (2.6) *and* (2.3) *are equivalent.*

References

[1] G. Hardy , J. Littlewood and G. Polya, *Inequalities*, Cambridge University Press Cambridge, 1952.

[2] J. Kuang, *Applied Inequalities,* Sandong Science and Technology Press, Jinan, 2004.

[3] Z. Wang and G. Dunren, *An Introduction to Special Function,* Science Press, Beijing,1979.

[4] G. Xi, *A Reverse Hardy-Hilbert-Type Inequality,* Journal of Inequalities and Applications, Volume 2007, Art. No. 79758.

[5] G. Xi, *On generalizations and reinforcements of a Hilbert's type inequality,* Rocky Mountain Journal of Mathematics, 42.1(2012): 329-337.

[6] G. Xi, *A generalization of the Hilbert's type inequality,* Mathematical Inequalities & Applications, 18.4(2015): 1501-1510.

[7] B. Yang and L. Debnath, *Some Inequalities Involing π and an Application to Hilbert's Inequality*, Applied Mathematics Leters, 12(1999), 101-105.

[8] B. Yang and L. Debnath, *On a New Generalization of Hardy-Hilbert's Inequality and Its Applications*, Journal of Mathematical Analysis and Applications, 233(1999), 484-497.

[9] B. Yang , *On a Strengthened Version of the More AccurateHardy-Hilbert's Inequality*, Acta Mathematica Sinica , 42:6(1999), 1103-1110.

[10] B. Yang, *Themistocles M. Rassias, On a New Extension of Hilbert's Inequality,* Mathematics Inequalities and Applications, 8:4(2005), 575-582.

[11] B. Yang, *On a New Extension of Hilbert's Inequality With Some Parameters,* Acta Math. Hungar, 108:4(2005), 337-350.

[12] B. Yang, *A Reverse of the Hardy-Hilbert's Type Inequality,* Journal of Southwest China Normal University(Natural Science Edition), 30:6(2005),1012-1015.

College of Mathematics and Physics, Chongqing University of Science and Technology, Chongqing, 401331, P. R. China,

E-mail address: xigaowen@163.com

Dunkl generalization of q-Szász-Mirakjan-Kantrovich type operators and approximation

Abdullah Alotaibi[1] and **M. Mursaleen**[1,2]
[1]Operator Theory and Applications Research Group, Department of Mathematics, Faculty of Science, King Abdulaziz University, P. O. Box 80203, Jeddah 21589, Saudi Arabia
[2]Department of Mathematics, Aligarh Muslim University, Aligarh–202002, India
mathker11@hotmail.com; mursaleenm@gmail.com

Abstract

The purpose of this paper is to introduce a modification of q-Dunkl generalization of exponential functions. These types of operators modification enables better error estimation on the interval $[\frac{1}{2},\infty)$ than the classical ones. We obtain some approximation results via well known Korovkin's type theorem. Convergence properties by using the modulus of continuity and the rate of convergence of the operators for functions belonging to the Lipschitz class are also presented.

Keywords and phrases: q-integers; Dunkl analogue; Szász operator; q- Szász-Mirakjan-Kantrovich; modulus of continuity; Peetre's K-functional.

AMS Subject Classification (2010): 41A25, 41A36, 33C45.

1. Introduction and preliminaries

In 1912, S.N Bernstein [1] introduced the following sequence of operators $B_n : C[0,1] \to C[0,1]$ defined by

$$B_n(f;x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} f\left(\frac{k}{n}\right),\ x \in [0,1] \tag{1.1}$$

for $n \in \mathbb{N}$ and $f \in C[0,1]$.

In 1950, for $x \geq 0$, Szász [17] introduced the operators

$$S_n(f;x) = e^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} f\left(\frac{k}{n}\right),\ f \in C[0,\infty). \tag{1.2}$$

In the field of approximation theory, the application of q-calculus emerged as a new area in the field of approximation theory. The first q-analogue of the well-known Bernstein polynomials was introduced by Lupaş by applying the idea of q-integers [5]. In 1997 Phillips [14] considered another q-analogue of the classical Bernstein polynomials. Later on, many authors introduced q-generalizations of various operators and investigated several approximation properties [6, 7, 8, 9, 10, 11, 12, 13].

We now present some basic definitions and concept details of the q-calculus which are used in this paper.

Let $k \in \mathbb{N}_0$ and $q \in (0,1)$ then q-integer $[k]_q$ is defined as

$$[k]_q = \begin{cases} \frac{1-q^k}{1-q} & \text{if } q \neq 1, \\ k & \text{if } q = 1. \end{cases}$$

The q-factorial $[k]_q!$ is defined as

$$[k]_q! = \begin{cases} [k]_q[k-1]_q \cdots [1]_q & \text{if } k \in \mathbb{N}, \\ 1 & \text{if } k = 0, \end{cases}$$

and for $k \in \mathbb{N}$, q-binomial coefficient $\left[\begin{array}{c} k \\ r \end{array}\right]_q$ is defined by

$$\left[\begin{array}{c} k \\ r \end{array}\right]_q = \frac{[k]_q!}{[r]_q![k-r]_q!}, \ 1 \leq r \leq k,$$

with $\left[\begin{array}{c} k \\ 0 \end{array}\right]_q = 1$ and $\left[\begin{array}{c} k \\ r \end{array}\right]_q = 0$ for $r > k$.

There are two q-analogue of the exponential function e^x

$$e_q(x) = \sum_{k=0}^{\infty} \frac{x^k}{[k]_q!} = \frac{1}{(1-(1-q)x)_q^{\infty}}, \ |x| < \frac{1}{1-q}, \ |q| < 1,$$

and

$$E_q(x) = \sum_{k=0}^{\infty} q^{\frac{k(k-1)}{2}} \frac{x^k}{[k]_q!} = (1+(1-q)x)_q^{\infty}, \ |q| < 1,$$

where

$$(1-x)_q^{\infty} = \prod_{j=0}^{\infty} (1-q^j x).$$

Our investigation is to construct a linear positive operators generated by generalization of exponential function for defined by [15]

$$e_{\mu}(x) = \sum_{n=0}^{\infty} \frac{x^n}{\gamma_{\mu}(n)}.$$

Here

$$\gamma_{\mu}(2k) = \frac{2^{2k} k! \Gamma\left(k + \mu + \frac{1}{2}\right)}{\Gamma\left(\mu + \frac{1}{2}\right)},$$

and

$$\gamma_{\mu}(2k+1) = \frac{2^{2k+1} k! \Gamma\left(k + \mu + \frac{3}{2}\right)}{\Gamma\left(\mu + \frac{1}{2}\right)}.$$

The recursion formula for γ_{μ} is given by

$$\gamma_{\mu}(k+1) = (k+1+2\mu\theta_{k+1})\gamma_{\mu}(k), \ k = 0, 1, 2, \cdots,$$

where $\mu > -\frac{1}{2}$ and

$$\theta_k = \begin{cases} 0 & \text{if } k \in 2\mathbb{N} \\ 1 & \text{if } k \in 2\mathbb{N}+1. \end{cases}$$

Sucu [16] defined a Dunkl analogue of Szász operators via a generalization of the exponential function [15] as follows:

$$S_n^*(f;x) := \frac{1}{e_\mu(nx)} \sum_{k=0}^{\infty} \frac{(nx)^k}{\gamma_\mu(k)} f\left(\frac{k+2\mu\theta_k}{n}\right), \tag{1.3}$$

where $x \geq 0,\ f \in C[0,\infty), \mu \geq 0,\ n \in \mathbb{N}$.
Cheikh et al., [2] stated the q-Dunkl classical q-Hermite type polynomials and gave definitions of q-Dunkl analogues of exponential functions and recursion relations for $\mu > -\frac{1}{2}$ and $0 < q < 1$.

$$e_{\mu,q}(x) = \sum_{n=0}^{\infty} \frac{x^n}{\gamma_{\mu,q}(n)},\ x \in [0,\infty) \tag{1.4}$$

$$E_{\mu,q}(x) = \sum_{n=0}^{\infty} \frac{q^{\frac{n(n-1)}{2}} x^n}{\gamma_{\mu,q}(n)},\ x \in [0,\infty) \tag{1.5}$$

$$\gamma_{\mu,q}(n+1) = \left(\frac{1-q^{2\mu\theta_{n+1}+n+1}}{1-q}\right)\gamma_{\mu,q}(n),\ n \in \mathbb{N}, \tag{1.6}$$

$$\theta_n = \begin{cases} 0 & \text{if } n \in 2\mathbb{N}, \\ 1 & \text{if } n \in 2\mathbb{N}+1. \end{cases}$$

An explicit formula for $\gamma_{\mu,q}(n)$ is

$$\gamma_{\mu,q}(n) = \frac{(q^{2\mu+1},q^2)_{[\frac{n+1}{2}]}(q^2,q^2)_{[\frac{n}{2}]}}{(1-q)^n}\gamma_{\mu,q}(n),\ n \in \mathbb{N}.$$

And some of the special cases of $\gamma_{\mu,q}(n)$ are defined as:

$$\gamma_{\mu,q}(0) = 1,\ \gamma_{\mu,q}(1) = \frac{1-q^{2\mu+1}}{1-q},\ \gamma_{\mu,q}(2) = \left(\frac{1-q^{2\mu+1}}{1-q}\right)\left(\frac{1-q^2}{1-q}\right),$$

$$\gamma_{\mu,q}(3) = \left(\frac{1-q^{2\mu+1}}{1-q}\right)\left(\frac{1-q^2}{1-q}\right)\left(\frac{1-q^{2\mu+3}}{1-q}\right),$$

$$\gamma_{\mu,q}(4) = \left(\frac{1-q^{2\mu+1}}{1-q}\right)\left(\frac{1-q^2}{1-q}\right)\left(\frac{1-q^{2\mu+3}}{1-q}\right)\left(\frac{1-q^4}{1-q}\right).$$

In [4], Gürhan Içöz gave the Dunkl generalization of Szász operators via q-calculus as:

$$D_{n,q}(f;x) = \frac{1}{e_{\mu,q}([n]_q x)} \sum_{k=0}^{\infty} \frac{([n]_q x)^k}{\gamma_{\mu,q}(k)} f\left(\frac{1-q^{2\mu\theta_k+k}}{1-q^n}\right), \tag{1.7}$$

for $\mu > \frac{1}{2},\ x \geq 0,\ 0 < q < 1$ and $f \in C[0,\infty)$.

Previous studies demonstrate that providing a better error estimation for positive linear operators plays an important role in approximation theory, which allows us to approximate much faster to the function being approximated.

Motivated essentially by by Içöz [4] the recent investigation of Dunkl generalization of Szász-Mirakjan operators via q-calculus we have showed that our modified operators have better error estimation than [4]. We have proved several approximation results. We have successfully extended these results and modifying the results of papers [4].

2. Construction of operators and moments estimation

We modify the q Dunkl analogue of Szász-operators [4].

Let $\{r_{[n]_q}(x)\}$ be a sequence of real-valued continuous functions defined on $[0,\infty)$ with $0 \leq r_{[n]_q} < \infty$. Then we define

$$D^{\star}_{n,q}(f;x) = \frac{1}{e_{\mu,q}([n]_q r_{[n]_q}(x))} \sum_{k=0}^{\infty} \frac{([n]_q r_{[n]_q}(x))^k}{\gamma_{\mu,q}(k)} f\left(\frac{1-q^{2\mu\theta_k+k}}{1-q^n}\right). \tag{2.1}$$

Now, if we replace $r_{[n]_q}(x)$ as

$$r_{[n]_q}(x) = x - \frac{1}{2[n]_q}, \text{ where } \frac{1}{2n} \leq x < \frac{1}{1-q^n} \text{ and } n \in \mathbb{N}. \tag{2.2}$$

Then for any $\frac{1}{2n} \leq x < \frac{1}{1-q^n},\ 0<q<1,\ \mu > \frac{1}{2n}$ and $n \in \mathbb{N}$ we have

$$D^{*}_{n,q}(f;x) = \frac{1}{e_{\mu,q}\left(\frac{2[n]_q x - 1}{2}\right)} \sum_{k=0}^{\infty} \frac{(2[n]_q x-1)^k}{2^k \gamma_{\mu,q}(k)} f\left(\frac{1-q^{2\mu\theta_k+k}}{1-q^n}\right). \tag{2.3}$$

where $e_{\mu,q}(x),\ \gamma_{\mu,q}$ are defined in (1.4),(1.6) by [16] and $f \in C_\zeta[0,\infty)$ with $\zeta \geq 0$ and

$$C_\zeta[0,\infty) = \{f \in C[0,\infty) : \mid f(t) \mid \leq M(1+t)^\zeta, \text{ for some } M>0,\ \zeta>0\}. \tag{2.4}$$

Lemma 2.1. *Let $D^*_{n,q}(.\ ;\ .)$ be the operators given by (2.3). Then for each $\frac{1}{2n} \leq x < \frac{1}{1-q^n},\ n \in \mathbb{N}$, we have we have the following identities:*

1. $D^*_{n,q}(1;x) = 1$,
2. $D^*_{n,q}(t;x) = r_{[n]_q}(x) = x - \frac{1}{2[n]_q}$,
3. $x^2 + \left(q^{2\mu}[1-2\mu]_q \frac{e_{\mu,q}(q[n]_q r_{[n]_q}(x))}{e_{\mu,q}([n]_q r_{[n]_q}(x))} - 1\right) \frac{x}{[n]_q} - \frac{1}{4[n]_q^2}\left(2q^{2\mu}[1-2\mu]_q \frac{e_{\mu,q}(q[n]_q r_{[n]_q}(x))}{e_{\mu,q}([n]_q r_{[n]_q}(x))} - 1\right) \leq D^*_{n,q}(t^2;x) \leq x^2 + ([1+2\mu]_q - 1)\frac{x}{[n]_q} - \frac{1}{4[n]_q^2}(2[1+2\mu]_q - 1)$.

Proof. (1) $D^*_{n,q}(1;x) = \frac{1}{e_{\mu,q}([n]_q r_{[n]_q}(x))} \sum_{k=0}^{\infty} \frac{([n]_q r_{[n]_q}(x))^k}{\gamma_\mu(k)} = 1$.

(2)

$$\begin{aligned} D_{n,q}^*(t;x) &= \frac{1}{e_{\mu,q}([n]_q r_{[n]_q}(x))} \sum_{k=0}^{\infty} \frac{([n]_q r_{[n]_q}(x))^k}{\gamma_\mu(k)} \left(\frac{1-q^{2\mu\theta_k+k}}{1-q^n} \right) \\ &= \frac{1}{[n]_q e_{\mu,q}([n]_q r_{[n]_q}(x))} \sum_{k=1}^{\infty} \frac{([n]_q r_{[n]_q}(x))^k}{\gamma_\mu(k-1)} \\ &= x - \frac{1}{2[n]_q} \end{aligned}$$

(3)

$$\begin{aligned} D_{n,q}^*(t^2;x) &= \frac{1}{e_{\mu,q}([n]_q r_{[n]_q}(x))} \sum_{k=0}^{\infty} \frac{([n]_q r_{[n]_q}(x))^k}{\gamma_\mu(k)} \left(\frac{1-q^{2\mu\theta_k+k}}{1-q^n} \right)^2 \\ &= \frac{1}{[n]_q^2 e_{\mu,q}([n]_q r_{[n]_q}(x))} \sum_{k=0}^{\infty} \frac{([n]_q r_{[n]_q}(x))^k}{\gamma_\mu(k-1)} \left(\frac{1-q^{2\mu\theta_k+k}}{1-q} \right) \\ &= \frac{1}{[n]_q^2 e_{\mu,q}([n]_q r_{[n]_q}(x))} \sum_{k=0}^{\infty} \frac{([n]_q r_{[n]_q}(x))^{k+1}}{\gamma_\mu(k)} \left(\frac{1-q^{2\mu\theta_{k+1}+k+1}}{1-q} \right). \end{aligned}$$

From [4] we know that

$$[2\mu\theta_{k+1}+k+1]_q = [2\mu\theta_k+k]_q + q^{2\mu\theta_k+k}[2\mu(-1)^k+1]_q, \tag{2.5}$$

Now by separating to the even and odd terms and using (2.5), we get

$$\begin{aligned} D_{n,q}^*(t^2;x) &= \frac{1}{[n]_q^2 e_{\mu,q}([n]_q r_{[n]_q}(x))} \sum_{k=0}^{\infty} \frac{([n]_q r_{[n]_q}(x))^{k+1}}{\gamma_\mu(k)} \left(\frac{1-q^{2\mu\theta_{k+1}+k+1}}{1-q} \right) \\ &+ \frac{[1+2\mu]_q}{[n]_q^2 e_{\mu,q}([n]_q r_{[n]_q}(x))} \sum_{k=0}^{\infty} \frac{([n]_q r_{[n]_q}(x))^{2k+1}}{\gamma_\mu(2k)} q^{2\mu\theta_{2k}+2k} \\ &+ \frac{[1-2\mu]_q}{[n]_q^2 e_{\mu,q}([n]_q r_{[n]_q}(x))} \sum_{k=0}^{\infty} \frac{([n]_q r_{[n]_q}(x))^{2k+2}}{\gamma_\mu(2k)} q^{2\mu\theta_{2k+1}+2k+1}. \end{aligned}$$

We know the inequality

$$[1-2\mu]_q \leq [1+2\mu]_q. \tag{2.6}$$

Therefore by using (2.6) we have

$$\begin{aligned} D_{n,q}^*(t^2;x) &\geq (r_{[n]_q}(x))^2 + \frac{r_{[n]_q}(x)[1-2\mu]_q}{[n]_q e_{\mu,q}([n]_q r_{[n]_q}(x))} \sum_{k=0}^{\infty} \frac{(q[n]_q r_{[n]_q}(x))^{2k}}{\gamma_\mu(2k)} \\ &+ \frac{q^{2\mu} r_{[n]_q}(x)[1-2\mu]_q}{[n]_q e_{\mu,q}([n]_q r_{[n]_q}(x))} \sum_{k=0}^{\infty} \frac{(q[n]_q r_{[n]_q}(x))^{2k+1}}{\gamma_\mu(2k+1)} \\ &\geq (r_{[n]_q}(x))^2 + q^{2\mu}[1-2\mu]_q \frac{e_{\mu,q}(q[n]_q r_{[n]_q}(x))}{e_{\mu,q}([n]_q r_{[n]_q}(x))} \frac{r_{[n]_q}(x)}{[n]_q}. \end{aligned}$$

Similarly on the other hand we have

$$D^*_{n,q}(t^2;x) \leq (r_{[n]_q}(x))^2 + [1+2\mu]_q \frac{r_{[n]_q}(x)}{[n]_q}.$$

Which completes the proof.

□

Lemma 2.2. *Let the operators $D^*_{n,q}(.\ ;\ .)$ be given by (2.3). Then for each $x \geq \frac{1}{2n}$, $n \in \mathbb{N}$, we have*

(1) $D^*_{n,q}(t-x;x) = -\frac{1}{2[n]_q}$,
(2) $D^*_{n,q}((t-x)^2;x) \leq [1+2\mu]_q \frac{x}{[n]_q} - \frac{1}{4[n]_q^2}\left(2[1+2\mu]_q - 1\right)$.

3. Main results

We obtain the Korovkin's type approximation properties for our operators defined by (2.3).

Let $C_B(\mathbb{R}^+)$ be the set of all bounded and continuous functions on $\mathbb{R}^+ = [0,\infty)$, which is linear normed space with

$$\| f \|_{C_B} = \sup_{x \geq 0} | f(x) | .$$

Let

$$H := \{f : x \in [0,\infty), \frac{f(x)}{1+x^2} \text{ is convergent as } x \to \infty\}.$$

Remark 3.1. By lemma 2.1, it is clear that the positive liner operators $D^*_{n,q}$ given by (2.3) preserve a linear functions, that is for $\phi(y) = cy + d,\ c,d \in \mathbb{R}(Real\ numbers)$, $D^*_{n,q}(\phi;x) = \phi(x)$ for all $x \geq \frac{1}{2n},\ n \in \mathbb{N}$.
Now, fix $b > \frac{1}{2}$ and consider the lattice homomorphism $H_b : C[0,\infty] \to C[0,b]$ defined by $H_b(f) = f\big|_{[0,b]}$ for every $f \in C[0,\infty]$, where $f\big|_{[0,b]}$ denotes the restriction of the domain of f to the interval $[0,b]$. In this case for each $j = 0,1,2$, we have

$$\lim H_b\left(D^*_{n,q}(e_j)\right) = H_b(e_j) \qquad uniformly \quad on \quad \left[\frac{1}{2}, b\right]. \tag{3.1}$$

Thus, by using (3.1) and with the universal Korovkin-type property with respect to the monotone operators. And hence we have the following Korovkin-type approximation result.

Theorem 3.2. *Let $D^*_{n,q}(.\ ;\ .)$ be the operators defined by (2.3). Then for any function $f \in C_\zeta[0,\infty) \cap H,\ \zeta \geq 2$,*

$$\lim_{n\to\infty} D^*_{n,q}(f;x) = f(x)$$

is uniformly on each compact subset of $[0,\infty)$, where $x \in \left[\frac{1}{2}, b\right],\ \frac{1}{2} < b < \infty$.

Proof. The proof is based on Lemma 2.1 and well known Korovkin's theorem regarding the convergence of a sequence of linear and positive operators, so it is enough to prove the conditions

$$\lim_{n\to\infty} D^*_{n,q}((t^j;x) = x^j,\ j = 0,1,2,\ \{\text{as } n\to\infty\}$$

uniformly on $[0,1]$.
Clearly $\frac{1}{[n]_q} \to 0$ $(n\to\infty)$ we have

$$\lim_{n\to\infty} D^*_{n,q}(t;x) = x,\ \lim_{n\to\infty} D^*_{n,q}(t^2;x) = x^2.$$

Which complete the proof. □

We recall the weighted spaces of the functions on $\mathbb{R}^+$, which are defined as follows:

$$\begin{aligned} P_\rho(\mathbb{R}^+) &= \{f : \mid f(x) \mid \leq M_f\rho(x)\}, \\ Q_\rho(\mathbb{R}^+) &= \left\{f : f \in P_\rho(\mathbb{R}^+) \cap C[0,\infty)\right\}, \\ Q^k_\rho(\mathbb{R}^+) &= \left\{f : f \in Q_\rho(\mathbb{R}^+) \text{ and } \lim_{x\to\infty}\frac{f(x)}{\rho(x)} = k(k \text{ is a constant})\right\}, \end{aligned}$$

where $\rho(x) = 1 + x^2$ is a weight function and M_f is a constant depending only on f. Note that $Q_\rho(\mathbb{R}^+)$ is a normed space with the norm $\| f \|_\rho = \sup_{x\geq 0}\frac{|f(x)|}{\rho(x)}$.

Lemma 3.3. *([3]) The linear positive operators L_n, $n \geq 1$ act from $Q_\rho(\mathbb{R}^+) \to P_\rho(\mathbb{R}^+)$ if and only if*

$$\| L_n(\varphi;x) \| \leq K\varphi(x),$$

where $\varphi(x) = 1 + x^2$, $x \in \mathbb{R}^+$ and K is a positive constant.

Theorem 3.4. *([3]) Let $\{L_n\}_{n\geq 1}$ be a sequence of positive linear operators acting from $Q_\rho(\mathbb{R}^+) \to P_\rho(\mathbb{R}^+)$ and satisfying the condition*

$$\lim_{n\to\infty} \| L_n(\rho^\tau) - \rho^\tau \|_\varphi = 0,\ \tau = 0,1,2.$$

Then for any function $f \in Q^k_\rho(\mathbb{R}^+)$, we have

$$\lim_{n\to\infty} \| L_n(f;x) - f \|_\varphi = 0.$$

Theorem 3.5. *Let $D^*_{n,q}(.\ ;\ .)$ be the operators defined by (2.3). Then for each function $f \in Q^k_\rho(\mathbb{R}^+)$ we have*

$$\lim_{n\to\infty} \| D^*_{n,q}(f;x) - f \|_\rho = 0.$$

Proof. From Lemma 2.1 and Theorem 3.4 for $\tau = 0$, the first condition is fulfilled. Therefore

$$\lim_{n\to\infty} \| D^*_{n,q}(1;x) - 1 \|_\rho = 0.$$

Similarly From Lemma 2.1 and Theorem 3.4 for $\tau = 1,2$ we have that

$$\begin{aligned} \sup_{x\in[0,\infty)} \frac{\mid D^*_{n,q}(t;x) - x \mid}{1+x^2} &\leq \frac{1}{2[n]_q} \sup_{x\in[0,\infty)} \frac{1}{1+x^2} \\ &= \frac{1}{2[n]_q}, \end{aligned}$$

which imply that

$$\lim_{n\to\infty} \parallel D^*_{n,q}(t;x) - x \parallel_\rho = 0.$$

$$\begin{aligned}\sup_{x\in[0,\infty)} \frac{\mid D^*_{n,q}(t^2;x) - x^2 \mid}{1+x^2} &\leq \frac{\mid [1+2\mu]_q - 1 \mid}{[n]_q} \sup_{x\in[0,\infty)} \frac{x}{1+x^2} \\ &+ \frac{1}{4[n]_q^2} \mid [1+2\mu]_q - 1 \mid \sup_{x\in[0,\infty)} \frac{1}{1+x^2}\end{aligned}$$

which imply that

$$\lim_{n\to\infty} \parallel D^*_{n,q}(t^2;x) - x^2 \parallel_\rho = 0.$$

This complete the proof. □

4. **Rate of Convergence**

Here we calculate the rate of convergence of operators (2.3) by means of modulus of continuity and Lipschitz type maximal functions.

Let $f \in C_B[0,\infty]$, the space of all bounded and continuous functions on $[0,\infty)$ and $x \geq \frac{1}{2n}$, $n \in \mathbb{N}$. Then for $\delta > 0$, the modulus of continuity of f denoted by $\omega(f,\delta)$ gives the maximum oscillation of f in any interval of length not exceeding $\delta > 0$ and it is given by

$$\omega(f,\delta) = \sup_{|t-x|\leq\delta} \mid f(t) - f(x) \mid,\ t \in [0,\infty). \tag{4.1}$$

It is known that $\lim_{\delta\to 0+} \omega(f,\delta) = 0$ for $f \in C_B[0,\infty)$ and for any $\delta > 0$ one has

$$\mid f(t) - f(x) \mid \leq \left(\frac{\mid t-x \mid}{\delta} + 1 \right) \omega(f,\delta). \tag{4.2}$$

Theorem 4.1. *Let $D^*_{n,q}(.\ ;\ .)$ be the operators defined by* (2.3)*. Then for $f \in C_B[0,\infty)$, $x \geq \frac{1}{2n}$ and $n \in \mathbb{N}$ we have*

$$\mid D^*_{n,q}(f;x) - f(x) \mid \leq 2\omega\left(f;\delta_{n,x}\right),$$

where $C_B[0,\infty)$ is the space of uniformly continuous bounded functions on $\mathbb{R}^+$, $\omega(f,\delta)$ is the modulus of continuity of the function $f \in C_B[0,\infty)$ defined in (4.1) *and*

$$\delta_{n,x} = \sqrt{[1+2\mu]_q \frac{x}{[n]_q} - \frac{1}{4[n]_q^2}\left(2[1+2\mu]_q - 1\right)}. \tag{4.3}$$

Proof. We prove it by using (4.1), (4.2) and Cauchy-Schwarz inequality we can easily get

$$\mid D^*_{n,q}(f;x) - f(x) \mid \ \leq \ \left\{ 1 + \frac{1}{\delta}\left(D^*_{n,q}(t-x)^2;x \right)^{\frac{1}{2}} \right\} \omega(f;\delta)$$

if we choose $\delta = \delta_{n,x}$ and by applying the result (2) of Lemma 2.2 complete the proof. □

Remark 4.2. For the operators $D_{n,q}(.\,;\ .)$ defined by (1.7) we may write that, for every $f \in C_B[0,\infty),\ x \geq 0$ and $n \in \mathbb{N}$

$$\mid D_{n,q}(f;x) - f(x) \mid \leq 2\omega\left(f;\lambda_{n,x}\right), \tag{4.4}$$

where by [4] we have

$$\lambda_{n,x} = \sqrt{D_{n,q}((t-x)^2;x)} \leq \sqrt{[1+2\mu]_q \frac{x}{[n]_q}}. \tag{4.5}$$

Now we claim that the error estimation in Theorem 4.1 is better than that of (4.4) provided $f \in C_B[0,\infty)$ and $x \geq \frac{1}{2n},\ n \in \mathbb{N}$. Indeed, for $x \geq \frac{1}{2n},\ \mu \geq \frac{1}{2n}$ and $n \in \mathbb{N}$, it is guarantees that

$$D^*_{n,q}((t-x)^2;x) \leq D_{n,q}((t-x)^2;x), \tag{4.6}$$

$$[1+2\mu]_q \frac{x}{[n]_q} - \frac{1}{4[n]_q^2}\left(2[1+2\mu]_q - 1\right) \leq [1+2\mu]_q \frac{x}{[n]_q}. \tag{4.7}$$

Which imply that

$$\sqrt{[1+2\mu]_q \frac{x}{[n]_q} - \frac{1}{4[n]_q^2}\left(2[1+2\mu]_q - 1\right)} \leq \sqrt{[1+2\mu]_q \frac{x}{[n]_q}}. \tag{4.8}$$

Now we give the rate of convergence of the operators $D^*_{n,q}(f;x)$ defined in (2.3) in terms of the elements of the usual Lipschitz class $Lip_M(\nu)$.

Let $f \in C_B[0,\infty)$, $M > 0$ and $0 < \nu \leq 1$. The class $Lip_M(\nu)$ is defined as

$$Lip_M(\nu) = \{f : \mid f(\zeta_1) - f(\zeta_2) \mid \leq M \mid \zeta_1 - \zeta_2 \mid^{\nu} \quad (\zeta_1, \zeta_2 \in [0,\infty))\} \tag{4.9}$$

Theorem 4.3. *Let $D^*_{n,q}(.\,;\ .)$ be the operator defined in* (2.3)*.Then for each $f \in Lip_M(\nu),\ (M > 0,\ 0 < \nu \leq 1)$ satisfying* (4.9) *we have*

$$\mid D^*_{n,q}(f;x) - f(x) \mid \leq M\left(\delta_{n,x}\right)^{\frac{\nu}{2}}$$

where $\delta_{n,x}$ is given in Theorem 4.1.

Proof. We prove it by using (4.9) and Hölder inequality.

$$\begin{aligned}
\mid D^*_{n,q}(f;x) - f(x) \mid &\leq \mid D^*_{n,q}(f(t) - f(x);x) \mid \\
&\leq D^*_{n,q}\left(\mid f(t) - f(x) \mid;x\right) \\
&\leq \mid MD^*_{n,q}\left(\mid t - x \mid^{\nu};x\right).
\end{aligned}$$

Therefore

$$
\begin{aligned}
&\mid D^{*}_{n,q}(f;x)-f(x)\mid \\
&\leq M\frac{[n]_q}{e_{\mu,q}([n]_q r_{[n]_q}(x))}\sum_{k=0}^{\infty}\frac{([n]_q r_{[n]_q}(x))^k}{\gamma_{\mu,q}(k)}\left|\frac{1-q^{2\mu\theta_k+k}}{1-q^n}-x\right|^{\nu}\mathrm{d}t \\
&\leq M\frac{[n]_q}{e_{\mu,q}([n]_q r_{[n]_q}(x))}\sum_{k=0}^{\infty}\left(\frac{([n]_q r_{[n]_q}(x))^k}{\gamma_{\mu,q}(k)}\right)^{\frac{2-\nu}{2}} \\
&\times \left(\frac{([n]_q r_{[n]_q}(x))^k}{\gamma_{\mu,q}(k)}\right)^{\frac{\nu}{2}}\left|\frac{1-q^{2\mu\theta_k+k}}{1-q^n}-x\right|^{\nu}\mathrm{d}t \\
&\leq M\left(\frac{n}{e_{\mu,q}([n]_q r_{[n]_q}(x))}\sum_{k=0}^{\infty}\frac{([n]_q r_{[n]_q}(x))^k}{\gamma_{\mu,q}(k)}\mathrm{d}t\right)^{\frac{2-\nu}{2}} \\
&\times \left(\frac{[n]_q}{e_{\mu,q}([n]_q r_{[n]_q}(x))}\sum_{k=0}^{\infty}\frac{([n]_q r_{[n]_q}(x))^k}{\gamma_{\mu,q}(k)}\left|\frac{1-q^{2\mu\theta_k+k}}{1-q^n}-x\right|^{2}\mathrm{d}t\right)^{\frac{\nu}{2}} \\
&= M\left(D^{*}_{n,q}(t-x)^2;x\right)^{\frac{\nu}{2}}.
\end{aligned}
$$

Which complete the proof. □

Let $C_B[0,\infty)$ denote the space of all bounded and continuous functions on $\mathbb{R}^+=[0,\infty)$ and

$$C_B^2(\mathbb{R}^+)=\{g\in C_B(\mathbb{R}^+):g',g''\in C_B(\mathbb{R}^+)\}, \tag{4.10}$$

with the norm

$$\parallel g\parallel_{C_B^2(\mathbb{R}^+)}=\parallel g\parallel_{C_B(\mathbb{R}^+)}+\parallel g'\parallel_{C_B(\mathbb{R}^+)}+\parallel g''\parallel_{C_B(\mathbb{R}^+)}, \tag{4.11}$$

also

$$\parallel g\parallel_{C_B(\mathbb{R}^+)}=\sup_{x\in\mathbb{R}^+}\mid g(x)\mid. \tag{4.12}$$

Theorem 4.4. *Let $D^{*}_{n,q}(.\ ;\ .)$ be the operator defined in* (2.3). *Then for any $g\in C_B^2(\mathbb{R}^+)$ we have*

$$\mid D^{*}_{n,q}(f;x)-f(x)\mid\leq\left\{\left(-\frac{1}{2[n]_q}\right)+\frac{\delta_{n,x}}{2}\right\}\parallel g\parallel_{C_B^2(\mathbb{R}^+)},$$

where $\delta_{n,x}$ is given in Theorem 4.1.

Proof. Let $g\in C_B^2(\mathbb{R}^+)$, then by using the generalized mean value theorem in the Taylor series expansion we have

$$g(t)=g(x)+g'(x)(t-x)+g''(\psi)\frac{(t-x)^2}{2},\ \psi\in(x,t).$$

By applying linearity property on $D^{*}_{n,q}$, we have

$$D^{*}_{n,q}(g,x)-g(x)=g'(x)D^{*}_{n,q}\left((t-x);x\right)+\frac{g''(\psi)}{2}D^{*}_{n,q}\left((t-x)^2;x\right),$$

which imply that

$$| D^*_{n,q}(g;x) - g(x) | \leq \left(-\frac{1}{2[n]_q}\right) \| g' \|_{C_B(\mathbb{R}^+)} + \left([1+2\mu]_q \frac{x}{[n]_q} - \frac{1}{4[n]_q^2}\left(2[1+2\mu]_q - 1\right)\right) \frac{\|g''\|_{C_B(\mathbb{R}^+)}}{2}.$$

From (4.11) we have $\| g' \|_{C_B[0,\infty)} \leq \| g \|_{C_B^2[0,\infty)}$.

$$| D^*_{n,q}(g;x) - g(x) | \leq \left(-\frac{1}{2[n]_q}\right) \| g \|_{C_B^2(\mathbb{R}^+)} + \left([1+2\mu]_q \frac{x}{[n]_q} - \frac{1}{4[n]_q^2}\left(2[1+2\mu]_q - 1\right)\right) \frac{\|g\|_{C_B^2(\mathbb{R}^+)}}{2}.$$

This completes the proof from 2 of Lemma 2.2. □

Acknowledgement

This project was funded by the Deanship of Scientific Research (DSR) at King Abdulaziz University, Jeddah, under grant no. G-312-130-38. The authors, therefore, acknowledge with thanks DSR for technical and financial support.

References

[1] S.N. Bernstein, Démonstration du théoréme de Weierstrass fondée sur le calcul des probabilités, Commun. Soc. Math. Kharkow, 2(13) (2012), 1–2.

[2] B. Cheikh, Y. Gaied, M. Zaghouani, A q-Dunkl-classical q-Hermite type polynomials, Georgian Math. J., 21(2) (2014), 125–137.

[3] A. D. Gadzhiev, The convergence problem for a sequence of positive linear operators on unbounded sets and theorems analogues to that of PP Korovkin. Soviet Mathematics Doklady, 15(5) (1974), 1453-1436.

[4] G. İçōz, B. Çekim, Dunkl generalization of Szász operators via q-calculus, Jour. Ineq. Appl., 284:(2015), 2015.

[5] A. Lupaş, A q-analogue of the Bernstein operator, In Seminar on Numerical and Statistical Calculus, University of Cluj-Napoca, Cluj-Napoca 9 (1987), 85–92.

[6] N.I. Mahmudov, V. Gupta, On certain q-analogue of Szász Kantorovich operators, J. Appl. Math. Comput., 37 (2011), 407–419.

[7] M. Mursaleen, K.J. Ansari, Approximation of q-Stancu-Beta operators which preserve x^2, Bull. Malaysian Math. Sci. Soc., DOI: 10.1007/s40840-015-0146-9.

[8] M. Mursaleen, A. Khan, Statistical approximation properties of modified q- Stancu-Beta operators, Bull. Malays. Math. Sci. Soc. (2), 36(3) (2013), 683–690.

[9] M. Mursaleen, A. Khan, Generalized q-Bernstein-Schurer operators and some approximation theorems, Jour. Function Spaces Appl., Volume (2013), Article ID 719834, 7 pages.

[10] M. Mursaleen, Faisal Khan, Asif Khan, Approximation properties for modified q-bernstein-kantorovich operators, Numerical Functional Analysis and Optimization, 36(9) (2015) 1178–1197.

[11] M. Mursaleen, Faisal Khan, Asif Khan, Approximation properties for King's type modified q-Bernstein-Kantorovich operators, Math. Meth. Appl. Sci., 38 (2015) 5242–5252.

[12] M. Örkcü, O. Doğru, Weighted statistical approximation by Kantorovich type q-Szász Mirakjan operators, Appl. Math. Comput., 217 (2011), 7913–7919.

[13] M. Örkcü, O. Doğru, q-Szász-Mirakyan-Kantorovich type operators preserving some test functions, Applied Mathematics Letters 24 (2011), 1588-1593.

[14] G.M. Phillips, Bernstein polynomials based on the q- integers, Ann. Numer. Math., 4 (1997), 511–518.

[15] M. Rosenblum, Generalized Hermite polynomials and the Bose-like oscillator calculus, Oper. Theory, Adv. Appl., 73 (1994), 369–396.

[16] S. Sucu, Dunkl analogue of Szász operators, Appl. Math. Comput., 244 (2014), 42–48.

[17] O. Szász, Generalization of S. Bernstein's polynomials to the infinite interval, J. Res. Natl. Bur. Stand., 45 (1950), 239–245.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 27, NO.1, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC

Pointwise error estimates for spherical hybrid interpolation *

Chunmei Ding Ming Li Feilong Cao

Department of Applied Mathematics, College of Sciences, China Jiliang University,
Hangzhou 310018, Zhejiang Province, P R China.
E-mail: feilongcao@gmail.com

Abstract

This paper studies pointwise error estimates for spherical hybrid interpolation, which combines spherical polynomials together with spherical radial basis functions constructed by a strictly positive definite zonal kernel. The study is first carried out in the native space associated with the kernel, and then refined error estimates for a target function in a still smaller space are established.

MSC(2000): 41A17, 41A30

Keywords: Sphere; Interpolation; Approximation; Pointwise Error

1 Introduction

In recent years, fitting a surface to scattered data arising from sampling an unknown function defined on an underlying manifold comes up frequently in applied problems. If the underlying manifold is $\mathbb{S}^2$, the unit sphere embedded in the Euclidean space $\mathbb{R}^3$, then there are applications to astrophysics, meteorology, geodesy, geophysics and other areas (see [5, 6, 27]). Amongst approaches for scattered data interpolation and approximation on $\mathbb{S}^2$, many authors have used spherical polynomials or spherical radial basis functions (see [5, 6, 9, 12, 18, 20, 25, 26, 27, 13, 2]). Motivated by the fact that the spherical radial basis functions are helpful to handle scattered data and rapid changes, at the same time, the spherical polynomials contribute to handle the slowly varying large-scale features, a hybrid interpolation scheme was given in [23].

The hybrid interpolation scheme combines spherical radial basis functions together with spherical polynomials, that is a little different from interpolation by radial basis functions constructed from conditionally positive definite kernels (in which case a polynomial part is needed to make the theory work, see [8]). Sloan and Sommariva [23] restricted their attention to the case of strictly positive definite kernels, so that the polynomial component is voluntary rather than forced.

This paper studies the hybrid interpolation problem in an appropriate native space $\mathcal{N}_\phi$ of continuous functions on $\mathbb{S}^2$, which is defined by an underlying strictly positive definite kernel ϕ. We use the method in [23] to get the pointwise error estimate for the hybrid interpolation.

It is known that if ϕ is smooth, the native space $\mathcal{N}_\phi$ is small in the sense that it is composed of very smooth functions. That is so called "native space barrier" problem and there are several literatures focus on it. We refer the readers to [10, 11, 15, 16, 17] for more details. In this paper, we combine the approach which was used by Levesley and Sun in [10] with the techniques in [24], and embed the smooth radial basis functions in a larger native space generated by a less smooth kernel ψ and still use the hybrid interpolation associated with the smooth kernel ϕ to interpolate the target function from the larger native space. In the process of obtaining the corresponding error estimates, we will use the "norming set" method developed by Jetter in [9] and a special case of the general Bernstein-type inequality established by Ditzian [4].

This paper is organized as follows. In Section 2, we give some notations and preliminary results. The hybrid interpolation is introduced and the crucial condition for the scheme to be well defined

*Supported by the National Natural Science Foundation of China (No. 61672477)

and discussed in Subsection 2.2, and native space and Sobolev space are introduced in Subsection 2.3. Finally, the pointwise errors are estimated in Section 3.

In the following, we adopt the following convention regarding symbols. Let C be a positive constant, whose value will be different at different occurrence even within the same formula. The symbol $A \sim B$ means that there exist positive constant C_1 and C_2 such that $C_1 B \leq A \leq C_2 B$.

2 Preliminaries

Let $\mathbb{S}^2$ be the unit sphere embedded in the Euclidean space $\mathbb{R}^3$, i.e.,

$$\mathbb{S}^2 := \left\{ x := (x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 + x_3^2 = 1 \right\}.$$

For integer $l \geq 0$, the restriction to $\mathbb{S}^2$ of a homogeneous harmonic polynomial with degree l is called a spherical harmonic of degree l. The class of all spherical harmonics with degree l is denoted by $\mathcal{H}_l$, and it is well know that spherical harmonics of different degrees are orthogonal with respect to the $L_2(\mathbb{S}^2)$ inner product

$$\langle f, g \rangle := \int_{\mathbb{S}^2} f(x) g(x) d\omega(x),$$

where $d\omega$ denotes surface measure on $\mathbb{S}^2$. Hence, if we choose an orthogonal basis $\{Y_{l,k} : k = 1, \ldots, 2l+1\}$ for each $\mathcal{H}_l$, then the set $\{Y_{l,k} : l = 0, 1, \ldots,\ k = 1, \ldots, 2l+1\}$ is an orthogonal basis for $L_2(\mathbb{S}^2)$. The class of all spherical harmonics with total degree $l \leq L$ is denoted by $\mathcal{P}_L$. Of course, $\mathcal{P}_L = \bigoplus_{l=0}^{L} \mathcal{H}_l$, and the dimension of $\mathcal{H}_l$ is $2l+1$ and that of $\mathcal{P}_L$ is $(L+1)^2$.

We denote by $L_p(\mathbb{S}^2)$ the space of p-integrable functions on $\mathbb{S}^2$ endowed with the norms

$$\|f\|_\infty := \|f\|_{L_\infty(\mathbb{S}^2)} := \operatorname*{ess\,sup}_{x \in \mathbb{S}^2} |f(x)|, \quad p = \infty,$$

and

$$\|f\|_p := \|f\|_{L_p(\mathbb{S}^2)} := \left\{ \int_{\mathbb{S}^2} |f(x)|^p d\omega(x) \right\}^{1/p} < \infty, \quad 1 \leq p < \infty.$$

The well known addition formula is given by (see [14])

$$\sum_{k=1}^{2l+1} Y_{l,k}(x) Y_{l,k}(y) = \frac{2l+1}{4\pi} P_l(x \cdot y),$$

where P_l is the Legendre polynomial with degree l and dimension three, which is normalized such that $P_l(1) = 1$, and satisfies the orthogonality relation (see [14])

$$\int_{-1}^{1} P_k(t) P_j(t) dt = \frac{2}{2l+1} \delta_{k,j},$$

where the symbol $\delta_{k,j}$ denotes the usual Kronecker symbol.

The addition formula also yields the following useful relation

$$\sum_{k=1}^{2l+1} |Y_{l,k}(x) Y_{l,k}(y)| \leq \sum_{k=1}^{2l+1} Y_{l,k}^2(x) = \frac{2l+1}{4\pi}, \quad x,\ y \in \mathbb{S}^2. \tag{2.1}$$

2.1 Strictly positive definite kernel

Definition 2.1 *(see [27]). A continuous and symmetric function $\phi : \mathbb{S}^2 \times \mathbb{S}^2 \longrightarrow \mathbb{R}$ is called positive definite kernel, if, for any $N \in \mathbb{N}_+$, $\alpha = (\alpha_i)_{i=1,\ldots,N} \in \mathbb{R}^N$ and $\{x_1, \ldots, x_N\} \subset \mathbb{S}^2$, we have*

$$\sum_{i=1}^{N} \sum_{j=1}^{N} \alpha_i \alpha_j \phi(x_i, x_j) \geq 0.$$

When for any N distinct points $\{x_1, \ldots, x_N\}$, the above quadratic form is positive for all $\alpha = (\alpha_i)_{i=1,\ldots,N} \in \mathbb{R}^N / \{0\}$, then ϕ is called strictly positive definite kernel.

A kernel ϕ is called rotational invariant, if $\phi(\rho x, \rho y) = \phi(x, y)$ for all $x, y \in \mathbb{S}^2$ and for all rotations ρ. It can be shown that a continuous rotational invariant kernel depends only on the distance between x and y, that is, there is a function $\varphi : [-1, 1] \to \mathbb{R}$, such that $\varphi(xy) = \phi(x, y)$ for all $x, y \in \mathbb{S}^2$ (see [22]). Therefore, a rotational invariant kernel is also called a zonal kernel in the literature.

Schoenberg characterized the positive definite zonal kernels in [21] and the notation of strictly positive definiteness on spheres was first introduced by Xu and Cheney [28]. It is important to characterize all the strictly positive definite functions on spheres and such an endeavor has been taken by Ron and Sun in [19]. In [3], Chen et al. established a necessary and sufficient condition for strictly positive definite zonal kernels: the kernel ϕ is strictly positive definite and zonal if and only if

$$\phi(x, y) = \sum_{l=0}^{\infty} a_l \sum_{k=1}^{2l+1} Y_{l,k}(x) Y_{l,k}(y) = \sum_{l=0}^{\infty} \frac{(2l + 1)a_l}{4\pi} P_l(x \cdot y),$$

with $a_l \geq 0$ for all l, $\sum_{l=0}^{\infty} l a_l < \infty$ and $a_l > 0$ for infinitely many even values of l and infinitely many odd values of l.

2.2 The hybrid interpolation

Assume that we are given a strictly positive definite kernel $\phi(\cdot, \cdot)$ and a set of distinct points $X = \{x_1, \ldots, x_N\} \subset \mathbb{S}^2$. Then for a target function $f \in C(\mathbb{S}^2)$ we can take the hybrid interpolation for f in the form

$$I_{X,L} f = \sum_{j=1}^{N} \alpha_j \phi(\cdot, x_j) + \sum_{l=0}^{L} \sum_{k=1}^{2l+1} \beta_{l,k} Y_{l,k},$$

where we fix $L \geq 0$ as the desired degree of the polynomial component of the hybrid interpolation and the coefficients $\{\alpha_j\}_{j=1}^{N}$, $\{\beta_{l,k}\}_{k=1,\ldots,2l+1,\, l=0,\ldots,L}$ are determined by the interpolation conditions

$$I_{X,L} f(x_i) = f(x_i), \;\; i = 1, \ldots, N, \tag{2.2}$$

and also (in order to give a square linear system) the side conditions

$$\sum_{j=1}^{N} \alpha_j p(x_j) = 0, \;\; \forall p \in \mathcal{P}_L.$$

In order to give the conditions which will make sure that the interpolation is exist and unique, we shall impose a condition on the point set X.

Definition 2.2 *(see [23, Definition 3.1]). The set* $X = \{x_1, \ldots, x_N\} \subset \mathbb{S}^2$ *is said to be* $\mathcal{P}_L$ -unisolvent *if*

$$p \in \mathcal{P}_L, \;\; p(x_j) = 0 \text{ for } j = 1, \ldots, N \Rightarrow p = 0.$$

For the analysis of the interpolation error in the later sections it is convenient to define a finite-dimensional space $V_{X,L}$ within the interpolation $I_{X,L} f$ lies.

$$V_{X,L} := \left\{ \sum_{j=1}^{N} \alpha_j \phi(\cdot, x_j) + q : \; q \in \mathcal{P}_L, \; \alpha_j \in \mathbb{R} \text{ for } j = 1, \ldots, N, \text{ and} \sum_{j=1}^{N} \alpha_j p(x_j) = 0, \; \forall p \in \mathcal{P}_L \right\}.$$

The following Theorem 2.1 gives a crucial condition for the interpolation to be well defined, whose proof can be find in [23].

Theorem 2.1 *Let* $\phi(\cdot, \cdot)$ *be a strictly positive definite kernel, let* $L \geq 0$ *and* $X = \{x_1, \ldots, x_N\} \subset \mathbb{S}^2$ *be a set of distinct points which is* $\mathcal{P}_L$ *-unisolvent. Then for each* $f \in C(\mathbb{S}^2)$ *there exists a unique* $I_{X,L} f \in V_{X,L}$ *that satisfies the interpolation conditions in* (2.2).

2.3 Native space and Sobolev space

Here and in the other sections we assume that the strictly positive definite kernel ϕ is zonal and has the expansion

$$\phi(x,y)=\sum_{l=0}^{\infty}a_l\sum_{k=1}^{2l+1}Y_{l,k}(x)Y_{l,k}(y) \tag{2.3}$$

with $a_l>0$ for all l, $\sum_{l=0}^{\infty}la_l<\infty$, in which case the series of the right side in (2.3) converges uniformly for $x,y\in\mathbb{S}^2$.

For $f,g\in L_2(\mathbb{S}^2)$, they can be represented by their Fourier series

$$f=\sum_{l=0}^{\infty}\sum_{k=1}^{2l+1}\hat{f}_{l,k}Y_{l,k},\qquad g=\sum_{l=0}^{\infty}\sum_{k=1}^{2l+1}\hat{g}_{l,k}Y_{l,k},$$

respectively. With respect to the inner product expressed as (see [27])

$$(f,g)_{\mathcal{N}_\phi}=\sum_{l=0}^{\infty}\sum_{k=1}^{2l+1}\frac{\hat{f}_{l,k}\hat{g}_{l,k}}{a_l},$$

the native space $\mathcal{N}_\phi$, which is the subspace of $L_2(\mathbb{S}^2)$, can be defined by

$$\mathcal{N}_\phi:=\left\{f\in L_2(\mathbb{S}^2):\|f\|^2_{\mathcal{N}_\phi}=\sum_{l=0}^{\infty}\sum_{k=1}^{2l+1}\frac{|\hat{f}_{l,k}|^2}{a_l}<\infty\right\}.$$

It is easy to verify that the native space $\mathcal{N}_\phi$ is a reproducing kernel Hilbert space with reproducing kernel $\phi(\cdot,\cdot)$, that is,

$$(f,\phi(\cdot,x))_{\mathcal{N}_\phi}=f(x),\ x\in\mathbb{S}^2,\ f\in\mathcal{N}_\phi.$$

When $a_l\sim(l+1)^{-2s}$ for $l=0,1,\ldots$, the native space $\mathcal{N}_\phi$ is norm equivalent to the Sobolev space H_s:

$$H_s:=\left\{f\in L_2(\mathbb{S}^2):\|f\|^2_{H_s}=\sum_{l=0}^{\infty}\sum_{k=1}^{2l+1}(l+1)^{2s}|\hat{f}_{l,k}|^2<\infty\right\},$$

and the Sobolev embedding theorem in [7] implies that if $s>1$, then the space H_s is continuously embedded in $C(\mathbb{S}^2)$, so that H_s is a reproducing kernel Hilbert space.

3 Pointwise error estimates

As we can see that the uniqueness result in Theorem 2.1 ensures the existence and uniqueness of the lagrangians $l_j:=l_{j,X,L}:\mathbb{S}^2\to\mathbb{R}$, which is defined by

$$l_j\in V_{X,L},\ \ l_j(x_i)=\delta_{i,j},\ \ i,j=1,\ldots,N.$$

The following Theorem 3.1 is a little different from the obtained result in [23] and it is the difference that helps us to extend the error estimates for hybrid interpolation to L_p norm in the next section.

Theorem 3.1 *Let $\phi\in C(\mathbb{S}^2\times\mathbb{S}^2)$ be a strictly positive definite kernel defined in (2.3), and let $X=\{x_1,\ldots,x_N\}\subset\mathbb{S}^2$ be a $\mathcal{P}_L$-unisolvent set of distinct points on $\mathbb{S}^2$. For $f\in\mathcal{N}_\phi$, let $I_{X,L}f\in V_{X,L}$ be the hybrid interpolation defined in Section 2.2. Then for a fixed $x\in\mathbb{S}^2$, we have*

$$|f(x)-I_{X,L}f(x)|\le\|f-I_{X,L}f\|_{\mathcal{N}_\phi}P_{\phi,X,L}(x),$$

where $P_{\phi,X,L}$ is the power function defined by

$$P_{\phi,X,L}(x) = \left(\phi(x,x) - 2\sum_{j=1}^{N} l_j(x)\phi(x,x_j) + \sum_{i=1}^{N}\sum_{j=1}^{N} l_i(x)l_j(x)\phi(x_i,x_j)\right)^{1/2}.$$

Proof. With the help of the reproducing property of ϕ, we can rewrite the form of $I_{X,L}f$ as

$$I_{X,L}f(x) = \sum_{j=1}^{N} f(x_j)l_j(x) = \sum_{j=1}^{N}(f,\phi(\cdot,x_j))_{\mathcal{N}_\phi} l_j(x) = \left(f, \sum_{j=1}^{N}\phi(\cdot,x_j)l_j(x)\right)_{\mathcal{N}_\phi}, \quad x \in \mathbb{S}^2.$$

Since we also have $f(x) = (f,\phi(\cdot,x))_{\mathcal{N}_\phi}, \quad x \in \mathbb{S}^2$, from the reproducing property of ϕ, for $I_{X,L}f \in V_{X,L}$, we have, since $V_{X,L} \subset \mathcal{N}_\phi$,

$$\left(I_{X,L}f, \phi(\cdot,x) - \sum_{j=1}^{N}\phi(\cdot,x_j)l_j(x)\right)_{\mathcal{N}_\phi} = I_{X,L}f(x) - \sum_{j=1}^{N} l_j(x)I_{X,L}f(x_j) = 0,$$

here the Lagrange representation of $I_{X,L}f \in V_{X,L}$ ensures that

$$\sum_{j=1}^{N} l_j(x)I_{X,L}f(x_j) = I_{X,L}f(x), \quad \forall x \in \mathbb{S}^2.$$

So the pointwise error turns into

$$f(x) - I_{X,L}f(x) = \left(f - I_{X,L}f, \phi(\cdot,x) - \sum_{j=1}^{N}\phi(\cdot,x_j)l_j(x)\right)_{\mathcal{N}_\phi}, \tag{3.4}$$

and by the Cauchy-Schwarz inequality, we have

$$|f(x) - I_{X,L}f(x)| \le \|f - I_{X,L}f\|_{\mathcal{N}_\phi} P_{\phi,X,L}(x), \quad x \in \mathbb{S}^2,$$

where $P_{\phi,X,L}$ is the power function defined by

$$P_{\phi,X,L}(x) = \left\|\phi(\cdot,x) - \sum_{j=1}^{N}\phi(\cdot,x_j)l_j(x)\right\|_{\mathcal{N}_\phi}, \quad x \in \mathbb{S}^2.$$

On using the definition $\|\cdot\| = (\cdot,\cdot)_{\mathcal{N}_\phi}^{1/2}$ and the reproducing property of ϕ, the power function turns into

$$P_{\phi,X,L}(x) = \left(\phi(x,x) - 2\sum_{j=1}^{N} l_j(x)\phi(x,x_j) + \sum_{i=1}^{N}\sum_{j=1}^{N} l_i(x)l_j(x)\phi(x_i,x_j)\right)^{1/2},$$

completing the proof of Theorem 3.1.

The following Lemma 3.1 is taken from [27] and it is also established by Sloan and Sommariva in [23].

Lemma 3.1 *(see [23, Lemma 5.3]). Let $\phi \in C(\mathbb{S}^2 \times \mathbb{S}^2)$ be a strictly positive definite kernel on $\mathbb{S}^2$, and assume that $X = \{x_1, \ldots, x_N\} \subset \mathbb{S}^2$ is a $\mathcal{P}_L$- unisolvent set of distinct points on $\mathbb{S}^2$. For a fixed $x \in \mathbb{S}^2$, we define the quadratic functional $\mathcal{L}_x : \mathbb{R}^N \to \mathbb{R}$ by*

$$\mathcal{L}_x(\alpha) := \phi(x,x) - 2\sum_{j=1}^{N}\alpha_j\phi(x,x_j) + \sum_{i=1}^{N}\sum_{j=1}^{N}\alpha_i\alpha_j\phi(x_i,x_j), \ \ \alpha = (\alpha_1, \ldots, \alpha_N).$$

Then the minimum of $\mathcal{L}_x(\alpha)$ on the set

$$M_{x,L} := \left\{ \alpha \in \mathbb{R}^N : \sum_{j=1}^N \alpha_j p(x_j) = p(x), \;\; \forall p \in \mathcal{P}_L \right\},$$

is achieved by the vector $(l_1(x), \ldots, l_N(x))$, that is, $\mathcal{L}_x(l_1(x), \ldots, l_N(x)) \le \mathcal{L}_x(\alpha)$, for all $\alpha \in M_{x,L}$.

Follows from Theorem 3.1 and 3.1, we can easily obtain the next Theorem 3.2.

Theorem 3.2 *Under the conditions of Theorem 3.1, for a fixed $x \in \mathbb{S}^2$, we have*

$$|f(x) - I_{X,L}f(x)| \le \|f - I_{X,L}f\|_{\mathcal{N}_\phi} \left(\mathcal{L}_x(\alpha)\right)^{1/2},$$

for any real number $\alpha_j := \alpha_j(x), \; j = 1, \ldots, N$, such that $\sum_{j=1}^N \alpha_j p(x_j) = p(x)$, for all $p \in \mathcal{P}_L$, and

$$\mathcal{L}_x(\alpha) := \phi(x,x) - 2\sum_{j=1}^N \alpha_j \phi(x, x_j) + \sum_{i=1}^N \sum_{j=1}^N \alpha_i \alpha_j \phi(x_i, x_j).$$

The error estimates are general expressed in terms of the mesh norm of $X = \{x_1, \ldots, x_N\} \subset \mathbb{S}^2$, which is defined by

$$h_X := \sup_{x \in \mathbb{S}^2} \inf_{x_j \in X} d(x, x_j),$$

where $d(x, x_j) = \arccos(x \cdot x_j)$ is the geodesic distance between x_j and x.

Next, we state the following Lemma 3.2, whose proof can be found in [27, Corollary 17.12].

Lemma 3.2 *Suppose that $X = \{x_1, \ldots, x_N\} \subset \mathbb{S}^2$ has mesh norm $h_X \le \frac{1}{2L}$ for some integer $L \ge 1$. Then there exist functions $\alpha_j : \mathbb{S}^2 \to \mathbb{R}$ for $j = 1, \ldots, N$ such that*
(i) $\sum_{j=1}^N \alpha_j(x) p(x_j) = p(x), \quad \forall p \in \mathcal{P}_L, \; \forall x \in \mathbb{S}^2$,
(ii) $\sum_{j=1}^N |\alpha_j(x)| \le 2, \quad \forall x \in \mathbb{S}^2$.

With the above obtained results we can provide the following crucial result about the pointwise error estimate for the hybrid interpolation.

Theorem 3.3 *Let $\phi \in C(\mathbb{S}^2 \times \mathbb{S}^2)$ be a strictly positive definite kernel defined by (2.3) and $a_l \sim (l+1)^{-2s}, s > 1$. Assume that integer $L \ge 1$ and that $X = \{x_1, \ldots, x_N\} \subset \mathbb{S}^2$ is a set of distinct points on $\mathbb{S}^2$ with mesh norm $1/(2L+2) < h_X \le 1/(2L)$. For $f \in \mathcal{N}_\phi$, let $I_{X,L}f \in V_{X,L}$ be the hybrid interpolation defined in Section 2.2. Then for a fixed $x \in \mathbb{S}^2$, we have*

$$|f(x) - I_{X,L}f(x)| \le C h_X^{s-1} \|f - I_{X,L}f\|_{\mathcal{N}_\phi}.$$

Proof. Because $h_X \le \frac{1}{2L}$, it follows that for each $x \in \mathbb{S}^2$ there exists $\alpha = \alpha(x) \in \mathbb{R}^N$ satisfying (i) and (ii) in Lemma 3.2. For (i), it means that a polynomial $p \in \mathcal{P}_L$ that vanishes at $x_1, \ldots, x_N$ must vanish identically, which verify that $X = \{x_1, \ldots, x_N\} \subset \mathbb{S}^2$ is a $\mathcal{P}_L$ *-unisolvent* set of distinct points on $\mathbb{S}^2$. By using Theorem 3.2, we only have to give the estimate of the factor $\big(\mathcal{L}_x(\alpha)\big)^{1/2}$,

$$\begin{aligned}
\mathcal{L}_x(\alpha) &:= \phi(x,x) - 2\sum_{j=1}^N \alpha_j \phi(x, x_j) + \sum_{i=1}^N \sum_{j=1}^N \alpha_i \alpha_j \phi(x_i, x_j) \\
&= \frac{1}{4\pi} \sum_{l=0}^\infty (2l+1) a_l \Big[P_l(x \cdot x) - \sum_{j=1}^N \alpha_j P_l(x \cdot x_j) - \sum_{j=1}^N \alpha_j \big(P_l(x \cdot x_j) - \sum_{i=1}^N \alpha_i P_l(x_i \cdot x_j) \big) \Big],
\end{aligned}$$

in which the terms with $l \le L$ vanish by property (i) of Lemma 3.2. Hence

$$\mathcal{L}_x(\alpha) := \frac{1}{4\pi} \sum_{l=L+1}^\infty (2l+1) a_l \Big(P_l(x \cdot x) - 2\sum_{j=1}^N \alpha_j P_l(x \cdot x_j) + \sum_{i=1}^N \sum_{j=1}^N \alpha_i \alpha_j P_l(x_i \cdot x_j) \Big),$$

and since $|P_l(z)| \leq 1$, $\sum_{j=1}^N |\alpha_j| \leq 2$ and $a_l \sim (l+1)^{-2s}$, we have

$$\begin{aligned}
|\mathcal{L}_x(\alpha)| &\leq \frac{1}{4\pi}\sum_{l=L+1}^{\infty}(2l+1)a_l\left(1+2\sum_{j=1}^N|\alpha_j|+\sum_{i=1}^N\sum_{j=1}^N|\alpha_i||\alpha_j|\right)\\
&\leq C\sum_{l=L+1}^{\infty}(2l+1)a_l \leq C\sum_{l=L+1}^{\infty}(l+1)^{-2s+1}\\
&\leq C\int_L^{\infty}(x+1)^{-2s+1}dx = C(L+1)^{-2s+2} \leq Ch_X^{2s-2}.
\end{aligned}$$

With the help of Theorem 3.2, we see that

$$|f(x) - I_{X,L}f(x)| \leq Ch_X^{s-1}\|f - I_{X,L}f\|_{\mathcal{N}_\phi}.$$

This completes the proof of Theorem 3.3.

References

[1] S. C. Brenner, R. L. Scott, The Mathematical Theory of Finite Element Methods, Springer, New York, 1994.

[2] F. Cao, M. Li, Spherical data fitting by multiscale moving least squares, Applied Math. Model., 39 (2015) 3448-3458.

[3] D. Chen, V. A. Menegatto, X. Sun, A necessary and sufficient condition for strictly positive definite functions on spheres, Proc. Amer. Math. Soc., 131 (2003) 2733-2740.

[4] Z. Ditzian, Fractional derivatives and best approximation, Acta. Math. Hungar., 81 (1998) 323-348.

[5] G. E. Fasshauer, L. L. Schumaker, Scattered data fitting on the sphere, in Mathematical Methods for Curves and Surfaces II (M. Dælen, T. Lyche, and L. L. Schumaker, eds.), Vanderbilt University Press, Nashville, TN, (1998) 117-166.

[6] W. Freeden, T. Gervens, M. Schreiner, Constructive Approximation on the Sphere, Oxford University Press Inc., New York, 1998.

[7] P. B. Gilkey, The Index Theorem and the Heat Equation, Publish or Perish, Boston, MA, 1974.

[8] M. v. Golitschek, W. A. Light, Interpolation by polynomials and radial basis functions on spheres, Constr. Approx., 17 (2001) 1-18.

[9] K. Jetter, J. Stöckler, J. D. Ward, Error estimates for scattered data interpolation on spheres, Math. Comp., 68 (1999) 733-747.

[10] J. Levesley, X. Sun, Approximation in rough native spaces by shifts of smooth kernels on spheres, J. Approx. Theory, 133 (2005) 269-283.

[11] J. Levesley, X. Sun, Corrigendum to and two open questions arising from the article "Approximation in rough native spaces by shifts of smooth kernels on spheres" [J. Approx. Theory, 133 (2005) 269-283], J. Approx. Theory, 138 (2006) 124-127.

[12] Q. T. Le Gia, F. J. Narcowich, J. D. Ward, H. Wendland, Continuous and discrete least-squares approximation by radial basis functions on spheres, J. Approx. Theory, 143 (2006) 124-133.

[13] M. Li, F. L. Cao, Local uniform error estimates for spherical basis functions interpolation, Math. Meth. Applied Sci., 37 (2014) 1364-1376.

[14] C. Müller, Spherical Harmonics, Lecture Notes in Mathematics, Vol. 17, Springer-Verlag, Berlin, 1966.

[15] F. J. Narcowich, R. Schaback, J. D. Ward, Approximation in Sobolev spaces by kernel expansions, J. Approx. Theory, 114 (2002) 70-83.

[16] F. J. Narcowich, J. D. Ward, Scattered data interpolation on spheres: Error estimates and locally supported basis functions, SIAM J. Math. Anal., 33 (2002) 1393-1410.

[17] F. J. Narcowich, X. Sun, J. D. Ward, H. Wendland, Direct and inverse sobolev error estimates for scattered data interpolation via spherical basis functions, Found. Comput. Math., (2007) 369-390.

[18] F. J. Narcowich, X. Sun, J. D. Ward, Approximation power of RBFs and their associated SBFs: A connection, Adv. Comput. Math., 27 (2007) 107-124.

[19] A. Ron, X. Sun, Strictly positive definite functions on spheres in Enclidean spaces, Math. Comp., 65 (1996) 1513-1530.

[20] R. Schaback, Improved error bounds for scattered data interpolation by radial basis functions, Math. Comp., 68 (1999) 201-216.

[21] I. J. Schoenberg, Positive definite functions on spheres, Duke Math. J., 9 (1942) 96-108.

[22] E. M. Stein, G. Weiss, Introduction to Fourier Analysis on Euclidean Spaces, Princeton University Press, Princetion, NJ, 1971.

[23] I. H. Sloan, A. Sommariva, Approximation on the sphere using radial basis function plus polynomials, Adv. Comput. Math., 29 (2008) 147-177.

[24] I. H. Sloan, H. Wendland, Inf-sup condition for spherical polynomials and radial basis functions on spheres, Math. Comp., 78 (2009) 1319-1331.

[25] I. H. Sloan, Polynomial interpolation and hyperinterpolation over general regions, J. Approx. Theory, 83 (1995) 238-254.

[26] I. H. Sloan, R. S. Womersley, Constructive polynomial approximation on the sphere, J. Approx. Theory, 103 (2000) 91-118.

[27] H. Wendland, Scattered Data Approximation, Cambridge University Press, Cambridge, Uk, 2005.

[28] Y. Xu, E. W. Cheney, Strictly positive definite functions on spheres, Proc. Amer. Math. Soc., 116 (1992) 977-981.

INVESTIGATING DYNAMICS OF THE RATIONAL DIFFERENCE EQUATION $x_{n+1} = \dfrac{x_{n-1}}{A + Bx_n x_{n-1}}$

MALEK GHAZEL, TAHER S. HASSAN, AND AHMED M. MOSALLEM

Abstract. This paper is devoted to investigate the dynamics of the rational difference equation

$$x_{n+1} = \frac{x_{n-1}}{A + Bx_n x_{n-1}}$$

with arbitrary initial conditions A and B as nonzero real numbers. The solution is obtained and analytical study and asymptotic behavior are investigated. The forbidden set is determined. The existence of periodic and oscillatory solutions are discussed. Our results are illustrated with numerical simulations.

1. Introduction

The study of difference equation has been of great interest and many spectacular developments have been witnessed in the last decade. They are also used to present many numerical schemes in an easiest manner [1–16]. This is largely due to the fact that it appears as direct mathematical models describing real life situations in physics and engineering [5, 6], biology [8], game theory [7, 9, 10, 12, 13, 19] and economy [14, 15]. Therefore, the study of behavior and global stability of nonlinear difference equations is of paramount importance and rational difference equations are one of the most practical classes of equations. Immense literature is available on the second order difference equations of the form

$$x_{n+1} = \frac{\alpha + \beta x_n + \gamma x_{n-1}}{A + Bx_n + Cx_{n-1}},$$

where α, β, γ, A, B and C and the initial conditions x_{-1}, x_0 are real numbers. In a particular case when $\gamma = C = 0$, this equation is known as the first order Riccati difference equation which can also be written in the form $x_{n+1} = a + \dfrac{b}{x_n}$. The results such as Agarwal *et al* [17], investigated the global stability, periodic nature and solved some particular cases of the difference equation

$$x_{n+1} = a + \frac{dx_{n-l}x_{n-k}}{b - cx_{n-s}}.$$

Elsayed [18] studied the dynamical behavior and gave the solution of the difference equation

$$x_{n+1} = \frac{x_{n-5}}{\pm 1 \pm x_{n-2}x_{n-5}}.$$

Aloqeili [11] found the solution of the difference equation

$$x_{n+1} = \frac{x_{n-1}}{a - x_n x_{n-1}}.$$

Cinar [20] determined the global stability and obtained the positive solutions of the following difference equation

$$x_{n+1} = \frac{ax_{n-1}}{1 + bx_n x_{n-1}}.$$

1991 *Mathematics Subject Classification.* 34K13, 34K05, 34K20, 39A10.

Key words and phrases. Rational difference equations, Stability, Infinite products, Forbidden set, Asymptotic behavior, Periodicity, Oscillation, Numerical Simulation.

Elabbasy *et al* [21, 22] obtained the solution in some particular cases and studied the global stability, periodicity of the following difference equations

$$x_{n+1} = ax_n - \frac{bx_n}{cx_n - dx_{n-1}} \text{ and } x_{n+1} = \frac{\alpha x_{n-k}}{\beta + \gamma \Pi_{p=0}^{k} x_{n-p}}.$$

The problem of existence of solutions for a given difference equation is of great importance. The primary aim is to find the set $\mathcal{F}$ of all initial values at which the solution of the given equation is not defined for all natural number n. The set of this nature is called the forbidden set of the equation. In order to avoid the appearance of the forbidden set, the common assumption used by some authors, while studying rational difference equations, is to choose positive initial values and coefficients. The interest of this problem has increased in the literature recently [23–25]. Azizi [26] found the forbidden set of the second order rational Riccati difference equation. Also, Balibrea *et al* [27] gave sufficient conditions for a rational difference equation of order two to be not uniformly eventually positive outside a bounded set. Camouzis *et al* [28] described the forbidden set of the difference equation

$$x_{n+1} = p + \frac{x_{n-1}}{x_n}.$$

In [29] Sedaghat studied the existence of solutions of certain singular difference equations. Stevič [30] studied the domains for which the solutions of some equations and systems of difference equations are not well-defined.

The study of existence of oscillatory solutions (periodic or aperiodic) of difference equations is in a great concern and it is extremely useful in the behavior of mathematical models describing real live situations, for some results in this area. Ladas [31] studied the oscillation of positive solutions about the positive steady state N in the delay logistic difference equation

$$N_{n+1} = N_n \exp\left(r - r\sum_{j=0}^{m} p_j N_{n-j}\right),$$

which describes that the population growth is not continuous but seasonal.
Matti [32] studied the oscillations in some nonlinear economic relationships modeled by a difference equations. Sedaghat [33] studied the oscillations and chaos in a discrete model of combat. See also related results [34–37].

Motivated by above, in this paper, we will present complete analytical study and asymptotic behavior of the solutions of the more general second order difference equation

$$x_{n+1} = \frac{x_{n-1}}{A + Bx_n x_{n-1}}, \; x_0 = c \text{ and } x_{-1} = d, \tag{1.1}$$

with arbitrary parameters A and B. To the best of our knowledge, the analysis for convergence, oscillation and periodicity of equation (1.1) have not been considered till now and other results extend and improve existing results in the literature, especially those established in [11, 20].

Throughout the paper we use the convention that $\mathbb{N} = \{0, 1, 2, \ldots\}$, $\prod_{p=n}^{m} a_p = 1$ and $\sum_{p=n}^{m} a_p = 0$, where $(a_p)_p$ is a sequence of real numbers and $m < n$ for $m, n \in \mathbb{Z}$ and the cases when $AB = 0$ and $A + B \neq 0$ are trivial, therefore we will assume that $A \neq 0$ and $B \neq 0$.

2. Stability analysis of the equilibrium points

Before stating stability analysis of the equilibrium points, we begin with the following theorem which will given equilibrium points of Eq. (1.1).

Theorem 1. *Let* $(x_n)_{n\geq -1}$ *be a solution of Eq.* (1.1).

(1) *If $B(1-A) \leq 0$, then the Eq.* (1.1) *has a unique equilibrium point* $\bar{x}_1 = 0$.
(2) *If $B(1-A) > 0$, then the Eq.* (1.1) *has exactly three equilibrium points*

$$\bar{x}_1 = 0 \quad \textit{and} \quad \bar{x}_{2,3} = \pm\sqrt{\frac{1-A}{B}}.$$

Proof. Let $\bar{x}$ be a equilibrium point of Eq. (1.1). It is easy to see that

$$\bar{x} = 0 \quad \text{or} \quad \bar{x}^2 = \frac{1-A}{B}.$$

This completes the proof. ■

Now, we will prove the following stability analysis of the equilibrium points for equation (1.1).

Theorem 2. *Let $(x_n)_{n \geq -1}$ be a solution of Eq.* (1.1). *Then:*

(1) *For $A < 0$, the characteristic equation about the equilibrium point $\bar{x}_1$ has no real roots.*
(2) *For $0 < A < 1$, the equilibrium point $\bar{x}_1$ is a repeller.*
(3) *For $A = 1$, the equilibrium point $\bar{x}_1$ is nonhyperbolic.*
(4) *For $A > 1$, the equilibrium point $\bar{x}_1$ is locally asymptotically stable.*

Moreover, for $B(1-A) > 0$,

(i) *The equilibrium points $\bar{x}_{2,3}$ are nonhyperbolic.*
(ii) *If $0 < |A| < 1$, then the equilibrium points $\bar{x}_{2,3}$ are unstable.*

Proof. Denote by $U := (u_0, u_1)$ an arbitrary point in the good set of Eq. (1.1) and $\bar{x}$ be an equilibrium point of Eq. (1.1), recall that the characteristic equation about the equilibrium point $\bar{x}$ is defined as

$$\lambda^2 - q_0\lambda - q_1 = 0, \tag{2.2}$$

where $q_k = \dfrac{\partial F}{\partial u_k}(\bar{x}, \bar{x})$, $k = 0, 1$ with $F(u_0, u_1) = \dfrac{u_1}{A + Bu_0u_1}$. Since the partial derivative of the function F are

$$\frac{\partial F}{\partial u_0} = \frac{-Bu_1}{(A + Bu_0u_1)^2} \quad \text{and} \quad \frac{\partial F}{\partial u_1} = \frac{A}{(A + Bu_0u_1)^2}, \tag{2.3}$$

so, for the equilibrium point $\bar{x}_1 = 0$, the coefficients of the characteristic equation are $q_0 = \dfrac{\partial F}{\partial u_0}(0,0) = 0$ and $q_1 = \dfrac{\partial F}{\partial u_1}(0,0) = \dfrac{1}{A}$. Hence the characteristic equation about the equilibrium point $\bar{x}_1$ is

$$\lambda^2 - \frac{1}{A} = 0. \tag{2.4}$$

Thus, we have the following cases:

(1) If $A < 0$, then the Eq. (2.4) has no real roots.
(2) If $0 < A < 1$, then the real roots of Eq. (2.4) are $\pm\sqrt{\dfrac{1}{A}}$, their absolute values are greater than one which implies that the equilibrium point $\bar{x}_1$ is a repeller.
(3) If $A = 1$, then the real roots of Eq. (2.4) are ± 1, so $\bar{x}_1$ is nonhyperbolic.
(4) If $A > 1$, then all real roots of Eq. (2.4) have absolute value less than one, so $\bar{x}_1$ is locally asymptotically stable.

In the case when $B(1-A) > 0$, two new equilibrium points appear $\bar{x}_2$ and $\bar{x}_3$. According to the Eq. (2.3), the coefficients q_0 and q_1 of their characteristic equations are the same and they are given as $q_0 = A - 1$ and $q_1 = A$, so the characteristic equation about $\bar{x}_k$, $k = 2, 3$ is

$$\lambda^2 - (A-1)\lambda - A = 0,$$

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 27, NO.1, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC

which has -1 and A as real roots, then $\bar{x}_2$ and $\bar{x}_3$ are nonhyperbolic. Furthermore, if $|A| < 1$, then $\bar{x}_2$ and $\bar{x}_3$ are unstable. This completes the proof. ■

3. Analytical expressions of $(x_n)_{n\geq -1}$

In this section, we give some analytical expressions of the sequence $(x_n)_{n\geq-1}$, where $(x_n)_{n\geq-1}$ is a solution of Eq. (1.1).

Theorem 3. *Let $(x_n)_{n\geq-1}$ be a solution of Eq. (1.1). Then for all integer $n \in \mathbb{N}$,*

$$x_{2n-1} = d\prod_{p=0}^{n-1}\left(\frac{A^{2p} + Bcd\sum_{k=0}^{2p-1} A^k}{A^{2p+1} + Bcd\sum_{k=0}^{2p} A^k}\right), \tag{3.5}$$

and

$$x_{2n} = c\prod_{p=0}^{n-1}\left(\frac{A^{2p+1} + Bcd\sum_{k=0}^{2p} A^k}{A^{2p+2} + Bcd\sum_{k=0}^{2p+1} A^k}\right). \tag{3.6}$$

Proof. We show it by induction. First we have

$$x_{-1} = d\prod_{p=0}^{-1}\left(\frac{A^{2p} + Bcd\sum_{k=0}^{2p-1} A^k}{A^{2p+1} + Bcd\sum_{k=0}^{2p} A^k}\right) = d$$

and

$$x_{0} = c\prod_{p=0}^{-1}\left(\frac{A^{2p+1} + Bcd\sum_{k=0}^{2p} A^k}{A^{2p+2} + Bcd\sum_{k=0}^{2p+1} A^k}\right) = c$$

This shows that (3.5) and (3.6) hold for $n=0$. Assume (3.5) and (3.6) hold with n replaced by some $k \in \mathbb{N}$. From Eq. (1.1) we get

$$\begin{aligned} x_{2(k+1)-1} &= x_{2k+1} = \frac{x_{2k-1}}{A + Bx_{2k}x_{2k-1}} \\ &= \Big(d\prod_{p=0}^{n-1}\Big(A^{2p} + Bcd\sum_{k=0}^{2p-1}A^k\Big)\prod_{p=0}^{n-1}\Big(A^{2p+2} + Bcd\sum_{k=0}^{2p+1}A^k\Big)\Big)\Big/ \\ &\quad \Big(\prod_{p=0}^{n-1}\Big(A^{2p+1} + Bcd\sum_{k=0}^{2p}A^k\Big) \\ &\quad \Big[A\prod_{p=0}^{n-1}\Big(A^{2p+2} + Bcd\sum_{k=0}^{2p+1}A^k\Big) + Bcd\prod_{p=0}^{n-1}\Big(A^{2p} + Bcd\sum_{k=0}^{2p-1}A^k\Big)\Big]\Big) \\ &= d\prod_{p=0}^{k}\left(\frac{A^{2p} + Bcd\sum_{k=0}^{2p-1}A^k}{A^{2p+1} + Bcd\sum_{k=0}^{2p}A^k}\right). \end{aligned}$$

and

$$\begin{aligned} x_{2(k+1)} &= x_{2k+1+1} = \frac{x_{2k}}{A + Bx_{2k+1}x_{2k}} \\ &= \Big(c\prod_{p=0}^{k-1}\Big(A^{2p+1} + Bcd\sum_{k=0}^{2p}A^k\Big)\prod_{p=0}^{k}\Big(A^{2p+1} + Bcd\sum_{k=0}^{2p}A^k\Big)\Big)\Big/ \\ &\quad \Big(\prod_{p=1}^{k}\Big(A^{2p} + Bcd\sum_{k=0}^{2p-1}A^k\Big) \\ &\quad \Big[A\prod_{p=0}^{k}\Big(A^{2p+1} + Bcd\sum_{k=0}^{2p}A^k\Big) + Bcd\prod_{p=0}^{k-1}\Big(A^{2p+1} + Bcd\sum_{k=0}^{2p}A^k\Big)\Big]\Big) \\ &= c\prod_{p=0}^{k}\Big(\frac{A^{2p+1} + Bcd\sum\limits_{k=0}^{2p}A^k}{A^{2p+2} + Bcd\sum\limits_{k=0}^{2p+1}A^k}\Big). \end{aligned}$$

This shows that (3.5) and (3.6) hold for $k+1$. Therefore, (3.5) and (3.6) hold for $n \in \mathbb{N}$. This completes the proof. ∎

Corollary 4. *Let $(x_n)_{n\geq -1}$ be a solution of Eq. (1.1). Then:*

(1) *for $A \neq 1$,*

$$x_{2n-1} = d\prod_{p=0}^{n-1}\frac{(A-1+Bcd)A^{2p} - Bcd}{(A-1+Bcd)A^{2p+1} - Bcd}, \tag{3.7}$$

and

$$x_{2n} = c\prod_{p=0}^{n-1}\frac{(A-1+Bcd)A^{2p+1} - Bcd}{(A-1+Bcd)A^{2p+2} - Bcd}.$$

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 27, NO.1, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC

(2) *for* $A = 1$,

$$x_{2n-1} = d \prod_{p=0}^{n-1} \Big(\frac{1 + 2pBcd}{1 + (2p+1)Bcd}\Big), \tag{3.8}$$

and

$$x_{2n} = c \prod_{p=0}^{n-1} \Big(\frac{1 + (2p+1)Bcd}{1 + (2p+2)Bcd}\Big).$$

Proof. It is sufficient to use in the (3.5) and (3.6), the identity

$$\sum_{k=0}^{p} x^k = \frac{1 - x^{p+1}}{1 - x},$$

where p is a nonnegative integer and x is a real numbers different of one, and the proof is directly obtained. ■

4. Main Results

4.1. **The forbidden set.** The determination of the set of all initial conditions through which the solution of a given difference equation is defined for all $n \in \mathbb{N}$ is in general a problem of great difficulty. This problem leads to introduce the notion of forbidden set.

Definition 1. Consider a difference equation of order k in $\mathbb{N}$

$$x_{n+1} = F(x_n, x_{n-1}, ..., x_{n-(k-1)}) \quad \text{for } n \in \mathbb{N}, \tag{4.9}$$

where $F = F(u_0, u_1, ..., u_{k-1})$ is a function that maps on some subset Ω in $\mathbb{R}^k$, and let $(x_0, x_{-1}, \dots, x_{-k+1}) \in \Omega$ be the vector of initial conditions of the Eq. (4.9). The forbidden set of Eq. (4.9) is the set denoted $\mathcal{F}$ defined as the set of all vectors of initial conditions $(x_0, x_{-1}, ..., x_{-k+1})$ through which the solution of Eq. (4.9) is not defined for all positive integer n. The good set $\mathcal{G}$ is the complementary in Ω of the forbidden set, consequently, the solution $(x_n)_n$ of Eq. (4.9) is well defined for all $n \in \mathbb{N}$ if and only if $(x_0, x_{-1}, ..., x_{-k+1}) \in \mathcal{G}$.

When we obtain the analytic expression of the solution for a given difference equation, the determination of the forbidden set becomes more easy to obtain. However it can be gotten in some particular cases by the mean of substitution, in the following Theorem, we give the forbidden set in the case when $A = 1$.

Theorem 5. *Let $(x_n)_{n\geq -1}$ be a solution of the Eq. (1.1) and $\mathcal{F}$ be the forbidden set of the sequence $(x_n)_{n\geq -1}$. If $A = 1$, then*

$$\mathcal{F} = \Big\{(c, d) \in \mathbb{R}^2 \text{ such that } cd \in \Big\{\frac{-1}{nB},\ n \in \mathbb{N}\Big\}\Big\}.$$

Proof. The sequence $(x_n)_{n\geq -1}$ satisfies the equation

$$Bx_{n+1}x_n = \frac{Bx_n x_{n-1}}{A + Bx_n x_{n-1}}, \quad x_{-1} = d,\ x_0 = c,$$

Hence,

$$A + Bx_{n+1}x_n = A + 1 - \frac{A}{A + Bx_n x_{n-1}}. \tag{4.10}$$

Let $(y_n)_{n\geq 0}$ be the sequence defined as

$$y_n := A + Bx_n x_{n-1}, \tag{4.11}$$

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 27, NO.1, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC

So the Eq. (4.10) can be written as

$$y_{n+1} = A + 1 - \frac{A}{y_n},$$

which is a first order Ricatti difference equation. If $A = 1$, then

$$y_{n+1} = 2 - \frac{1}{y_n} \quad \text{for } n \in \mathbb{N}.$$

Let $n \in \mathbb{N}$, for y_n to exist a necessary and sufficient condition are that for all integer $0 \le k \le n-1$, $y_k \neq 0$,

$$y_0 \neq 0 \text{ is equivalent to } cd \neq \frac{-1}{B},$$

$$y_1 \neq 0 \text{ is equivalent to } y_0 \neq 0 \text{ and } y_0 \neq \frac{1}{2},$$

and

$$y_2 \neq 0 \textit{ iff } y_0 \notin \left\{0, \frac{1}{2}, \frac{2}{3}\right\}.$$

By induction, we can easily prove that for all $n \in \mathbb{N}$,

$$y_n \neq 0 \textit{ iff for all } k \le n+1, \quad y_0 \neq \frac{k-1}{k}.$$

So the forbidden set of the sequence $Y = (y_n)_{n\ge 0}$ is $\mathcal{F}_Y = \{\frac{n-1}{n}, \quad n \in \mathbb{N}\}$. Now, let $n \in \mathbb{N}$, $y_0 = \frac{n-1}{n}$ is equivalent to $1 + Bcd = \frac{n-1}{n}$ which is equivalent to $cd = \frac{-1}{nB}$. Thus, the forbidden set of the sequence $(x_n)_{n\ge -1}$ is given by

$$\mathcal{F} = \left\{(c,d) \in \mathbb{R}^2 \text{ such that } cd \in \left\{\frac{-1}{nB}, \ n \in \mathbb{N}\right\}\right\}.$$

The proof is complete. ■

This results can be immediately found by using Corollary 4. Also, in the case when $A \neq 1$, the forbidden set $\mathcal{F}$ of Eq. (1.1) can be easily obtained by using Corollary 4 as in the following theorem.

Theorem 6. *Let $(x_n)_{n\ge -1}$ be a solution of the Eq.* (1.1). *Suppose that $A \neq 1$, then the forbidden set of the sequence $(x_n)_{n\ge -1}$ is*

$$\mathcal{F} = \left\{ (c,d) \in \mathbb{R}^2 \textit{ such that } \left\{ \begin{array}{l} A = -1 \quad \textit{and} \quad cd = \dfrac{1}{B} \\ \textit{or} \\ A \neq -1 \quad \textit{and} \quad cd \in \left\{ \dfrac{(1-A)A^n}{B(A^n - 1)}, \ n \in \mathbb{N} \right\}, \end{array} \right. \right\}.$$

4.2. **Convergence.** In this section, we study the asymptotic behavior of a solution of the difference Eq. (1.1).

4.2.1. *The case when $0 < |A| < 1$.*

Theorem 7. *Let $(x_n)_{n\ge -1}$ be a solution of the Eq.* (1.1). *Assume that $|A| < 1$, then the subsequences (x_{2n-1}) and (x_{2n}) converge.*

Proof. Using Corollary 4, we obtain

$$\begin{aligned} x_{2n-1} &= d\prod_{p=0}^{n-1}\frac{(A-1+Bcd)A^{2p}-Bcd}{(A-1+Bcd)A^{2p+1}-Bcd} \\ &= d\prod_{p=0}^{n-1}\frac{1-\frac{A-1+Bcd}{Bcd}A^{2p}}{1-\frac{A-1+Bcd}{Bcd}A^{2p+1}} \\ &= d\prod_{p=0}^{n-1}U_p, \end{aligned}$$

where

$$U_p := \frac{1-\alpha A^{2p}}{1-\alpha A^{2p+1}} \text{ with } \alpha := \frac{A-1+Bcd}{Bcd}.$$

One of the following cases holds: For p big enough, U_p is always is in $(0,1)$ or lies greater than one, this allows us to apply of the Taylor expansion to the sequence $(U_p)_{p\geq 0}$ which gives that

$$U_p \text{ is asymptotically equivalent to } 1-\alpha(A-1)A^{2p},$$

which is the general term of convergent infinite product, thus (x_{2n-1}) converges.
Again by using Corollary 4, we get

$$\begin{aligned} x_{2n} &= c\prod_{p=0}^{n-1}\frac{(A-1+Bcd)A^{2p+1}-Bcd}{(A-1+Bcd)A^{2p+2}-Bcd} \\ &= c\prod_{p=0}^{n-1}\frac{1-\frac{A-1+Bcd}{Bcd}A^{2p+1}}{1-\frac{A-1+Bcd}{Bcd}A^{2p+2}} \\ &= c\prod_{p=0}^{n-1}T_p, \end{aligned}$$

where

$$T_p := \frac{1-\alpha A^{2p+1}}{1-\alpha A^{2p+2}} \text{ with } \alpha = \frac{A-1+Bcd}{Bcd}.$$

Hence,

$$T_p \text{ is asymptotically equivalent to } 1-\alpha(1-A)A^{2p+1},$$

the last term is the general term of convergent infinite product, then $(x_{2n})_n$ converges. This completed the proof. ■

4.2.2. *The case when $A=-1$.*

Lemma 8. *Let $(x_n)_{n\geq -1}$ be a solution of the Eq. (1.1). Assume that $A=-1$, then*

(1) *The subsequence $(x_{2n-1})_n$ converges iff $Bcd \in (-\infty,0)\cup[2,\infty)$.*
(2) *The subsequence $(x_{2n})_n$ converges iff $Bcd \in (0,2]$.*

Proof. 1. Replacing A by -1 in Corollary (4), for the subsequence $(x_{2n-1})_n$, we obtain

$$\begin{aligned} x_{2n-1} &= d\Big(\frac{-2}{2-2Bcd}\Big)^n \\ &= \frac{d}{(Bcd-1)^n}, \end{aligned}$$

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 27, NO.1, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC

then

$$(x_{2n-1})_n \textit{ converges iff } \begin{cases} |Bcd-1|>1, \\ or \\ Bcd-1=1, \end{cases}$$

the last system is equivalent to $Bcd \in (-\infty, 0) \cup [2, \infty)$.

2. To prove the second part of the Theorem, we replace A by (-1) in Corollary (4) for the subsequence $(x_{2n})_n$, we get

$$\begin{aligned} x_{2n} &= c\Big(\frac{2Bcd-2}{-2}\Big)^n \\ &= c(1-Bcd)^n, \end{aligned}$$

then

$$(x_{2n})_n \textit{ converges iff } \begin{cases} |Bcd-1|<1, \\ or \\ Bcd-1=1, \end{cases}$$

the last system holds iff $Bcd \in (0, 2]$. As a result, the proof is completed.

■

Remark 1. Using the computation in the proof of Lemma (8), we can easily deduce that when $A=-1$, we have

(1) If $Bcd \in (-\infty, 0) \cup (2, \infty)$, then (x_{2n-1}) converges to zero and $(|x_{2n}|)$ goes to infinity.
(2) If $Bcd \in (0, 2)$, then $(|x_{2n-1}|)$ goes to infinity and (x_{2n}) converges to zero.
(3) If $Bcd = 2$, then the subsequences (x_{2n-1}) and (x_{2n}) are constant, $x_{2n-1} = d$ and $x_{2n} = c$.

The following theorem is now proved.

Theorem 9. *Let $(x_n)_{n\geq -1}$ be a solution of the Eq. (1.1). Assume that $A=-1$, then*

$$\textit{The whole sequence } (x_n)_{n\geq -1} \textit{ converges iff } B>0 \textit{ and } c=d=\pm\sqrt{\frac{2}{B}}.$$

In this case, $(x_n)_{n\geq -1}$ is constant and equal $\pm\sqrt{\frac{2}{B}}$.

4.2.3. *The case when $A=1$.*

Theorem 10. *Let $(x_n)_{n\geq -1}$ be a solution of the Eq. (1.1). Assume that $A=1$, then $(x_n)_{n\geq -1}$ converges to zero.*

Proof. Replacing A by 1, then by Eq. (3.8),

$$\begin{aligned} x_{2n-1} &= d\prod_{p=0}^{n-1}\Big(\frac{1+2pBcd}{1+(2p+1)Bcd}\Big) \\ &= d\prod_{p=0}^{n-1} V_p, \end{aligned}$$

where $(V_p)_{p\geq 1}$ is the sequence defined as

$$V_p = 1 - \frac{Bcd}{1+(2p+1)Bcd}. \tag{4.12}$$

It can be easily verified that there exists a positive integer r_0 such that for all $p \geq r_0$, we have $V_p \in (0,1)$. Therefore, if p is big enough, the x_{2n-1} is then written in infinite series form as

$$x_{2n-1} = d\Big(\prod_{p=0}^{r_0-1} V_p\Big)\exp\Big(\sum_{p=r_0}^{n-1} \ln V_p\Big). \tag{4.13}$$

We have $\ln V_p$ is equivalent to $\dfrac{-1}{2p}$ which is a general term divergence infinite series, since for all $p \geq r_0$ $V_p \in (0,1)$, then the infinite series $\sum\limits_{p\geq r_0} \ln V_p$ goes to $-\infty$, consequently $(x_{2n-1})_n$ converges to zero.
Although the proof of the convergence of the subsequence $(x_{2n})_n$ to zero can be done similarly, we describe in order to use its notations in the sequel, from Eq. (3.7), we can see that

$$\begin{aligned} x_{2n} &= c\prod_{p=0}^{n-1}\Big(\frac{1+(2p+1)Bcd}{1+(2p+2)Bcd}\Big) \\ &= c\prod_{p=0}^{n-1} W_p, \end{aligned}$$

where $(W_p)_{p\geq 0}$ is the sequence defined as

$$W_p = 1 - \frac{Bcd}{1+(2p+2)Bcd}. \tag{4.14}$$

Similarly, it can be easily checked that there exist a positive integer s_0 such that for all $p \geq s_0$, we have $W_p \in (0,1)$. Hence if p is big enough, the subsequence x_{2n} is then written as

$$x_{2n} = c\Big(\prod_{p=0}^{s_0-1} W_p\Big)\exp\Big(\sum_{p=s_0}^{n-1} \ln W_p\Big). \tag{4.15}$$

We have $\ln W_p$ is equivalent to $\dfrac{-1}{2p}$ which is a general term divergence infinite series, since for all $p \geq s_0$ $W_p \in (0,1)$, then the infinite series $\sum\limits_{p=s_0}^{n-1} \ln W_p$ goes to $-\infty$, consequently $(x_{2n})_n$ converges to zero. This complete the proof of Theorem. ■

4.2.4. *The case when* $|A| > 1$.

Theorem 11. *Let* $(x_n)_{n\geq -1}$ *be a solution of the Eq. (1.1). Assume that* $|A| > 1$, *then*

1. *The subsequences* $(x_{2n-1})_n$ *and* $(x_{2n})_n$ *converges.*
2. *The whole sequence* $(x_n)_{n\geq -1}$ *converge iff*

$$\begin{cases} A - 1 + Bcd \neq 0 \\ \textit{or} \\ (1-A)B > 0 \ \textit{and} \ c = d = \pm\sqrt{\dfrac{1-A}{B}}. \end{cases}$$

Proof. We distinguish two cases:

(1) (I) If $A-1+Bcd \neq 0$, then using Corollary (4),

$$\begin{aligned} x_{2n-1} &= d\prod_{p=0}^{n-1}\Big(\frac{(A-1+Bcd)A^{2p}-Bcd}{(A-1+Bcd)A^{2p+1}-Bcd}\Big) \\ &= d\prod_{p=0}^{n-1}\Big(\frac{1-\dfrac{Bcd}{(A-1+Bcd)A^{2p}}}{A(1-\dfrac{Bcd}{(A-1+Bcd)A^{2p+1}})}\Big) \\ &= \frac{d}{A^n}\prod_{p=0}^{n-1}Y_p, \end{aligned}$$

where $(Y_p)_{p\geq 0}$ is the sequence defined as

$$Y_p = \frac{1-\dfrac{\beta}{A^{2p}}}{1-\dfrac{\beta}{A^{2p+1}}} \text{ and } \beta = \frac{Bcd}{A-1+Bcd}.$$

It can be easily verified that for p big enough, always Y_p is in the interval $(0,1)$ or lies in the interval $(1,\infty)$. The Taylor expansion applied to the sequence $(Y_p)_{p\geq 0}$ gives

$$(Y_p)_{p\geq 0} \textit{ is equivalent to } 1+\beta(\frac{1}{A}-1)\frac{1}{A^{2p}},$$

the last term is a general term of convergent infinite product so $(x_{2n-1})_n$ converges to zero. An easy calculus gives that

$$x_{2n} = \frac{c}{A^n}\prod_{p=0}^{n-1}Z_p,$$

where $(Z_p)_{p\geq 0}$ is the sequence defined as

$$Z_p = \frac{1-\dfrac{\beta}{A^{2p+1}}}{1-\dfrac{\beta}{A^{2p+2}}},$$

we have

$$(Z_p)_{p\geq 0} \textit{ is asymptotically equivalent to } 1+\beta(\frac{1}{A}-1)\frac{1}{A^{2p+1}},$$

the last term is a general term of convergent infinite product, so $(x_{2n})_n$ converges to zero.

(II) If $A-1+Bcd=0$, then the subsequences $(x_{2n-1})_n$ and $(x_{2n})_n$ are constant $x_{2n-1}=d$ and $x_{2n}=c$, so they converge. By the calculus in the preview part of the proof, if $A-1+Bcd\neq 0$, then the whole sequence $(x_n)_{n\geq -1}$ converges to zero. When $A-1+Bcd=0$ that is

$$cd = \frac{1-A}{B}, \tag{4.16}$$

the subsequences $(x_{2n-1})_n$ and $(x_{2n})_n$ are constant equal d and c respectively, then the whole sequence $(x_n)_{n\geq -1}$ converges if and only if $c=d$, using Eq. (4.16) the last proposition is equivalent to $c=d=\pm\sqrt{\dfrac{1-A}{B}}$, for this to can hold it is necessary and sufficient that $(1-A)B>0$. Hence, the proof is achieved.

■

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 27, NO.1, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC

4.3. **Oscillation about the equilibrium point $\overline{x}_1 = 0$.** In this section, we study the oscillation the solution of difference Eq. (1.1) about the equilibrium point $\overline{x}_1 = 0$.

Theorem 12. *Let $(x_n)_{n\geq -1}$ be a solution of the Eq. (1.1). Assume that $|A| < 1$, then the subsequences $(x_{2n-1})_n$ and $(x_{2n})_n$ converge, then*

(1) *For $|A| < 1$*

$$(x_n)_{n\geq -1} \text{ is oscillatory about zero iff } cd\Big(\prod_{p=0}^{n_0-1} U_p\Big)\Big(\prod_{p=0}^{m_0-1} T_p\Big) < 0,$$

where $(U_p)_p$, $(T_p)_p$ are the sequences defined in the proof of Theorem (7) and n_0, m_0, are integers such that, for all $p \geq n_0$, U_p is positive and for all $p \geq m_0$, T_p is positive.

(2) *For $A = -1$, $(x_n)_{n\geq -1}$ is oscillatory about zero.*

(3) *For $A = 1$,*

$$(x_n)_{n\geq -1} \text{ is oscillatory about zero iff } cd\prod_{p=0}^{r_0-1} V_p \prod_{p=0}^{s_0-1} W_p < 0,$$

where $(V_p)_{p\geq 1}$, $(W_p)_{p\geq 0}$, r_0 and s_0 are defined in the proof of Theorem (10).

(4) *For $|A| > 1$,*

$$(x_n)_{n\geq -1} \text{ is oscillatory about zero iff}$$
$$\begin{cases} A - 1 + Bcd = 0 \text{ and } cd < 0, \\ \text{or} \\ A - 1 + Bcd \neq 0 \text{ and } \begin{cases} A < -1, \\ \text{or} \\ A > 1 \text{ and } \quad cd\displaystyle\prod_{p=0}^{p_0-1} Y_p \prod_{p=0}^{q_0-1} Z_p < 0, \end{cases} \end{cases}$$

where $(Y_p)_{p\geq 1}$, $(Z_p)_{p\geq 0}$, p_0 and q_0 are defined in the proof of Theorem (11).

Proof. 1. For $|A| < 1$ The sequences $(x_{2n-1})_n$ and $(x_{2n})_n$ have a constant signs which are these of

$$d\prod_{p=0}^{n_0-1} U_p \text{ and } c\prod_{p=0}^{m_0-1} T_p,$$

respectively, so we can immediately obtain the aimed result.

2. For $A = -1$, in this case $x_{2n-1} = \dfrac{d}{(Bcd-1)^n}$ and $x_{2n-1} = c(1 - Bcd)^n$. Hence, if $Bcd - 1 < 0$, then $(x_{2n-1})_n$ and therefore $(x_n)_n$ is oscillatory about zero. If $Bcd - 1 > 0$, then $(x_{2n})_n$ and therefore $(x_n)_n$ is oscillatory about zero.

3. For $A = 1$, the Eq. (4.13) and (4.15) give

$$x_{2n-1} = d\prod_{p=0}^{r_0-1} V_p \exp\Big(\sum_{p=r_0}^{n-1} \ln V_p\Big),$$

and

$$x_{2n} = c\prod_{p=0}^{q_0-1} W_p \exp\Big(\sum_{p=q_0}^{n-1} \ln W_p\Big),$$

in this case the sequences $(x_{2n-1})_n$ and $(x_{2n})_n$ have a constant signs which are these of

$$d\prod_{p=0}^{r_0-1} V_p \text{ and } c\prod_{p=0}^{q_0-1} W_p,$$

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 27, NO.1, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC

respectively, we find that $(x_n)_n$ is oscillatory about zero iff

$$cd \prod_{p=0}^{r_0-1} V_p \prod_{p=0}^{q_0-1} W_p < 0.$$

4. For $|A| > 1$, if $A - 1 + Bcd = 0$, then the subsequences are constant $(x_{2n-1})_n$ and $(x_{2n})_n$ equal d and c respectively, so $(x_n)_n$ is oscillatory about zero iff $cd < 0$. If $A - 1 + Bcd \neq 0$, the the sequence $(x_n)_{n\geq -1}$ converges to zero and we have

$$x_{2n-1} = \frac{d}{A^n} \prod_{p=0}^{n-1} Y_p \text{ and } x_{2n} = \frac{c}{A^n} \prod_{p=0}^{n-1} Z_p,$$

where $(Y_p)_p$ and $(Z_p)_p$ are the sequences defined in the proof of Theorem (11). It has been seen that there exists integers p_0 and q_0 such that for all $p \geq p_0$, Y_p is positive and for all $p \geq q_0$, Z_p is positive, then for n big enough, the sign of $d \prod_{p=0}^{n-1} Y_p$ and $c \prod_{p=0}^{n-1} Z_p$ are constant. Then, we have the following cases:
   (a) When $A < -1$, the sequence $(x_{2n-1})_n$ and consequently $(x_n)_{n\geq -1}$ are oscillatory about zero.
   (b) When $A > 1$, the sign of x_{2n-1} is that of $d \prod_{p=0}^{p_0-1} Y_p$ and the sign of x_{2n} is that of $c \prod_{p=0}^{q_0-1} Z_p$.
   Thus, we can immediately have the target result and the proof is complete.

■

4.4. **Periodicity.** Firstly, we recall the following Lemma, which describes sufficient conditions for Eq. (1.1) to have a periodic solution.

Lemma 13. *Let $(x_n)_{n\geq -k+1}$ be a solution of Eq. (1.1). Suppose that there are real numbers l_r, $r = 0, 1, ..., p-1$ such that*

$$\lim_{n\to\infty} x_{pn+r} = l_r \quad \textit{for all} \quad r = 0, 1, ..., p-1.$$

Finally, let $(y_n)_{n\geq -k+1}$ be the periodic-p sequence such that

$$y_r = l_r \quad \textit{for all} \quad r = 0, 1, ..., p-1.$$

Then $(y_n)_{n\geq -k+1}$ is a periodic-p solution of Eq. (1.1).

Note that the zero sequence is a solution of Eq. (1.1) corresponding to the initial conditions $x_{-1} = 0$ and $x_0 = 0$, this solution is called trivial solution of of Eq. (1.1). The periodicity results are given by the following Theorem

Theorem 14. *Let $(x_n)_{n\geq -1}$ be a solution of the Eq. (1.1).*
(1) *For $|A| < 1$, Eq. (1.1) has a nontrivial periodic-2 solution.*
(2) *For $A = -1$, Eq. (1.1) has a nontrivial periodic-2 solution if and only if $cd = \frac{2}{B}$.*
(3) *For $A = 1$, Eq. (1.1) has no nontrivial periodic-2 solution.*
(4) *For $|A| > 1$, Eq. (1.1) has a nontrivial periodic-2 solution if and only if $cd = \frac{1-A}{B}$.*

Proof. 1. If $|A| < 1$, then by Theorem (7), the subsequences $(x_{2n-1})_n$ and $(x_{2n})_n$ converge, let l_1 and l_0 be their limits respectively. Applying Lemma (13), it follow that the sequence

$$l_1, l_0, l_1, l_0, ...$$

is a periodic-2 solution of Eq. (1.1).

2. Suppose that $A = -1$, we distinguish two cases:

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 27, NO.1, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC

(a) If $Bcd \neq 2$, then using Lemma (8), every solution of Eq. (1.1) is unbounded, so Eq. (1.1) has no periodic solutions.
(b) If $Bcd = 2$, then using Lemma (8), the subsequences $(x_{2n-1})_n$ and $(x_{2n})_n$ are constant $x_{2n-1} = d$ and $x_{2n} = c$, therefore $(x_n)_{n\geq -1}$ is the periodic-2 solution

$$d, c, d, c, \dots .$$

3. If $A = 1$, then by using the proof of Theorem (10), every solution of of Eq. (1.1) converges to zero, so Eq. (1.1) has no nontrivial solution.
4. If $|A| > 1$, we distinguish two cases:
   (a) If $A - 1 + Bcd \neq 0$, then by using the proof of Theorem (11), every solution of Eq. (1.1) converges to zero, so Eq. (1.1) has no nontrivial solution.
   (b) If $A-1+Bcd = 0$, then by Theorem (11), the subsequences $(x_{2n-1})_n$ and $(x_{2n})_n$ are constant $x_{2n-1} = d$ and $x_{2n} = c$, consequently $(x_n)_{n\geq -1}$ is the periodic-2 solution $d, c, d, c, \dots$

This achieves the proof. ∎

5. Numerical simulation

(1) The case $|A| < 1$ is illustrated in Fig. (1), in which we set $A = \frac{1}{2}$, $B = 4$, $c = 3$ and $d = 2$. The subsequences $(x_{2n-1})_n$ and $(x_{2n})_n$ converge. This is coherent with Theorem (7).

(2) In Fig. (2) (case $A = -1$ and $Bcd \in (-\infty, 0) \cup (2, \infty)$, we choose $A = -1$, $B = \frac{1}{2}$, $c = 1$ and $d = -2$. The subsequence $(x_{2n-1})_n$ converges to zero and the subsequence $(|x_{2n}|)_n$ goes to infinity and oscillates about zero which matches Lemma (8), Remark (1) and Theorem (12).

(3) The case $A = -1$ and $Bcd \in (0, 2)$ is studied using the parameters values $A = -1$, $B = \frac{1}{2}$, $c = 3$ and $d = 1$. The subsequence $(|x_{2n-1}|)_n$ goes to infinity and the subsequence $(x_{2n})_n$ converges to zero as depicted in Fig. (3) which is coherent to Lemma (8), Remark (1) and Theorem (12).

(4) In order to illustrate the case $A = -1$ and $Bcd = 2$, we choose $A = -1$, $B = \frac{1}{2}$, $c = 1$ and $d = 4$. In Fig. (4), it is shown that the subsequences $(x_{2n-1})_n$ and $(x_{2n})_n$ are constant $x_{2n-1} = d$ and $x_{2n} = c$ which agrees Lemma (8) and Remark (1), consequently $(x_n)_{n\geq -1}$ is the periodic-2 solution

$$d, c, d, c, \dots .$$

This is in harmony with Theorem (14).

(5) The case $A = 1$ is investigated using the parameters values $A = 1$, $B = 3$, $c = 0.5$ and $d = 3$. In Fig. (5), the simulation results show that the whole sequence $(x_n)_{n\geq -1}$ converges to zero which matches Theorem (10).

(6) The case $|A| > 1$ and $A-1+Bcd \neq 0$ can be taken by choosing $A = 5$, $B = 1$, $c = 3$ and $d = 0.5$. The whole sequence $(x_n)_{n\geq -1}$ converges to zero as depicted in Fig. (6) which is coherent to Theorem (11).

(7) Fig. (7) illustrates the case $|A| > 1$ and $A - 1 + Bcd = 0$, we choose $A = 5$, $B = 1$, $c = 2$ and $d = -2$, the subsequences $(x_{2n-1})_n$ and $(x_{2n})_n$ are constant: $x_{2n-1} = d$ and $x_{2n} = c$, we obtain a periodic-2 solution. This case is justified analytically in the proofs of Theorems (11), (12) and (14).

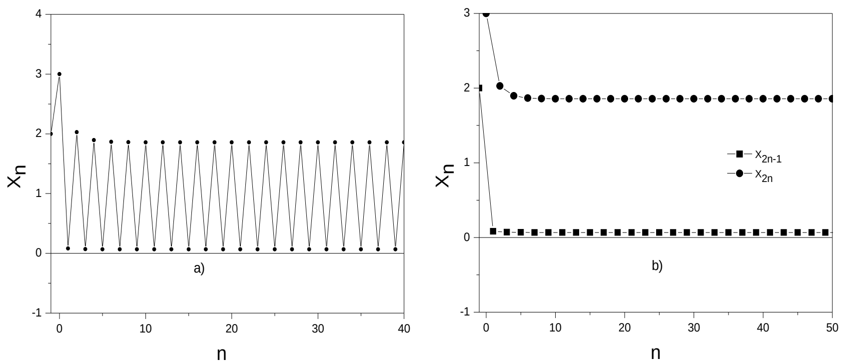


FIGURE 1. $|A| < 1$, $A - 1 + Bcd \neq 0$: $(x_{2n-1})_n$ and $(x_{2n})_n$ converge.

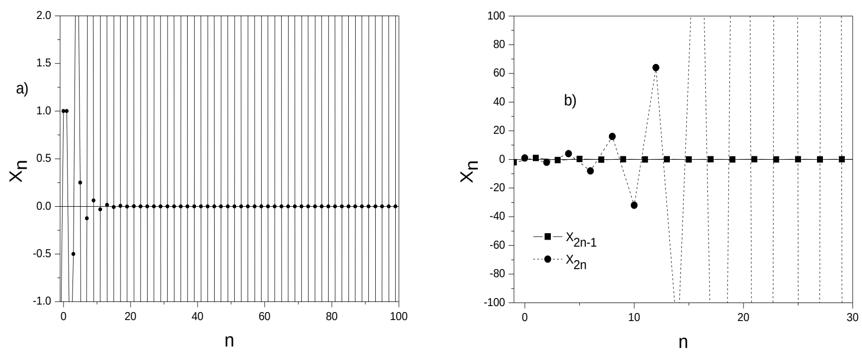


FIGURE 2. $A = -1$ and $Bcd \in (-\infty, 0) \cup (2, \infty)$: $(x_{2n-1})_n$ converges to zero and $(|x_{2n}|)_n$ goes to infinity, the solution is unbounded.

Acknowledgement

The authors thank the Deanship of Research at the University of Hail, Saudi Arabia, for funding this work under Grant no. 0150287.

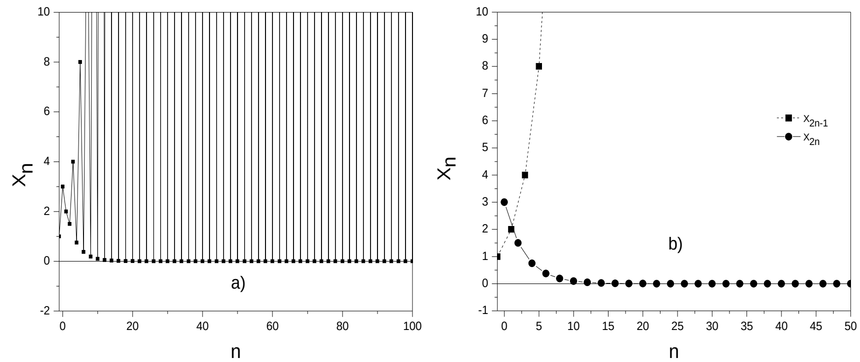


FIGURE 3. $A = -1$ and $Bcd \in (0, 2)$: $(|x_{2n-1}|)_n$ goes to infinity and $(x_{2n})_n$ converges to zero, the solution is unbounded.

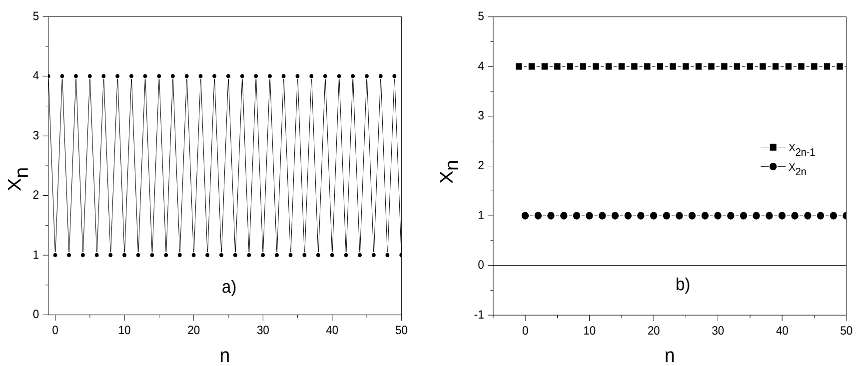


FIGURE 4. $A = -1$ and $Bcd = 2$: $(x_{2n-1})_n$ and $(x_{2n})_n$ are constants, $(x_n)_n$ is periodic-2 solution.

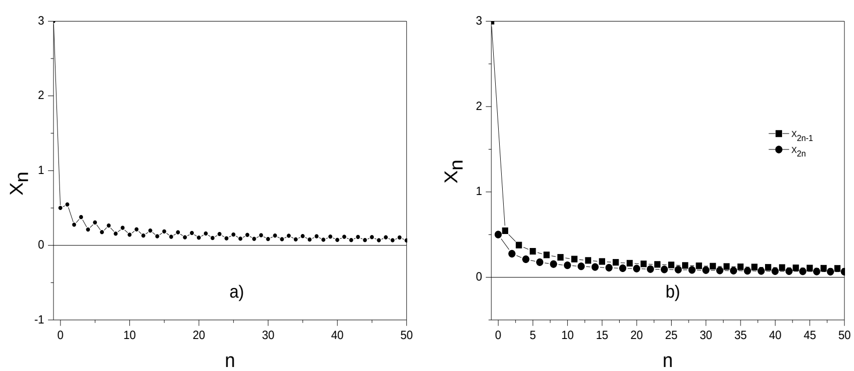


FIGURE 5. $A = 1$: the solution converges to zero.

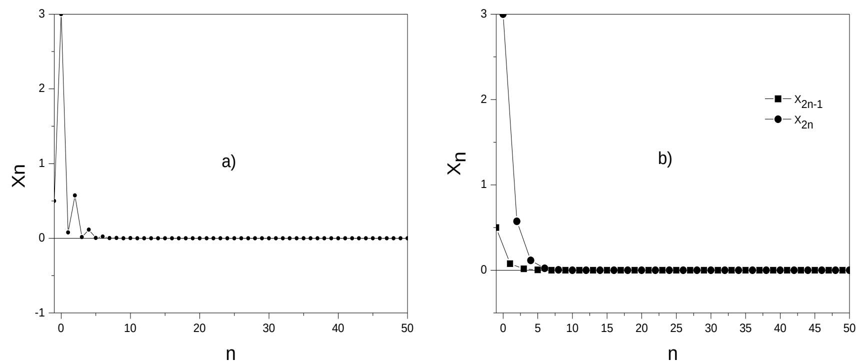


FIGURE 6. $|A| > 1$ and $A - 1 + Bcd \neq 0$: the solution converges to zero.

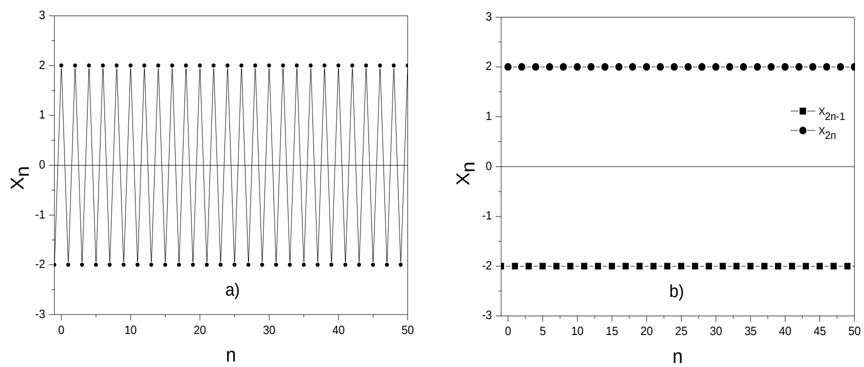


FIGURE 7. $|A| > 1$ and $A - 1 + Bcd = 0$: $(x_{2n-1})_n$ and $(x_{2n})_n$ are constants, $(x_n)_n$ is periodic-2 solution.

REFERENCES

[1] R. Karatas, C. Cinar, D. Simsek, On positive solutions of the difference equation $x_{n+1} = x_{n-5}/(1 + x_{n-2}x_{n-5})$, Int. Journal of Contemp. Math. Sciences 1(10) (2006) 494–500.

[2] E. M. Elsayed, Bratislav D. Iričanin, On a max-type and a min-type difference equation, Appl. Math. Comput. 215 (2009) 608–614.

[3] H. A. El-Morshedy, E. Liz, Globally attracting fixed points in higher order discrete population models, J. Math. Biology 53 (2006) 365–384.

[4] H. El-Metwally, E. M. Elsayed, H. El-Morshedy, Dynamics of some rational difference equations, J. Comput. Anal. Applic. 18 (2015) 993–1003.

[5] E. M. Elabbasy, A. A. Elsadany, Y. Zhang, Bifurcation analysis and chaos in a discrete reduced Lorenz system, Appl. Math. Comput. 228(1) (2014) 184–194.

[6] R. P. Agarwal, A. M. A. El-Sayed, S. M. Salman, Fractional-order Chua's system discretization, bifurcation and chaos, Advances in Difference Equations 320 (2013) 13 pages.

[7] S. S. Askar, A. M. Alshamrani, K. Alnowibet, The arising of cooperation in Cournot duopoly games, Appl. Math. Comput. 273 (2016) 535–542.

[8] A. E. Matouk, A. A. Elsadany, E. Ahmed, H. N. Agiza, Dynamical behavior of fractional-order Hastings-Powell food chain model and its discretization, Communications in Nonlinear Science and Numerical Simulation 27(1-3) (2015) 153–167.

[9] E. Ahmed, A. S. Hegazi, On dynamical multi-team and signaling games, Appl. Math. Comput. 172(1) (2006) 524–530.

[10] S. S. Askar, The impact of cost uncertainty on Cournot oligopoly game with concave demand function. Appl. Math. Comput. 232 (2014) 144–149.
[11] A. A. Elsadany, A. E. Matouk, Dynamic Cournot duopoly game with delay, Journal of Complex Systems 2014Article ID 384843 (2014) 7 pages.
[12] A. A. Elsadany, H. N. Agiza, E. M. Elabbasy, Complex dynamics and chaos control of heterogeneous quadropoly game, Appl. Math. Comput. 219(24) (2013) 11110–11118.
[13] E. Ahmed, A. A. Elsadany, Tonu Puu, On Bertrand duopoly game with differentiated goods, Appl. Math. Comput. 251(15) (2015) 169–179.
[14] V. L. Kocic, G. Ladas, I.W. Rodrigues, On the rational recursive sequences, J. Math. Anal. Appl. 173 (1993) 127–157, Chapman and Hall/CRC Boca Raton (2002).
[15] L. A. Moye, A.S. Kapadia, Difference equations with public health applications, (2000)Marcel Dekker, Inc.
[16] A. A. Elsadany, A dynamic Cournot duopoly model with different strategies, Journal of the Egyptian Mathematical Society 23(1) (2015) 56–61.
[17] R. P. Agarwal, E. M. Elsayed, Periodicity and stability of solutions of higher order rational difference equation, Advanced Studies in Contemporary Mathematics 17(2) (2008) 181–201.
[18] E. M. Elsayed, Dynamics of a rational recursive sequences, International Journal of Difference Equations 4(2) (2009) 185–200.
[19] M. Aloqeili, Dynamics of a rational difference equation, Appl. Math. Comput. 176(2) (2006) 768–774.
[20] C. Cinar, On the positive solutions of the difference equation $x_{n+1} = \frac{ax_{n-1}}{1 + bx_n x_{n-1}}$, Appl. Math. Comput. 156 (2004) 587–590.
[21] E. M. Elabbasy, H. El-Metwalli, E. M. Elsayed, On the difference equation $x_{n+1} = ax_n - \frac{bx_n}{cx_n - dx_{n-1}}$, Adv. Differ. Equ. 2006 Article ID 82579 (2006) 10 pages.
[22] E. M. Elabbasy, H. El-Metwalli, E. M. Elsayed, On the difference equation $x_{n+1} = \frac{\alpha x_{n-k}}{\beta + \gamma \prod_{i=0}^{k} x_{n-i}}$, J. Conc. Appl. Math. 5(2) (2007) 101–113.
[23] E. A. Grove, G. Ladas, Periodicities in Nonlinear Difference Equations, Chapman and Hall/CRC Press, London/Boca Raton (2005).
[24] M. R. S. Kulenovic, G. Ladas, Dynamics of Second Order Rational Difference Equations with Open Problems and Conjectures, Chapman and Hall/CRC Press, London/Boca Raton (2002).
[25] J. Rubi-Masseg, Global periodicity and openness of the set of solutions for discrete dynamical systems, J. Differ. Equ. Appl. 15 (2009) 569–578.
[26] R. Azizi, Global behaviour of the rational Riccati difference equation of order two: the general case, J. Differ. Equ. Appl. 18 (2012) 947–961.
[27] F Balibrea and A Cascales, Eventually positive solutions in rational difference equations, Comp and Math with Appl 64(7) (2012) 2275–2281.
[28] E. Camouzis, R. Devault, The forbidden set of $x_{n+1} = p + \frac{x_{n-1}}{x_n}$, Special Session of the American Mathematical Society Meeying, Part II, San Diego (2002).
[29] H Sedaghat, Existence of solutions of certain singular difference equations, J. Differ. Equ. Appl., 6 535–561 (2000).
[30] S Stevič, Domains of undefinable solutions of some equations and systems of difference equations, Appl Math Comput, 219 11206–11213 (2013).
[31] G. Ladas, Recent developments in the oscillation of delay difference equations, In: Int Conf. on Differential Equations, Theory Appl. Stab. Control, pp (1989) 7–10.
[32] L. Matti, Oscillations in some nonlinear economic relationships, Chaos, Solitons and Fractals, 7 (1996) 2235–2245.
[33] H. Sedaghat, Converges, oscillations, and chaos in a discrete model of combat, SIAM 44 (2002) 74–92.
[34] L. Erbe, T. S. Hassan, A. Peterson, S. H. Saker, Interval oscillation criteria for forced second-order nonlinear delay dynamic equations with oscillatory potential. Dyn. Contin. Discrete Impuls. Syst. Ser. A Math. Anal. 17 (2010) no. 4 533–542.
[35] T. S. Hassan, Interval oscillation for second order nonlinear differential equations with a damping term. Serdica Math. J. 34 (2008) no. 4 715–732.
[36] E. M. Elabbasy, T. S. Hassan, Interval oscillation for second order sublinear differential equations with a damping term. Int. J. Dyn. Syst. Differ. Equ. 1 (2008) no. 4 291–299.
[37] T. S. Hassan, Oscillation criteria for second-order nonlinear dynamic equations. Adv. Difference Equ. 2012, 2012:171 13 pp.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 27, NO.1, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC

Department of Mathematics, Faculty of Science,University of Hail, Hail 2440, Saudi Arabia.
E-mail address: malek_-ghazel@yahoo.fr

Department of Mathematics, Faculty of Science, Mansoura University Mansoura, 35516, Egypt.
E-mail address: tshassan@mans.edu.eg

Department of Mathematics, Faculty of Science,University of Hail, Hail 2440, Saudi Arabia.
E-mail address: ahmedmetwally77@hotmail.com

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 27, NO.1, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC

L_p approximation errors for hybrid interpolation on the unit sphere *

Chunmei Ding Ming Li Feilong Cao

Department of Applied Mathematics, College of Sciences, China Jiliang University,
Hangzhou 310018, Zhejiang Province, P R China.
E-mail: feilongcao@gmail.com

Abstract

This paper discusses L_p approximation error estimates for hybrid interpolation on the unit sphere. This interpolation scheme is integrated by spherical polynomials and radial basis functions. The smooth radial basis functions generated by a strictly positive definite zonal kernel are embedded in a larger native space generated by a less smooth kernel, and the error estimates for hybrid interpolation to a target function from the larger native space are given. In a sense, the results of this paper show that the hybrid interpolation associated with the smooth kernel enjoys the same order of error estimate as hybrid interpolation associated with the less smooth kernel for a target function from the rough native space.

MSC(2000): 41A17, 41A30

Keywords: Sphere; Interpolation; Approximation; Error

1 Introduction

Recently, fitting spherical scattered data comes up in many application areas, such as astrophysics, meteorology, geodesy, geophysics, and so on [5, 6, 29]. As interpolation or approximation tools, spherical polynomials or spherical radial basis functions were used to handle spherical scattered data in more studies [5, 6, 11, 14, 20, 22, 27, 28, 29, 15, 2]. Since spherical polynomials can handle the slowly varying large-scale features, and spherical radial basis functions are helpful to handle scattered and rapidly changed data, Sloan and Sommariv [25] introduced a hybrid interpolation scheme, which combines spherical radial basis functions together with spherical polynomials, and restricts the radial basis functions to the case of strictly positive definite kernels, so that the polynomial component is voluntary rather than forced.

This paper studies the hybrid interpolation in an appropriate native space $\mathcal{N}_\phi$ of continuous functions on the unit sphere, which is defined by a underlying strictly positive definite kernel ϕ. We apply the approach used by Hubbert and Morton [9, 10] to obtain error estimates in L_p norm. However, if the target function is from a subspace of the native space $\mathcal{N}_\phi$, we then adopt the inf-sup condition [26] and the method of constructing a convolution kernel to improve the error estimates.

So called “native space barrier” problem means that if ϕ is smooth, then the native space $\mathcal{N}_\phi$ is small. There have been much literature to focus on it, for example, [12, 13, 17, 18, 19]. In this paper, we employ the approach in [12] and the techniques in [26], and embed the smooth radial basis functions in a larger native space generated by a less smooth kernel ψ. At same time, we utilize the hybrid interpolation associated with the smooth kernel ϕ to interpolate the target function from the larger native space. In the process of error estimates, the “norming set” method developed by Jetter [11] and a special case of the general Bernstein-type inequality in [4] are used.

This paper is organized as follows. Section 2 is preliminary, which is related to introducing notations, hybrid interpolation and its crucial condition, native space, and Sobolev space. The L_p approximation error estimates are established in Section 3. In Section 4, for a target function f

*Supported by the National Natural Science Foundation of China (No. 61672477)

in a subspace of the original native space, we improve the global L_p-error estimates. In Section 5, we still use the hybrid interpolation defined in Section 2 to interpolate and approximate a target function f from a larger native space generated by a less smooth kernel.

2 Preliminaries

This paper uses C to denote a positive constant, whose value may be different at different occurrence even within the same formula. The symbol $A \sim B$ means that there exist positive constant C_1 and C_2 such that $C_1 B \le A \le C_2 B$.

We use $\mathbb{S}^2 := \left\{x := (x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 + x_3^2 = 1\right\}$ to denote the unit sphere embedded in the Euclidean space $\mathbb{R}^3$, and denote by $L_p(\mathbb{S}^2)$ the space of p-integrable functions on $\mathbb{S}^2$ endowed with the norms $\|f\|_\infty := \|f\|_{L_\infty(\mathbb{S}^2)} := \operatorname{esssup}_{x\in\mathbb{S}^2} |f(x)| (p = \infty)$, and $\|f\|_p := \|f\|_{L_p(\mathbb{S}^2)} := \left\{\int_{\mathbb{S}^2} |f(x)|^p d\omega(x)\right\}^{1/p} < \infty (1 \le p < \infty)$. The so called spherical harmonic with degree l is the restriction to $\mathbb{S}^2$ of a homogeneous harmonic polynomial with degree $l \ge 0$. The class of all spherical harmonics with degree l is denoted by $\mathcal{H}_l$, and the class of all spherical harmonics with total degree $l \le L$ is denoted by $\mathcal{P}_L$. Clearly, spherical harmonics with different degrees are orthogonal with respect to the $L_2(\mathbb{S}^2)$ inner product: $\langle f, g\rangle := \int_{\mathbb{S}^2} f(x)g(x)d\omega(x)$, where $d\omega$ is surface measure on $\mathbb{S}^2$.

The famous addition formula $\sum_{k=1}^{2l+1} Y_{l,k}(x)Y_{l,k}(y) = \frac{2l+1}{4\pi} P_l(x \cdot y)$ yields the following useful relation [16]:

$$\sum_{k=1}^{2l+1} |Y_{l,k}(x)Y_{l,k}(y)| \le \sum_{k=1}^{2l+1} Y_{l,k}^2(x) = \frac{2l+1}{4\pi}, \quad x,\, y \in \mathbb{S}^2. \tag{2.1}$$

Here P_l is the Legendre polynomial with degree l and dimension three, which is normalized such that $P_l(1) = 1$, and satisfies the orthogonality relation: $\int_{-1}^{1} P_k(t)P_j(t)dt = \frac{2}{2l+1}\delta_{k,j}$, where the symbol $\delta_{k,j}$ denotes the usual Kronecker symbol.

The definition of strictly positive definite kernel is given by

Definition 2.1 *(see [29]). A continuous and symmetric function $\phi : \mathbb{S}^2 \times \mathbb{S}^2 \longrightarrow \mathbb{R}$ is called positive definite kernel, if, for any $N \in \mathbb{N}_+$, $\alpha = (\alpha_i)_{i=1,\dots,N} \in \mathbb{R}^N$ and $\{x_1, \dots, x_N\} \subset \mathbb{S}^2$, we have*

$$\sum_{i=1}^{N}\sum_{j=1}^{N} \alpha_i\alpha_j\phi(x_i, x_j) \ge 0.$$

When for any N distinct points $\{x_1, \dots, x_N\}$, the above quadratic form is positive for all $\alpha = (\alpha_i)_{i=1,\dots,N} \in \mathbb{R}^N/\{0\}$, then ϕ is called strictly positive definite kernel.

We say that a kernel ϕ is called rotational invariant if $\phi(\rho x, \rho y) = \phi(x, y)$ for all $x, y \in \mathbb{S}^2$ and for all rotations ρ. So a continuous rotational invariant kernel depends only on the distance between x and y [24], that is, there is a function $\varphi : [-1, 1] \to \mathbb{R}$, such that $\varphi(xy) = \phi(x, y)$ for all $x, y \in \mathbb{S}^2$. Therefore, a rotational invariant kernel is also called a zonal kernel. In [23], Schoenberg characterized the positive definite zonal kernels. In [30], Xu and Cheney introduced the notation of strictly positive definiteness on the sphere. Clearly, it is important to characterize all the strictly positive definite functions on the sphere, and such an endeavor has been taken by Ron and Sun in [21]. In [3], Chen et al. established a necessary and sufficient condition for strictly positive definite zonal kernels: the kernel ϕ is strictly positive definite and zonal if and only if

$$\phi(x, y) = \sum_{l=0}^{\infty} a_l \sum_{k=1}^{2l+1} Y_{l,k}(x)Y_{l,k}(y) = \sum_{l=0}^{\infty} \frac{(2l+1)a_l}{4\pi} P_l(x \cdot y),$$

with $a_l \ge 0$ for all l, $\sum_{l=0}^{\infty} l a_l < \infty$ and $a_l > 0$ for infinitely many even values of l and infinitely many odd values of l.

For given strictly positive definite kernel $\phi(\cdot,\cdot)$, a set of distinct points $X=\{x_1,\dots,x_N\}\subset\mathbb{S}^2$, and target function $f\in C(\mathbb{S}^2)$, we take the hybrid interpolation for f in the form

$$I_{X,L}f=\sum_{j=1}^{N}\alpha_j\phi(\cdot,x_j)+\sum_{l=0}^{L}\sum_{k=1}^{2l+1}\beta_{l,k}Y_{l,k},$$

where we fix $L\geq 0$ as the desired degree of the polynomial component of the hybrid interpolation and the coefficients $\{\alpha_j\}_{j=1}^N$, $\{\beta_{l,k}\}_{k=1,\dots,2l+1,\,l=0,\dots,L}$ are determined by the interpolation conditions

$$I_{X,L}f(x_i)=f(x_i),\ \ i=1,\dots,N, \tag{2.2}$$

and also (in order to give a square linear system) the side conditions $\sum_{j=1}^{N}\alpha_j p(x_j)=0,\ \forall p\in\mathcal{P}_L$.

Now we give a condition on the point set X, which makes sure that the interpolation is exist and unique.

Definition 2.2 *(see [25, Definition 3.1]). The set* $X=\{x_1,\dots,x_N\}\subset\mathbb{S}^2$ *is said to be* $\mathcal{P}_L$ -unisolvent *if*

$$p\in\mathcal{P}_L,\ \ p(x_j)=0 \text{ for } j=1,\dots,N\Rightarrow p=0.$$

In order to analyze the interpolation error in the later sections it is convenient to define a finite-dimensional space $V_{X,L}$ within the interpolation $I_{X,L}f$ lies.

$$V_{X,L}:=\left\{\sum_{j=1}^{N}\alpha_j\phi(\cdot,x_j)+q:\ q\in\mathcal{P}_L,\ \alpha_j\in\mathbb{R}\text{ for } j=1,\dots,N,\ \text{and}\sum_{j=1}^{N}\alpha_j p(x_j)=0,\ \forall p\in\mathcal{P}_L\right\}.$$

The following Theorem 2.1 gives a crucial condition for the interpolation to be well defined, whose proof can be find in [25].

Theorem 2.1 *Let $\phi(\cdot,\cdot)$ be a strictly positive definite kernel, and $X=\{x_1,\dots,x_N\}\subset\mathbb{S}^2$ be a set of distinct points which is $\mathcal{P}_L$ -unisolvent for $L\geq 0$. Then for each $f\in C(\mathbb{S}^2)$ there exists a unique $I_{X,L}f\in V_{X,L}$ that satisfies the interpolation conditions in* (2.2).

In this paper, we assume that the strictly positive definite kernel ϕ is zonal and has the expansion

$$\phi(x,y)=\sum_{l=0}^{\infty}a_l\sum_{k=1}^{2l+1}Y_{l,k}(x)Y_{l,k}(y) \tag{2.3}$$

with $a_l>0$ for all l, $\sum_{l=0}^{\infty}la_l<\infty$, in which case the series of the right side in (2.3) converges uniformly for $x,y\in\mathbb{S}^2$.

For $f,g\in L_2(\mathbb{S}^2)$, they can be represented by their Fourier series $f=\sum_{l=0}^{\infty}\sum_{k=1}^{2l+1}\hat{f}_{l,k}Y_{l,k}$ and $g=\sum_{l=0}^{\infty}\sum_{k=1}^{2l+1}\hat{g}_{l,k}Y_{l,k}$, respectively. With respect to the inner product expressed as (see [29]) $(f,g)_{\mathcal{N}_\phi}=\sum_{l=0}^{\infty}\sum_{k=1}^{2l+1}\frac{\hat{f}_{l,k}\hat{g}_{l,k}}{a_l}$, the native space $\mathcal{N}_\phi$, which is the subspace of $L_2(\mathbb{S}^2)$, can be defined by

$$\mathcal{N}_\phi:=\left\{f\in L_2(\mathbb{S}^2):\|f\|^2_{\mathcal{N}_\phi}=\sum_{l=0}^{\infty}\sum_{k=1}^{2l+1}\frac{|\hat{f}_{l,k}|^2}{a_l}<\infty\right\}.$$

It is easy to verify that the native space $\mathcal{N}_\phi$ is a reproducing kernel Hilbert space with reproducing kernel $\phi(\cdot,\cdot)$, that is, $(f,\phi(\cdot,x))_{\mathcal{N}_\phi}=f(x),\ x\in\mathbb{S}^2,\ f\in\mathcal{N}_\phi$.

When $a_l\sim(l+1)^{-2s}$ for $l=0,1,\dots$, the native space $\mathcal{N}_\phi$ is norm equivalent to the Sobolev space H_s:

$$H_s:=\left\{f\in L_2(\mathbb{S}^2):\|f\|^2_{H_s}=\sum_{l=0}^{\infty}\sum_{k=1}^{2l+1}(l+1)^{2s}|\hat{f}_{l,k}|^2<\infty\right\},$$

and the Sobolev embedding theorem in [7] implies that if $s>1$, then the space H_s is continuously embedded in $C(\mathbb{S}^2)$, so that H_s is a reproducing kernel Hilbert space.

The error estimates are general expressed in terms of the mesh norm of $X=\{x_1,\dots,x_N\}\subset\mathbb{S}^2$, which is defined by $h_X:=\sup_{x\in\mathbb{S}^2}\inf_{x_j\in X}d(x,x_j)$, where $d(x,x_j)=\arccos(x\cdot x_j)$ is the geodesic distance between x_j and x.

3 Global error estimates for L_p norm

We first give the following three lemmas, which can be found in [9] and [10].

Lemma 3.1 *Let $d \geq 1$ be an integer and set $M := 2\sqrt{d}$ and $\delta_d := \frac{1}{4(d+1)^{3/2}}$. Let M_1 be an arbitrary positive number, $\theta \in (0, \frac{\pi}{3})$ and set $h_0 := \frac{\theta}{M+M_1+\delta_d}$. Then for any $h \in (0, h_0)$, there exists a set of points $Z_h \subset \mathbb{S}^d$ such that $\mathbb{S}^d = \bigcup_{z \in Z_h} D(z, Mh)$. Here we denote by $D(x_0, \gamma)$ the spherical cap with center x_0 and angle γ, i.e., $D(x_0, \gamma) := \left\{x \in \mathbb{S}^d : x \cdot x_0 > \cos\gamma\right\}$, and then denote by $A(x_0, \gamma)$ the volume of $D(x_0, \gamma)$, i.e., $A(x_0, \gamma) := \Omega_d \int_0^\gamma \sin^{d-1}\theta d\theta$, where Ω_d denotes the volume of $\mathbb{S}^d$. Let F_A denote the characteristic function of a set $A \subset \mathbb{S}^d$. There exists a positive integer Q independent of h such that*

$$\sum_{z \in Z_h} F_{D(z, M'h)} \leq Q, \ \text{where } M' = M + M_1.$$

Further, the cardinality of Z_h is bounded above by $C_Q h^{-d}$, where C_Q is independent of h.

Lemma 3.2 *Let $z \in \mathbb{S}^d$ and $X = \{x_i\}_{i=1}^N$ denote a set of distinct points on $\mathbb{S}^d$. Let $s \in [k, k+1]$, where $k > \frac{d}{2}$ is a positive integer. There exist positive numbers C_1 and C_2 such that if we let $M_1 > \max\{C_1 - 2d^{1/2}, 0\}$ be a fixed positive number and let*

$$h_0 = \frac{C_2}{3M_2}, \ \text{where } M_2 = 2d^{1/2} + M_1,$$

then, assuming that X has mesh norm $h := h_X \in (0, h_0)$, there exists an extension operator $E_{D(z,M_2h)} : H_s(D(z, M_2h)) \longrightarrow H_s(\mathbb{S}^d)$ satisfying
(1) $(E_{D(z,M_2h)}f)|_{D(z,M_2h)} = f$, for all $f \in H_s(D(z, M_2h))$,
(2) there exists a positive constant C, independent of h and z such that

$$\|E_{D(z,M_2h)}f\|_{H_s(\mathbb{S}^d)} \leq C\|f\|_{H_s(D(z,M_2h))},$$

for all $f \in H_s(D(z, M_2h))$ such that $f(\xi) = 0$ for $\xi \in X \bigcap D(z, M_2h)$.

Lemma 3.3 *Let $s > 0$ and let M_1 be any positive number. Let $h \in (0, h_0)$ and let Z_h denote the corresponding quasi-uniform mesh for $\mathbb{S}^d$ from Lemma 3.1. Then, for any $f \in H_s(\mathbb{S}^d)$, we have*

$$\sum_{z \in Z_h} \|f\|^2_{H_s(D(z,M_2h))} \leq Q\|f\|^2_{H_s(\mathbb{S}^d)},$$

where Q is the constant (independent of h) from Lemma 3.1.

We are now ready to state the main results for the error estimates of the hybrid interpolation in L_p norm.

Theorem 3.1 *Let $\phi \in C(\mathbb{S}^2 \times \mathbb{S}^2)$ be a strictly positive definite kernel on $\mathbb{S}^2$, having the representation in (2.3) and $a_l \sim (l+1)^{-2s}$. Assume that integer $L \geq 1$ and $X = \{x_1, \ldots, x_N\} \subset \mathbb{S}^2$ is a set of distinct points on $\mathbb{S}^2$ with mesh norm $1/(2L+2) < h_X \leq 1/(2L)$. For $f \in \mathcal{N}_\phi$, let $I_{X,L}f \in V_{X,L}$ be the hybrid interpolation defined in Section 2. Then we have*

$$\|f - I_{X,L}f\|_{L_p(\mathbb{S}^2)} \leq Ch_X^{\frac{2}{p}+s-1}\|f - I_{X,L}f\|_{\mathcal{N}_\phi}, \quad p \in [2, +\infty),$$

and

$$\|f - I_{X,L}f\|_{L_p(\mathbb{S}^2)} \leq Ch_X^s\|f - I_{X,L}f\|_{\mathcal{N}_\phi}, \quad p \in [1, 2),$$

where the constant C is independent of f and h_X.

Proof. For the case $\mathbb{S}^2$, we can take $d=2$ in Lemma 3.1, Lemma 3.2 and Lemma 3.3. By using Lemma 3.1, for arbitrary $1 \le p < \infty$, we have

$$\|f - I_{X,L}f\|^p_{L_p(\mathbb{S}^2)} = \int_{\mathbb{S}^2} |(f - I_{X,L}f)(\xi)|^p d\omega(\xi) \le \sum_{z\in Z_h} \int_{D(z,Mh)} |(f - I_{X,L}f)(\xi)|^p d\omega(\xi), \quad (3.4)$$

where $M = 2^{3/2}$. This step motivates us to consider the error estimates locally. In particular, $f - I_{X,L}f$ is continuous on $\overline{D(z,Mh)}$, which is a compact subset of $\mathbb{S}^2$, so there exists a point $\xi_z \in \overline{D(z,Mh)}$ where $f - I_{X,L}f$ attains its *maximum*. Now we can write

$$\|f - I_{X,L}f\|^p_{L_p(\mathbb{S}^2)} \le \sum_{z\in Z_h} |(f - I_{X,L}f)(\xi)|^p \int_{D(z,Mh)} d\omega(\xi) \le Ch_X^2 \sum_{z\in Z_h} |(f - I_{X,L}f)(\xi)|^p, \quad (3.5)$$

where the constant C satisfies $A(z,Mh) \le Ch_X^2$. We know that $f - I_{X,L}f \in \mathcal{N}_\phi$ and $\mathcal{N}_\phi$ is norm equivalent to the Sobolev space H_s. Now, rather than consider $f - I_{X,L}f$, we choose instead to consider the restriction $f - I_{X,L}f\big|_{D(z,M_2h)}$, where $M_2 = 2^{3/2} + M_1$.

We should choose a suitable M_1 to fit the conditions of Lemma 3.2, because we can find constant C_1, C_2 such that

$$\frac{2C_2L}{3} > C_1, \quad \frac{2C_2L}{3} > 2^{3/2}.$$

So set $h_0 = \frac{1}{2L}$, $M_1 = \frac{2C_2L}{3} - 2^{3/2}$, and $M_2 = \frac{2C_2L}{3}$, then it is easy to prove that Lemma 3.2 holds. If we let $v_z := f - I_{X,L}f\big|_{D(z,M_2h)}$ and use Lemma 3.2, we have

(E1) $E_{D(z,M_2h)}v_z \in H_s(\mathbb{S}^2)$,

(E2) $E_{D(z,M_2h)}v_z(\xi) = 0$ for all $\xi \in X \bigcap D(z,M_2h)$,

(E3) there exists a positive constant C, independent of h_X and z such that

$$\left\|E_{D(z,M_2h)}v_z\right\|_{H_s(\mathbb{S}^2)} \le C\|v_z\|_{H_s(D(z,M_2h))}.$$

Hence, with the help of Theorem and (E3) we can obtain

$$\begin{aligned} |(f - I_{X,L}f)(\xi_z)| &= \left|E_{D(z,M_2h)}v_z(\xi_z)\right| \le Ch_X^{s-1}\left\|E_{D(z,M_2h)}v_z\right\|_{\mathcal{N}_\phi} \\ &\le Ch_X^{s-1}\left\|E_{D(z,M_2h)}v_z\right\|_{H_s} \le Ch_X^{s-1}\|v_z\|_{H_s(D(z,M_2h))}. \end{aligned}$$

Substituting this into (3.5) gives

$$\|f - I_{X,L}f\|^p_{L_p(\mathbb{S}^2)} \le Ch_X^{2+p(s-1)} \sum_{z\in Z_h} \|v_z\|^p_{H_s(D(z,M_2h))}. \quad (3.6)$$

For $p \in [2,\infty)$ we use Jensen's inequality [1] $\sum_{i=1}^N a_i^p \le \left(\sum_{i=1}^N a_i^2\right)^{\frac{p}{2}}$ followed by Lemma 3.3 to give

$$\begin{aligned} \|f - I_{X,L}f\|^p_{L_p(\mathbb{S}^2)} &\le Ch_X^{2+p(s-1)}\left(\sum_{z\in Z_h}\left\|f - I_{X,L}f\big|_{D(z,M_2h)}\right\|^2_{H_s(D(z,M_2h))}\right)^{p/2} \\ &\le Ch_X^{2+p(s-1)}\left(\|f - I_{X,L}f\|^2_{H_s(\mathbb{S}^2)}\right)^{p/2} \le Ch_X^{2+p(s-1)}\|f - I_{X,L}f\|^p_{\mathcal{N}_\phi}. \end{aligned}$$

Finally, taking the p-th root gives

$$\|f - I_{X,L}f\|_{L_p(\mathbb{S}^2)} \le Ch_X^{\frac{2}{p}+s-1}\|f - I_{X,L}f\|_{\mathcal{N}_\phi}, \quad p \in [2,+\infty), \quad (3.7)$$

where the constant C is independent of f and h_X.

For $p \in [1,2)$ we conduct the same steps as above, however we replace Jensen's inequality with

$$\sum_{i=1}^N a_i^p \le N^{1-\frac{p}{2}}\left(\sum_{i=1}^N a_i^2\right)^{\frac{p}{2}}.$$

Moreover, we use the fact that the cardinality of Z_h is bounded by $C_Q h^{-2}$ (see Lemma 3.1), and we obtain

$$\begin{aligned} \|f - I_{X,L}f\|^p_{L_p(\mathbb{S}^2)} &\le Ch_X^{ps}\left(\sum_{z\in Z_h}\left\|f - I_{X,L}f\big|_{D(z,M_2h)}\right\|^2_{H_s(D(z,M_2h))}\right)^{p/2} \\ &\le Ch_X^{ps}\left(\|f - I_{X,L}f\|^2_{H_s(\mathbb{S}^2)}\right)^{p/2} \le Ch_X^{ps}\|f - I_{X,L}f\|^p_{\mathcal{N}_\phi}. \end{aligned}$$

Finally, taking the p-th root gives

$$\|f - I_{X,L}f\|_{L_p(\mathbb{S}^2)} \le Ch_X^{s}\|f - I_{X,L}f\|_{\mathcal{N}_\phi}, \quad p \in [1,2), \tag{3.8}$$

where the constant C is independent of f and h_X.

Combining (3.7) and (3.8) yields Theorem 3.1.

4 Inf-sup condition and improved global error estimates

As we can see that the factor $\|f - I_{X,L}f\|_{\mathcal{N}_\phi}$ in Theorem 3.1 may be harder to estimate than factor $\|f\|_{\mathcal{N}_\phi}$. Considering the fact that the hybrid interpolation defined in Section 2 is different from the interpolation scheme only by radial basis functions constructed from strictly positive definite kernels or conditionally positive definite kernels (see [10,]), we should find the other method to characterize the relationship between $\|f - I_{X,L}f\|_{\mathcal{N}_\phi}$ and $\|f\|_{\mathcal{N}_\phi}$. The following Inf-sup condition is quoted from [26], whose method is helpful to "tidy up" the error results in Theorem 3.1.

Theorem 4.1 *(see [26, Theorem 6.1]). Let $\phi \in C(\mathbb{S}^2 \times \mathbb{S}^2)$ be a strictly positive definite kernel on $\mathbb{S}^2$, having the representation in (2.3) and $a_l \sim (l+1)^{-2s}$. Then there exist constants $\gamma > 0$ and $\tau > 0$ depending only on s such that for all $L \ge 1$ and all $X = \{x_1, \ldots, x_N\} \subset \mathbb{S}^2$ satisfying $h_X \le \tau/L$, the following inequality holds:*

$$\sup_{v\in R_X\setminus\{0\}} \frac{(p,v)_{\mathcal{N}_\phi}}{\|v\|_{\mathcal{N}_\phi}} \ge \gamma\|p\|_{\mathcal{N}_\phi}, \ \ p \in \mathcal{P}_L, \tag{4.9}$$

where $R_X = span\{\phi(\cdot,x_1), \ldots, \phi(\cdot,x_N)\}$.

In order to use the same method in [26], we simply denote that $I_{X,L}f = u_{X,L} + p_{X,L}$, where $p_{X,L} = \sum_{l=0}^{L}\sum_{k=1}^{2l+1}\beta_{l,k}Y_{l,k}$, and $u_{X,L} = \sum_{j=1}^{N}\phi(\cdot,x_j)$.

For a given $f \in \mathcal{N}_\phi$, the interpolation conditions $I_{X,L}f(x_i) = f(x_i), \ \ i = 1, \ldots, N$, and the side conditions $\sum_{j=1}^{N}\alpha_j q(x_j) = 0, \ \ \forall q \in \mathcal{P}_L$, are equivalent to

$$(u_{X,L}, v_X)_{\mathcal{N}_\phi} + (p_{X,L}, v_X)_{\mathcal{N}_\phi} = (f, v_X)_{\mathcal{N}_\phi}, \ \ v_X \in R_X, \tag{4.10}$$

and

$$(q, u_{X,L})_{\mathcal{N}_\phi} = 0, \ \ \forall q \in \mathcal{P}_L. \tag{4.11}$$

Now we can write the target function $f \in \mathcal{N}_\phi$ in an analogous way to $I_{X,L}f$ as $f := u + p$, where $p \in \mathcal{P}_L$ and $u \in \mathcal{N}_\phi$ are defined by $(p,q)_{\mathcal{N}_\phi} = (f,q)_{\mathcal{N}_\phi}, q \in \mathcal{P}_L$, which means that p is the $\mathcal{N}_\phi$-orthogonal project of f onto $\mathcal{P}_L$.

Similar to (4.10) and (4.11), we have

$$(u, v)_{\mathcal{N}_\phi} + (p, v)_{\mathcal{N}_\phi} = (f, v)_{\mathcal{N}_\phi}, \ \ v_X \in R_X, \tag{4.12}$$

and

$$(q, u)_{\mathcal{N}_\phi} = 0, \ \ q \in \mathcal{P}_L. \tag{4.13}$$

By subtracting (4.10) from (4.12) (with v replaced by v_X) and (4.11) from (4.13), we can obtain

$$(u - u_{X,L}, v_X)_{\mathcal{N}_\phi} + (p - p_{X,L}, v_X)_{\mathcal{N}_\phi} = 0, \ \ v_X \in R_X, \tag{4.14}$$

and

$$(q, u - u_{X,L})_{\mathcal{N}_\phi} = 0, \quad q \in \mathcal{P}_L. \tag{4.15}$$

Now we define $\tilde{u}_X \in R_X$ to be the $\mathcal{N}_\phi$-orthogonal project of u onto R_X, that is,

$$(\tilde{u}_X, v_X)_{\mathcal{N}_\phi} = (u, v_X)_{\mathcal{N}_\phi}, \quad v_X \in R_X. \tag{4.16}$$

From (4.14), (4.15) and (4.16), we clearly have

$$(\tilde{u}_X - u_{X,L}, v_X)_{\mathcal{N}_\phi} + (p - p_{X,L}, v_X)_{\mathcal{N}_\phi} = 0, \quad v_X \in R_X, \tag{4.17}$$

and

$$(q, \tilde{u}_X - u_{X,L})_{\mathcal{N}_\phi} = (q, \tilde{u}_X - u)_{\mathcal{N}_\phi}, \quad q \in \mathcal{P}_L. \tag{4.18}$$

With the help of Theorem 4.1, we have

$$\begin{aligned} \|p - p_{X,L}\|_{\mathcal{N}_\phi} &\le \frac{1}{\gamma} \sup_{v_X \in R_X \setminus \{0\}} \frac{(p - p_{X,L}, v_X)_{\mathcal{N}_\phi}}{\|v_X\|_{\mathcal{N}_\phi}} \\ &= \frac{1}{\gamma} \sup_{v_X \in R_X \setminus \{0\}} \frac{(u_{X,L} - \tilde{u}_X, v_X)_{\mathcal{N}_\phi}}{\|v_X\|_{\mathcal{N}_\phi}} \le \frac{1}{\gamma} \|u_{X,L} - \tilde{u}_X\|_{\mathcal{N}_\phi}. \end{aligned}$$

By using (4.17) with $v_X = \tilde{u}_X - u_{X,L}$ and (4.18), we also have

$$\begin{aligned} \|\tilde{u}_X - u_{X,L}\|^2_{\mathcal{N}_\phi} &= -(p - p_{X,L}, \tilde{u}_X - u_{X,L})_{\mathcal{N}_\phi} = -(p - p_{X,L}, \tilde{u}_X - u)_{\mathcal{N}_\phi} \\ &\le \|p - p_{X,L}\|_{\mathcal{N}_\phi} \|\tilde{u}_X - u\|_{\mathcal{N}_\phi} \le \frac{1}{\gamma} \|u_{X,L} - \tilde{u}_X\|_{\mathcal{N}_\phi} \|\tilde{u}_X - u\|_{\mathcal{N}_\phi}. \end{aligned}$$

So we obtain that

$$\|\tilde{u}_X - u_{X,L}\|_{\mathcal{N}_\phi} \le \frac{1}{\gamma} \|\tilde{u}_X - u\|_{\mathcal{N}_\phi} \le C \|\tilde{u}_X - u\|_{\mathcal{N}_\phi}, \tag{4.19}$$

and

$$\|p - p_{X,L}\|_{\mathcal{N}_\phi} \le \frac{1}{\gamma^2} \|\tilde{u}_X - u\|_{\mathcal{N}_\phi} \le C \|\tilde{u}_X - u\|_{\mathcal{N}_\phi}. \tag{4.20}$$

With the above inequalities (4.19) and (4.20), we can establish the following Theorem 4.2, which indicates the relationship between $\|f - I_{X,L} f\|_{\mathcal{N}_\phi}$ and $\|f\|_{\mathcal{N}_\phi}$.

Theorem 4.2 *Let $\phi \in C(\mathbb{S}^2 \times \mathbb{S}^2)$ be a strictly positive definite kernel on $\mathbb{S}^2$, having the representation in* (2.3) *and $a_l \sim (l+1)^{-2s}$, $s > 1$. Assume that integer $L \ge 1$ and $X = \{x_1, \ldots, x_N\} \subset \mathbb{S}^2$ is a set of distinct points on $\mathbb{S}^2$ with mesh norm $h_X \le \tau/L$, where τ is as in Theorem 4.1. For $f \in \mathcal{N}_\phi$, let $I_{X,L} f \in V_{X,L}$ be the hybrid interpolation defined in Section 2. Then we have*

$$\|f - I_{X,L} f\|_{\mathcal{N}_\phi} \le C \inf_{q \in \mathcal{P}_L} \|f - q\|_{\mathcal{N}_\phi} \le C \|f\|_{\mathcal{N}_\phi}.$$

Proof. Using the representation $I_{X,L} f = u_{X,L} + p_{X,L}$, $f = u + p$ and (4.19), (4.20) we have

$$\begin{aligned} \|f - I_{X,L} f\|_{\mathcal{N}_\phi} &\le \|u - u_{X,L}\|_{\mathcal{N}_\phi} + \|p - p_{X,L}\|_{\mathcal{N}_\phi} \\ &\le \|\tilde{u}_X - u\|_{\mathcal{N}_\phi} + \|\tilde{u}_X - u_{X,L}\|_{\mathcal{N}_\phi} + \|p - p_{X,L}\|_{\mathcal{N}_\phi} \le C \|\tilde{u}_X - u\|_{\mathcal{N}_\phi}, \end{aligned}$$

and also we have $\|\tilde{u}_X - u\|_{\mathcal{N}_\phi} \le \|u\|_{\mathcal{N}_\phi} = \|f - p\|_{\mathcal{N}_\phi} = \inf_{q \in \mathcal{P}_L} \|f - q\|_{\mathcal{N}_\phi}$, which yields $\|f - I_{X,L} f\|_{\mathcal{N}_\phi} \le C \inf_{q \in \mathcal{P}_L} \|f - q\|_{\mathcal{N}_\phi} \le C \|f\|_{\mathcal{N}_\phi}$, and the proof of Theorem 4.2 is completed.

Combining Theorem 4.2 with Theorem 3.1, we can easily verify the following Corollary 4.1.

Corollary 4.1 *Under the conditions of Theorem 3.1 apart from the mesh norm $1/(2L+2) < h_X \le \min\{1/(2L), \tau/L\}$, where τ is as in Theorem 4.1. For $f \in \mathcal{N}_\phi$, let $I_{X,L} f \in V_{X,L}$ be the hybrid interpolation defined in Section 2. Then we have*

$$\|f - I_{X,L} f\|_{L_p(\mathbb{S}^2)} \le C h_X^{\frac{2}{p} + s - 1} \|f\|_{\mathcal{N}_\phi}, \quad p \in [2, +\infty),$$

and

$$\|f - I_{X,L} f\|_{L_p(\mathbb{S}^2)} \le C h_X^{s} \|f\|_{\mathcal{N}_\phi}, \quad p \in [1, 2),$$

where the constant C is independent of f and h_X.

In the rest part of this section, unlike the above arguments we used to perform the "cleaner" error estimates in Corollary 4.1, we will show that improved global error estimates are available, provided that the target function f belongs to a certain subspace of $\mathcal{N}_\phi$, which defined by $\mathcal{N}_{\phi*\phi}$. This procedure is the same as in [10] and the following Definition 4.1 is about the convolution kernel of ϕ, which generates the corresponding native space $\mathcal{N}_{\phi*\phi}$.

Definition 4.1 *Let ϕ be a strictly positive definite zonal kernel that defined in* (2.3) *We define the convolution kernel of ϕ by*

$$(\phi * \phi)(x,y) := \int_{\mathbb{S}^2} \phi(x,z)\phi(y,z)d\omega(z), \quad x,y \in \mathbb{S}^2.$$

Working in terms of Fourier expansions, we have $(\phi*\phi)(x,y) := \sum_{l=0}^{\infty} a_l^2 \sum_{k=1}^{2l+1} Y_{l,k}(x)Y_{l,k}(y)$. Executing the same arguments as in Section 2, we know that the native space $\mathcal{N}_{\phi*\phi}$ associated with kernel $(\phi*\phi)(\cdot,\cdot)$ can be defined by

$$\mathcal{N}_{\phi*\phi} := \left\{ f \in L_2(S^2) : \|f\|^2_{\mathcal{N}_{\phi*\phi}} = \sum_{l=0}^{\infty}\sum_{k=1}^{2l+1} \frac{|\hat{f}_{l,k}|^2}{a_l^2} < \infty \right\},$$

and it is a reproducing kernel Hilbert space with the reproducing kernel $(\phi*\phi)(\cdot,\cdot)$.

When $a_l \sim (l+1)^{-2s}$ for $l = 0,1,\ldots$ and $s > 1$, we know that the native space $\mathcal{N}_\phi$ is norm equivalent to the Sobolev space H_s. So $\mathcal{N}_{\phi*\phi} \cong H_{2s} \subset H_s \cong \mathcal{N}_\phi$, where $\cong$ denotes norm equivalence. Obviously, we see $\mathcal{N}_{\phi*\phi} \subset \mathcal{N}_\phi$.

The following Lemma 4.1 gives a crucial inequality, which is helpful to improve the global error estimates of the hybrid interpolation for a target function $f \in \mathcal{N}_{\phi*\phi}$.

Lemma 4.1 *Let $u \in \mathcal{N}_{\phi*\phi}$ and $\tilde{u}_X \in R_X$ be the $\mathcal{N}_\phi$-orthogonal project of u onto R_X, which has the property as in (4.16), then we have*

$$\|\tilde{u}_X - u\|^2_{\mathcal{N}_\phi} \leq \|u\|_{\mathcal{N}_{\phi*\phi}} \cdot \|\tilde{u}_X - u\|_{L_2(\mathbb{S}^2)}, \tag{4.21}$$

where R_X is the same as in Theorem 4.1.

Proof. By using (4.16), the definition of $(\cdot,\cdot)_{\mathcal{N}_{\phi*\phi}}$, and Cauchy-Schwarz inequality respectively, we have

$$\begin{aligned}
\|\tilde{u}_X - u\|^2_{\mathcal{N}_\phi} &= (u, \tilde{u}_X - u)_{\mathcal{N}_\phi} = \sum_{l=0}^{\infty}\sum_{k=1}^{2l+1} \frac{\hat{u}_{l,k}\cdot\left(\hat{u}_{l,k} - \left(\widehat{\tilde{u}_X}\right)_{l,k}\right)}{a_l} \\
&\leq \left(\sum_{l=0}^{\infty}\sum_{k=1}^{2l+1} \frac{|\hat{u}_{l,k}|^2}{a_l^2}\right)^{1/2} \left(\sum_{l=0}^{\infty}\sum_{k=1}^{2l+1} \left(\hat{u}_{l,k} - \left(\widehat{\tilde{u}_X}\right)_{l,k}\right)^2\right)^{1/2} \leq \|u\|_{\mathcal{N}_{\phi*\phi}} \cdot \|\tilde{u}_X - u\|_{L_2(\mathbb{S}^2)}
\end{aligned}$$

With this in place we can provide the following improved global error estimates.

Theorem 4.3 *Under the conditions of Corollary 4.1 and assume further that $f \in \mathcal{N}_{\phi*\phi}$, we have*

$$\|f - I_{X,L}f\|_{L_p(\mathbb{S}^2)} \leq Ch_X^{\frac{2}{p}+2s-1}\|f\|_{\mathcal{N}_{\phi*\phi}}, \quad p \in [2,+\infty), \tag{4.22}$$

and

$$\|f - I_{X,L}f\|_{L_p(\mathbb{S}^2)} \leq Ch_X^{2s}\|f\|_{\mathcal{N}_{\phi*\phi}}, \quad p \in [1,2), \tag{4.23}$$

where the constant C is independent of f and h_X.

Proof. First we have, from Theorem 3.1, with $p = 2$, that

$$\|\tilde{u}_X - u\|_{L_2(\mathbb{S}^2)} \leq Ch_X^s\|\tilde{u}_X - u\|_{\mathcal{N}_\phi}.$$

Substituting this into (4.21) gives

$$\|\tilde{u}_X - u\|_{\mathcal{N}_\phi}^2 \leq Ch_X^s \|u\|_{\mathcal{N}_{\phi*\phi}} \cdot \|\tilde{u}_X - u\|_{\mathcal{N}_\phi}. \tag{4.24}$$

So,

$$\|\tilde{u}_X - u\|_{\mathcal{N}_\phi} \leq Ch_X^s \|u\|_{\mathcal{N}_{\phi*\phi}}. \tag{4.25}$$

Using the same procedure as in the proof of Theorem 4.2, we see that

$$\|f - I_{X,L}f\|_{\mathcal{N}_\phi} \leq \|u - u_{X,L}\|_{\mathcal{N}_\phi} + \|p - p_{X,L}\|_{\mathcal{N}_\phi} \leq C\|\tilde{u}_X - u\|_{\mathcal{N}_\phi} \leq Ch_X^s \|u\|_{\mathcal{N}_{\phi*\phi}}.$$

Clearly,

$$\|u\|_{\mathcal{N}_{\phi*\phi}} = \|f - p\|_{\mathcal{N}_{\phi*\phi}} = \inf_{q \in \mathcal{P}_L} \|f - q\|_{\mathcal{N}_{\phi*\phi}} \leq \|f\|_{\mathcal{N}_{\phi*\phi}}, \tag{4.26}$$

which implies

$$\|f - I_{X,L}f\|_{\mathcal{N}_\phi} \leq Ch_X^s \|f\|_{\mathcal{N}_{\phi*\phi}}. \tag{4.27}$$

With the help of Theorem 3.1 we see

$$\|f - I_{X,L}f\|_{L_p(\mathbb{S}^2)} \leq Ch_X^{\frac{2}{p}+2s-1} \|f\|_{\mathcal{N}_{\phi*\phi}}, \;\; p \in [2, +\infty),$$

and

$$\|f - I_{X,L}f\|_{L_p(\mathbb{S}^2)} \leq Ch_X^{2s} \|f\|_{\mathcal{N}_{\phi*\phi}}, \;\; p \in [1, 2),$$

where the constant C is independent of f and h_X.

5 Hybrid interpolation for rough native space

In order to generate a larger native space than $\mathcal{N}_\phi$, we should give a new kernel defined in the form

$$\psi(x, y) = \sum_{l=0}^{\infty} b_l \sum_{k=1}^{2l+1} Y_{l,k}(x) Y_{l,k}(y), \tag{5.28}$$

with $b_l > 0$ for all l, and $\sum_{l=0}^{\infty} l b_l < \infty$.

With respect to the inner product expressed as $(f, g)_{\mathcal{N}_\psi} = \sum_{l=0}^{\infty} \sum_{k=1}^{2l+1} \frac{\hat{f}_{l,k}\hat{g}_{l,k}}{b_l}$, the native space $\mathcal{N}_\psi$ may alternatively be characterized as the following set

$$\mathcal{N}_\psi := \left\{ f \in L_2(\mathbb{S}^2) : \|f\|_{\mathcal{N}_\psi}^2 = \sum_{l=0}^{\infty} \sum_{k=1}^{2l+1} \frac{|\hat{f}_{l,k}|^2}{b_l} < \infty \right\}.$$

Assuming that $b_l > a_l$, for all $l = 0, 1, \ldots$, we then see that $\mathcal{N}_\phi \subset \mathcal{N}_\psi$.

Next, we will consider the error estimates for the hybrid interpolation of a target function $f \in \mathcal{N}_\psi \supset \mathcal{N}_\phi$. Obviously, if we take the hybrid interpolation associated with the less smooth kernel ψ in the form $I_{X,L,\psi}f = \sum_{j=1}^{N} \alpha_j \psi(\cdot, x_j) + \sum_{l=0}^{L} \sum_{k=1}^{2l+1} \beta_{l,k} Y_{l,k}$, then Theorem 3.1 above still holds for $f \in \mathcal{N}_\psi$. However, motivated by the idea in [12], we still take the initial hybrid interpolation $I_{X,L,\phi}f$ in the form

$$I_{X,L,\phi}f = \sum_{j=1}^{N} \alpha_j \phi(\cdot, x_j) + \sum_{l=0}^{L} \sum_{k=1}^{2l+1} \beta_{l,k} Y_{l,k}, \tag{5.29}$$

and consider the error estimate $\|f - I_{X,L,\phi}f\|_{L_p(\mathbb{S}^2)}$.

Lemma 5.1 *Let α be a nonnegative real number, and let M be the multiplier operator defined on $\mathcal{P}_L$ (embedded in $C(\mathbb{S}^2)$) by $M(p) = \sum_{l=0}^{L} (\lambda_l)^\alpha \sum_{k=1}^{2l+1} c_{l,k} Y_{l,k}$, where $p = \sum_{l=0}^{L} \sum_{k=1}^{2l+1} c_{l,k} Y_{l,k}$. Then we have $\|M(p)\| \leq C(\lambda_L)^\alpha \|p\|$, where C is a constant independent of p and L.*

Lemma 5.1 is a special case of Theorem 3.2 in [4].

Lemma 5.2 *Let $X = \{x_1, \ldots, x_N\} \subset \mathbb{S}^2$ satisfying $h_X \le 1/(2L)$, then for any linear functional σ on $\mathcal{P}_L$ (embedded in $C(\mathbb{S}^2)$), such that $\|\sigma\|_* = 1$, there exist N real numbers $\alpha_j := \alpha_j(x)$ (x is fixed) with $\sum_{j=1}^N |\alpha_j| \le 2$, so that $\sigma(f) = \sum_{j=1}^N \alpha_j \delta_j(f)$ for all $f \in \mathcal{P}_L$, where δ_j denotes the point evaluation functional at the point x_j in X.*

The proof of the following Lemma 5.3 can be found in [29, Corollary 17.12].

Lemma 5.3 *Suppose that $X = \{x_1, \ldots, x_N\} \subset \mathbb{S}^2$ has mesh norm $h_X \le \frac{1}{2L}$ for some integer $L \ge 1$. Then there exist functions $\alpha_j : \mathbb{S}^2 \to \mathbb{R}$ for $j = 1, \ldots, N$ such that*
(i) $\sum_{j=1}^N \alpha_j(x) p(x_j) = p(x), \quad \forall p \in \mathcal{P}_L, \ \forall x \in \mathbb{S}^2$,
(ii) $\sum_{j=1}^N |\alpha_j(x)| \le 2, \quad \forall x \in \mathbb{S}^2$.

The following Theorem 5.1 is about the pointwise error estimate $|f(x) - I_{X,L,\phi} f(x)|$, by which we can obtain the global error estimate $\|f - I_{X,L,\phi} f\|_{L_p(\mathbb{S}^2)}$.

Theorem 5.1 *Let $\phi \in C(\mathbb{S}^2 \times \mathbb{S}^2)$ be a strictly positive definite kernel defined by (2.3), let $\psi \in C(\mathbb{S}^2 \times \mathbb{S}^2)$ be a strictly positive definite kernel on $\mathbb{S}^2$, having the representation in (5.28), $b_l/a_l = \lambda_l$ for $l \ge 1$ and $b_l \sim (l+1)^{-2s}, s > 1$, $l \ge 0$. Assume that integer $L \ge 1$ and $X = \{x_1, \ldots, x_N\} \subset \mathbb{S}^2$ is a set of distinct points on $\mathbb{S}^2$ with mesh norm $1/(2L+2) < h_X \le 1/(2L)$. For $f \in \mathcal{N}_\psi$, let $I_{X,L,\phi} f \in V_{X,L}$ be the hybrid interpolation defined in (5.29). Then for a fixed $x \in \mathbb{S}^2$, we have*

$$|f(x) - I_{X,L,\phi} f(x)| \le C h_X^{s-1} \|f - I_{X,L,\phi} f\|_{\mathcal{N}_\psi}.$$

Proof. For $\forall f \in \mathcal{N}_\psi$, we simply take the hybrid interpolation associated with the smooth kernel ϕ by $I_{X,L,\phi} f(x) = u_{X,L,\phi} + p_{X,L}$, where $u_{X,L,\phi} = \sum_{j=1}^N \alpha_j \phi(\cdot, x_j), \quad x_j \in X = \{x_1, x_2, \ldots, x_N\}$, and $p_{X,L} = \sum_{l=0}^L \sum_{k=1}^{2l+1} \beta_{l,k} Y_{l,k}$, such that $I_{X,L,\phi} f(x_j) = f(x_j) (j = 1, 2, \ldots, N)$.

However, if we just use the hybrid interpolation associated with the less smooth kernel ψ, we have $I_{X,L,\psi} f(x) = u_{X,L,\psi} + p'_{X,L}$, where $u_{X,L,\psi} = \sum_{j=1}^N \gamma_j \psi(\cdot, x_j), x_j \in X = \{x_1, x_2, \ldots, x_N\}$, and $p'_{X,L} = \sum_{l=0}^L \sum_{k=1}^{2l+1} \beta'_{l,k} Y_{l,k}$.

First, we consider the estimate of $\|\psi(\cdot, x) - u_{X,L,\psi}\|_{\mathcal{N}_\psi}$. Using the same method as that in [12], we have

$$\|\psi(\cdot, x) - u_{X,L,\psi}\|_{\mathcal{N}_\psi} = \sup_{\substack{v \in \mathcal{N}_\psi \\ \|v\|_{\mathcal{N}_\psi} = 1}} (\psi(\cdot, x) - u_{X,L,\psi}, v)_{\mathcal{N}_\psi}$$

$$= \sup_{\substack{v \in \mathcal{N}_\psi \\ \|v\|_{\mathcal{N}_\psi} = 1}} \sum_{l=0}^\infty b_l^{-1} \sum_{k=1}^{2l+1} \hat{v}_{l,k} \left(b_l \sum_{j=1}^N \gamma_j Y_{l,k}(x_j) - b_l Y_{l,k}(x) \right)$$

$$= \sup_{\substack{v \in \mathcal{N}_\psi \\ \|v\|_{\mathcal{N}_\psi} = 1}} \sum_{l=0}^\infty \sum_{k=1}^{2l+1} \hat{v}_{l,k} \left(\sum_{j=1}^N \gamma_j Y_{l,k}(x_j) - Y_{l,k}(x) \right).$$

By using Lemma 5.3, for a fixed x, we see that there exist N real numbers γ_j with $\sum_{j=1}^N |\gamma_j| \le 2$ such that

$$\sum_{j=1}^N \gamma_j Y_{l,k}(x_j) = Y_{l,k}(x), \quad l = 0, 1, \ldots, L, \tag{5.30}$$

which yields

$$\|\psi(\cdot,x)-u_{X,L,\psi}\|_{\mathcal{N}_\psi} = \sup_{\substack{v\in\mathcal{N}_\psi\\ \|v\|_{\mathcal{N}_\psi}=1}} \sum_{l=L+1}^{\infty}\sum_{k=1}^{2l+1}\hat{v}_{l,k}\left(\sum_{j=1}^{N}\gamma_j Y_{l,k}(x_j)-Y_{l,k}(x)\right)$$

$$= \sup_{\substack{v\in\mathcal{N}_\psi\\ \|v\|_{\mathcal{N}_\psi}=1}} \left[\sum_{j=1}^{N}\gamma_j\sum_{l=L+1}^{\infty}\sum_{k=1}^{2l+1}\hat{v}_{l,k}Y_{l,k}(x_j)-\sum_{l=L+1}^{\infty}\sum_{k=1}^{2l+1}\hat{v}_{l,k}Y_{l,k}(x)\right]$$

$$\leq \sup_{\substack{v\in\mathcal{N}_\psi\\ \|v\|_{\mathcal{N}_\psi}=1}} \left[\sum_{j=1}^{N}|\gamma_j|\left|\sum_{l=L+1}^{\infty}\sum_{k=1}^{2l+1}\hat{v}_{l,k}Y_{l,k}(x_j)\right|\right]+\sup_{\substack{v\in\mathcal{N}_\psi\\ \|v\|_{\mathcal{N}_\psi}=1}}\left|\sum_{l=L+1}^{\infty}\sum_{k=1}^{2l+1}\hat{v}_{l,k}Y_{l,k}(x)\right|.$$

By using the Cauchy-Schwarz inequality, (5.30) and the relation in (2.1), we see that

$$\|\psi(\cdot,x)-u_{X,L,\psi}\|_{\mathcal{N}_\psi} \leq \sum_{j=1}^{N}|\gamma_j|\max_{x_j\in X}\left(\sum_{l=L+1}^{\infty}b_l\sum_{k=1}^{2l+1}Y_{l,k}^2(x_j)\right)^{\frac{1}{2}}\sup_{\substack{v\in\mathcal{N}_\psi\\ \|v\|_{\mathcal{N}_\psi}=1}}\left(\sum_{l=L+1}^{\infty}\sum_{k=1}^{2l+1}\frac{\hat{v}_{l,k}^2}{b_l}\right)^{\frac{1}{2}}$$

$$+\left(\sum_{l=L+1}^{\infty}b_l\sum_{k=1}^{2l+1}Y_{l,k}^2(x)\right)^{\frac{1}{2}} t \sup_{\substack{v\in\mathcal{N}_\psi\\ \|v\|_{\mathcal{N}_\psi}=1}}\left(\sum_{l=L+1}^{\infty}\sum_{k=1}^{2l+1}\frac{\hat{v}_{l,k}^2}{b_l}\right)^{\frac{1}{2}}$$

$$\leq 2\left(\sum_{l=L+1}^{\infty}b_l\,\frac{2l+1}{4\pi}\right)^{\frac{1}{2}}+\left(\sum_{l=L+1}^{\infty}b_l\,\frac{2l+1}{4\pi}\right)^{\frac{1}{2}}\leq C_1\left(\sum_{l=L+1}^{\infty}(2l+1)b_l\right)^{\frac{1}{2}}$$

$$\leq C_1\left(\sum_{l=L+1}^{\infty}(l+1)^{-2s+1}\right)^{\frac{1}{2}}\leq C_1(L+1)^{-s+1}\leq C_1 h_X^{s-1}. \tag{5.31}$$

Next we consider the estimate of $\|\psi(\cdot,x)-u_{X,L,\phi}\|_{\mathcal{N}_\psi}$, in which we will use Lemma 5.1 and Lemma 5.2.

$$\begin{aligned}\|\psi(\cdot,x)-u_{X,L,\phi}\|_{\mathcal{N}_\psi} &= \sup_{\substack{v\in\mathcal{N}_\psi\\ \|v\|_{\mathcal{N}_\psi}=1}}(\psi(\cdot,x)-u_{X,L,\phi},v)_{\mathcal{N}_\psi}\\ &= \sup_{\substack{v\in\mathcal{N}_\psi\\ \|v\|_{\mathcal{N}_\psi}=1}}\sum_{l=0}^{\infty}b_l^{-1}\sum_{k=1}^{2l+1}\hat{v}_{l,k}\left(a_l\sum_{j=1}^{N}\alpha_j Y_{l,k}(x_j)-b_lY_{l,k}(x)\right)\\ &= \sup_{\substack{v\in\mathcal{N}_\psi\\ \|v\|_{\mathcal{N}_\psi}=1}}\sum_{l=0}^{\infty}b_l^{-1}a_l\sum_{k=1}^{2l+1}\hat{v}_{l,k}\left(\sum_{j=1}^{N}\alpha_jY_{l,k}(x_j)-\frac{b_l}{a_l}Y_{l,k}(x)\right).\end{aligned}$$

Let T_L be the multiplier operator defined on $\mathcal{P}_L$ (embedded in $C(\mathbb{S}^2)$) by $T_L(Y_{l,k})=\frac{b_l}{a_l}Y_{l,k}$, for each $l=0,1,\ldots,L$ and all $k=1,2,\ldots,2l+1$, and extended linearly throughout $\mathcal{P}_L$. Let σ be the linear functional on $\mathcal{P}_L$ defined by $\sigma=\delta_x\circ T_L$. That is $\sigma(p)=(T_L(p))(x)$ for each $p\in\mathcal{P}_L$. By Lemma 5.1 with $\alpha=1$ and the assumption that $b_l/a_l=\lambda_l$, $l\geq 1$, we have

$$|\sigma(p)|=|(T_L(p))(x)|\leq\|T_L(p)\|\leq C\lambda_L\|p\|=C\frac{b_L}{a_L}\|p\|,$$

in which C is a constant independent of p and L. Then by Lemma 5.2, there exist N real numbers α_j with $\sum_{j=1}^{N}|\alpha_j|\leq 2C\frac{b_L}{a_L}$ such that

$$\sum_{j=1}^{N}\alpha_jY_{l,k}(x_j)=\frac{b_l}{a_l}Y_{l,k}(x),\quad l=0,1,\ldots,L. \tag{5.32}$$

With the help of (5.32), we see that

$$\|\psi(\cdot,x)-u_{X,L,\phi}\|_{\mathcal{N}_\psi} = \sup_{\substack{v\in\mathcal{N}_\psi\\ \|v\|_{\mathcal{N}_\psi}=1}} \sum_{l=L+1}^{\infty} b_l^{-1}a_l \sum_{k=1}^{2l+1}\hat{v}_{l,k}\left(\sum_{j=1}^{N}\alpha_j Y_{l,k}(x_j)-\frac{b_l}{a_l}Y_{l,k}(x)\right)$$

$$= \sup_{\substack{v\in\mathcal{N}_\psi\\ \|v\|_{\mathcal{N}_\psi}=1}} \left[\sum_{j=1}^{N}\alpha_j \sum_{l=L+1}^{\infty} b_l^{-1}a_l\sum_{k=1}^{2l+1}\hat{v}_{l,k}Y_{l,k}(x_j)-\sum_{l=L+1}^{\infty}\sum_{k=1}^{2l+1}\hat{v}_{l,k}Y_{l,k}(x)\right]$$

$$\leq \sup_{\substack{v\in\mathcal{N}_\psi\\ \|v\|_{\mathcal{N}_\psi}=1}} \left[\sum_{j=1}^{N}|\alpha_j|\left|\sum_{l=L+1}^{\infty}\frac{a_l}{b_l}\sum_{k=1}^{2l+1}\hat{v}_{l,k}Y_{l,k}(x_j)\right|\right]+\sup_{\substack{v\in\mathcal{N}_\psi\\ \|v\|_{\mathcal{N}_\psi}=1}}\left|\sum_{l=L+1}^{\infty}\sum_{k=1}^{2l+1}\hat{v}_{l,k}Y_{l,k}(x)\right|.$$

Using the Cauchy-Schwarz inequality, we see that

$$\|\psi(\cdot,x)-u_{X,L,\phi}\|_{\mathcal{N}_\psi} \leq \sum_{j=1}^{N}|\alpha_j|\max_{x_j\in X}\left(\sum_{l=L+1}^{\infty}\frac{a_l^2}{b_l}\sum_{k=1}^{2l+1}Y_{l,k}^2(x_j)\right)^{\frac12}\sup_{\substack{v\in\mathcal{N}_\psi\\ \|v\|_{\mathcal{N}_\psi}=1}}\left(\sum_{l=L+1}^{\infty}\sum_{k=1}^{2l+1}\frac{\hat{v}_{l,k}^2}{b_l}\right)^{\frac12}$$

$$+\left(\sum_{l=L+1}^{\infty}b_l\sum_{k=1}^{2l+1}Y_{l,k}^2(x)\right)^{\frac12}\sup_{\substack{v\in\mathcal{N}_\psi\\ \|v\|_{\mathcal{N}_\psi}=1}}\left(\sum_{l=L+1}^{\infty}\sum_{k=1}^{2l+1}\frac{\hat{v}_{l,k}^2}{b_l}\right)^{\frac12}.$$

With the help of $\sum_{j=1}^{N}|\alpha_j|\leq 2C\frac{b_L}{a_L}$, $b_l>a_l$ and the relation in (2.1), we have

$$\begin{aligned}
\|\psi(\cdot,x)-u_{X,L,\phi}\|_{\mathcal{N}_\psi} &\leq 2C\frac{b_L}{a_L}\left(\sum_{l=L+1}^{\infty}\frac{a_l^2}{b_l}\frac{2l+1}{4\pi}\right)^{\frac12}+\left(\sum_{l=L+1}^{\infty}b_l\,\frac{2l+1}{4\pi}\right)^{\frac12}\\
&\leq 2C\frac{b_L}{a_L}\left(\sum_{l=L+1}^{\infty}b_l\,\frac{2l+1}{4\pi}\right)^{\frac12}+\left(\sum_{l=L+1}^{\infty}b_l\,\frac{2l+1}{4\pi}\right)^{\frac12}\\
&\leq C_2\left(\sum_{l=L+1}^{\infty}(2l+1)b_l\right)^{\frac12}\leq C_2\left(\sum_{l=L+1}^{\infty}(l+1)^{-2s+1}\right)^{\frac12}\\
&\leq C_2(L+1)^{-s+1}\leq C_2h_X^{s-1}.
\end{aligned}\tag{5.33}$$

With the above obtained results, we can provide the following pointwise error estimate:

$$\begin{aligned}
|f(x)-I_{X,L,\phi}f(x)| &= \left|(f-I_{X,L,\phi}f,\psi(\cdot,x))_{\mathcal{N}_\psi}\right|\\
&= \Big|(f-I_{X,L,\phi}f,\psi(\cdot,x)-u_{X,L,\phi})_{\mathcal{N}_\psi}+(f-I_{X,L,\phi}f,u_{X,L,\psi})_{\mathcal{N}_\psi}\\
&\quad+(f-I_{X,L,\phi}f,u_{X,L,\phi}-u_{X,L,\psi})_{\mathcal{N}_\psi}\Big| := |I_1+I_2+I_3|.
\end{aligned}$$

It is easy to verify that

$$\begin{aligned}
I_2: &= (f-I_{X,L,\phi}f,u_{X,L,\psi})_{\mathcal{N}_\psi}=\left(f-I_{X,L,\phi}f,\sum_{j=1}^{N}\gamma_j\psi(\cdot,x_j)\right)_{\mathcal{N}_\psi}\\
&= \sum_{j=1}^{N}\gamma_j\left(f(x_j)-I_{X,L,\phi}f(x_j)\right)=0.
\end{aligned}$$

With the help of (5.33), we have

$$|I_1|:=\left|(f-I_{X,L,\phi}f,\psi(\cdot,x)-u_{X,L,\phi})_{\mathcal{N}_\psi}\right|\leq\|f-I_{X,L,\phi}f\|_{\mathcal{N}_\psi}\|\psi(\cdot,x)-u_{X,L,\phi}\|_{\mathcal{N}_\psi}$$
$$\leq C_2h_X^{s-1}\|f-I_{X,L,\phi}f\|_{\mathcal{N}_\psi}.$$

We denote $I_4 := (f - I_{X,L,\phi}f, \psi(\cdot,x) - u_{X,L,\psi})_{\mathcal{N}_\psi}$ so that we have $I_3 = I_4 - I_1$.

With the help of (5.31) we can see that

$$\begin{aligned}|I_4| &= \left|(f - I_{X,L,\phi}f, \psi(\cdot,x) - u_{X,L,\psi})_{\mathcal{N}_\psi}\right| \le \|f - I_{X,L,\phi}f\|_{\mathcal{N}_\psi}\|\psi(\cdot,x) - u_{X,L,\psi}\|_{\mathcal{N}_\psi}\\ &\le C_1 h_X^{s-1}\|f - I_{X,L,\phi}f\|_{\mathcal{N}_\psi},\end{aligned}$$

which yields $|I_3| \le (C_1 + C_2)h_X^{s-1}\|f - I_{X,L,\phi}f\|_{\mathcal{N}_\psi}$. Then

$$\begin{aligned}|f(x) - I_{X,L,\phi}f(x)| &\le |I_1| + |I_2| + |I_3| \le (C_1 + 2C_2)h_X^{s-1}\|f - I_{X,L,\phi}f\|_{\mathcal{N}_\psi}\\ &\le Ch_X^{s-1}\|f - I_{X,L,\phi}f\|_{\mathcal{N}_\psi}.\end{aligned}$$

This completes the proof of Theorem 5.1.

Having the pointwise error estimate in Theorem 5.1, we can perform the same steps in Theorem 3.1, where the local-global strategy is the key to establish the error estimates. So we are now ready to state the error estimates of the hybrid interpolation for a target function $f \in \mathcal{N}_\psi$ for L_p norm.

Theorem 5.2 *Under the conditions of Theorem 5.1, we have*

$$\|f - I_{X,L,\phi}f\|_{L_p(\mathbb{S}^2)} \le Ch_X^{\frac{2}{p}+s-1}\|f - I_{X,L,\phi}f\|_{\mathcal{N}_\psi}, \quad p \in [2, +\infty), \tag{5.34}$$

and

$$\| f - I_{X,L,\phi}f \|_{L_p(\mathbb{S}^2)} \le Ch_X^s \| f - I_{X,L,\phi}f \|_{\mathcal{N}_\psi}, \quad p \in [1, 2), \tag{5.35}$$

where the constant C is independent of f and h_X.

References

[1] S. C. Brenner, R. L. Scott, The Mathematical Theory of Finite Element Methods, Springer, New York, 1994.

[2] F. Cao, M. Li, Spherical data fitting by multiscale moving least squares, Applied Math. Model., 39 (2015) 3448-3458.

[3] D. Chen, V. A. Menegatto, X. Sun, A necessary and sufficient condition for strictly positive definite functions on spheres, Proc. Amer. Math. Soc., 131 (2003) 2733-2740.

[4] Z. Ditzian, Fractional derivatives and best approximation, Acta. Math. Hungar., 81 (1998) 323-348.

[5] G. E. Fasshauer, L. L. Schumaker, Scattered data fitting on the sphere, in Mathematical Methods for Curves and Surfaces II (M. Dælen, T. Lyche, and L. L. Schumaker, eds.), Vanderbilt University Press, Nashville, TN, (1998) 117-166.

[6] W. Freeden, T. Gervens, M. Schreiner, Constructive Approximation on the Sphere, Oxford University Press Inc., New York, 1998.

[7] P. B. Gilkey, The Index Theorem and the Heat Equation, Publish or Perish, Boston, MA, 1974.

[8] M. v. Golitschek, W. A. Light, Interpolation by polynomials and radial basis functions on spheres, Constr. Approx., 17 (2001) 1-18.

[9] S. Hubbert, T. M. Morton, A Duchon framework for the sphere, J. Approx. Theory, 129 (2004) 28-57.

[10] S. Hubbert, T. M. Morton, L_p-error estimates for radial basis function interpolation on the sphere, J. Approx. Theory, 129 (2004) 58-77.

[11] K. Jetter, J. Stöckler, J. D. Ward, Error estimates for scattered data interpolation on spheres, Math. Comp., 68 (1999) 733-747.

[12] J. Levesley, X. Sun, Approximation in rough native spaces by shifts of smooth kernels on spheres, J. Approx. Theory, 133 (2005) 269-283.

[13] J. Levesley, X. Sun, Corrigendum to and two open questions arising from the article "Approximation in rough native spaces by shifts of smooth kernels on spheres" [J. Approx. Theory, 133 (2005) 269-283], J. Approx. Theory, 138 (2006) 124-127.

[14] Q. T. Le Gia, F. J. Narcowich, J. D. Ward, H. Wendland, Continuous and discrete least-squares approximation by radial basis functions on spheres, J. Approx. Theory, 143 (2006) 124-133.

[15] M. Li, F. Cao, Local uniform error estimates for spherical basis functions interpolation, Math. Meth. Applied Sci., 37 (2014) 1364-1376.

[16] C. Müller, Spherical Harmonics, Lecture Notes in Mathematics, Vol. 17, Springer-Verlag, Berlin, 1966.

[17] F. J. Narcowich, R. Schaback, J. D. Ward, Approximation in Sobolev spaces by kernel expansions, J. Approx. Theory, 114 (2002) 70-83.

[18] F. J. Narcowich, J. D. Ward, Scattered data interpolation on spheres: Error estimates and locally supported basis functions, SIAM J. Math. Anal., 33 (2002) 1393-1410.

[19] F. J. Narcowich, X. Sun, J. D. Ward, H. Wendland, Direct and inverse sobolev error estimates for scattered data interpolation via spherical basis functions, Found. Comput. Math., (2007) 369-390.

[20] F. J. Narcowich, X. Sun, J. D. Ward, Approximation power of RBFs and their associated SBFs: A connection, Adv. Comput. Math., 27 (2007) 107-124.

[21] A. Ron, X. Sun, Strictly positive definite functions on spheres in Enclidean spaces, Math. Comp., 65 (1996) 1513-1530.

[22] R. Schaback, Improved error bounds for scattered data interpolation by radial basis functions, Math. Comp., 68 (1999) 201-216.

[23] I. J. Schoenberg, Positive definite functions on spheres, Duke Math. J., 9 (1942) 96-108.

[24] E. M. Stein, G. Weiss, Introduction to Fourier Analysis on Euclidean Spaces, Princeton University Press, Princetion, NJ, 1971.

[25] I. H. Sloan, A. Sommariva, Approximation on the sphere using radial basis function plus polynomials, Adv. Comput. Math., 29 (2008) 147-177.

[26] I. H. Sloan, H. Wendland, Inf-sup condition for spherical polynomials and radial basis functions on spheres, Math. Comp., 78 (2009) 1319-1331.

[27] I. H. Sloan, Polynomial interpolation and hyperinterpolation over general regions, J. Approx. Theory, 83 (1995) 238-254.

[28] I. H. Sloan, R. S. Womersley, Constructive polynomial approximation on the sphere, J. Approx. Theory, 103 (2000) 91-118.

[29] H. Wendland, Scattered Data Approximation, Cambridge University Press, Cambridge, Uk, 2005.

[30] Y. Xu, E. W. Cheney, Strictly positive definite functions on spheres, Proc. Amer. Math. Soc., 116 (1992) 977-981.

Some best approximation formulas and inequalities for the Bateman's $G-$function

Ahmed Hegazi[1], Mansour Mahmoud[2], Ahmed Talat[3] and Hesham Moustafa[4]

[1,2,4]Mansoura University, Faculty of Science, Mathematics Department, Mansoura 35516, Egypt.

[3]Port Said University, Faculty of Science, Mathematics and Computer Sciences Department, Port Said, Egypt.

[1] hegazi@mans.edu.eg, [2]mansour@mans.edu.eg,

[3]a_t_amer@yahoo.com, [4]heshammoustafa14@gmail.com

.

.

Abstract

In the paper, the authors established two best approximation formulas for the Bateman's $G-$function. Also, they studied the completely monotonicity of some functions involving $G(x)$. Some new inequalities are deduced for the function and its derivative such as

$$\frac{1}{2}\ln\left[1+\frac{2x+a}{x^2+2x+\frac{4}{3}}\right] < G(x+2) < \frac{1}{2}\ln\left[1+\frac{2x+b}{x^2+2x+\frac{4}{3}}\right], \qquad x>0$$

where $a=3$ and $b=\frac{e^4-16}{12}$ are the best possible constants. Our results improve some recent inequalities about the function $G(x)$.

2010 Mathematics Subject Classification: 33B15, 26D15, 41A25, 26A48.

Key Words: Digamma function, Bateman $G-$function , best approximation, completely monotonic, monotonicity, bounds, rate of convergence, best possible constant.

1 Introduction.

In 2010, Mortici [21] presented the following Lemma which is considered as a powerful tool to constructing asymptotic expansions and to measure the rate of convergence:

Lemma 1.1. *If $\{\tau_s\}_{s\in N}$ is convergent to zero and there exist h in R and $k>1$ such that*

$$\lim_{s\to\infty} s^k(\tau_s-\tau_{s+1})=h, \tag{1}$$

then we get

$$\lim_{s\to\infty} s^{k-1}\tau_s = \frac{h}{k-1}.$$

It is clear from lemma (1.1) that, the sequence $\{\tau_s\}_{s\in N}$ will converge more quickly when the value of k is large in the relation (1). This Lemma has been applied successfully to produce several approximations and inequalities in several papers such as [6], [7], [11], [15], [16], [22], [24], [28]. In this paper, Lemma (1.1) will be an effectively tool in producing best approximations of the Bateman's $G-$function defined by [9]

$$G(x) = \psi\left(\frac{x+1}{2}\right) - \psi\left(\frac{x}{2}\right), \qquad x \neq 0, -1, -2, ... \tag{2}$$

where $\psi(x)$ is the digamma or Psi function which is defined by

$$\psi(x) = \frac{d}{dx}\ln\Gamma(x)$$

and $\Gamma(x)$ is the classical Euler gamma given for $x > 0$ by

$$\Gamma(x) = \int_0^\infty e^{-w}w^{x-1}dw.$$

The hypergeometric representation of the function $G(x)$ is given by

$$G(x) = \frac{1}{x}\,{}_2F_1\left(1, 1; 1+x; 1/2\right) \tag{3}$$

and it satisfies the following relations [9]:

$$G(x+1) + G(x) = \frac{2}{x} \tag{4}$$

and

$$G(x) = 2\int_0^\infty \frac{e^{-xw}}{1+e^{-w}}dw, \qquad x > 0. \tag{5}$$

Qiu and Vuorinen [30] established the inequality

$$\frac{x + (6 - 4\ln 4)}{x^2} < G(x) < \frac{x + 1/2}{x^2}, \qquad x > 1/2 \tag{6}$$

and Mortici [23] presented the general inequality

$$0 < \psi(x+j) - \psi(x) \leq \psi(j) + \gamma - j + j^{-1}, \qquad x \geq 1;\ j \in (0,1) \tag{7}$$

where γ is Euler$-$Mascheroni constant (also called Euler's constant) defined by

$$\gamma = \lim_{m\to\infty}\left(-\ln m + \sum_{w=1}^{m}\frac{1}{w}\right).$$

Mahmoud and Agarwal [17] deduced the following asymptotic formula for $x \to \infty$

$$G(x) - \frac{1}{x} \sim \sum_{w=1}^{\infty} \frac{(2^{2w}-1)B_{2w}}{w} x^{-2w}, \tag{8}$$

where $B_w{}'s$ are the Bernoulli numbers [1] defined by the generating series

$$\sum_{w=0}^{\infty} B_w \frac{v^w}{w!} = \frac{v}{e^v - 1}.$$

They also presented the following double inequality

$$\frac{1}{x} + \frac{1}{2x^2 + \frac{3}{2}} < G(x) < \frac{x + 1/2}{x^2}, \qquad x > 0 \tag{9}$$

which improves the lower bound of the inequality (6) for $x > \left(\frac{9-12\ln 2}{16\ln 2 - 11}\right)^{1/2}$. In [18] Mahmoud and Almuashi proved the following inequality

$$\sum_{w=1}^{2r} \frac{(2^{2w}-1)}{w} \frac{B_{2w}}{x^{2w}} < G(x) - x^{-1} < \sum_{w=1}^{2r-1} \frac{(2^{2w}-1)}{w} \frac{B_{2w}}{x^{2w}}, \qquad r \in \mathbb{N} \tag{10}$$

where $\frac{(2^{2w}-1)}{w} B_{2w}$ are the best possible constants. In [19], Mahmoud, Talat and Moustafa presented the following approximations of the Bateman's $G-$function

$$G(x) \approx \ln\left(1 + \frac{1}{x+c}\right) + \frac{2}{x(x+1)}, \qquad c \in [1,2], \qquad x > 0 \tag{11}$$

and they deduced the following double inequality

$$\ln\left(1 + \frac{1}{x+\alpha_2}\right) + \frac{2}{x(x+1)} < G(x) < \ln\left(1 + \frac{1}{x+\alpha_1}\right) + \frac{2}{x(x+1)}, \qquad x > 0 \tag{12}$$

where the constants $\alpha_1 = 1$ and $\alpha_2 = \frac{4}{e^2-4}$ are the best possible constants.

Recently, Mahmoud, Talat, Moustafa and Agarwal [20] improved the double inequality (9) by

$$\frac{1}{x} + \frac{1}{2x^2 + a} < G(x) < \frac{1}{x} + \frac{1}{2x^2 + b}, \qquad x > 0 \tag{13}$$

where $a = 1$ and $b = 0$ are the best possible constants.

A function T defined on an interval I is said to be completely monotonic if it possesses derivatives $T^{(s)}(x)$ for all $s = 0, 1, 2, ...$ such that

$$(-1)^s T^{(s)}(x) \geq 0 \qquad x \in I;\ s = 0, 1, 2, \ldots . \tag{14}$$

Such functions occur in several areas such as numerical analysis, elasticity and probability theory, for more details see [2], [5], [12]-[14], [26], [27], [29]. According to Bernstein theorem [31],

the necessary and sufficient condition for the function $T(x)$ to be completely monotonic for $0 < x < \infty$ is that

$$T(x) = \int_0^\infty e^{-xt} d\lambda(t), \tag{15}$$

where $\lambda(t)$ is non-decreasing and the integral converges for $0 < x < \infty$.

In this paper, we presented two best approximation formulas of the Bateman's $G-$function and some completely monotonic functions involving it. Some new inequalities of $G(x)$ and its derivative will be deduced, which improve some pervious results.

2 Auxiliary Results

We can easily prove the following simple modification of Lemma (1.1):

Lemma 2.1. *If $\{\tau_s\}_{s\in N}$ is convergent to zero and there exist $h \in R$, $m \in N$ and $k > 1$ such that $\lim_{s\to\infty} s^k(\tau_s - \tau_{s+m}) = h$, then we get $\lim_{s\to\infty} s^{k-1}\tau_s = \frac{h/m}{k-1}$.*

Proof. Using the relation

$$\begin{aligned}
\lim_{s\to\infty} s^k(\tau_s - \tau_{s+m}) &= \lim_{s\to\infty} s^k \sum_{i=0}^{m-1}(\tau_{s+i} - \tau_{s+i+1}) = \lim_{s\to\infty} \sum_{i=0}^{m-1}(\frac{s}{s+i})^k(s+i)^k(\tau_{s+i} - \tau_{s+i+1}) \\
&= \sum_{i=0}^{m-1} \lim_{s\to\infty}(\frac{s}{s+i})^k(s+i)^k(\tau_{s+i} - \tau_{s+i+1}) = \sum_{i=0}^{m-1} \lim_{s\to\infty}(s+i)^k(\tau_{s+i} - \tau_{s+i+1}) \\
&= \sum_{i=0}^{m-1} \lim_{v\to\infty} v^k(\tau_v - \tau_{v+1}) = m \lim_{v\to\infty} v^k(\tau_v - \tau_{v+1}),
\end{aligned}$$

then $\lim\limits_{v\to\infty} v^k(\tau_v - \tau_{v+1}) = \frac{h}{m}$. Applying Lemma (1.1) to get $\lim_{s\to\infty} s^{k-1}\tau_s = \frac{h/m}{k-1}$. ☐

Lemma 2.2.

1. *For $x > x_0 \approx 4.02361$, we have $N(x) = \ln\left(\frac{(x+1)(3-\sqrt{6}+3x)}{(x+2)(-\sqrt{6}+3x)}\right) - \frac{\left(1+\sqrt{\frac{2}{3}}\right)}{x(x+1)} < 0$.*

2. *For $x > x_\lambda \approx 2.02059$, we have $M(x) = \ln\left(\frac{(x+\frac{4}{e^2-4})(3+\sqrt{6}+3x)}{(x+1+\frac{4}{e^2-4})(\sqrt{6}+3x)}\right) - \frac{\left(1-\sqrt{\frac{2}{3}}\right)}{x(x+1)} > 0$.*

3. *For $x > 0$, we have $H(x) = \ln\left(\frac{(\sqrt{6}+3x)^2(13+12x+3x^2)}{(3+\sqrt{6}+3x)^2(4+6x+3x^2)}\right) + 2\frac{\left(1-\sqrt{\frac{2}{3}}\right)}{x(x+1)} > 0$.*

Proof.

1. For $x > \sqrt{\frac{2}{3}}$, $N'(x) = \frac{9x^3-(9+14\sqrt{6})x^2-(6+11\sqrt{6})x-2\sqrt{6}}{x^2(x+1)^2(x+2)(3x-\sqrt{6})(3x+3-\sqrt{6})} \triangleq \frac{n_1(x)}{n_2(x)}$, where the polynomial $n_2(x)$ is positive for $x > \sqrt{\frac{2}{3}}$ and $n_1(x)$ is a polynomial of degree 3 has only one positive real root $x_1 \approx 5.49455$ and $n_1(x) > (<)\ 0$ for $x > (<)\ x_1$. Then $N'(x) > 0$ for $x > x_1$ with

$\lim_{x\to\infty} N(x) = 0$ and hence $N(x) < 0$ for $x > x_1$. Also, $N(x)$ is decreasing on $\left(\sqrt{\frac{2}{3}}, x_1\right)$ with $N(4.023) \approx 0.0000005 > 0$ and $N(4.024) \approx -0.0000003 < 0$. Then $N(x)$ has only one real root $x_0 \approx 4.02361 \in \left(\sqrt{\frac{2}{3}}, x_1\right)$ and $N(x) < 0$ for $x_0 < x < x_1$. Hence, $N(x) < 0$ for $x > x_0$.

2. For $x > 0$, $M'(x) = \frac{m(x)}{x^2(1+x)^2(\sqrt{6}+3x)(3+\sqrt{6}+3x)(4+(e^2-4)x)(e^2+(e^2-4)x)}$, where

$$\begin{aligned} m(x) &= 4\sqrt{6}e^2 + \left(-16\sqrt{6} + (-12+20\sqrt{6})e^2 + \sqrt{6}e^4\right)x + \left(576 - 216e^2 + 18e^4\right)x^5 \\ &\quad + \left(144 - 16\sqrt{6} - (72+20\sqrt{6})e^2 + (3+9\sqrt{6})e^4\right)x^2 \\ &\quad + \left(384 + 224\sqrt{6} - (144+160\sqrt{6})e^2 + (12+20\sqrt{6})e^4\right)x^3 \\ &\quad + \left(432 + 384\sqrt{6} - (252+144\sqrt{6})e^2 + (27+12\sqrt{6})e^4\right)x^4 \end{aligned}$$

and

$$\begin{aligned} m'(x) &= 5\left(576 - 216e^2 + 18e^4\right)x^4 + \left(-16\sqrt{6} + (-12+20\sqrt{6})e^2 + \sqrt{6}e^4\right) \\ &+ 4\left(432 + 384\sqrt{6} + (-252-144\sqrt{6})e^2 + (27+12\sqrt{6})e^4\right)x^3 \\ &+ 3\left(384 + 224\sqrt{6} + (-144-160\sqrt{6})e^2 + (12+20\sqrt{6})e^4\right)x^2 \\ &+ 2\left(144 - 16\sqrt{6} + (-72-20\sqrt{6})e^2 + (3+9\sqrt{6})e^4\right)x. \end{aligned}$$

The polynomial $m'(x)$ of fourth degree has only one positive real root $x_\alpha \approx 2.57862$ also $m'(x) < 0$ for $x > x_\alpha$ and $m'(x) > 0$ for $0 < x < x_\alpha$. Hence $m(x)$ is increasing on $(0, x_\alpha)$ and is decreasing on (x_α, ∞) with $m(0) > 0$, $m(3.453) \approx 22.157 > 0$ and $m(3.455) \approx -6.01919 < 0$. Then $m(x)$ has only one positive real root $x_\beta \approx 3.45457$ with $m(x) < 0$ for $x > x_\beta$ and $m(x) > 0$ for $0 < x < x_\beta$. Now $M(x)$ is decreasing on (x_β, ∞) and $\lim\limits_{x\to\infty} M(x) = 0$, then $M(x) > 0$ for $x > x_\beta$. Also, $M(x)$ is increasing on $(0, x_\beta)$ with $M(2.0205) \approx -0.0000006 < 0$ and $M(2.0206) \approx 0.0000001 > 0$, then $M(x)$ has only one positive real root $x_\lambda \approx 2.02059$. Hence, $M(x) > 0$ for $x > x_\lambda$.

3.
$$H'(x) = \frac{-4h(x)}{x^2(1+x)^2(\sqrt{6}+3x)(3+\sqrt{6}+3x)(4+6x+3x^2)(13+12x+3x^2)},$$

where

$$h(x) = 26\sqrt{6} + (-78+193\sqrt{6})x + (-300+477\sqrt{6})x^2 + (-324+498\sqrt{6})x^3$$
$$+(-126+234\sqrt{6})x^4 + 36\sqrt{6}x^5 > 0, \qquad x > 0.$$

Hence $H'(x) < 0$ for all $x > 0$ with $\lim\limits_{x\to\infty} H(x) = 0$, then $H(x) > 0$ for $x > 0$.

□

The following result is considered as a method presented by Elbert and Laforgia in [8] (see also, [4], [25] and [32]):

Corollary 2.3. *Let K be a real-valued function defined on $x > a$, $a \in \mathbb{R}$ with $\lim_{x\to\infty} K(x) = 0$. Then $K(x) > 0$, if $K(x) > K(x+1)$ for all $x > a$ and $K(x) < 0$, if $K(x) < K(x+1)$ for all $x > a$.*

This result has the following simple modification [20]:

Corollary 2.4. *Let K be a real-valued function defined on $x > a$, $a \in \mathbb{R}$ with $\lim_{x\to\infty} K(x) = 0$. Then for $m \in \mathbb{N}$, $K(x) > 0$, if $K(x) > K(x+m)$ for all $x > a$ and $K(x) < 0$, if $K(x) < K(x+m)$ for all $x > a$.*

3 First formula of the best approximations and some its related inequalities

With the help of Mortici's technique in Lemma(1.1), we provide the first best approximation formula of the Bateman's $G-$function.

Lemma 3.1. *The best approximation*

$$G(n) \approx \ln(1+\frac{1}{n+a}) + \frac{b}{n(n+c)} \tag{16}$$

holds for

$$a = \frac{\theta_1+\theta_2-5}{9},\ b = a+1 \ \ and \ \ c = \frac{\theta_1^2+\theta_2^2+2\theta_1+2\theta_2-21}{54}, \tag{17}$$

where $\theta_1, \theta_2 = \sqrt[3]{91 \pm 63\sqrt{2}}$ and the sequence $G(n) - \ln(1+\frac{1}{n+a}) - \frac{b}{n(n+c)}$ converges to zero with speed estimated by n^{-5}.

Proof. Define the error sequence by $\upsilon_n = G(n) - \ln\left(1+\frac{1}{n+a}\right) - \frac{b}{n(n+c)}$. Using the functional equation (4), we get

$$\begin{aligned}
\upsilon_n - \upsilon_{n+2} &= \sum_{r=3}^{\infty} \frac{(-1)^{r-1}}{n^r}\left\{b[c^{r-1}+2^{r-1}-(2+c)^{r-1}]/c + [(a+3)^r-(a+2)^r-(a+1)^r\right. \\
&+ \left. a^r]/r - 2\right\} \\
&= \frac{4(a-b+1)}{n^3} - \frac{2(7+3a(a+3)-3b(c+2))}{n^4} \\
&+ \frac{4(a(16+a(9+2a))-2(-5+b(4+c(3+c))))}{n^5} \\
&- \frac{326/3+10a(3+a)(7+a(3+a))-10b(2+c)(4+c(2+c))}{n^6} + O(n^{-7}).
\end{aligned}$$

According to Lemma (2.1), the three parameters a, b and c which produce the fastest convergence of the sequence υ_n satisfy the system

$$\begin{aligned}
a-b+1 &= 0 \\
3a^2+9a-3b(c+2)+7 &= 0 \\
a^3+\frac{9}{2}a^2+8a+5-b(c^2+3c+4) &= 0.
\end{aligned}$$

Now, the values of a, b and c determined in (17) form solution of this system and the sequence υ_n converges to zero with speed estimated by n^{-5}. $\square$

Now we will prove the complete monotonicity of some functions involving the function $G(x)$ depending on the approximation formula (16).

Lemma 3.2.

1. *For the values of a and c in (17), the function $L_1(x) = \ln\left(1 + \frac{1}{x+a}\right) + \frac{a+1}{x(x+c)} - G(x)$ is completely monotonic on $(0, \infty)$.*

2. *The function $L_2(x) = \ln\left(1 + \frac{1}{x-\sqrt{\frac{2}{3}}}\right) + \frac{(1-\sqrt{\frac{2}{3}})}{x(x+1)} - G(x)$ is completely monotonic on $\left(\sqrt{\frac{2}{3}}, \infty\right)$.*

3. *The function $L_3(x) = G(x) - \ln\left(1 + \frac{1}{x+\sqrt{\frac{2}{3}}}\right) - \frac{(1+\sqrt{\frac{2}{3}})}{x(x+1)}$ is completely monotonic on $(0, \infty)$.*

Proof. 1. Using the formula [1]

$$\frac{1}{x^k} = \frac{1}{(k-1)!}\int_0^\infty t^{k-1}e^{-xt}dt, \qquad k \in \mathbb{N} \tag{18}$$

and the integral representation (5) of $G'(x)$, we get

$$L_1'(x) = \int_0^\infty \frac{e^{-(x+a+1)t}}{1+e^t}\nu_1(t)dt,$$

where

$$\begin{aligned}
\nu_1(t) &= \sum_{k=0}^{\infty} \frac{\left[(2-\frac{(a+1)}{c})(a+2)^k + \frac{(a+1)}{c}[(a+1-c)^k + (a+2-c)^k - (a+1)^k] - \frac{2^{k+1}}{(k+1)}\right]}{k!}t^{k+1} \\
&= -0.0316t^5 - 0.0381t^6 - 0.243t^7 + \sum_{k=7}^{\infty} \frac{(a+2)^k\left[C_1(k)\right] - \frac{(a+1)}{c}(a+1)^k - \frac{2^{k+1}}{(k+1)}}{k!}t^{k+1}
\end{aligned}$$

with

$$C_1(k) = \left(2 - \frac{a+1}{c}\right) + \frac{a+1}{c}\left[\left(\frac{a+1-c}{a+2}\right)^k + \left(\frac{a+2-c}{a+2}\right)^k\right].$$

The sequences $(\frac{a+1-c}{a+2})^k$ and $(\frac{a+2-c}{a+2})^k$ are decreasing for $k \geq 7$, hence

$$C_1(k) < \left(2 - \frac{a+1}{c}\right) + \frac{a+1}{c}\left[\left(\frac{a+1-c}{a+2}\right)^7 + \left(\frac{a+2-c}{a+2}\right)^7\right] \approx -0.05248 < 0$$

and consequently $\nu_1(t) < 0$. Then $-L_1'(x)$ is completely monotonic. The function $L_1(x)$ is decreasing on $(0, \infty)$ and $\lim_{x\to\infty} L_1(x) = 0$, then $L_1(x) > 0$ and hence $L_1(x)$ is completely monotonic on $(0, \infty)$.

2.
$$L_2'(x)=\int_0^\infty \frac{e^{-(x+1)t}}{1+e^t}\nu_2(t)dt,$$

where

$$\nu_2(t) = \sum_{k=3}^{\infty}\frac{2^{k+1}\left[\left(\frac{1}{\sqrt{6}}\right)^{k+1}-\left(\frac{1}{\sqrt{6}}+1\right)^{k+1}+(k+1)\left(\frac{1}{\sqrt{6}}+\frac{1}{2}\right)\right]}{(k+1)!}t^{k+1}.$$

Now, consider the following sequence for $k=3,4,5,...$

$$\begin{aligned} C_2(k) &= \left(\frac{1}{\sqrt{6}}\right)^{k+1}-\left(\frac{1}{\sqrt{6}}+1\right)^{k+1}+(k+1)\left(\frac{1}{\sqrt{6}}+\frac{1}{2}\right)\\ &= -\sum_{r=0}^{k}\tbinom{k+1}{r}\left(\frac{1}{\sqrt{6}}\right)^{r}+(k+1)\left(\frac{1}{\sqrt{6}}+\frac{1}{2}\right)\\ &< -\sum_{r=0}^{2}\tbinom{k+1}{r}\left(\frac{1}{\sqrt{6}}\right)^{r}+(k+1)\left(\frac{1}{\sqrt{6}}+\frac{1}{2}\right)<-\frac{1}{12}(k-2)(k-3)<0. \end{aligned}$$

Hence $\nu_2(t)<0$ and $-L_2'(x)$ is completely monotonic. The function $L_2(x)$ is decreasing on $\left(\sqrt{\frac{2}{3}},\infty\right)$ with $\lim_{x\to\infty}L_2(x)=0$ and then $L_2(x)>0$. Hence $L_2(x)$ is completely monotonic on $\left(\sqrt{\frac{2}{3}},\infty\right)$.

3.
$$L_3'(x)=\int_0^\infty \frac{e^{-\left(x+\sqrt{\frac{2}{3}}+1\right)t}}{1+e^t}\nu_3(t)dt,$$

where

$$\nu_3(t) = \sum_{k=3}^{\infty}\frac{2^{k+1}\left[\left(\frac{-1}{2}+\frac{1}{\sqrt{6}}\right)\left(1+\frac{1}{\sqrt{6}}\right)^{k}+\frac{1}{(k+1)}-\left(\frac{1}{2}+\frac{1}{\sqrt{6}}\right)\left(\frac{1}{\sqrt{6}}\right)^{k}\right]}{k!}t^{k+1}.$$

The sequence

$$\begin{aligned} C_3(k) &= \left(\frac{-1}{2}+\frac{1}{\sqrt{6}}\right)\left(1+\frac{1}{\sqrt{6}}\right)^{k}+\frac{1}{k+1}=\left(\frac{-1}{2}+\frac{1}{\sqrt{6}}\right)\sum_{r=0}^{k}\tbinom{k}{r}\left(\frac{1}{\sqrt{6}}\right)^{r}+\frac{1}{k+1}\\ &< \left(\frac{-1}{2}+\frac{1}{\sqrt{6}}\right)\sum_{r=0}^{2}\tbinom{k}{r}\left(\frac{1}{\sqrt{6}}\right)^{r}+\frac{1}{k+1}\\ &< \frac{(k-3)((\sqrt{6}-3)k^2+(3-3\sqrt{6})k-(12+4\sqrt{6}))}{72(k+1)}<0, \qquad k=3,4,5,\ldots. \end{aligned}$$

Then $\nu_3(t)<0$ and hence $-L_3'(x)$ is completely monotonic . The function $L_3(x)$ is decreasing for all $x>0$ with $\lim_{x\to\infty}L_3(x)=0.$ and hence $L_3(x)>0$ for all $x>0$. Then $L_3(x)$ is completely monotonic on $(0,\infty)$.

$\square$

From the complete monotonicity of the functions $L_1(x)$, $L_2(x)$ and $L_3(x)$, we deduce the following result:

Lemma 3.3. *1.*

$$\ln\left(1+\frac{1}{x+\sqrt{\frac{2}{3}}}\right)+\frac{\left(1+\sqrt{\frac{2}{3}}\right)}{x(x+1)}<G(x)<\ln\left(1-\frac{1}{x-\sqrt{\frac{2}{3}}}\right)+\frac{\left(1-\sqrt{\frac{2}{3}}\right)}{x(x+1)}, \tag{19}$$

where the upper bound holds for $x>\sqrt{\frac{2}{3}}$ and the lower bound holds for $x>0$.

2.

$$G(x)<\ln\left(1+\frac{1}{x+a}\right)+\frac{1+a}{x(x+c)}, \qquad x>0 \tag{20}$$

where the values of a and c are in (17)

Remark 1. From Lemma (2.2), we can conclude that the inequality (19) improves the lower bound of the inequality (12) for $x>x_\lambda\simeq 2.02059$ and improves its upper bound for $x>x_0\simeq 4.02361$.

Lemma 3.4. *The following inequality holds*

$$\ln\left(1+\frac{1}{x+\sqrt{\frac{2}{3}}}\right)+\frac{\left(1+\sqrt{\frac{2}{3}}\right)}{x(x+1)}+\frac{1}{6\sqrt{6}x^4}<G(x)<\ln\left(1+\frac{1}{x-\sqrt{\frac{2}{3}}}\right)+\frac{\left(1-\sqrt{\frac{2}{3}}\right)}{x(x+1)}-\frac{1}{6\sqrt{6}x^4},$$

where the upper bound holds for $x>\sqrt{\frac{2}{3}}$ and the lower bound holds for $x\geq 2$.

Proof. Consider the function

$$T(x)=\ln\left(1+\frac{1}{x-\sqrt{\frac{2}{3}}}\right)+\frac{\left(1-\sqrt{\frac{2}{3}}\right)}{x(x+1)}-\frac{1}{6\sqrt{6}x^4}-G(x), \qquad x>\sqrt{\frac{2}{3}}$$

and use the functional equation (4) to obtain

$$T'(x+2)-T'(x)=\frac{2l(x)}{81x^5(1+x)^2(2+x)^5(3+x)^2\sum_{i=0}^{3}\left(x+i-\sqrt{\frac{2}{3}}\right)},$$

where

$$\begin{aligned}
l(x) \;=\;& \left(25920-10080\sqrt{6}\right)+\left(197856-75408\sqrt{6}\right)x \\
&+\left(677952-257008\sqrt{6}\right)x^2+\left(1367472-520800\sqrt{6}\right)x^3 \\
&+\left(1777284-666478\sqrt{6}\right)x^4+\left(1535268-502094\sqrt{6}\right)x^5 \\
&+\left(879720-127639\sqrt{6}\right)x^6+\left(321960+145938\sqrt{6}\right)x^7 \\
&+\left(66960+180403\sqrt{6}\right)x^8+\left(4596+95742\sqrt{6}\right)x^9 \\
&+\left(-864+28431\sqrt{6}\right)x^{10}+\left(-144+4590\sqrt{6}\right)x^{11}+315\sqrt{6}x^{12}>0, \quad x>0.
\end{aligned}$$

Then $T'(x+2)-T'(x)>0$ for $x>\sqrt{\frac{2}{3}}$ and also $\lim\limits_{x\to\infty}T'(x)=0$. Using Corollary (2.4), we get that $T'(x)<0$ for all $x>\sqrt{\frac{2}{3}}$. Hence $T(x)$ is decreasing on $\left(\sqrt{\frac{2}{3}},\infty\right)$ with $\lim\limits_{x\to\infty}T(x)=0$, thus $T(x)>0$ for all $x\in\left(\sqrt{\frac{2}{3}},\infty\right)$. Now consider the function

$$Q(x)=G(x)-\ln(1+\frac{1}{x+\sqrt{\frac{2}{3}}})-\frac{(1+\sqrt{\frac{2}{3}})}{x(x+1)}-\frac{1}{6\sqrt{6}x^4},\quad x>0.$$

Then

$$Q'(x+2)-Q'(x)=\frac{2u(x-2)}{81x^5(1+x)^2(2+x)^5(3+x)^2\sum_{i=0}^{3}\left(x+i+\sqrt{\frac{2}{3}}\right)},$$

where

$$\begin{aligned}
u(x) \;=\; & \left(-207466560+113432160\sqrt{6}\right)+\left(-585268704+582357840\sqrt{6}\right)x\\
&+\left(-729011328+1250421968\sqrt{6}\right)x^2+\left(-523396080+1539421184\sqrt{6}\right)x^3\\
&+\left(-235893516+1231511026\sqrt{6}\right)x^4+\left(-67175076+680979590\sqrt{6}\right)x^5\\
&+\left(-10943256+268473813\sqrt{6}\right)x^6+\left(-465384+76331554\sqrt{6}\right)x^7\\
&+\left(195912+15574939\sqrt{6}\right)x^8+\left(44364+2228562\sqrt{6}\right)x^9\\
&+\left(4032+212571\sqrt{6}\right)x^{10}+\left(144+12150\sqrt{6}\right)x^{11}+315\sqrt{6}x^{12}>0,\quad x\geq 0.
\end{aligned}$$

Thus $Q'(x+2)-Q'(x)>0$ for $x\geq 2$ and also $\lim\limits_{x\to\infty}Q'(x)=0$. Using Corollary (2.4), we obtain that $Q'(x)<0$ for all $x\geq 2$. and then $Q(x)$ is decreasing on $[2,\infty)$ with $\lim\limits_{x\to\infty}Q(x)=0$. Then $Q(x)>0$ for all $x\geq 2$. □

4 Second formula of the best approximations and some of its related inequalities

In this section, we will present the best constants of the approximation of formula

$$G(n)\approx\frac{1}{2}\ln\left[1+\frac{P_1(n)}{P_2(n)}\right]+\frac{2}{n(n+1)},\quad n\in\mathbb{N}$$

where $P_1(n)$ and $P_2(n)$ are two polynomials of degrees one and two (resp.). Also, some inequalities of the function $G(x)$ will provided, which improve some results of the previous section.

Lemma 4.1. *The best approximation of the formula*

$$G(n)\approx\frac{1}{2}\ln\left[1+\frac{\alpha n+\beta}{n^2+\rho x+\sigma}\right]+\frac{2}{n(n+1)},\quad n\in\mathbb{N}\tag{21}$$

holds for $\alpha = 2$, $\beta = 3$, $\rho = 2$ and $\sigma = 4/3$ and the sequence $G(n) - \frac{1}{2}\ln\left[1 + \frac{\alpha n+\beta}{n^2+\rho x+\sigma}\right] - \frac{2}{n(n+1)}$ converges to zero with speed estimated by n^{-5}.

Proof. Consider the error sequence $\chi_n = G(n) - \frac{1}{2}\ln\left[1 + \frac{\alpha n+\beta}{n^2+\rho x+\sigma}\right] - \frac{2}{n(n+1)}$, then we have

$$\begin{aligned}
\chi_n - \chi_{n+2} &= \frac{1}{n^2}(2-\alpha) + \frac{1}{n^3}(-2(5+\beta) + \alpha(2+\alpha+2\rho))\\
&+ \frac{1}{n^4}(38 - \alpha^3 - 3\alpha^2(1+\rho) + 3\beta(2+\rho) + \alpha(-4+3\beta-3\rho(2+\rho)+3\sigma))\\
&+ \frac{1}{n^5}(\alpha^4 + 4\alpha^3(1+\rho) + 2(-65+\beta^2-2\beta(4+\rho(3+\rho)-\sigma))\\
&+ \alpha^2\ (8 - 4\beta + 6\rho(2+\rho) - 4\sigma) + 4\alpha(2 - \beta(3+2\rho) + \rho(4+\rho(3+\rho)-2\sigma) - 3\sigma))\\
&+ \frac{1}{n^6}(422 - \alpha^5 - 5\alpha^4(1+\rho) + 5\beta(2+\rho)(4-\beta+\rho(2+\rho)-2\sigma)\\
&+ 5/3\alpha^3(-8+3\beta-6\rho(2+\rho)+3\sigma) + 5\alpha^2(\beta(4+3\rho) - 2(1+\rho)(2+\rho(2+\rho))\\
&+ (4+3\rho)\sigma) - \alpha(16+5\beta^2-5\beta(8+\rho(8+3\rho)-2\sigma) - 40\sigma + 5(\rho(2+\rho)(4\\
&+ \rho(2+\rho)) - \rho(8+3\rho)\sigma + \sigma^2))) + O(n^{-7}).
\end{aligned}$$

According to Lemma (2.1), the fastest convergence of the sequence χ_n satisfies if $\alpha = 2$, $\beta = 3$, $\rho = 2$ and $\sigma = 4/3$ with speed estimated by n^{-5}. $\square$

Lemma 4.2. *For $x > -1$, the function*

$$R(x) = (e^{2G(x+2)} - 1)(x^2 + 2x + \frac{4}{3}) - 2x \tag{22}$$

is strictly decreasing and convex. As consequence, we have

$$\frac{1}{2}\ln\left[1 + \frac{2x+3}{x^2+2x+\frac{4}{3}}\right] < G(x+2) < \frac{1}{2}\ln\left[1 + \frac{2x+\frac{e^4-16}{12}}{x^2+2x+\frac{4}{3}}\right], \qquad x > 0 \tag{23}$$

where the constants 3 and $\frac{e^4-16}{12}$ are the best possible.

Proof.

$$\frac{1}{2}R'(x) = -x - 2 + [(x^2+2x+\frac{4}{3})G'(x+2) + (x+1)]e^{2G(x+2)},$$

$$\begin{aligned}
\frac{1}{2}R''(x) &= -1 + 2[(x^2+2x+\frac{4}{3})G'(x+2) + (x+1)]G'(x+2)e^{2G(x+2)}\\
&\quad +[2(x+1)G'(x+2) + (x^2+2x+\frac{4}{3})G''(x+2) + 1]e^{2G(x+2)}
\end{aligned}$$

and

$$\begin{aligned}
\frac{1}{2e^{2G(x+2)}}R'''(x) &= 4(x^2+2x+\frac{4}{3})(G'(x+2))^3 + 12(x+1)(G'(x+2))^2\\
&+ 6(x^2+2x+\frac{4}{3})G'(x+2)G''(x+2)\\
&+ 6(x+1)G''(x+2) + 6G'(x+2) + (x^2+2x+\frac{4}{3})G'''(x+2) \triangleq U(x).
\end{aligned}$$

Now,

$$\begin{aligned}
U(x+2)-U(x) &= 16(x+2)(G'(x+2))^3+4(x+2)G'''(x+2)+24(x+2)G'(x+2)G''(x+2)\\
&+ \frac{8(4+x)(62+66x+24x^2+3x^3)}{(x+2)^2(x+3)^2}(G'(x+2))^2\\
&+ \frac{8(1448+2564x+2008x^2+915x^3+258x^4+42x^5+3x^6)}{(x+2)^4(x+3)^4}G'(x+2)\\
&+ \frac{4(4+x)(62+66x+24x^2+3x^3)}{(x+2)^2(x+3)^2}G''(x+2)\\
&+ \frac{1}{3(2+x)^6(3+x)^6}4(-33056-88128x-92112x^2-46976x^3-10548x^4\\
&+ 360x^5+705x^6+138x^7+9x^8).
\end{aligned}$$

Also, let $V(x)=\frac{U(x+2)-U(x)}{4(x+2)}$, then

$$\begin{aligned}
V(x+2)-V(x) &= \frac{-4}{(2+x)^3(3+x)^2(4+x)^3(5+x)^2}[34576+91136x+105392x^2+68520x^3\\
&\quad +27152x^4+6698x^5+1005x^6+84x^7+3x^8](G'(x+2))^2\\
&- \frac{4}{(2+x)^5(3+x)^4(4+x)^5(5+x)^4}[376528768+1642942016x+3297590048x^2\\
&+ 4031614688x^3+3354474592x^4+2010658592x^5+896184192x^6\\
&+ 302070808x^7+77457190x^8+15051780x^9\\
&+ 2183975x^{10}+229624x^{11}+16548x^{12}+732x^{13}+15x^{14}]G'(x+2)\\
&- \frac{2}{(2+x)^3(3+x)^2(4+x)^3(5+x)^2}(34576+91136x+105392x^2\\
&+ 68520x^3+27152x^4+6698x^5+1005x^6+84x^7+3x^8)G''(x+2)\\
&+ \frac{2}{3(2+x)^7(3+x)^6(4+x)^7(5+x)^6}[(331346962432+5157549202432x\\
&+ 24078469545984x^2+60544326323200x^3+99175110059776x^4\\
&+ 116067402353280x^5+102360341211232x^6+70356164081536x^7\\
&+ 38541166023024x^8+17080773307136x^9+6184124004420x^{10}\\
&+ 1839407553792x^{11}+450403876283x^{12}+90669453918x^{13}+14930598072x^{14}\\
&+ 1992453932x^{15}+212255598x^{16}+17635104x^{17}+1101756x^{18}+48708x^{19}\\
&+ 1359x^{20}+18x^{21})]
\end{aligned}$$

Using the completely monotonicity of the functions $X_1(x)=\frac{1}{x}-G(x)+\sum_{k=1}^{2m-1}\frac{(2^{2k}-1)B_{2k}}{kx^{2k}}$ and $X_2(x)=G(x)-\frac{1}{x}-\frac{1}{2x^2+1}$ for $x>0$ (see [20]), we get the following inequalities: $G'(x)>-\frac{x+1}{x^3}$, $(G'(x))^2>\left[\frac{4x^4+4x^3+4x^2+1}{x^2(2x^2+1)^2}\right]^2$ and $G''(x)>\frac{2(8x^6+12x^5+12x^4-2x^3+6x^2+1)}{x^3(2x^2+1)^3}$ for $x>0$. Hence,

$$V(x+2)-V(x)<F(x), \qquad x>0$$

where

$$F(x) = \frac{-2A(x+1)}{3(2+x)^8(3+x)^6(4+x)^7(5+x)^6(9+8x+2x^2)^4}$$

with

$$\begin{aligned} A(x) &= 74586749184 + 1263034988928x + 9398610597600x^2 + 42943724513952x^3 \\ &+ 138441396784472x^4 + 339577282357568x^5 + 663837528239296x^6 \\ &+ 1065966556249164x^7 + 1434284269631783x^8 + 1638094930661455x^9 \\ &+ 1600662870950288x^{10} + 1344078197917456x^{11} + 971670315225407x^{12} \\ &+ 604754235543331x^{13} + 323563802759956x^{14} + 148404632743888x^{15} \\ &+ 58109372496201x^{16} + 19315095938361x^{17} + 5408999070416x^{18} \\ &+ 1263403407224x^{19} + 242838160053x^{20} + 37707313393x^{21} + 4608156812x^{22} \\ &+ 426325500x^{23} + 28044100x^{24} + 1167972x^{25} + 23136x^{26}. \end{aligned}$$

Using $A(x) > 0$ for all $x > 0$, then we obtain $F(x) < 0$ for all $x > -1$ and hence $V(x+2) - V(x) < 0$ for all $x > -1$. Using the asymptotic expansion (8) and its derivatives, we have

$$G'(x) = \frac{-1}{x^2} - \frac{1}{x^3} + \frac{1}{x^5} - \frac{3}{x^7} + \frac{17}{x^9} + O(x^{-11}) \tag{24}$$

$$G''(x) = \frac{2}{x^3} + \frac{3}{x^4} - \frac{5}{x^6} + \frac{21}{x^8} - \frac{153}{x^{10}} + O(x^{-12}), \tag{25}$$

and

$$G'''(x) = \frac{-6}{x^4} - \frac{12}{x^5} + \frac{30}{x^7} - \frac{168}{x^9} + \frac{1530}{x^{11}} + O(x^{-13}). \tag{26}$$

Then

$$\lim_{x\to\infty} V(x) = \lim_{x\to\infty} \left(\frac{64x^{25}}{(2+x)^{27}(3+x)^6} + O(x^{-9}) \right) = 0$$

and hence $V(x) > 0$ for all $x > -1$. Now, $U(x+2) - U(x) > 0$ with

$$\lim_{x\to\infty} U(x) = \lim_{x\to\infty} \left(\frac{-64x^{21}}{3(x+2)^{27}} + O(x^{-7}) \right) = 0$$

and so $U(x) < 0$ for all $x > -1$. Thus, $R'''(x) < 0$ and

$$\lim_{x\to\infty} R''(x) = \lim_{x\to\infty} \left(\frac{128}{15x^5} - \frac{448}{9x^6} + \frac{2368}{21x^7} + O(x^{-8}) \right) = 0.$$

Then $R''(x) > 0$ for all $x > -1$ and so the function $R(x)$ is convex for $x \in (-1, \infty)$. Also,

$$\lim_{x\to\infty} R'(x) = \lim_{x\to\infty} \left(\frac{-32}{15x^4} + \frac{448}{45x^5} - \frac{1184}{63x^6} - \frac{688}{63x^7} + \frac{11104}{81x^8} + O(x^{-9}) \right) = 0$$

and thus $R'(x) < 0$ for all $x > -1$. Hence we conclude that $R(x)$ is decreasing on $(-1, \infty)$ with $R(0) = \frac{e^4-16}{12}$ and

$$\lim_{x\to\infty} R(x) = \lim_{x\to\infty} \left(3 + \frac{32}{45x^3} - \frac{112}{45x^4} + \frac{1184}{315x^5} + O(x^{-6}) \right) = 3.$$

Then

$$3 < (e^{2G(x+2)} - 1)(x^2 + 2x + \frac{4}{3}) - 2x < \frac{e^4 - 16}{12},$$

where the constants 3 and $\frac{e^4-16}{12}$ are the best possible. □

Lemma 4.3. *For every $x \geq 0$, we have*

$$\frac{1}{2}\ln[\frac{4x+a}{(x^2+6x+\frac{28}{3})e^{-\frac{4}{(x+2)(x+3)}} - (x^2+2x+\frac{4}{3})}] \leq G(x+2)$$
$$< \frac{1}{2}\ln[\frac{4x+b}{(x^2+6x+\frac{28}{3})e^{-\frac{4}{(x+2)(x+3)}} - (x^2+2x+\frac{4}{3})}] \tag{27}$$

where $a = \frac{7e^{\frac{10}{3}}-e^4}{12}$ and $b = 12$ are the best possible constants.

Proof. For $x \geq 0$, consider $f(x) = R(x+2) - R(x)$, where $R(x)$ defined in (22). Then $f'(x) = R'(x+2) - R'(x)$ and $R(x)$ is convex function for $x \in (-1, \infty)$. Hence $f(x)$ is increasing with $f(0) = \frac{7e^{\frac{10}{3}}-e^4}{12} - 12$ and $\lim_{x\to\infty} f(x) = 0$, where $\lim_{x\to\infty} R(x) = 3$. Then $\frac{7e^{\frac{10}{3}}-e^4}{12} - 12 \leq f(x) < 0$ or

$$\frac{7e^{\frac{10}{3}} - e^4}{12} \leq e^{2G(x+2)}[(x^2+6x+\frac{28}{3})e^{-\frac{4}{(x+2)(x+3)}} - (x^2+2x+\frac{4}{3})] - 4x < 12,$$

where $\frac{7e^{\frac{10}{3}}-e^4}{12}$ and 12 are the best possible constants. □

Lemma 4.4. *For every $x > 0$, then we have*

$$\frac{(x+\alpha)e^{-2G(x+2)} - (x+1)}{(x^2+2x+\frac{4}{3})} < G'(x+2) < \frac{(x+\beta)e^{-2G(x+2)} - (x+1)}{(x^2+2x+\frac{4}{3})}, \tag{28}$$

where $\alpha = \frac{(2\pi^2-15)e^4}{144}$ and $\beta = 2$ are the best possible constants.

Proof. The function $R(x)$ defined in (22) is convex for $x \in (-1, \infty)$ and hence $R'(x)$ is increasing. Then $R'(0) < R'(x) < \lim_{x\to\infty} R'(x)$ with $R'(0) = \frac{(2\pi^2-15)e^4}{72} - 4$ and $\lim_{x\to\infty} R'(x) = 0$. Hence

$$\frac{(2\pi^2-15)e^4}{144} < -x + [(x^2+2x+\frac{4}{3})G'(x+2) + (x+1)]e^{2G(x+2)} < 2,$$

where $\frac{(2\pi^2-15)e^4}{144}$ and 2 are the best possible constants. □

Lemma 4.5. *The following inequality holds*

$$\frac{1}{2}\ln\left[1 + \frac{2x+3}{x^2+2x+\frac{4}{3}}\right] + \frac{2}{x(x+1)} < G(x) < \frac{1}{2}\ln\left[1 + \frac{2x+3}{x^2+2x+\frac{48}{e^4-16}}\right] + \frac{2}{x(x+1)}, \tag{29}$$

where the upper bound holds for $x > x_\delta \approx 0.575833$ and the lower bound holds for $x > 0$.

Proof. Consider the function $F(x) = G(x+2) - \frac{1}{2}\ln\left[1 + \frac{2x+3}{x^2+2x+\frac{4}{3}}\right]$, then

$$\begin{aligned} F'(x+2) - F'(x) &= G'(x+4) - G'(x+2) \\ &- \frac{54(5+2x)(56+140x+103x^2+30x^3+3x^4)}{(4+6x+3x^2)(13+12x+3x^2)(28+18x+3x^2)(49+24x+3x^2)}. \end{aligned}$$

Using the functional equation (4) and its derivative, we get

$$F'(x+2) - F'(x) = \frac{32(5+2x)(1057+1680x+1011x^2+270x^3+27x^4)}{(2+x)^2(3+x)^2(4+6x+3x^2)(13+12x+3x^2)(28+18x+3x^2)(49+24x+3x^2)}.$$

Thus $F'(x+2) - F'(x) > 0$, for $x > 0$ and also $\lim_{x\to\infty} F'(x) = 0$. Using Corollary (2.4), we get that $F'(x) < 0$ for all $x > 0$. Then $F(x)$ is decreasing function on $(0, \infty)$ with $\lim_{x\to\infty} F(x) = 0$, thus $F(x) > 0$ for $x > 0$. Now, let

$$S(x) = G(x+2) - \frac{1}{2}\ln\left[1 + \frac{2x+3}{x^2+2x+\frac{48}{e^4-16}}\right]$$

and then

$$S'(x+2) - S'(x) = \frac{-8(5+2x)W(x)}{(e^4-16)^4(2+x)^2(3+x)^2(x^2+2x+\frac{48}{e^4-16})(x^2+4x+\frac{3e^4}{e^4-16})D(x)},$$

where

$$D(x) = \left((x+2)^2 + 2(x+2) + \frac{48}{e^4-16}\right)\left((x+2)^2 + 4(x+2) + \frac{3e^4}{e^4-16}\right) > 0, \qquad x > 0$$

and

$$\begin{aligned} W(x) &= \left(21233664 - 6856704e^4 + 720576e^8 - 28944e^{12} + 324e^{16}\right) \\ &+ \left(100270080 - 26173440e^4 + 2361600e^8 - 84240e^{12} + 900e^{16}\right)x \\ &+ \left(152764416 - 35570688e^4 + 2920320e^8 - 96948e^{12} + 1005e^{16}\right)x^2 \\ &+ \left(106332160 - 23142400e^4 + 1795200e^8 - 57040e^{12} + 580e^{16}\right)x^3 \\ &+ \left(37257216 - 7818240e^4 + 587520e^8 - 18204e^{12} + 183e^{16}\right)x^4 \\ &+ \left(6389760 - 1320960e^4 + 97920e^8 - 3000e^{12} + 30e^{16}\right)x^5 \\ &+ \left(425984 - 88064e^4 + 6528e^8 - 200e^{12} + 2e^{16}\right)x^6. \end{aligned}$$

Then

$$\begin{aligned} W'(x) &= \left(100270080 - 26173440e^4 + 2361600e^8 - 84240e^{12} + 900e^{16}\right) \\ &+2\left(152764416 - 35570688e^4 + 2920320e^8 - 96948e^{12} + 1005e^{16}\right)x \\ &+3\left(106332160 - 23142400e^4 + 1795200e^8 - 57040e^{12} + 580e^{16}\right)x^2 \\ &+4\left(37257216 - 7818240e^4 + 587520e^8 - 18204e^{12} + 183e^{16}\right)x^3 \\ &+5\left(6389760 - 1320960e^4 + 97920e^8 - 3000e^{12} + 30e^{16}\right)x^4 \\ &+6\left(425984 - 88064e^4 + 6528e^8 - 200e^{12} + 2e^{16}\right)x^5 \end{aligned}$$

and

$$\begin{aligned} W''(x) &= 2\left(152764416 - 35570688e^4 + 2920320e^8 - 96948e^{12} + 1005e^{16}\right) \\ &+6\left(106332160 - 23142400e^4 + 1795200e^8 - 57040e^{12} + 580e^{16}\right)x \\ &+12\left(37257216 - 7818240e^4 + 587520e^8 - 18204e^{12} + 183e^{16}\right)x^2 \\ &+20\left(6389760 - 1320960e^4 + 97920e^8 - 3000e^{12} + 30e^{16}\right)x^3 \\ &+30\left(425984 - 88064e^4 + 6528e^8 - 200e^{12} + 2e^{16}\right)x^4 > 0, \quad x > 0. \end{aligned}$$

Thus $W'(x)$ is increasing on $(0,\infty)$ which implies that $W'(x) > W'(0.1) > 0$. Then $W(x)$ is increasing on $(0.1,\infty)$ with $W(0.57583) \approx -475.425 < 0$ and $W(0.57584) \approx 1147.33 > 0$. Hence $W(x)$ has only one positive root on $(0.57583,\infty)$ say $x_\delta \approx 0.575833$ and then $W(x) > 0$ for $x > 0.575833$. Now, $S'(x+2) - S'(x) < 0$ for $x > 0.575833$ and also $\lim_{x\to\infty} S'(x) = 0$. Using Corollary (2.4), then $S'(x) > 0$ for all $x > 0.575833$ or $S(x)$ is increasing on $(0.575833,\infty)$ with $\lim_{x\to\infty} S(x) = 0$. Thus $S(x) < 0$ for all $x > 0.575833$. □

Remark 2. Using the inequalities

$$1 + (2x+3)/(x^2 + 2x + 48/(e^4 - 16)) < (1 + 1/(x+1))^2, \qquad x > 0$$

and

$$1 + (2x+3)/(x^2 + 2x + 4/3) > (1 + 1/(x + 4/(e^2 - 4)))^2, \qquad x > x_\mu$$

where $x_\mu = \frac{-112+68e^2-7e^4-\sqrt{52480-30208e^2+6672e^4-664e^6+25e^8}}{6(32-12e^2+e^4)} \simeq 0.465586$, we can conclude that the inequality (29) improves the lower bound of the inequality (12) for $x > x_\mu$ and improves its upper bound for $x > 0$.

Remark 3. The inequality (29) improves the lower bound of the inequality (19) for $x > 0$.

References

[1] M. Abramowitz and I. A. Stegun (Eds.), Handbook of Mathematical Functions with Formulas, Graphs and Mathematical Tables, Dover, New York, 1965.

[2] H. Alzer and C. Berg, Some classes of completely monotonic functions, Annales Acad. Sci. Fenn. Math. 27(2), 445-460, 2002.

[3] G. E. Andrews, R. Askey and R. Roy, Special Functions, Cambridge Univ. Press, 1999.

[4] N. Batir, Sharp bounds for the psi function and harmonic numbers, Math. Inequal.Appl, Vol. 14, No. 4, 917-925, 2011.

[5] T. Burić, N. Elezović, Some completely monotonic functions related to the psi function, Math. Inequal. Appl., 14(3), 679-691, 2011.

[6] C.-P. Chen and H. M. Srivastava, New representations for the Lugo and Euler-Mascheroni constants, *II*, Appl. Math. Lett. 25, no. 3, 333-338, 2012.

[7] C.-P. Chen and C. Mortici, Sharpness of Muqattash-Yahdi problem, Comput. Appl. Math., Vol. 31, No.1 , 85-93, 2012.

[8] Á. Elbert and A. Laforgia, On some properties of the gamma function, Proc. Am. Math. Soc. 128(9), 2667-2673, 2000.

[9] A. Erdélyi et al., Higher Transcendental Functions Vol. I-III, California Institute of Technology - Bateman Manuscript Project, 1953-1955 McGraw-Hill Inc., reprinted by Krieger Inc. 1981.

[10] W. Feller, An Introduction to probability theory and its applications, Vol. 2, 3rd ed. New York, Wiley, 1971.

[11] O. Furdui, A class of fractional part integrals and zeta function values, Integral Transforms Spec. Funct. 24, no. 6, 485-490, 2013.

[12] A. Z. Grinshpan and M. E. H. Ismail, Completely monotonic functions involving the gamma and $q-$gamma functions, Proc. Amer. Math. Soc., 134, 1153-1160, 2006.

[13] H. Van Haeringen, Completely monotonic and related functions, J. Math. Anal. Appl., 204, 389-408, 1996.

[14] M. E. H. Ismail, L. Lorch, and M. E. Muldon, Completely monotonic functions associated with the gamma function and its $q-$analogues, J. Math. Anal. Appl. 116, 1-9, 1986.

[15] A. Laforgia and P. Natalini, On the asymptotic expansion of a ratio of gamma functions, J. Math. Anal. Appl. 389 , no. 2, 833837, 2012.

[16] M. Mahmoud, M. A. Alghamdi and R. P. Agarwal, New upper bounds of $n!$, J. Inequal. Appl. 2012;2012 doi: 10.1186/1029-242X-2012-27.

[17] M. Mahmoud and R. P. Agarwal, Bounds for Bateman's G-function and its applications, Georgian Mathematical Journal, Vol. 23, Issue 4, 579-586, 2016.

[18] M. Mahmoud and H. Almuashi, On some inequalities of the Bateman's $G-$function, J. Comput. Anal. Appl., Vol. 22, No.4, , 672-683, 2017.

[19] M. Mahmoud, A. Talat and H. Moustafa, Some approximations of the Bateman's $G-$function, J. Comput. Anal. Appl., Vol. 23, No. 6, 1165-1178, 2017.

[20] M. Mahmoud, A. Talat, H. Moustafa and R. P. Agarwal, Completely monotonic functions involving Bateman's $G-$function, Submitted for publication.

[21] C. Mortici, New approximations of the gamma function in terms of the digamma function, Appl. Math. Lett., Vol. 23, Issue 1, 97-100, 2010.

[22] C. Mortici, The proof of Muqattash-Yahdi conjecture, Math. Comput. Mod., Vol. 51, Issue 9, 1154-1159, 2010.

[23] C. Mortici, A sharp inequality involving the psi function, Acta Universitatis Apulensis, Vol. 22, 41-45, 2010.

[24] C. Mortici, A new Stirling series as continued fraction, Numer. Algorithms 56, no. 1, 1726, 2011.

[25] F. Qi, The best bounds in Kershaw's inequality and two completely monotonic functions, RGMIA Res. Rep. Coll. 9 (2006), no. 4, Art. 2.

[26] F. Qi, S. Guo, B.-N. Guo, Complete monotonicity of some functions involving polygamma functions, J. Comput. Appl. Math., 233(9), 2149-2160, 2010.

[27] F. Qi and S.-H. Wang, Complete monotonicity, completely monotonic degree, integral representations, and an inequality related to the exponential, trigamma, and modified Bessel functions, Glob. J. Math. Anal. 2, no. 3, 91-97, 2014.

[28] F. Qi and C. Mortici, Some best approximation formulas and inequalities for the Wallis ratio, Applied Mathematics and Computation, Vol. 253 (15), 363-368, 2015.

[29] F. Qi and W.-H. Lic, Integral representations and properties of some functions involving the logarithmic function, Filomat 30:7, 1659-1674, 2016.

[30] S.-L. Qiu and M. Vuorinen, Some properties of the gamma and psi functions with applications, Math. Comp., Vol.74, No. 250, 723-742, 2004.

[31] D. V. Widder, The Laplace Transform, Princeton University Press, Princeton, 1946.

[32] T.-H. Zhao, Z.-H. Yang and Y.-M. Chu, Monotonicity properties of a function involving the Psi function with applications, Journal of Inequalities and Applications (2015) 2015:193.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 27, NO.1, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC

A new q-extension of Euler polynomial of the second kind and some related polynomials

R. P. Agarwal [1], J. Y. Kang[2*], C. S. Ryoo[3]

Abstract : We define q-Euler polynomials of the second kind using q-analogue within exponential function. We have some basic properties of this polynomials such as addition, alternative finite sum, and symmetry property. We also investigate some relations of q-Euler, q-Bernoulli, and q-tangent polynomials using q-Euler polynomials of the second kind including two parameters.

Key words : q-Euler polynomials of the second kind, q-Euler polynomials, q-Bernoulli polynomials, q-tangent polynomials

2000 Mathematics Subject Classification : 11B68, 11B75, 12D10

1. Introduction

The main aim of this paper is to extend Euler numbers and polynomials of the second kind, and study some of their properties. Our paper is organised as follows: in Section 2, we define q-Euler numbers and polynomials of the second kind. From this definition we investigate some interesting properties of these numbers and polynomials using q-analogue of exponential function. In Section 3, we consider q-Euler polynomials of the second kind in two parameters and make some relations between q-Euler polynomials of the second kind and q-Euler , q-Bernoulli, q-tangent polynomials. Furthermore, we derive a symmetric relation, multiple q-derivative, and multiple q-integral.

For any $n \in \mathbb{C}$, the q-number is defined by

$$[n]_q = \frac{1-q^n}{1-q} = \sum_{0 \le i \le n} q^i = 1 + q + q^2 + \cdots + q^{n-1}.$$

An intensive and somewhat surprising interest in q-numbers appeared in many areas of mathematics and applications including q-difference equations, special functions, q-combinatorics, q-integrable systems, variational q-calculus, q-series, and so on. In this paper, we introduce some basic definitions and theorems(see [1-18]).

Definition 1.1.[1,3-5,10-13] The Gaussian binomial coefficients are defined by

$$\binom{m}{r}_q = \begin{bmatrix} m \\ r \end{bmatrix}_q = \begin{cases} 0 & \text{if } r > m \\ \frac{(1-q^m)(1-q^{m-1})\cdots(1-q^{m-r+1})}{(1-q)(1-q^2)\cdots(1-q^r)} & \text{if } r \le m \end{cases},$$

where m and r are non-negative integers. For $r = 0$ the value is 1 since the numerator and the denominator are both empty products. Like the classical binomial coefficients, the Gaussian binomial coefficients are center-symmetric. There are analogues of the binomial formula, and this definition has a number of properties.

[1] *Department of Mathematics, Texas A & M University, Kingsville, USA*
[2] *Department of Information and Statistics, Anyang University, Anyang, KOREA*
[3] *Department of Mathematics, Hanman University, Daejeon, KOREA*

Theorem 1.2.[5] Let n, k be non-negative integers. Then we get

$$\text{(i)} \qquad \prod_{k=0}^{n-1}(1+q^k t) = \sum_{k=0}^{n} q^{\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q t^k,$$

$$\text{(ii)} \qquad \prod_{k=0}^{n-1} \frac{1}{(1-q^k t)} = \sum_{k=0}^{\infty} \begin{bmatrix} n+k-1 \\ k \end{bmatrix}_q t^k.$$

Definition 1.3.[1,4,12-13] Let z be any complex numbers with $|z| < 1$. Two forms of q-exponential functions are defined by

$$e_q(z) = \sum_{n=0}^{\infty} \frac{z^n}{[n]_q!}, \quad e_{q^{-1}}(z) = \sum_{n=0}^{\infty} \frac{z^n}{[n]_{q^{-1}}!} = \sum_{n=0}^{\infty} q^{\binom{n}{2}} \frac{z^n}{[n]_q!}.$$

Definition 1.4.[4,10-11,13] The q-derivative operator of any function f is defined by

$$D_q f(x) = \frac{f(x) - f(qx)}{(1-q)x}, \qquad x \neq 0,$$

and $D_q f(0) = f'(0)$. We can prove that f is differentiable at 0, and it is clear that $D_q x^n = [n]_q x^{n-1}$.

Definition 1.5.[4,10-11,13] We define the q-integral as

$$\int_0^b f(x) d_q x = (1-q) b \sum_{j=0}^{\infty} q^j f(q^j b).$$

If this function, $f(x)$, is differentiable on the point x, the q-derivative in Definition 1.4 goes to the ordinary derivative in the classical analysis when $q \to 1$.

In 1961, L.Calitz introduced several properties of the Bernoulli and Euler polynomials of the second kind(see [6]). Euler numbers of the second kind was expanded, and C. S. Ryoo have studied these numbers and polynomials of the second kind in [17]. He also developed several properties of these numbers and polynomials.

Definition 1.6.[7-8, 6, 17] The classical Euler numbers, $\widetilde{E}_n$, and the classical Euler polynomials, $\widetilde{E}_n(x)$, of the second kind are defined by means of the generating functions

$$\sum_{n=0}^{\infty} \widetilde{E}_n \frac{t^n}{n!} = \frac{2}{e^t + e^{-t}}, \quad \sum_{n=0}^{\infty} \widetilde{E}_n(x) \frac{t^n}{n!} = \frac{2}{e^t + e^{-t}} e^{tx}.$$

Theorem 1.7.[17] For any positive integer n, we have

(i) For any positive integer m(=odd),

$$\widetilde{E}_n(x) = m^n \sum_{i=0}^{m-1} (-1)^i \widetilde{E}_n \left(\frac{2i + x + 1 - m}{m} \right) \text{ for } n \geq 0,$$

$$\text{(ii)} \qquad \widetilde{E}_l(x+y) = \sum_{n=0}^{l} \binom{l}{n} \widetilde{E}_n(x) y^{l-n},$$

$$\text{(iii)} \qquad \widetilde{E}_n(x) = (-1)^n \widetilde{E}_n(-x).$$

2. Some basic properties of the q-Euler polynomials of the second kind

In this section, we define the q-Euler numbers and polynomials of the second kind, and investigate basic properties of these numbers and polynomials. Furthermore, we find the alternative finite sum which is related to the q-Euler numbers and polynomials of second kind.

Definition 2.1. Let n be any non-negative integer. For $|q| < 1, x \in \mathbb{C}$, we define q-Euler numbers and polynomials of the second kind as

$$\begin{aligned}\sum_{n=0}^{\infty}\widetilde{\mathcal{E}}_{n,q}\frac{t^n}{[n]_q!} &= \frac{2}{e_q(t)+e_q(-t)},\\ \sum_{n=0}^{\infty}\widetilde{\mathcal{E}}_{n,q}(x)\frac{t^n}{[n]_q!} &= \frac{2}{e_q(t)+e_q(-t)}e_q(tx).\end{aligned}$$

Substituting $x=0$ in the q-Euler polynomials of the second kind, they can be simplified as follows:

$$\sum_{n=0}^{\infty}\widetilde{\mathcal{E}}_{n,q}(0)\frac{t^n}{[n]_q!} = \sum_{n=0}^{\infty}\widetilde{\mathcal{E}}_{n,q}\frac{t^n}{[n]_q!} = \frac{2}{e_q(t)+e_q(-t)} = \frac{1}{\cosh_q(t)},$$

where $\widetilde{\mathcal{E}}_{n,q}$ is q-Euler numbers of the second kind. If $q \to 1$, then we can find the classical Euler polynomials of the second kind in $\widetilde{\mathcal{E}}_{n,q}(x)$(see [6,17]).

Theorem 2.2. Let $|q| < 1, x$ be any complex numbers. Then, we have

$$\widetilde{\mathcal{E}}_{n,q}(x) = \sum_{k=0}^{n}\begin{bmatrix} n\\ k\end{bmatrix}_q \widetilde{\mathcal{E}}_{k,q}x^{n-k}.$$

Proof. From the generating function of the q-Euler polynomials of second kind, $\widetilde{\mathcal{E}}_{n,q}(x)$, we can find

$$\begin{aligned}\sum_{n=0}^{\infty}\widetilde{\mathcal{E}}_{n,q}(x)\frac{t^n}{[n]_q!} = \frac{2}{e_q(t)+e_q(-t)}e_q(tx) &= \sum_{n=0}^{\infty}\widetilde{\mathcal{E}}_{n,q}\frac{t^n}{[n]_q!}\sum_{n=0}^{\infty}x^n\frac{t^n}{[n]_q!}\\ &= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\begin{bmatrix} n\\ k\end{bmatrix}_q \widetilde{\mathcal{E}}_{k,q}(x)x^{n-k}\right)\frac{t^n}{[n]_q!},\end{aligned}$$

which gives the required result.

Theorem 2.3. For $|q| < 1$, the following holds:

$$D_q\widetilde{\mathcal{E}}_{n,q}(x) = [n]_q\widetilde{\mathcal{E}}_{n-1,q}(x).$$

Proof. Considering q-derivative of x^{n-k} in Theorem 2.2, we get

$$D_q\widetilde{\mathcal{E}}_{n,q}(x) = \sum_{k=0}^{n-1}\begin{bmatrix} n\\ k\end{bmatrix}_q [n-k]_q\widetilde{\mathcal{E}}_{k,q}x^{n-k-1}.$$

Transforming a binomial operation of q and using Theorem 2.2 again, we obtain

$$D_q\widetilde{\mathcal{E}}_{n+1,q}(x) = [n+1]_q\sum_{k=0}^{n}\begin{bmatrix} n\\ k\end{bmatrix}_q \widetilde{\mathcal{E}}_{k,q}x^{n-k} = [n+1]_q\widetilde{\mathcal{E}}_{n,q}(x).$$

The required relation now follows at once.

Theorem 2.4. Let n be any non-negative integer. Then, the following holds:

$$\int_0^x \widetilde{\mathcal{E}}_{n,q}(x)d_qx = \frac{\widetilde{\mathcal{E}}_{n+1,q}(x) - \widetilde{\mathcal{E}}_{n+1,q}}{[n+1]_q},$$

where $\widetilde{\mathcal{E}}_{n,q}(0) = \widetilde{\mathcal{E}}_{n,q}$ is q-Euler numbers of the second kind.

Proof. Using q-integral in Theorem 2.2, we have

$$\begin{aligned}
\int_0^x \widetilde{\mathcal{E}}_{n,q}(x)d_qx &= \int_0^x \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \widetilde{\mathcal{E}}_{k,q}x^{n-k}d_qx = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \widetilde{\mathcal{E}}_{k,q}\frac{1}{[n-k+1]_q}x^{n-k+1}\Big|_0^x \\
&= \frac{1}{[n+1]_q}\sum_{k=0}^{n+1} \begin{bmatrix} n+1 \\ k \end{bmatrix}_q \widetilde{\mathcal{E}}_{k,q}x^{n-k+1}\Big|_0^x = \frac{1}{[n+1]_q}\left(\widetilde{\mathcal{E}}_{n+1,q}(x) - \widetilde{\mathcal{E}}_{n+1,q}\right),
\end{aligned}$$

and we obtain the required relation at once.

Corollary 2.5. In Theorem 2.4, we get

$$\int_a^b \widetilde{\mathcal{E}}_{n,q}(x)d_qx = \frac{\widetilde{\mathcal{E}}_{n+1,q}(b) - \widetilde{\mathcal{E}}_{n+1,q}(a)}{[n+1]_q}.$$

Now we find some properties of q-exponential function to obtain the next theorem. From Definition 1.3 and Theorem 1.2, we find that

(i) $$[n]_{q^{-1}}! = q^{-\binom{n}{2}}[n]_q!,$$

(ii) $$\begin{aligned}
e_q(t)e_{q^{-1}}(t) &= \sum_{n=0}^{\infty}\frac{t^n}{[n]_q!}\sum_{n=0}^{\infty}\frac{t^n}{[n]_{q^{-1}}!} = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q q^{\binom{k}{2}}\right)\frac{t^n}{[n]_q!} \\
&= \sum_{n=0}^{\infty}\left(\prod_{k=0}^{n-1}(1+q^k)\right)\frac{t^n}{[n]_q!},
\end{aligned}$$

(iii) $$\begin{aligned}
e_q(t)e_{q^{-1}}(-t) &= \sum_{n=0}^{\infty}\frac{t^n}{[n]_q!}\sum_{n=0}^{\infty}\frac{(-t)^n}{[n]_{q^{-1}}!} = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q (-1)^k q^{\binom{k}{2}}\right)\frac{t^n}{[n]_q!} \\
&= \sum_{n=0}^{\infty}\left(\prod_{k=0}^{n-1}(1-q^k)\right)\frac{t^n}{[n]_q!},
\end{aligned}$$

(iv) $$\begin{aligned}
e_q(-t)e_{q^{-1}}(t) &= \sum_{n=0}^{\infty}\frac{(-t)^n}{[n]_q!}\sum_{n=0}^{\infty}\frac{t^n}{[n]_{q^{-1}}!} = \sum_{n=0}^{\infty}\left((-1)^n\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q q^{\binom{k}{2}}(-1)^k\right)\frac{t^n}{[n]_q!} \\
&= \sum_{n=0}^{\infty}\left((-1)^n\prod_{k=0}^{n-1}(1-q^k)\right)\frac{t^n}{[n]_q!},
\end{aligned}$$

(v) $$\begin{aligned}
e_q(-t)e_{q^{-1}}(-t) &= \sum_{n=0}^{\infty}\frac{(-t)^n}{[n]_q!}\sum_{n=0}^{\infty}\frac{(-t)^n}{[n]_{q^{-1}}!} = \sum_{n=0}^{\infty}\left((-1)^n\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q q^{\binom{k}{2}}\right)\frac{t^n}{[n]_q!} \\
&= \sum_{n=0}^{\infty}\left((-1)^n\prod_{k=0}^{n-1}(1+q^k)\right)\frac{t^n}{[n]_q!}.
\end{aligned}$$

Theorem 2.6. For $|q|<1$, we find

$$\text{(i)}\qquad \sum_{l=0}^{n}\begin{bmatrix} n \\ l \end{bmatrix}_q\left(\sum_{k=0}^{n-l}\begin{bmatrix} n-l \\ k \end{bmatrix}_q(-1)^k+\sum_{k=0}^{n-l}\begin{bmatrix} n-l \\ k \end{bmatrix}_q(-1)^{n-l}\right)\widetilde{\mathcal{E}}_{l,q}=2(-1)^n,$$

$$\begin{aligned}\text{(ii)}\qquad &\sum_{l=0}^{n}\begin{bmatrix} n \\ l \end{bmatrix}_q\left(\sum_{k=0}^{n-l}\begin{bmatrix} n-l \\ k \end{bmatrix}_q(-1)^k+\sum_{k=0}^{n-l}\begin{bmatrix} n-l \\ k \end{bmatrix}_q(-1)^{n-l}\right)\widetilde{\mathcal{E}}_{l,q}(x)\\ &=2\sum_{l=0}^{n}\begin{bmatrix} n \\ l \end{bmatrix}_q(-1)^{n-l}x^l.\end{aligned}$$

Proof. (i) Loading $e_q(t)e_q(-t)+e_q(-t)e_q(-t)\neq 0$ for the generating function of q-Euler numbers of the second kind, one obtains

$$\sum_{n=0}^{\infty}\widetilde{\mathcal{E}}_{n,q}\frac{t^n}{[n]_q!}(e_q(t)e_q(-t)+e_q(-t)e_q(-t))=2e_q(-t),$$

and we can transform such as

$$\begin{aligned}&\sum_{n=0}^{\infty}\widetilde{\mathcal{E}}_{n,q}\frac{t^n}{[n]_q!}(e_q(t)e_q(-t)+e_q(-t)e_q(-t))\\ &=\sum_{n=0}^{\infty}\widetilde{\mathcal{E}}_{n,q}\frac{t^n}{[n]_q!}\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q(-1)^k+\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q(-1)^n\right)\frac{t^n}{[n]_q!}\\ &=\sum_{n=0}^{\infty}\left\{\sum_{l=0}^{n}\begin{bmatrix} n \\ l \end{bmatrix}_q\left(\sum_{k=0}^{n-l}\begin{bmatrix} n-l \\ k \end{bmatrix}_q(-1)^k+\sum_{k=0}^{n-l}\begin{bmatrix} n-l \\ k \end{bmatrix}_q(-1)^{n-l}\right)\widetilde{\mathcal{E}}_{l,q}\right\}\frac{t^n}{[n]_q!}\\ &=2\sum_{n=0}^{\infty}(-1)^n\frac{t^n}{[n]_q!}.\end{aligned}$$

The required relation now follows at once.
(ii) We omit a proof of the q-Euler polynomials of the second kind due to its similarity to (i).

Corollary 2.7. For $q\to 1$, in Theorem 2.6, one holds

$$\text{(i)}\qquad \sum_{l=0}^{n}\binom{n}{l}\left(\sum_{k=0}^{n-l}\binom{n-l}{k}(-1)^k+\sum_{k=0}^{n-l}\binom{n-l}{k}(-1)^{n-l}\right)\widetilde{E}_l=2(-1)^n,$$

$$\text{(ii)}\qquad \sum_{l=0}^{n}\binom{n}{l}\left(\sum_{k=0}^{n-l}\binom{n-l}{k}(-1)^k+\sum_{k=0}^{n-l}\binom{n-l}{k}(-1)^{n-l}\right)\widetilde{E}_l(x)=2\sum_{l=0}^{n}\binom{n}{k}(-1)^{n-l}x^l,$$

where $\widetilde{E}_n(x)$ is the classical Euler polynomials of the second kind and $\widetilde{E}_n$ is the classical Euler numbers of the second kind(see [16]).

Theorem 2.8. Let $|q|<1$. Then we have

$$\text{(i)}\qquad \sum_{l=0}^{n}\begin{bmatrix} n \\ l \end{bmatrix}_q\left(\prod_{k=0}^{n-l-1}(1+q^k)+(-1)^{n-l}\prod_{k=0}^{n-l-1}(1-q^k)\right)\widetilde{\mathcal{E}}_{l,q}=2q^{\binom{n}{2}},$$

$$\text{(ii)}\qquad \sum_{l=0}^{n}\begin{bmatrix} n \\ l \end{bmatrix}_q\left(\prod_{k=0}^{n-l-1}(1+q^k)+(-1)^{n-l}\prod_{k=0}^{n-l-1}(1-q^k)\right)\widetilde{\mathcal{E}}_{l,q}(x)=2\sum_{l=0}^{n}\begin{bmatrix} n \\ l \end{bmatrix}_q q^{\binom{n-l}{2}}x^l.$$

Proof. (i) For $e_q(t)e_{q^{-1}}(t)+e_q(-t)e_{q^{-1}}(t)\neq 0$, we have

$$\sum_{n=0}^{\infty}\widetilde{\mathcal{E}}_{n,q}\frac{t^n}{[n]_q!}(e_q(t)e_{q^{-1}}(t)+e_q(-t)e_{q^{-1}}(t))=2e_{q^{-1}}(t).$$

To obtain the result, we can calculate the above equation as

$$\begin{aligned}
&\sum_{n=0}^{\infty}\widetilde{\mathcal{E}}_{n,q}\frac{t^n}{[n]_q!}(e_q(t)e_{q^{-1}}(t)+e_q(-t)e_{q^{-1}}(t))\\
&=\sum_{n=0}^{\infty}\widetilde{\mathcal{E}}_{n,q}\frac{t^n}{[n]_q!}\sum_{n=0}^{\infty}\left(\prod_{k=0}^{n-1}(1+q^k)+(-1)^n\prod_{k=0}^{n-1}(1-q^k)\right)\frac{t^n}{[n]_q!}\\
&=\sum_{n=0}^{\infty}\sum_{l=0}^{n}\begin{bmatrix} n\\ l\end{bmatrix}_q\left(\prod_{k=0}^{n-l-1}(1+q^k)+(-1)^{n-l}\prod_{k=0}^{n-l-1}(1-q^k)\right)\widetilde{\mathcal{E}}_{l,q}\frac{t^n}{[n]_q!}\\
&=2\sum_{n=0}^{\infty}q^{\binom{n}{2}}\frac{t^n}{[n]_q!}.
\end{aligned}$$

The required relation now follows on comparing the coefficients of t^n on both sides.
(ii) Using the same method as (i) we can find the required result, so we omit the proof.

Corollary 2.9. In Theorem 2.8, we can see

(i) $$q^{\binom{n}{2}}=\frac{1}{2}\sum_{l=0}^{n}\begin{bmatrix} n\\ l\end{bmatrix}_q\left(\prod_{k=0}^{n-l-1}(1+q^k)+(-1)^{n-l}\prod_{k=0}^{n-l-1}(1-q^k)\right)\widetilde{\mathcal{E}}_{l,q},$$

(ii) $$\sum_{l=0}^{n}\begin{bmatrix} n\\ l\end{bmatrix}_q q^{\binom{n-l}{2}}x^l=\frac{1}{2}\sum_{l=0}^{n}\begin{bmatrix} n\\ l\end{bmatrix}_q\left(\prod_{k=0}^{n-l-1}(1+q^k)+(-1)^{n-l}\prod_{k=0}^{n-l-1}(1-q^k)\right)\widetilde{\mathcal{E}}_{l,q}(x).$$

Theorem 2.10. For $|q|<1, k\in\mathbb{N}$, one holds

(i) $$\sum_{l=0}^{n}\begin{bmatrix} n\\ l\end{bmatrix}_q\left(\prod_{k=0}^{n-l-1}(1-q^k)+(-1)^{n-l}\prod_{k=0}^{n-l-1}(1+q^k)\right)\widetilde{\mathcal{E}}_{l,q}=2(-1)^n q^{\binom{n}{2}},$$

(ii) $$\sum_{l=0}^{n}\begin{bmatrix} n\\ l\end{bmatrix}_q\left(\prod_{k=0}^{n-l-1}(1-q^k)+(-1)^{n-l}\prod_{k=0}^{n-l-1}(1+q^k)\right)\widetilde{\mathcal{E}}_{l,q}(x)=2\sum_{l=0}^{n}\begin{bmatrix} n\\ l\end{bmatrix}_q(-1)^{n-l}q^{\binom{n-l}{2}}x^l.$$

Proof. (i) Let $e_q(t)e_{q^{-1}}(-t)+e_q(-t)e_{q^{-1}}(-t)\neq 0$. From the generating function of q-Euler numbers of the second kind, we can find

$$\sum_{n=0}^{\infty}\widetilde{\mathcal{E}}_{n,q}\frac{t^n}{[n]_q!}(e_q(t)e_{q^{-1}}(-t)+e_q(-t)e_{q^{-1}}(-t))=2e_{q^{-1}}(-t),$$

or, equivalently,

$$\begin{aligned}
&\sum_{n=0}^{\infty}\widetilde{\mathcal{E}}_{n,q}\frac{t^n}{[n]_q!}(e_q(t)e_{q^{-1}}(-t)+e_q(-t)e_{q^{-1}}(-t))\\
&=\sum_{n=0}^{\infty}\widetilde{\mathcal{E}}_{n,q}\frac{t^n}{[n]_q!}\sum_{n=0}^{\infty}\left(\prod_{k=0}^{n-1}(1-q^k)+(-1)^n\prod_{k=0}^{n-1}(1+q^k)\right)\frac{t^n}{[n]_q!}\\
&=\sum_{n=0}^{\infty}\left\{\sum_{l=0}^{n}\begin{bmatrix} n\\ l\end{bmatrix}_q\left(\prod_{k=0}^{n-l-1}(1-q^k)+(-1)^{n-l}\prod_{k=0}^{n-l-1}(1+q^k)\right)\widetilde{\mathcal{E}}_{l,q}\right\}\frac{t^n}{[n]_q!}=2\sum_{n=0}^{\infty}(-1)^n q^{\binom{n}{2}}\frac{t^n}{[n]_q!}.
\end{aligned}$$

Comparing the coefficients of $\frac{t^n}{[n]_q!}$, the proof is complete.
(ii) We omit the proof of the q-Euler polynomials because we can derive it in the same method as (i).

Corollary 2.11. In Theorem 2.10, we get

$$\text{(i)} \qquad (-1)^n q^{\binom{n}{2}} = \frac{1}{2}\sum_{l=0}^{n} \begin{bmatrix} n \\ l \end{bmatrix}_q \left(\prod_{k=0}^{n-l-1}(1-q^k) + (-1)^{n-l}\prod_{k=0}^{n-l-1}(1+q^k)\right)\widetilde{\mathcal{E}}_{l,q},$$

$$\text{(ii)} \qquad \sum_{l=0}^{n} \begin{bmatrix} n \\ l \end{bmatrix}_q (-1)^{n-l} q^{\binom{n-l}{2}} x^l = \frac{1}{2}\sum_{l=0}^{n} \begin{bmatrix} n \\ l \end{bmatrix}_q \left(\prod_{k=0}^{n-l-1}(1-q^k) + (-1)^{n-l}\prod_{k=0}^{n-l-1}(1+q^k)\right)\widetilde{\mathcal{E}}_{l,q}(x).$$

Theorem 2.12. For $x \in \mathbb{C}$, we hold

$$\text{(i)} \qquad \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q (1+(-1)^k)\widetilde{\mathcal{E}}_{n-k,q} = \begin{cases} 2 & \text{if } n = 0 \\ 0 & \text{if } n \neq 0 \end{cases},$$

$$\text{(ii)} \qquad \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q (1+(-1)^k)\widetilde{\mathcal{E}}_{n-k,q}(x) = 2x^n.$$

Proof. (i) From Definition 2.1, we can represent q-Euler numbers, $\widetilde{\mathcal{E}}_{n,q}$, as

$$\sum_{n=0}^{\infty}\widetilde{\mathcal{E}}_{n,q}\frac{t^n}{[n]_q!}\sum_{n=0}^{\infty}(1+(-1)^n)\frac{t^n}{[n]_q!} = 2.$$

Now using the Cauchy's product, we find the relation,

$$\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q (1+(-1)^k)\widetilde{\mathcal{E}}_{n-k,q}\right)\frac{t^n}{[n]_q!} = 2,$$

and the proof is done.
(ii) We omit a proof of (ii) since we can obtain (ii) using Cauchy's product and the method of coefficient comparison for Definition 2.1 using the same method (i).

Theorem 2.13. Let $x \in \mathbb{C}$ and $|q| < 1$. Then, the following holds:

$$\text{(i)} \qquad \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q (1+(-1)^k)\widetilde{\mathcal{E}}_{n-k,q} = 2\sum_{k=0}^{[\frac{n}{2}]} \begin{bmatrix} n \\ n-2k \end{bmatrix}_q \widetilde{\mathcal{E}}_{n-2k,q},$$

$$\text{(ii)} \qquad \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q (1+(-1)^k)\widetilde{\mathcal{E}}_{n-k,q}(x) = 2\sum_{k=0}^{[\frac{n}{2}]} \begin{bmatrix} n \\ n-2k \end{bmatrix}_q \widetilde{\mathcal{E}}_{n-2k,q}(x),$$

where $[x]$ is the greatest integer not exceeding x.
Proof. (i) In Theorem 2.12. (i), the left-side is changed as:

$$\begin{aligned}
&\sum_{n=0}^{\infty}\widetilde{\mathcal{E}}_{n,q}\frac{t^n}{[n]_q!}\sum_{n=0}^{\infty}(1+(-1)^n)\frac{t^n}{[n]_q!} = 2\sum_{n=0}^{\infty}\widetilde{\mathcal{E}}_{n,q}\frac{t^n}{[n]_q!}\sum_{n=0}^{\infty}\frac{t^{2n}}{[2n]_q!} \\
&= 2\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\begin{bmatrix} 2n-k \\ k \end{bmatrix}_q \widetilde{\mathcal{E}}_{k,q}\right)\frac{t^{2n-k}}{[2n-k]_q!} = 2\sum_{n=0}^{\infty}\left(\sum_{k=0}^{[\frac{n}{2}]}\begin{bmatrix} n \\ n-2k \end{bmatrix}_q \widetilde{\mathcal{E}}_{n-2k,q}\right)\frac{t^n}{[n]_q!}.
\end{aligned}$$

The required relation now follows on comparing the coefficients of t^n on both sides.

(ii) Now following the same procedure as (i), we find (ii).

Corollary 2.14. From the Theorem 2.12 and Theorem 2.13, the following relations hold:

$$\text{(i)}\qquad \sum_{k=0}^{\left[\frac{n}{2}\right]} \begin{bmatrix} n \\ n-2k \end{bmatrix}_q \widetilde{\mathcal{E}}_{n-2k,q} = \begin{cases} 1 & \text{if } n=0 \\ 0 & \text{if } n \neq 0 \end{cases},$$

$$\text{(ii)}\qquad \sum_{k=0}^{\left[\frac{n}{2}\right]} \begin{bmatrix} n \\ n-2k \end{bmatrix}_q \widetilde{\mathcal{E}}_{n-2k,q}(x) = x^n,$$

where $[x]$ is the greatest integer not exceeding x.

Theorem 2.15. For $x \in \mathbb{C}$, the following relation holds

$$\widetilde{\mathcal{E}}_{n,q}(x) = (-1)^n \widetilde{\mathcal{E}}_{n,q}(-x).$$

Proof. Replacing t, x with $-t, -x$, respectively, we get

$$\sum_{n=0}^{\infty} \widetilde{\mathcal{E}}_{n,q}(-x) \frac{(-t)^n}{[n]_q!} = \frac{2}{e_q(-t)+e_q(t)} e_q(tx) = \sum_{n=0}^{\infty} \widetilde{\mathcal{E}}_{n,q}(x) \frac{t^n}{[n]_q!},$$

which on comparing the coefficients immediately gives the required relation.

Corollary 2.16. Putting $x = 1$ in Theorem 2.15, we see

$$\widetilde{\mathcal{E}}_{n,q}(1) = (-1)^n \widetilde{\mathcal{E}}_{n,q}(-1).$$

3. Some special properties of the q-Euler polynomials of the second kind

In this section, we define the q-Euler polynomials of the second kind in two parameters. From these polynomials, we can find some relations between these polynomials and other polynomials. We can also observe a symmetric property of the q-Euler polynomials of the second kind.

Definition 3.1. Let $x, y \in \mathbb{C}$. We then define the q-Euler polynomials of the second kind in two parameters as:

$$\sum_{n=0}^{\infty} \widetilde{\mathcal{E}}_{n,q}(x,y) \frac{t^n}{[n]_q!} = \frac{2}{e_q(t)+e_q(-t)} e_q(tx) e_q(ty).$$

For $y = 0$, we can see that $\widetilde{\mathcal{E}}_{n,q}(x, 0) = \widetilde{\mathcal{E}}_{n,q}(x)$.

Theorem 3.2. Let x be any complex numbers. Then we hold

$$\text{(i)}\qquad \widetilde{\mathcal{E}}_{n,q}(x,y) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \widetilde{\mathcal{E}}_{k,q}(x) y^{n-k},$$

$$\text{(ii)}\qquad \widetilde{\mathcal{E}}_{n,q}(x,y) = \sum_{l=0}^{n} \begin{bmatrix} n \\ l \end{bmatrix}_q \widetilde{\mathcal{E}}_{n-l} \sum_{k=0}^{l} \begin{bmatrix} l \\ k \end{bmatrix}_q x^{l-k} y^k.$$

Proof. From Definition 3.1, we find

$$\begin{aligned} \sum_{n=0}^{\infty} \widetilde{\mathcal{E}}_{n,q}(x,y) \frac{t^n}{[n]_q!} &= \frac{2}{e_q(t)+e_q(-t)} e_q(tx) e_q(ty) \\ &= \sum_{n=0}^{\infty} \widetilde{\mathcal{E}}_{n,q}(x) \frac{t^n}{[n]_q!} \sum_{n=0}^{\infty} y^n \frac{t^n}{[n]_q!} = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \widetilde{\mathcal{E}}_{k,q}(x) y^{n-k} \right) \frac{t^n}{[n]_q!}. \end{aligned}$$

The required relation now follows immediately.

Theorem 3.3. For $x \in \mathbb{C}$, we hold

$$\widetilde{\mathcal{E}}_{n,q}(x,1) + \widetilde{\mathcal{E}}_{n,q}(x,-1) = 2x^n.$$

Proof. Setting $y = 1$ and -1, we can get

$$\begin{aligned}
\sum_{n=0}^{\infty} \left(\widetilde{\mathcal{E}}_{n,q}(x,1) + \widetilde{\mathcal{E}}_{n,q}(x,-1)\right) \frac{t^n}{[n]_q!} &= \frac{2}{e_q(t)+e_q(-t)} e_q(tx)e_q(t) + \frac{2}{e_q(t)+e_q(-t)} e_q(tx)e_q(-t) \\
&= \frac{2}{e_q(t)+e_q(-t)} e_q(tx)\left(e_q(t)+e_q(-t)\right) = 2e_q(tx) = 2\sum_{n=0}^{\infty} x^n \frac{t^n}{[n]_q!},
\end{aligned}$$

and the proof is complete on comparing the coefficient of both sides.

Corollary 3.4. From Theorem 3.3, we see

$$x^n = \frac{1}{2}\left(\widetilde{\mathcal{E}}_{n,q}(x,1) + \widetilde{\mathcal{E}}_{n,q}(x,-1)\right).$$

To investigate some relations of other polynomials, we define q-Euler, q-Bernoulli, and q-tangent polynomials. These polynomials have a lot of properties, applications, and identities.

Definition 3.5. We define q-tangent polynomials, $\mathcal{T}(x)$; q-Euler polynomials, $\mathcal{E}(x)$; and q-Bernoulli polynomials, $\mathcal{B}(x)$ as

$$\begin{aligned}
\sum_{n=0}^{\infty} \mathcal{E}_{n,q}(x) \frac{t^n}{[n]_q!} &= \frac{[2]_q}{e_q(t)+1} e_q(tx), \quad |t| < \pi, \\
\sum_{n=0}^{\infty} \mathcal{T}_{n,q}(x) \frac{t^n}{[n]_q!} &= \frac{[2]_q}{e_q(2t)+1} e_q(tx), \quad |t| < \frac{\pi}{2}, \\
\sum_{n=0}^{\infty} \mathcal{B}_{n,q}(x) \frac{t^n}{[n]_q!} &= \frac{t}{e_q(t)-1} e_q(tx), \quad |t| < 2\pi, \quad \text{(see [7, 14-15])}.
\end{aligned}$$

Theorem 3.6. For $x, y \in \mathbb{C}$, the following relation holds:

$$\widetilde{\mathcal{E}}_{n,q}(x,y) = \frac{1}{[2]_q} \sum_{l=0}^{n} \begin{bmatrix} n \\ l \end{bmatrix}_q \left(\frac{\widetilde{\mathcal{E}}_{n-l,q}(x)}{m^l} + \sum_{k=0}^{n-l} \begin{bmatrix} n-l \\ k \end{bmatrix}_q \frac{\widetilde{\mathcal{E}}_{k,q}(x)}{m^{n-k}} \right) \mathcal{E}_{l,q}(my),$$

where $\mathcal{E}_{n,q}(x)$ is q-Euler polynomials.

Proof. Transforming the q-Euler polynomials of the second kind containing two parameters, we have

$$\frac{2}{e_q(t)+e_q(-t)} e_q(tx)e_q(ty) = \left(\frac{[2]_q}{e_q(\frac{t}{m})+1} e_q(ty)\right) \left(\frac{e_q(\frac{t}{m})+1}{[2]_q}\right) \left(\frac{2}{e_q(t)+e_q(-t)} e_q(tx)\right).$$

For the relation between q-Euler polynomials of the second kind containing two parameters and q-Euler polynomials, we get

$$\begin{aligned}
\sum_{n=0}^{\infty} \widetilde{\mathcal{E}}_{n,q}(x,y) \frac{t^n}{[n]_q!} &= \frac{1}{[2]_q} \sum_{n=0}^{\infty} \left(\sum_{l=0}^{n} \begin{bmatrix} n \\ l \end{bmatrix}_q \mathcal{E}_{l,q}(my) \sum_{k=0}^{n-l} \begin{bmatrix} n-l \\ k \end{bmatrix}_q \frac{\widetilde{\mathcal{E}}_{k,q}(x)}{m^{n-k}} \right) \frac{t^n}{[n]_q!} \\
&\quad + \frac{1}{[2]_q} \sum_{n=0}^{\infty} \left(\sum_{l=0}^{n} \begin{bmatrix} n \\ l \end{bmatrix}_q \mathcal{E}_{l,q}(my) \frac{\widetilde{\mathcal{E}}_{n-l,q}(x)}{m^l} \right) \frac{t^n}{[n]_q!},
\end{aligned}$$

which, on comparing the coefficients, immediately gives the required relation.

Corollary 3.7. For $q \to 1$ in Theorem 3.6, we have

$$\widetilde{E}_n(x,y) = \frac{1}{2}\sum_{l=0}^{n}\binom{n}{l}\left(\frac{\widetilde{E}_{n-l}(x)}{m^l} + \sum_{k=0}^{n-l}\binom{n-l}{k}\frac{\widetilde{E}_k(x)}{m^{n-k}}\right)E_l(my),$$

where $\widetilde{E}_n(x)$ is the classical Euler polynomials of the second kind, and $E_n(x)$ is the classical Euler polynomials(see [17]).

Theorem 3.8. Let $x, y \in \mathbb{C}$ and $|q| < 1$. Then we get

$$\widetilde{\mathcal{E}}_{n,q}(x,y) = \frac{1}{[2]_q}\sum_{l=0}^{n}\begin{bmatrix} n \\ l \end{bmatrix}_q\left(\frac{2^l\widetilde{\mathcal{E}}_{n-l,q}(x)}{m^l} + \sum_{k=0}^{n-l}\begin{bmatrix} n-l \\ k \end{bmatrix}_q\frac{2^{n-k}\widetilde{\mathcal{E}}_{k,q}(x)}{m^{n-k}}\right)\mathcal{T}_{l,q}(\frac{m}{2}y),$$

where $\mathcal{T}_{n,q}(x)$ is q-tangent polynomials.

Proof. To obtain the relation between q-Euler polynomials of the second kind and q-tangent polynomials, we can make

$$\sum_{n=0}^{\infty}\widetilde{\mathcal{E}}_{n,q}(x,y)\frac{t^n}{[n]_q!} = \left(\frac{[2]_q}{e_q(\frac{2t}{m})+1}e_q(ty)\right)\left(\frac{e_q(\frac{2t}{m})+1}{[2]_q}\right)\left(\frac{2}{e_q(t)+e_q(-t)}e_q(tx)\right).$$

From the above equation, we hold

$$\begin{aligned}\sum_{n=0}^{\infty}\widetilde{\mathcal{E}}_{n,q}(x,y)\frac{t^n}{[n]_q!} &= \frac{1}{[2]_q}\sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\begin{bmatrix} n \\ l \end{bmatrix}_q\mathcal{T}_{l,q}(\frac{m}{2}y)\sum_{k=0}^{n-l}\begin{bmatrix} n-l \\ k \end{bmatrix}_q\frac{2^{n-k}\widetilde{\mathcal{E}}_{k,q}(x)}{m^{n-k}}\right)\frac{t^n}{[n]_q!} \\ &\quad + \frac{1}{[2]_q}\sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\begin{bmatrix} n \\ l \end{bmatrix}_q\widetilde{\mathcal{E}}_{n-l,q}(x)\frac{2^l\mathcal{T}_{l,q}(\frac{m}{2}y)}{m^l}\right)\frac{t^n}{[n]_q!},\end{aligned}$$

which, on comparing the coefficients, the required relation at once.

Corollary 3.9. For $q \to 1$ in Theorem 3.8, we see

$$\widetilde{E}_n(x,y) = \frac{1}{2}\sum_{l=0}^{n}\binom{n}{l}\left(\frac{2^l\widetilde{E}_{n-l}(x)}{m^l} + \sum_{k=0}^{n-l}\binom{n-l}{k}\frac{2^{n-k}\widetilde{E}_k(x)}{m^{n-k}}\right)T_l(\frac{m}{2}y),$$

where $\widetilde{E}_n(x)$ is the classical Euler polynomials of the second kind, and $T_l(x)$ is the classical tangent polynomials(see [6, 13-14, 17]).

Theorem 3.10. Let $|q| < 1$ and x, y be any complex numbers. Then we hold

$$[n]_q\widetilde{\mathcal{E}}_{n-1,q}(x,y) = \sum_{l=0}^{n}\begin{bmatrix} n \\ l \end{bmatrix}_q\left(\sum_{k=0}^{n-l}\begin{bmatrix} n-l \\ k \end{bmatrix}_q\frac{\widetilde{\mathcal{E}}_{k,q}(x)}{m^{n-k}} - \frac{[n-l]_q\widetilde{\mathcal{E}}_{n-l,q}(x)}{m^l}\right)\mathcal{B}_{l,q}(my).$$

Proof. Multiplying the generating function of q-Euler polynomials of the second kind in two param-

eters by a suitable function, we have

$$\begin{aligned}
\sum_{n=0}^{\infty}\widetilde{\mathcal{E}}_{n,q}(x,y)\frac{t^n}{[n]_q!} &= \left(\frac{t}{e_q(\frac{t}{m})-1}e_q(ty)\right)\left(\frac{e_q(\frac{t}{m})-1}{t}\right)\left(\frac{2}{e_q(t)+e_q(-t)}e_q(tx)\right)\\
&= \left(\sum_{n=0}^{\infty}B_{n,q}(my)\frac{t^n}{m^n[n]_q!}\right)\left(\sum_{n=0}^{\infty}\frac{t^{n-1}}{m^n[n]_q!}-\frac{1}{t}\right)\left(\sum_{n=0}^{\infty}\widetilde{\mathcal{E}}_{n,q}(x)\frac{t^n}{[n]_q!}\right)\\
&= \sum_{n=0}^{\infty}\left(\frac{1}{[n]_q}\sum_{l=0}^{n}\begin{bmatrix} n\\ l\end{bmatrix}_q\mathcal{B}_{l,q}(my)\sum_{k=0}^{n-l}\begin{bmatrix} n-l\\ k\end{bmatrix}_q\frac{\widetilde{\mathcal{E}}_{k,q}(x)}{m^{n-k}}\right)\frac{t^{n-1}}{[n-1]_q!}\\
&- \sum_{n=0}^{\infty}\left(\frac{1}{[n]_q}\sum_{l=0}^{n}\begin{bmatrix} n\\ l\end{bmatrix}_q\frac{[n-l]_q\widetilde{\mathcal{E}}_{n-l,q}(x)}{m^l}\mathcal{B}_{l,q}(my)\right)\frac{t^{n-1}}{[n-1]_q!}.
\end{aligned}$$

Comparing the coefficients of both sides leads to

$$\sum_{n=0}^{\infty}[n]_q\widetilde{\mathcal{E}}_{n-1,q}(x,y)\frac{t^n}{[n]_q!} = \sum_{n=0}^{\infty}\left\{\sum_{l=0}^{n}\begin{bmatrix} n\\ l\end{bmatrix}_q\left(\sum_{k=0}^{n-l}\begin{bmatrix} n-l\\ k\end{bmatrix}_q\frac{\widetilde{\mathcal{E}}_{k,q}(x)}{m^{n-k}}-\frac{[n-l]_q\widetilde{\mathcal{E}}_{n-l,q}(x)}{m^l}\right)\mathcal{B}_{l,q}(my)\right\}\frac{t^n}{[n]_q!},$$

which gives the required result.

Corollary 3.11. For $q\to 1$, in Theorem 3.10, we see

$$[n]_q\widetilde{E}_{n-1}(x,y) = \sum_{l=0}^{n}\binom{n}{l}\left(\sum_{k=0}^{n-l}\binom{n-l}{k}\frac{\widetilde{\mathcal{E}}_k(x)}{m^{n-k}}-\frac{(n-l)\widetilde{\mathcal{E}}_{n-l}(x)}{m^l}\right)B_l(my),$$

where $\widetilde{E}_n(x)$ is the classical Euler polynomials of the second kind, and $B_n(x)$ is the classical Bernoulli polynomials(see [6-9, 12, 15, 17]).

Theorem 3.12. For q, x and $y\in\mathbb{C}$, the q-Euler polynomials of the second kind have

$$\sum_{k=0}^{n}\begin{bmatrix} n\\ k\end{bmatrix}_q\left(\frac{b}{a}\right)^{n-2k}\widetilde{\mathcal{E}}_{n-k,q}\left(\frac{a}{b}x\right)\widetilde{\mathcal{E}}_{k,q}\left(\frac{b}{a}y\right) = \sum_{k=0}^{n}\begin{bmatrix} n\\ k\end{bmatrix}_q\left(\frac{a}{b}\right)^{n-2k}\widetilde{\mathcal{E}}_{n-k,q}\left(\frac{b}{a}x\right)\widetilde{\mathcal{E}}_{k,q}\left(\frac{a}{b}y\right).$$

Proof. Consider that

$$A = \frac{4e_q(tx)e_q(ty)}{\left(e_q(\frac{b}{a}t)+e_q(-\frac{b}{a}t)\right)\left(e_q(\frac{a}{b}t)+e_q(-\frac{a}{b}t)\right)}.$$

The form A can turn to

$$\begin{aligned}
A &= \frac{2e_q(tx)}{\left(e_q(\frac{b}{a}t)+e_q(-\frac{b}{a}t)\right)}\frac{2e_q(ty)}{\left(e_q(\frac{a}{b}t)+e_q(-\frac{a}{b}t)\right)}\\
&= \sum_{n=0}^{\infty}\widetilde{\mathcal{E}}_{n,q}\left(\frac{a}{b}x\right)\frac{\left(\frac{b}{a}t\right)^n}{[n]_q!}\sum_{n=0}^{\infty}\widetilde{\mathcal{E}}_{n,q}\left(\frac{b}{a}y\right)\frac{\left(\frac{a}{b}t\right)^n}{[n]_q!} \qquad (3.1)\\
&= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\begin{bmatrix} n\\ k\end{bmatrix}_q\left(\frac{b}{a}\right)^{n-2k}\widetilde{\mathcal{E}}_{n-k,q}\left(\frac{a}{b}x\right)\widetilde{\mathcal{E}}_{k,q}\left(\frac{b}{a}y\right)\right)\frac{t^n}{[n]_q!},
\end{aligned}$$

or, equivalently,

$$\begin{aligned}A &= \frac{2e_q(tx)}{\left(e_q(\frac{a}{b}t)+e_q(-\frac{a}{b}t)\right)}\frac{2e_q(ty)}{\left(e_q(\frac{b}{a}t)+e_q(-\frac{b}{a}t)\right)}\\ &= \sum_{n=0}^{\infty}\widetilde{\mathcal{E}}_{n,q}\left(\frac{b}{a}x\right)\frac{\left(\frac{a}{b}t\right)^n}{[n]_q!}\sum_{n=0}^{\infty}\widetilde{\mathcal{E}}_{n,q}\left(\frac{a}{b}y\right)\frac{\left(\frac{b}{a}t\right)^n}{[n]_q!}\\ &= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q\left(\frac{a}{b}\right)^{n-2k}\widetilde{\mathcal{E}}_{n-k,q}\left(\frac{b}{a}x\right)\widetilde{\mathcal{E}}_{k,q}\left(\frac{a}{b}y\right)\right)\frac{t^n}{[n]_q!},\end{aligned} \tag{3.2}$$

and the theorem is proved in (3.1) and (3.2).

Corollary 3.13. If $q \to 1$ in Theorem 3.12, then we see

$$\sum_{k=0}^{n}\binom{n}{k}\left(\frac{b}{a}\right)^{n-2k}\widetilde{E}_{n-k}\left(\frac{a}{b}x\right)\widetilde{E}_k\left(\frac{b}{a}y\right)=\sum_{k=0}^{n}\binom{n}{k}\left(\frac{a}{b}\right)^{n-2k}\widetilde{E}_{n-k}\left(\frac{b}{a}x\right)\widetilde{E}_k\left(\frac{a}{b}y\right),$$

where $\widetilde{E}_n(x)$ is the classical Euler polynomials of the second kind(see [6, 17]).

Theorem 3.14. For $k \in \mathbb{N}$, we get

$$D_q^{(k)}\widetilde{\mathcal{E}}_{n,q}(x)=\begin{bmatrix}n\\k\end{bmatrix}_q[k]_q!\widetilde{\mathcal{E}}_{n-k,q}(x).$$

Proof. To make this result we have to remember Theorem 2.2,

$$\widetilde{\mathcal{E}}_{n,q}(x)=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q\widetilde{\mathcal{E}}_{k,q}x^{n-k},$$

and Theorem 2.3,

$$D_q^{(1)}\widetilde{\mathcal{E}}_{n,q}(x)=[n]_q\widetilde{\mathcal{E}}_{n-1,q}(x).$$

Repeating q-derivative for Theorem 2.3, we can see

$$\begin{aligned}&D_q^{(2)}\widetilde{\mathcal{E}}_{n,q}(x)=[n]_q[n-1]_q\widetilde{\mathcal{E}}_{n-2,q}(x)\\ &D_q^{(3)}\widetilde{\mathcal{E}}_{n,q}(x)=[n]_q[n-1]_q[n-2]_q\widetilde{\mathcal{E}}_{n-3,q}(x)\\ &D_q^{(4)}\widetilde{\mathcal{E}}_{n,q}(x)=[n]_q[n-1]_q[n-2]_q[n-3]_q\widetilde{\mathcal{E}}_{n-4,q}(x)\\ &\cdots.\end{aligned}$$

For $k \in \mathbb{N}$, we find q-derivative of k-times such as

$$\begin{aligned}D_q^{(k)}\widetilde{\mathcal{E}}_{n,q}(x)&=[n]_q[n-1]_q[n-2]_q[n-3]_q\cdots[n-k+1]_q\widetilde{\mathcal{E}}_{n-k,q}(x)\\ &=\begin{bmatrix}n\\k\end{bmatrix}_q[k]_q!\widetilde{\mathcal{E}}_{n-k,q}(x),\end{aligned}$$

and it is proved.

Acknowledgement: This work was supported by the National Research Foundation of Korea(NRF) grant funded by the Korea government(MEST) (No. 2017070483).

REFERENCES

1. G. E. Andrews, R. Askey, R. Roy, *Special functions*, Cambridge Press, Cambridge, UK 1999.
2. R. Ayoub, *Euler and zeta function*, Amer. Math. Monthly, 81(1974), 1067-1086.
3. M. Arik, E. Demircan, T. Turgut, L. Ekinci, M. Mungan, *Fibonacci oscillators*, Z. Phys. C:particles and Fields, 55 (1992), 89-95.
4. G. Bangerezako, *An introduction to q-difference equations*, preprint, Bujumbura 2008.
5. L. Comtet, *Advanced Combinatorics*, Reidel, Dordrecht, The Netherlands, 1974.
6. L. Carlitz, , *A note on Bernoulli and Euler polynomials of the second kind*, Scripta Math., 25 (1961) 323-330.
7. N. I. Mahmudov, A Akkele, A. neren, *On a class of two dimensional* (w,q)*-Bernoulli and* (w,q)*-Euler polynomials: Properties and location of zeros*, Journal of Computational Analysis and Applications, 16 (1)(2014) 282-292.
8. Q. M. Luo, H. M. Srivastava, *q-Extensions of some relationships between the Bernoulli and Euler polynomials*, Taiwan. J. Math, 15 (2011) 241-257.
9. H. Ozden, Y. Simsek, *A new extension of q-Euler numbers and polynomials related to their interpolation functions.*, Appl. Math. Lett., 21 (2008) 934-939.
10. R. Jagannathan, K. S. Rao, *Two-parameter quantum algebras, twin-basic numbers, and associated generalized hypergeometric series*, arXiv:math/0602613[math.NT].
11. H. F. Jackson, *q-Difference equations*, Am. J. Math, 32 (1910), 305-314.
12. N. I. Mahmudov, *A new class of generalized Bernoulli polynomials and Euler polynomials*, arXive:1201.6633v1[math. CA], 31 Jan 2012.
13. C.S. Ryoo, *A numerical investigation on the zeros of the tangent polynomials*, J. Appl. Math. & Informatics, 32 (2014), 315-322.
14. C.S. Ryoo, *Differential equations associated with tangent numbers*, J. Appl. Math. & Informatics, 34 (2016), 487-494.
15. C.S. Ryoo, *A Note on the Zeros of the q-Bernoulli Polynomials*, J. Appl. Math. & Informatics, 28 (2010), 805-811.
16. C.S. Ryoo, *Reflection Symmetries of the q-Genocchi Polynomials*, J. Appl. Math. & Informatics, 28 (1010), 1277-1284.
17. C. S. Ryoo, *Calculating zeros of the second kind Euler polynomials*, J. Comput. Anal. Appl., 12 (2010), 828-833.
18. C. S. Ryoo, T. Kim, R. P. Agarwal, *A numerical investigation of the roots of q-polynomials*, Inter. J. Comput. Math., 83 (2006), 223-234.

Regularized moving least squares approximation with Laplace-Beltrami operator on the sphere *

Chunmei Ding Yongli Zhang Feilong Cao

Department of Applied Mathematics, College of Sciences, China Jiliang University,
Hangzhou 310018, Zhejiang Province, P R China.
E-mail: feilongcao@gmail.com

Abstract

This paper proposes a pointwise approximation scheme on the unit sphere $\mathbb{S}^2$, which aims to handle spherical scattered data with high level noise by using a regularized moving least squares (RMLS) approximation with Laplace-Beltrami operator. The pointwise errors of approximation by the RMLS are estimated, and some numerical examples are designed to examine the obtained theoretical results. Also, the given numerical examples illustrate that the different choices for the order of Laplace-Beltrami operator and regularized parameter can provide different pointwise approximation results.

MSC(2000): 65D32, 65H10, 41A17

Keywords Sphere; Moving least squares; Pointwise approximation; Laplace-Beltrami operator

1 Introduction

In recent years, scattered data approximation on the unit sphere $\mathbb{S}^2$ has been widely applied to astrophysics, meteorology, geodesy, geophysics, and other areas (see [9, 11, 36]). In the approximation, spherical polynomials and spherical radial basis functions are usually taken as approximation tools. We refer the reader to [12, 13, 14, 16, 17, 25, 24, 26, 36, 30, 31, 32].

Moving least squares (MLS) scheme for scattered data approximation in Euclidean space $\mathbb{R}^n$ was proposed in [34] and [36] several years ago, which is actually a scheme to approximate target functions by using polynomials. Its simplest form coincides with Shepards interpolation method [29]. And, in terms of Backus-Gilbert method and constructive way of computing MLS approximation, it is also called as Backus-Gilbert optimal [4, 5, 6, 8]. Afterwards MLS was extended by McLain [21, 22], Franke [10], and Lancaster [15]. Meanwhile, MLS approximation has been frequently applied to potential energy surfaces [20], surface reconstruction [15], and partial differential equations [7]. In addition, several errors of approximation of MLS were estimated by different ways in [2, 3, 18, 34, 36].

In the MLS method, the highlight is process of MLS approximation involving the local polynomial reproduction property and the key ingredient in depriving error estimates. Based on the idea, Wendland [35] studied MLS approximation on the sphere to estimate the function value $f(x)$ by solving a local weighted least squares problem for every point x on the sphere which seems to be better than the method of spherical radial basis functions (SBFs) interpolation, because it does not require solving a large linear system. Li [19] developed a theoretical analysis of the generalization performances of regularized least square regression algorithm with spherical polynomial kernels. An et al. [1] considered regularized least squares scheme by using spherical designs, which can be used to handle the data set with high level noise when the regularized operator is chosen as spherical Laplace-Beltrami operator.

Based on the denoising effect of the Laplace-Beltrami operator and the superiority of MLS approximation, this paper proposes an approximation scheme of regularized moving least squares

*Supported by the National Natural Science Foundation of China (No. 61672477)

(RMLS) with the Laplace-Beltrami operator. A pointwise approximation issue on the unit sphere $\mathbb{S}^2$, which aims to handle data set with high level noise, is considered, and the errors of approximation by the proposed RMLS are estimated. The numerical experiments are designed to further show the effectiveness of the proposed new method.

This paper is organized as follows. In Section 2, the necessary backgrounds about spherical harmonics, sphere function spaces, and Laplace-Beltrami operator are reviewed. The MLS approximation on the sphere is stated in Section 3. In Section 4, we propose the RMLS approximation scheme. Section 5 devotes to give the error estimation for RMLS approximation. Numerical experiments are given in Section 6 to demonstrate the effectiveness of RMLS approximation scheme for data with high level noise.

2 Preliminaries

2.1 Spherical harmonics, sphere function spaces, and sphere point sets

In this section, we introduce some notations about spherical harmonics, sphere function spaces, and sphere point sets, which can be found in [23] and [33].

Let $\mathbb{S}^2$ be the unit sphere embedded in the Euclidean space $\mathbb{R}^3$, i.e.,

$$\mathbb{S}^2 := \left\{ x := (x_1, x_2, x_3) \in \mathbb{R}^3 : \|x\|_2^2 := x_1^2 + x_2^2 + x_3^2 = 1 \right\}.$$

For integer $l \geq 0$, the restriction to $\mathbb{S}^2$ of a homogeneous harmonic polynomial with degree l is called a spherical harmonic with degree l. The class of all spherical harmonics with degree l is denoted by $\mathcal{H}_l$, and it is well-known that spherical harmonics of different degree are orthogonal with respect to the inner product $\langle f, g \rangle_{L_2} := \int_{\mathbb{S}^2} f(x)g(x)d\omega(x)$,where $d\omega$ denotes surface measure on $\mathbb{S}^2$. Hence, if we choose an orthogonal basis $\{Y_{l,k} : k = 1, \ldots, 2l+1\}$ for each $\mathcal{H}_l$, then the set $\{Y_{l,k} : l = 0, 1, \ldots,\ k = 1, \ldots, 2l+1\}$ is an orthogonal basis for $L_2(\mathbb{S}^2)$. The class of all spherical harmonics with total degree $l \leq L$ is denoted by $\mathcal{P}_L$. Of course, $\mathcal{P}_L = \bigoplus_{l=0}^{L} \mathcal{H}_l$, and the dimension of $\mathcal{H}_l$ is $2l+1$ and that of $\mathcal{P}_L$ is $(L+1)^2$. The well-known addition formula is given by (see [23])

$$\sum_{k=1}^{2l+1} Y_{l,k}(x)Y_{l,k}(y) = \frac{2l+1}{4\pi} P_l(x \cdot y), \tag{2.1}$$

where P_l is the Legendre polynomial with degree l and dimension 3, which is normalized such that $P_l(1) = 1$.

For every $p \in \mathcal{P}_L$, we have

$$p(x) = \sum_{l=0}^{L} \sum_{k=1}^{2l+1} \alpha_{l,k} Y_{l,k}(x), \tag{2.2}$$

where $\alpha_{l,k} = \int_{\mathbb{S}^2} p(y)Y_{l,k}(y)d\omega(y)$.

We denote by $C(\mathbb{S}^2)$ the space of continuous functions on $\mathbb{S}^2$ endowed with the uniform (supremum) norm $\|f\|_{C(\mathbb{S}^2)} := \sup_{x\in\mathbb{S}^2} |f(x)|$.The geodesic distance between two points on the unit sphere $\mathbb{S}^2$ is defined by $d(x, y) := \arccos(\langle x, y \rangle)(x, y \in \mathbb{S}^2)$,where $\langle x, y \rangle$ denotes the Euclidean inner product.

Let $x_1, x_2, \ldots, x_N \in \mathbb{S}^2$, which are pairwise distinct. Then $X := \{x_1, x_2, \ldots, x_N\}$ is called as a centers set. The mesh norm of X is denoted by

$$h_{X,\mathbb{S}^2} := \sup_{x\in\mathbb{S}^2} \min_{x_i\in X} d(x, x_i),$$

and the separation radius is defined to be

$$q_{X,\mathbb{S}^2} := \frac{1}{2} \min_{i\neq j} d(x_i, x_j)$$

that is half of the smallest geodesic distance between any two distinct points in X. It is easy to see that $h_{X,\mathbb{S}^2} \geq q_{X,\mathbb{S}^2}$, where equality can hold only for a uniform distribution of points on the circle $\mathbb{S}^1$.

For a given point set X, if there exists a constant $c_1 > 1$, such that

$$q_{X,\mathbb{S}^2} \leq h_{X,\mathbb{S}^2} \leq c_1 q_{X,\mathbb{S}^2}, \tag{2.3}$$

then X is called quasi-uniform.

In addition, the set X is said to be $\mathcal{P}_L$-unisolvent (see [32]), if

$$p \in \mathcal{P}_L, \quad p(x_i) = 0 \text{ for } i = 1, 2, \ldots, N \Rightarrow p = 0.$$

2.2 Laplace-Beltrami operator

In this subsection, we introduce Laplace-Beltrami operator on $\mathbb{S}^2$ (see [23, 33]). The Laplace-Beltrami operator is defined by

$$\Delta f := \sum_{i=1}^{3} \frac{\partial^2 g(x)}{\partial x_i^2}\bigg|_{\|x\|_2=1}, \quad g(x) := f\left(\frac{x}{\|x\|_2}\right).$$

In fact, the Laplace-Beltrami operator is the angular part of the Laplace operator in three dimensions

$$\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}.$$

Giving point $x := (x_1, x_2, x_3)$ on $\mathbb{S}^2$, then the related spherical polar coordinate system is (θ, φ), $0 \leq \theta \leq \pi, 0 \leq \varphi < 2\pi$, in terms of polar coordinate transformation

$$x_1 = \sin\theta\cos\varphi, \quad x_2 = \sin\theta\sin\varphi, \quad x_3 = \cos\theta,$$

the Laplace-Beltrami operator acting as a differential operator can be written by

$$\Delta := \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial}{\partial\theta}) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}.$$

The literature has pointed an intrinsic characterization of spherical harmonics, which is every element of $\mathcal{H}_l$ is an eigenfunction corresponding to the eigenvalue $-l(l+1)$ of the Laplace-Beltrami operator Δ, namely that

$$\Delta Y_{l,k}(x) = -l(l+1)Y_{l,k}(x).$$

In fact, Δ is a semi-positive operator, and for any $s > 0$ we can define $(-\Delta)^s$ as

$$(-\Delta)^s Y_{l,k} = (l(l+1))^s Y_{l,k}(x) = \beta_l Y_{l,k}(x).$$

So, for $p(x) \in \mathcal{P}_L$, $(-\Delta)^s p(x)$ can be represented by

$$(-\Delta)^s p(x) = \sum_{l=0}^{L} \beta_l \sum_{k=1}^{2l+1} Y_{l,k}(x)\langle Y_{l,k}, p\rangle = \sum_{l=0}^{L} \beta_l \int_{\mathbb{S}^2} \frac{2l+1}{4\pi} P_l(x,y)p(y)d\omega(y),$$

where $\beta_\mu = (\mu(\mu+1))^s$, $\mu = 0, 1, \ldots, L$, and P_l is the Legendre polynomial with degree l.

3 Moving least squares

Moving least squares (MLS) approximation has been frequently applied to potential energy surfaces [20], surface reconstruction [15], and partial differential equations [7]. In order to propose regularized moving least squares (RMLS) in the next section, we should first review some details about MLS approximation on the sphere [35].

The issue of MLS approximation on the sphere has been given some detailed discussions by Wendland in [34, 35, 36]. Suppose an unknown continuous function $f \in C(\mathbb{S}^2)$ and $x \in \mathbb{S}^2$, we can construct an approximation of $f(x)$ from values $\{f(x_i)\}_{i=1}^N$ of f on a given point set

$X = \{x_1, \ldots, x_N\} \subseteq \mathbb{S}^2$. Then the approximate value $p^*(x)$ of $f(x)$ can be obtained by the solution of following minimization problem

$$\min \left\{ \sum_{i=1}^{N} (f(x_i) - p(x_i))^2 w(x, x_i) : p \in \mathcal{P} \right\}, \tag{3.4}$$

where $\mathcal{P} \subseteq C(\mathbb{S}^2)$ is a finite dimensional subspace, usually spanned by spherical harmonics, and $w : \mathbb{S}^2 \times \mathbb{S}^2 \to [0, \infty)$ is a continuous function. Since we consider a local process, we choose $w(x, y)$ as a radial and compactly supported function, even if it is not really necessary. So Wendland [34, 35, 36] chose continuous function $\phi : [0, \infty) \to [0, \infty)$ with

- $\phi(r) > 0, 0 \leq r < 1$,
- $\phi(r) = 0, r \geq 1$,

and define

$$\theta_\delta(x, y) := \frac{1}{\phi\left(\frac{d(x,y)}{\delta}\right)}, \quad x, y \in \mathbb{S}^2, \tag{3.5}$$

where $\delta > 0$ is a scale. Then above weight function $w(x, x_i)$ has the following form

$$w(x, x_i) = \frac{1}{\theta_\delta(x, x_i)} = \phi\left(\frac{d(x, y)}{\delta}\right).$$

For $X = \{x_1, x_2, \ldots, x_N\}$, we further define the index set $I(x)$ as

$$I(x) := I(x, \delta, X) = \{i \in \{1, 2, \ldots, N\} : d(x, x_i) < \delta\}, \tag{3.6}$$

which contains the subscripts of points within the spherical cap of radius δ centered at x. And we choose $\mathcal{P} = \mathcal{P}_L$. Then the MLS approximation (3.4) takes the form (see [18, 34, 35, 36])

$$s_{f,X}(x) = \sum_{i \in I(x)} a_i^*(x) f(x_i),$$

where the coefficients $a_i^*(x)$ are determined by minimizing

$$\frac{1}{2} \sum_{i \in I(x)} a_i^2(x) \theta_\delta(x, x_i) \tag{3.7}$$

under the constraints

$$\sum_{i \in I(x)} a_i(x) p(x_i) = p(x), \quad p \in \mathcal{P}_L. \tag{3.8}$$

If X satisfies certain conditions, then we have the following theorem [35].

Theorem 3.1 *Assume that $Z = \{x_i \in X : i \in I(x, \delta, X)\}$ is $\mathcal{P}_L$-unisolvent. Then the minimization problem (3.7) with constraint (3.8) has an unique solution $a_i^*(x)$:*

$$a_i^*(x) = \phi\left(\frac{d(x, x_i)}{\delta}\right) \sum_{\mu=0}^{L} \sum_{\nu=1}^{2\mu+1} \lambda_{\mu,\nu} Y_{\mu,\nu}(x_i),$$

where $i \in I(x), x_i \in Z$, and the Lagrange multipliers $\lambda_{l,k}$ have unique solution by solving the following system of equations:

$$\sum_{\mu=0}^{L} \sum_{\nu=1}^{2\mu+1} \lambda_{\mu,\nu} \sum_{i \in I(x)} \phi\left(\frac{d(x, x_i)}{\delta}\right) Y_{\mu,\nu}(x_i) Y_{l,k}(x_i) = Y_{l,k}(x)$$

with $0 \leq l \leq L, 1 \leq k \leq 2\mu + 1$.

Since $Z = \{x_{i_1}, x_{i_2}, \ldots, x_{i_M}\} = \{x_i, i \in I(x)\} \subseteq X$ involves the choice of scale δ, so it is also an interesting research direction. From [36] we know that if x lies in a region with a high data density, then the δ should be chosen small. However, we should choose a bigger δ, since our method is local. Therefore, we often choose

$$\delta = \delta_X = C_1 h_X, \tag{3.9}$$

where C_1 is a constant.

4 Regularized moving least squares with Laplace-Beltrami operator

In this section, we propose a category of local polynomial approximation on the unit sphere $\mathbb{S}^2$ in terms of an improvement of MLS, and give the model of the RMLS.

For an unknown continuous function $f \in C(\mathbb{S}^2), X = \{x_1, x_2, \cdots, x_N\} \subseteq \mathbb{S}^2$, and $x \in \mathbb{S}^2$, we can get an approximate value $p(x)$ of $f(x)$ from values $\{f(x_i)\}_{i=1}^N$ by the solution p of following minimization problem

$$\min \left\{ \sum_{i=1}^{N} \left((f(x_i) - p(x_i))^2 + \lambda \left((-\Delta)^s p(x_i) \right)^2 \right) \phi \left(\frac{d(x, x_i)}{\delta} \right) : p \in \mathcal{P}_L \right\}, \tag{4.10}$$

where $(-\Delta)^s$ and ϕ is defined as above, $\lambda > 0$ is a regularization parameter.

Similar to [18, 34, 35, 36], we want to use polynomial local reconstruction to estimate approximation order. So, the new approximation form can be constructed and it is the same as the solution of (4.10). We construct the new approximation form:

$$s_{f,X}(x) = \sum_{i \in I(x)} a_i^*(x) f(x_i), \tag{4.11}$$

where the coefficients are determined by minimizing

$$\frac{1}{2} \sum_{i \in I(x)} a_i^2(x) \theta_\delta(x, x_i) \tag{4.12}$$

under the constraints

$$\sum_{i \in I(x)} a_i(x) p(x_i) = q(x), \;\; p \in \mathcal{P}_L, \tag{4.13}$$

where

$$q(x) = \sum_{\mu=0}^{L} \sum_{\nu=1}^{2\mu+1} (1 + \lambda \beta_\mu^2)^{-1} \hat{p}_{\mu,\nu} Y_{\mu,\nu}(x),$$

$\hat{p}_{\mu,\nu}$ is the Fourier coefficient of p, and $\beta_\mu = (\mu(\mu+1))^s$. The following (2) of Theorem 4.1 shows that the constructed approximation form (4.11) and constrained optimization problems (4.12)-(4.13) are valid.

In the following, we focus on how to solve the new constrained optimization problem, where $Z = \{x_{i_1}, x_{i_2}, \ldots, x_{i_M}\} = \{x_i, i \in I(x)\} \subseteq X$. We need the following notations:

$$f = (f(x_1), f(x_2), \ldots, f(x_M))^T;$$
$$a = (a_1^*(x), a_2^*(x), \ldots, a_M^*(x))^T;$$
$$\alpha = (\alpha_{0,1}, \ldots, \alpha_{L,2L+1})^T;$$
$$\varphi = (Y_{0,1}(x), \ldots, Y_{L,2L+1}(x))^T;$$
$$W = \operatorname{diag}\left\{ \phi\left(\frac{d(x,x_1)}{\delta}\right), \phi\left(\frac{d(x,x_2)}{\delta}\right), \ldots, \phi\left(\frac{d(x,x_M)}{\delta}\right) \right\};$$
$$B = \operatorname{diag}\{\beta_0, \beta_1, \beta_1, \beta_1, \ldots, \underbrace{\beta_\mu, \ldots, \beta_\mu}_{2\mu+1}, \ldots, \beta_{2L+1}, \ldots, \beta_{2L+1}\};$$

$$Y = \begin{pmatrix} Y_{0,1}(x_1) & Y_{1,1}(x_1) & Y_{1,2}(x_1) & \cdots & Y_{L,2L+1}(x_1) \\ Y_{0,1}(x_2) & Y_{1,1}(x_2) & Y_{1,2}(x_2) & \cdots & Y_{L,2L+1}(x_2) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ Y_{0,1}(x_M) & Y_{1,1}(x_M) & Y_{1,2}(x_M) & \cdots & Y_{L,2L+1}(x_M) \end{pmatrix}.$$

The following Theorem 4.1 will give the concrete form of the solution of RMLS approximation, and proves that the solution of (4.10) is equivalent to the solution of the minimization problem (4.12) with constraint (4.13). Namely, the constructions (4.11)-(4.13) are valid.

Theorem 4.1 *The following statements hold.*

(1). For a given point set $X = \{x_1, \ldots, x_N\} \subset \mathbb{S}^2$, if $Z = \{x_i \in X : i \in I(x, \delta, X)\}$ is $\mathcal{P}_L$-unisolvent, then the minimization problem (4.12) with constraint (4.13) has unique solution $a_i^*(x)$:

$$a_i^*(x) = \phi\left(\frac{d(x, x_i)}{\delta}\right) \sum_{\mu=0}^{L} \sum_{\nu=1}^{2\mu+1} z_{\mu,\nu} Y_{\mu,\nu}(x_i), \ i \in I(x), \tag{4.14}$$

where $z_{l,k}$ can be obtained by solving the following system of equations:

$$\sum_{\mu=0}^{L} \sum_{\nu=1}^{2\mu+1} z_{\mu,\nu} \sum_{i \in I(x)} \phi\left(\frac{d(x, x_i)}{\delta}\right) Y_{\mu,\nu}(x_i) Y_{l,k}(x_i) = (1 + \lambda \beta_l^2)^{-1} Y_{l,k}(x) \tag{4.15}$$

with $0 \le l \le L, 1 \le k \le 2l+1$, $\lambda > 0$;

(2). The solution of (4.10) is equivalent to the solution of the minimization problem (4.12) with constraint (4.13).

Proof. We first prove (1). Similar to [34], [35] and [36], we introduce Lagrange multiplies $z = (\hat{z}_{0,1}, \ldots, \hat{z}_{L,2L+1})$ to solve the optimal problem (4.12) with constraint (4.13). Let

$$J = \frac{1}{2} \sum_{i \in I(x)} a_i^2(x) \theta_\delta(x, x_i) - \sum_{\mu=0}^{L} \sum_{\nu=1}^{2\mu+1} z_{\mu,\nu} \left(\sum_{i \in I(x)} a_i(x) Y_{\mu,\nu}(x_i) - (1 + \lambda \beta_\mu^2)^{-1} Y_{\mu,\nu}(x) \right),$$

where $z_{\mu,\nu} = \hat{z}_{\mu,\nu} \hat{p}_{\mu,\nu}$. We solve partial derivatives about $a_i(x)$ and $z_{l,k}$ for J, respectively,

$$\frac{\partial J}{\partial a_i(x)} = a_i(x) \theta_\delta(x, x_i) - \sum_{\mu=0}^{L} \sum_{\nu=1}^{2\mu+1} z_{\mu,\nu} Y_{\mu,\nu}(x_i) = 0, \ i \in I(x),$$

$$\frac{\partial J}{\partial z_{l,k}} = -\sum_{i \in I(x)} a_i(x) Y_{l,k}(x_i) + (1 + \lambda \beta_l^2)^{-1} Y_{l,k}(x) = 0, \ 0 \le l \le L, 1 \le k \le 2l+1,$$

then, solving the above equations, we can get (4.14) and (4.15).

In order to prove equivalent conditions, the solution (4.14) of the optimal problem (4.12) under constraint (4.13) can be written as matrix form

$$a = WY(Y^T WY + \lambda B^T Y^T WYB)^{-1} \varphi.$$

Next, we prove the uniqueness of the solution. In fact, we only need to prove that $Y^T WY + \lambda B^T Y^T WYB$ is a positive definite matrix. For any vector

$$r = (r_{0,1}, \ldots, r_{L,2L+1})^T \in \mathbb{R}^{(L+1)^2},$$

and for $i \in I(x), w(x, x_i) > 0$,

$$\begin{aligned} & r^T(Y^T WY + \lambda B^T Y^T WYB) r = (Yr)^T W(Yr) + r^T \lambda B^T Y^T WYBr \\ = & \sum_{i \in I(x)} \phi\left(\frac{d(x, x_i)}{\delta}\right) \left(\sum_{l=0}^{L} \sum_{k=1}^{2l+1} Y_{l,k} \right)^2 + \sum_{i \in I(x)} \phi\left(\frac{d(x, x_i)}{\delta}\right) \left(\sum_{l=0}^{L} \sum_{k=1}^{2l+1} r_{l,k} \beta_l Y_{l,k} \right)^2 \\ \ge & \ 0. \end{aligned}$$

Since the entries of diagonal line of matrix B are not all 0, and Z is $\mathcal{P}_L$-unisolvent, we see that

$$\sum_{i \in I(x)} \phi\left(\frac{d(x, x_i)}{\delta}\right) \left(\sum_{l=0}^{L} \sum_{k=1}^{2l+1} Y_{l,k} \right)^2 + \sum_{i \in I(x)} \phi\left(\frac{d(x, x_i)}{\delta}\right) \left(\sum_{l=0}^{L} \sum_{k=1}^{2l+1} r_{l,k} \beta_l Y_{l,k} \right)^2 = 0,$$

implies $r = 0$. So $Y^T WY + \lambda B^T Y^T WYB$ is a positive definite matrix.

We now prove property (2). Let $\{Y_{0,1},\dots,Y_{L,2L+1}\}$ be a set of the spherical harmonics in $\mathcal{P}_L$. Then the minimizer of (4.10) can be written as

$$p^*(x)=\sum_{\mu=0}^{L}\sum_{\nu=1}^{2\mu+1}\alpha_{\mu,\nu}Y_{\mu,\nu}(x).$$

Thus

$$\alpha^T=f^TWY(Y^TWY+\lambda B^TY^TWYB)^{-1},$$

hence

$$\begin{aligned}p^*(x)&=\sum_{\mu=0}^{L}\sum_{\nu=1}^{2\mu+1}\alpha_{\mu,\nu}Y_{\mu,\nu}(x)=\alpha^T\varphi\\&=f^TWY(Y^TWY+\lambda B^TY^TWYB)^{-1}\varphi=f^Ta\\&=s_{f,X}(x).\end{aligned}$$

The proof of Theorem 4.1 is complete. □

The following Theorem 4.2 devotes that $p(x)$ of the solution of the minimization problem (4.12) with constraint (4.13) uniformly converges to "s-smoothed" solution f_s:

$$f_s(x):=\sum_{l=0}^{\infty}\frac{1}{1+\lambda(l(l+1))^{2s}}\sum_{k=1}^{2l+1}\hat{f}_{l,k}Y_{l,k}(x)=\sum_{l=0}^{\infty}\frac{1}{1+\lambda(l(l+1))^{2s}}\frac{1}{4\pi}\int_{\mathbb{S}^2}P_l(x\cdot y)f(y)d\omega(y),$$

where the last equation uses the addition theorem (2.1).

Theorem 4.2 *Assume that the order of Laplace-Beltrami operator $s>1/2$, $p(x)$ is the solution of the minimization problem (4.12) with constraints (4.13), and L is the order of $p(x)$, then we have* $\lim_{L\to\infty}\|p-f_s\|_{C(\mathbb{S}^2)}=0$.

This theorem has been proved in [1].

5 Error estimates

In this section, we will give an error estimate for RMLS approximation, which ensures the fact that RMLS approximation scheme is reasonable (see Theorem 6 below). But before starting the error analysis, we need to collect a few auxiliary results. The following Lemma 4 indicates the local polynomial reproduction on the sphere, which is quoted from [35]. We also refer the reader to [18] and [34] for a general form of the local polynomial reproduction property, and it plays an important role in the error estimates for RMLS approximation.

Lemma 5.1 *There exist constants $h_0,C_2,C_3>0$ such that for every point set $X=\{x_1,x_2,\dots,x_N\}\subseteq\mathbb{S}^2$ with $h_{X,\mathbb{S}^2}\le h_0$ and every $x\in\mathbb{S}^2$, there exist $a_1^X(x),a_2^X(x),\dots,a_N^X(x)$ satisfying that*
(1) $\sum_{i=1}^{N}a_i^X(x)p(x_i)=q(x)$, *for any* $p\in\mathcal{P}_L$;
(2) $a_i^X(x)=0$, *if* $d(x,x_i)>C_2h_{X,\mathbb{S}^2}$;
(3) $\sum_{i=1}^{N}|a_i^X(x)|\le C_3$,
where

$$q(x)=\sum_{\mu=0}^{L}\sum_{\nu=1}^{2\mu+1}(1+\lambda\beta_\mu^2)^{-1}\hat{p}_{\mu,\nu}Y_{\mu,\nu}(x).$$

The following Lemma 5.2 is quoted from [35], which shows that $|I(x)|$ is uniformly bounded in terms of packing argument from [27], and it plays an important role in the error estimates for RMLS and MLS approximation.

Lemma 5.2 *Assume that $X = \{x_1, x_2, \ldots, x_N\} \subset \mathbb{S}^2$ is quasi-uniform with $h_{X,\mathbb{S}^2} \le h_0$, $I(x) := \{i \in \{1, 2, \ldots, N\} : d(x, x_i) < \delta\}$, and $\delta = C_1 h_{X,\mathbb{S}^2}$. Then*

$$|I(x)| \le \frac{q_X + \delta}{q_X} \le (1 + c_1 C_1),$$

where the c_1 and C_1 are constants which associate with (2.3) and (3.9), respectively.

From Lemma 5.1 and Lemma 5.2, we use the similar techniques of [35], and obtain the following Theorem 5.1, which is an error estimate for RMLS approximation.

Theorem 5.1 *Let C_1, C_3, δ be given in (3.9), Lemma 5.1, and Lemma 5.2. Suppose that $X = \{x_1, x_2, \ldots, x_N\} \subseteq \mathbb{S}^2$ is quasi-uniform, and $s_{X,f}$ is the RMLS approximation of $f \in C(\mathbb{S}^2)$ by minimization (4.12) under the constraint (4.13). Then there exist constants h_0 and C which are independent of f and X, such that for every X with $h_{X,\mathbb{S}^2} \le h_0$ and every $x \in \mathbb{S}^2$, the error between f and $s_{X,f}$ can be bounded by*

$$|f(x) - s_{X,f}(x)| \le C c_f C_1^{l+1} h_{X,\mathbb{S}^2}^{l+1}.$$

Proof. Let $q \in \mathcal{P}_L$ and $B(x, \delta) = \{y \in \mathbb{S}^2; d(x, y) \le \delta\}$. We adopt the standard arguments to estimate the error of RMLS approximation:

$$\begin{aligned} |f(x) - s_{X,f}(x)| &= |f(x) - q(x) + q(x) - s_{X,f}(x)| \le |f(x) - q(x)| + |q(x) - s_{X,f}(x)| \\ &\le \|f(x) - q(x)\|_{\infty,B(x,\delta)} + \sum_{i \in I(x)} |a_i^*(x)| \|f(x) - p(x)\|_{\infty,B(x,\delta)} \end{aligned}$$

where the relationship between $p(x)$ and $q(x)$ is $q(x) = \sum_{\mu=0}^{L} \sum_{\nu=1}^{2\mu+1} (1 + \lambda\beta_\mu^2)^{-1} \hat{p}_{\mu,\nu} Y_{\mu,\nu}(x)$, $\hat{p}_{\mu,\nu}$ is the Fourier coefficient of p, and $\beta_\mu = (\mu(\mu + 1))^s$. So we can write

$$\max\left\{\|f(x) - q(x)\|_{\infty,B(x,\delta)}, \|f(x) - p(x)\|_{\infty,B(x,\delta)}\right\} := \|f(x) - G(x)\|_{\infty,B(x,\delta)},$$

then

$$|f(x) - s_{X,f}(x)| \le (1 + \sum_{i \in I(x)} |a_i^*(x)|) \|f(x) - G(x)\|_{\infty,B(x,\delta)}.$$

For $\sum_{i \in I(x)} |a_i^*(x)|$, using Cauchy inequality we have

$$\sum_{i \in I(x)} |a_i^*(x)| \le \left(\sum_{i \in I(x)} |a_i^*(x)|^2 \theta_\delta(x, x_i)\right)^{1/2} \left(\sum_{i \in I(x)} \phi(\frac{d(x, x_i)}{\delta})\right)^{1/2}. \tag{5.16}$$

Now we prove that the first term of the right of (5.16) is bounded. According to $h_{X,\mathbb{S}^2} \le h_0$, we can get $a_i(x)$ that reproduces spherical harmonics and vanishes if $d(x, x_i) > \frac{\delta}{2}$. We set $\widetilde{I(x)} = \{i : d(x, x_i) \le \frac{\delta}{2}\}$, then, by the minimization condition it is not difficult for us to obtain that

$$\begin{aligned} \sum_{i \in I(x)} |a_i^*(x)|^2 \theta_\delta(x, x_i) &\le \sum_{i \in \widetilde{I(x)}} |a_i(x)|^2 \theta_\delta(x, x_i) \le \frac{1}{\min_{i \in \widetilde{I(x)}} \phi(\frac{d(x,x_i)}{\delta})} \sum_{i \in \widetilde{I(x)}} |a_i(x)|^2 \\ &\le \left(\sum_{i=1}^{N} |a_i(x)|\right)^2 \frac{1}{\min_{y \in Z} \phi(\frac{d(x,y)}{\delta})} \le C_3^2 \frac{1}{\min_{y \in Z} \phi(\frac{d(x,y)}{\delta})}, \end{aligned}$$

and

$$\sum_{i \in \widetilde{I(x)}} \phi(\frac{d(x, x_i)}{\delta}) \le |I(x)| \|\phi\|_\infty.$$

From Lemma 5.2, we see that $|I(x)|$ is uniformly bounded, which implies that (5.16) is bounded. Therefore, there exists a constant C, such that $\left(1 + \sum_{i \in I(x)} |a_i^*(x)|\right) < C$.

Next, we prove that $\|f(x)-G(x)\|_{\infty,Z}$ is bounded. According to [35], without loss of generality, we suppose that $x=(0,0,1)^T$. Then

$$B(x,\delta)=\{y\in\mathbb{S}^2 : d(x,y)<\delta\}=\{y\in\mathbb{S}^2 : y_3>\cos\delta\}.$$

We define the bijective map $T:U\to B(x,\delta)$ by $\tilde{y}\to(\tilde{y},\sqrt{1-\|\tilde{y}\|_2^2})^T$, where $U=\{\tilde{y}\in R^2 : \|\tilde{y}\|_2^2<1-\cos^2\delta\}$. Obviously, the inverse of T is $T^{-1}(y)=\tilde{y}=(y_1,y_2)^T$. Then, the Taylor expansion of g around $\tilde{x}=0$ is

$$g(\tilde{y})=\sum_{|\alpha|\le l}\frac{g^{(\alpha)}(0)}{\alpha!}\tilde{y}^\alpha+\sum_{|\alpha|=l+1}\frac{g^{(\alpha)}(\xi)}{\alpha!}\tilde{y}^\alpha.$$

So

$$f(y)=g\circ T^{-1}(y)=\sum_{|\alpha|\le l}c_\alpha y^\alpha+\sum_{|\alpha|=l+1}\frac{g^{(\alpha)}(\xi)}{\alpha!}\tilde{y}^\alpha,$$

and

$$G(y)=\sum_{|\alpha|\le l}c_\alpha y^\alpha.$$

Hence

$$\begin{aligned}|f(y)-G(y)| &\le c_f\|\tilde{y}\|_2^{l+1}=c_f(1-y_3^2)^{(l+1)/2}\le c_f(1-\cos^2\delta)^{(l+1)/2}=c_f(\sin\delta)^{l+1}\\ &\le c_f\delta^{l+1}=c_fC_1h_{X,\mathbb{S}^2}^{l+1}.\end{aligned}$$

Therefore,

$$|f(x)-s_{X,f}(x)|\le C\|f-G\|_{\infty,B(x,\delta)}\le Cc_fC_1^{l+1}h_{X,\mathbb{S}^2}^{l+1}.$$

The proof of Theorem 5.1 is complete. □

6 Numerical experiments

In order to further validate our theoretical results derived in the previous sections, this section presents some numerical experiments handling data set with high level noise. In our experiments, we choose two test functions, where the Franke function $f(x,y,z)$ is chosen as the first test function which has been frequently used in the other literature (for example, [28, 35]),

$$\begin{aligned}f_1(x,y,z) &= \frac{3}{4}\exp\left(-\frac{(9x-2)^2}{4}-\frac{(9y-2)^2}{4}-\frac{(9z-2)^2}{4}\right)\\ &+\frac{3}{4}\exp\left(-\frac{(9x+1)^2}{49}-\frac{(9y+1)^2}{10}-\frac{(9z+1)^2}{10}\right)\\ &+\frac{1}{2}\exp\left(-\frac{(9x-7)^2}{4}-\frac{(9y-3)^2}{4}-\frac{(9z-5)^2}{4}\right)\\ &-\frac{1}{5}\exp\left(-(9x-4)^2-(9y-7)^2-(9z-5)^2\right),\quad (x,y,z)\in\mathbb{S}^2.\end{aligned}$$

This function is shown in the Figure 2 (a), and it is $C^\infty(\mathbb{S}^2)$. The second test function is spherical cap function which is a sum of the Franke function f_1 and an other function f_{cap} (see [38]), which is defined by $f_2:=f_1+f_{\text{cap}}$,where

$$f_{\text{cap}}:=\begin{cases}\rho\cos\left(\dfrac{\pi\arccos(\langle x_c,x\rangle)}{2r}\right), & x\in C(x_c,r);\\ 0, & \text{otherwise},\end{cases}$$

and ρ is a positive number. We set $x_c=(-\frac{1}{2},-\frac{1}{2},\sqrt{\frac{1}{2}})$, $\rho=2$, and $r=\frac{1}{2}$ in the experiment. This function is shown in the Figure 3 (a).

In the RMLS approximation, the weight function plays an important role. We choose a famous radial basis function $\phi(r)$ as weight function in our numerical experiments, that is

$$\phi(r) = (1-r)_+^4(4r+1),$$

which is called Wendland function (see [37]). The uniform error of the approximation is estimated by

$$\|f-p\|_{C(\mathbb{S}^2)} \approx \max_{x_i \in X} |f(x_i) - p(x_i)|.$$

In our numerical experiments, we choose X to be a set of 1024 points generated from the equal area algorithm [30], which is shown in Figure 1.

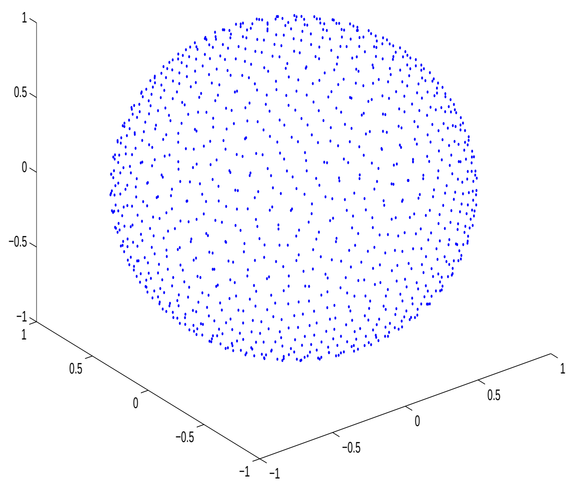


Figure 1: A set of 1024 points generated from the equal area algorithm

Next, we consider two groups of numerical experiments reconstructing the test function f_1 and f_2 in terms of RMLS and MLS, where the data set X has been contaminated by high levels of noise. In the experiment 1 and 2, $X = \{x_1, x_2, \ldots, x_N\}$, and $N = 1024$, meanwhile, 30% noise have been used in X, where the noise is a sample of a normal random variable with mean 0 and standard deviation $\sigma = 0.1$. In order to achieve uniform standard of comparison, we take polynomial degree $L = 2$ and scale $\delta = 0.25$.

Experiment 1. We want to reconstruct the Frank function f_1 from contaminated data and compare approximation results of RMLS ($\lambda = 0.2$) and MLS ($\lambda = 0$), meanwhile, s is set as 2.

Figure 2 illustrates that RMLS exists more obvious advantages than MLS when we reconstruct test function f_1 from data set with high level noise. The Figure 2 (a) shows original function f_1, the Figure 2 (b) reports f_1 with high level noise, and the Figure 2 (c) reveals approximation result of RMLS for reconstructing f_1, and the uniform error of RMLS approximation is 0.0868. At last, the Figure 2 (d) shows approximation result of MLS for reconstructing f_1, and the uniform error of MLS approximation is 0.1363.

As we known, the test function f_1 called Franke function is $C^\infty(\mathbb{S}^2)$, however, test function f_2 is continuous on the unit sphere $\mathbb{S}^2$ but not differentiable on the boundary of spherical cap $C(x_c, r)$. In order to show the effect of RMLS approximation for reconstructing function, we reconstruct f_2 from data set with high level noise in the following experiments.

Experiment 2. Test function f_2 is reconstructed from data set with high level noise, and its designing approach is similar with Experiment 1. First of all, we fix the order of Laplace-Beltrami

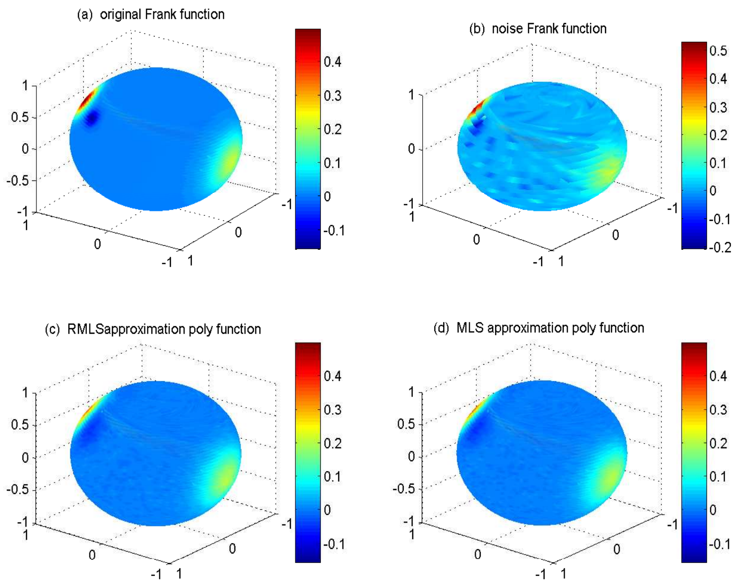


Figure 2: A result of test function f_1 in experiment 1

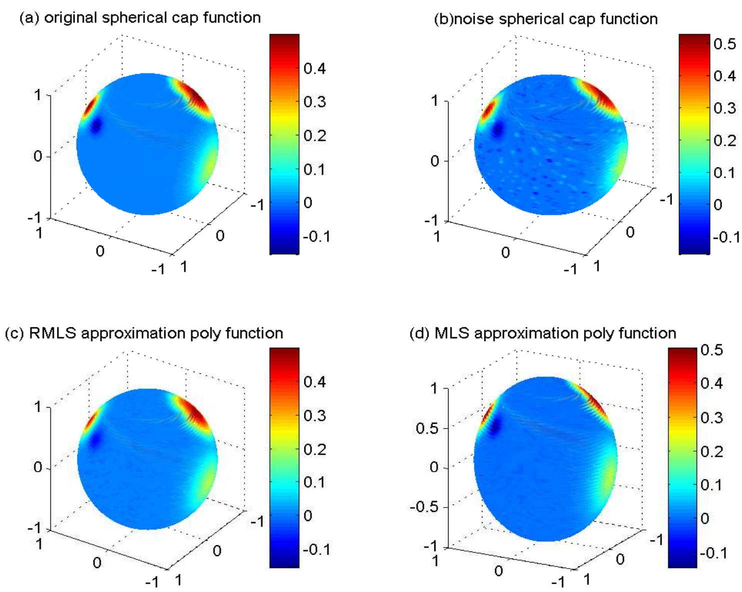


Figure 3: A result of test function f_2 in experiment 2

Table 1: The uniform error of MLS and RMLS for f_1 and f_2 when λ and s were changed

λ	uniform error for f_1		uniform error for f_2	
	$s=2$	$s=4$	$s=2$	$s=4$
0	0.1363	0.1363	0.1303	0.1303
0.2	0.0868	0.0805	0.0991	0.0958
0.4	0.0909	0.0881	0.0990	0.0818
0.6	0.0936	0.0839	0.0851	0.0755
0.8	0.0936	0.0832	0.0836	0.0801
1.0	0.0900	0.0933	0.0905	0.0829
1.2	0.0983	0.0929	0.0932	0.0932
1.4	0.0847	0.0902	0.0934	0.0758
1.6	0.0834	0.0845	0.0899	0.0791
1.8	0.1033	0.0854	0.0982	0.0863
2.0	0.0900	0.0844	0.0833	0.0830

operator $s = 4$. Secondly, in order to compare RMLS with MLS, we let $\lambda = 0$ (MLS) and $\lambda = 1.4$ (RMLS). At last, we show the superiority in terms of uniform error.

Figure 3 illustrates that RMLS exists more obvious advantages than MLS when we reconstruct test function f_2 from data set with high level noise. The Figure 3 (a) shows original function f_2, the Figure 3 (b) reports f_2 with high level noise, and the Figure 3 (c) reveals approximation result of RMLS for reconstructing f_2, and the uniform error of RMLS approximation 0.0758. Finally, the Figure 3 (d) shows approximation result of MLS for reconstructing f_2, and the uniform error of MLS approximation 0.1330.

Table 1 gives the values of uniform error for Experiment 1 and 2, when we choose different regularized parameter λ and order of Lplace-Beltrami operator s for (4.10). The results indicate that the choosing method of λ and s are uncertain, and the optimal combination of λ and s is $\lambda = 0.2, s = 4$ for approximation f_1. However, the optimal combination of λ and s is $\lambda = 1.4, s = 4$ for approximation f_2 . The different choices for the order of Lplace-Beltrami operator and regularlized parameter can provide different pointwise approximation results.

From what has been discussed above, the RMLS is better than the MLS for recoving a function from data set with high level noise. However, the choice of λ and s is critical. How to automatically choose the proper λ and s is a challenging problem.

References

[1] An C, Chen X, Sloan I H, Womersley R S. Regularized least squares approximations on the sphere using spherical designs. SIAM J. Num. Anal., 2012, 50(3): 1513-1534.

[2] Armentano M G. Error estimates in Sobolev spaces for moving least square approximations. SIAM J. Num. Anal., 2001, 39(1): 38-51.

[3] Armentano M G, Durán R G. Error estimates for moving least square approximations. Appl. Num. Math., 2001, 37(3): 397-416.

[4] Backus G, Gilbert F. Uniqueness in the inversion of inaccurate gross earth data. Phil. Trans. Roy. Soc. London, Ser. A, Math. & Phys. Sci., 1970: 123-192.

[5] Backus G, Gilbert F. Numerical applications of a formalism for geophysical inverse problems. Geophy. J. Int., 1967, 13(1-3): 247-276.

[6] Backus G, Gilbert F. The resolving power of gross earth data. Geophy. J. Roy. Astr. Soc., 2007, 16(2): 169-205.

[7] Belytschko T, Krongauz Y, Organ D, Fleming M, Krysl P. Meshless methods: an overview and recent developments. Comp. Meth. Appl. Mech. Eng., 1996, 139(1): 3-47.

[8] Bos L P, Salkauskas K. Moving least-squares are Backus-Gilbert optimal. J. Approx. Theory, 1989, 59(3): 267-275.

[9] Fasshauer G E, Schumaker L L. Scattered data fitting on the sphere. Math. Meth. Curv. Surf. II, 1998: 117-166.

[10] Franke R, Nielson G. Smooth interpolation of large sets of scattered data. Int. J. Num. Meth. Eng., 1980, 15(11): 1691-1704.

[11] Freeden W, Gervens T, Schreiner M. Constructive approximation on the sphere: with applications to geomathematics. Oxford: Clarendon Press, 1998.

[12] Golitschek M, Light W A. Interpolation by polynomials and radial basis functions on spheres. Constr. Approx., 2001, 17(1): 1-18.

[13] Hubbert S, Morton T M. L_p-error estimates for radial basis function interpolation on the sphere. J. Approx. Theory, 2004, 129(1): 58-77.

[14] Jetter K, Stöckler J, Ward J. Error estimates for scattered data interpolation on spheres. Math. Comp., 1999, 68(226): 733-747.

[15] Lancaster P, Salkauskas K. Surfaces generated by moving least squares methods. Math. Comp., 1981, 37(155): 141-158.

[16] Le Gia Q T, Narcowich F J, Ward J D, Wendland, H. Continuous and discrete least-squares approximation by radial basis functions on spheres. J. Approx. Theory, 2006, 143(1): 124-133.

[17] Levesley J, Sun X. Approximation in rough native spaces by shifts of smooth kernels on spheres. J. Approx. Theory, 2005, 133(2): 269-283.

[18] Levin D. The approximation power of moving least-squares. Math. Comp., 1998, 67(224): 1517-1531.

[19] Li L Q. Regularized least square regression with spherical polynomial kernels. Int. J. Wav. Multires. Inf. Proc., 2009, 7(06): 781-801.

[20] Maisuradze G G, Thompson D L, Wagner A F, Minkoff, M. Interpolating moving least-squares methods for fitting potential energy surfaces: Detailed analysis of one-dimensional applications. The J. Chem. Phys., 2003, 119(19): 10002-10014.

[21] McLain D H. Drawing contours from arbitrary data points. The Comp. J., 1974, 17(4): 318-324.

[22] McLain D H. Two dimensional interpolation from random data. The Comp. J., 1976, 19(2): 178-181.

[23] Müller C. Spherical harmonics. Springer, 1966.

[24] Narcowich F J, Ward J D. Scattered data interpolation on spheres: error estimates and locally supported basis functions. SIAM J. Math. Anal., 2002, 33(6): 1393-1410.

[25] Narcowich F J, Sun X, Ward J D, Wendland H. Direct and inverse Sobolev error estimates for scattered data interpolation via spherical basis functions. Found. Comp. Math., 2007, 7(3): 369-390.

[26] Narcowich F J, Sun X, Ward J D. Approximation power of RBFs and their associated SBFs: a connection. Adv. Comp. Math., 2007, 27(1): 107-124.

[27] Narcowich F J, Sivakumar N, Ward J D. Stability results for scattered-data interpolation on Euclidean spheres. Adv. Comp. Math., 1998, 8(3): 137-163.

[28] Renka R J. Multivariate interpolation of large sets of scattered data. ACM Trans. Math. Softw., 1988, 14(2): 139-148.

[29] Shepard D. A two-dimensional interpolation function for irregularly-spaced data. In: Proc. the 1968 23rd ACM Nat. Conf. ACM, 1968: 517-524.

[30] Sloan I H. Polynomial interpolation and hyperinterpolation over general regions. J. Approx. Theory, 1995, 83(2): 238-254.

[31] Sloan I H, Womersley R S. Constructive polynomial approximation on the sphere. J. Approx. Theory, 2000, 103(1): 91-118.

[32] Sloan I H, Sommariva A. Approximation on the sphere using radial basis functions plus polynomials. Adv. Comp. Math., 2008, 29(2): 147-177.

[33] Wang K Y, Li L Q. Harmonic Analysis and Approximation on the Unit Sphere. Sci. Press, Beijing, 2000.

[34] Wendland H. Local polynomial reproduction and moving least squares approximation. IMA J. Num. Anal., 2001, 21(1): 285-300.

[35] Wendland H. Moving least squares approximation on the sphere. Mathematical Methods for Curves and Surfaces, Vanderbilt Univ. Press, Nashville, TN, 2001: 517-526.

[36] Wendland H. Scattered data approximation. Cambridge: Cambr. Univ. Press, 2005.

[37] Wendland H. Piecewise polynomial, positive definite and compactly supported radial functions of minimal degree. Adv. Comp. Math., 1995, 4(1): 389-396.

[38] Williamson D L, Drake J B, Hack J J, Jakob R, Swarztrauber P N. A standard test set for numerical approximations to the shallow water equations in spherical geometry. J. Comp. Phys., 1992, 102(1): 211-224.

Chaos Control and Function Projective Synchronization of Noval Chaotic Dynamical System

M. M. El-Dessoky[1,2], E. O. Alzahrani[1] and N.A. Almohammadi[3]

[1]Mathematics Department, Faculty of Science, King Abdulaziz University,
P. O. Box 80203, Jeddah 21589, Saudi Arabia.
[2]Department of Mathematics, Faculty of Science, Mansoura University,
Mansoura 35516, Egypt.
[3]Mathematics Department, Faculty of science - AL Salmania Campus,
King Abdulaziz University, Jeddah, Saudi Arabia.

E-mail: dessokym@mans.edu.eg; eoalzahrani@gmail.com;
nalmohammadi0010@stu.kau.edu.sa

ABSTRACT

In this paper, a Noval chaotic dynamical system is proposed and the basic properties of the system are investigated. Linear feedback control technique is used to suppress chaos. The controlled system is stable under some conditions on the parameters of the system determined by Lyapunov direct method. In addition, a function projective synchronization of two identical Noval system is presented. Lyapunov method of stability is used to prove the asymptotic stability of solutions for the error dynamical system. Numerical simulations results are included to show the effectiveness of the proposed schemes.

1. INTRODUCTION

Chaos has been developed and thoroughly studied over the past two decades. A chaotic system is a nonlinear deterministic system that displays complex and unpredictable behavior. The sensitive dependence on the initial conditions and on the system's parameter variation is a prominent characteristic of chaotic behavior. Research efforts have investigated chaos control and chaos synchronization problems in many physical chaotic systems.

Controlling chaos has become a challenging topic in nonlinear dynamics. Feedback control methods are used to control chaos by stabilizing a desired unstable periodic solution which is embedded in a chaotic attractor [1-12].

Generalized synchronization is another interesting chaos synchronization technique. Li considered a new type of projective synchronization method, called a modified projective synchronization (MPS). Chen et al. introduced another new projective synchronization which is called a function projective synchronization (FPS), where the response of the synchronized dynamical states synchronizes up to scaling function factor [11-29].

The object of this paper is to study the function project synchronization (FPS) of two identical Noval chaotic system with known parameters.

The paper is organized as follows. In Section 2, presented the model of Noval chaotic system. In Section 3, the dissipation, symmetry, equilibrium points and lyapunov exponents. In Section 4, the feedback control method is applied to Noval system and numerical simulations are presented to show the effectiveness of the proposed method. In Section 5, the proposed scheme is applied to function projective synchronize two identical Noval chaotic systems. Also numerical simulations are presented in order to validate the proposed synchronization approach. Finally, in Section 6 the conclusion of the paper is given.

2. THE MODEL OF NOVAL CHAOTIC SYSTEM

The Noval chaotic system [30] is described by the following system of differential equations:

$$\begin{aligned} \dot{x} &= (-a+\frac{1}{b})x + xy + z \\ \dot{y} &= -by - x^2 \\ \dot{z} &= -x - cz \end{aligned} \tag{1}$$

Where the parameters $a,\ b,\ c$ are positive real constants.
A new chaotic attractor for the parameters $a=2,\ b=0.1,\ c=1$ is shown in Fig. 1.

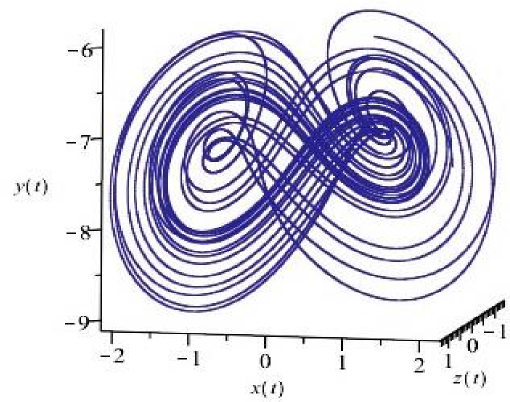


Figure 1: Noval Chaotic System at $a=2, b=0.1, c=1$.

3. DYNAMICAL BEHAVIOR OF THE NOVAL CHAOTIC SYSTEM

3.1. The dissipation

The divergence of Noval system is given by;

$$\nabla V = \frac{\partial \dot{x}}{\partial x} + \frac{\partial \dot{y}}{\partial y} + \frac{\partial \dot{z}}{\partial z} = -a + \frac{1}{b} + y - b - c.$$

When $y < a+b+c-\frac{1}{b}$, then Noval system is dissipative.

3.2. Symmetry

The relation of $(x,y,z) \to (-x,y,-z)$ is transformed, the system remains unchanged. The system trajectory in the x,z plane symmetry of y axis.

3.3. Equilibrium points and stability

By putting the right side of equation of system (1) equal to zero, that is;

$$\begin{aligned} (-a+\frac{1}{b})x + xy + z &= 0 \\ -by - x^2 &= 0 \\ -x - cz &= 0 \end{aligned}$$

This system has three equilibrium points:
$\mathbf{P}_1 = (0,\ 0,\ 0)$, $\mathbf{P}_{2,3} = (\pm\sqrt{1-ab-b/c},\ a-1/b+1/c,\ \mp\frac{1}{c}\sqrt{1-ab-b/c})$

The eigenvalues at each equilibrium point can be obtained as shown in Table 1. And all the equilibrium points are unstable, since at least one eigenvalue has positive real part for each equilibrium point.

Table 1. Eigenvalues and stability of equilibrium points

Equilibrium points	Eigenvalues	Stable/Unstable
$\mathbf{P}_1$	$\lambda_1 = -0.1$, $\lambda_2 = -0.887$, $\lambda_3 = 7.887$	Unstable
$\mathbf{P}_2$	$\lambda_1 = -0.7446$, $\lambda_2 = 0.32 + 1.33i$, $\lambda_3 = 0.32 - 1.33i$	Unstable
$\mathbf{P}_3$	$\lambda_1 = -0.7446$, $\lambda_2 = 0.32 + 1.33i$, $\lambda_3 = 0.32 - 1.33i$	Unstable

3.4. Lyapunov exponents and its dimension

By using singular value decomposition method and we may get three Lyapunov exponents of system,$\lambda_1 = 0.13,\ \lambda_2 = 0,\ \lambda_3 = -0.52$. and the Lyapunov dimension of the new chaotic system is as follows:

$$D_L = j + \frac{1}{|\lambda_{j+1}|}\sum_{i=1}^{j}\lambda_i = 2 + \frac{\lambda_1 + \lambda_2}{|\lambda_3|} = 2 + \frac{0.13 + 0}{|-0.52|} = 2.254$$

Thus, the Lyapunov dimension is the fractal dimension, shows that the system is a chaotic system

4. CONTROLLING NOVAL SYSTEM

In order to control the Noval system to the unstable fixed point $(x_i,\ y_i,\ z_i)$, we introduce the feedback control to guide the chaotic trajectory $(x(t),\ y(t),\ z(t))$ to the unstable fixed point $(x_i,\ y_i,\ z_i)$.
let system (1) be controlled by the following form:

$$\begin{aligned} \dot{x} &= (-a + \frac{1}{b})x + xy + z - k_{i1}(x - x_i) \\ \dot{y} &= -by - x^2 - k_{i2}(y - y_i) \\ \dot{z} &= -x - cz - k_{i3}(z - z_i) \end{aligned} \tag{2}$$

where $i = 1,\ 2,\ 3$.

4.1. *First*

For $i = 1$, the controlled system (2) has one equilibrium point $(x_1,\ y_1,\ z_1) = (0,\ 0,\ 0)$. Let system (2) be controlled by a linear feedback control of the form:

$$\begin{aligned} \dot{x} &= (-a + \frac{1}{b})x + xy + z - k_{11}(x - x_1) \\ \dot{y} &= -by - x^2 - k_{12}(y - y_1) \\ \dot{z} &= -x - cz - k_{13}(z - z_1) \end{aligned} \tag{3}$$

The controlled system (3) has one equilibrium point $(x_1,\ y_1,\ z_1)$. We linearize (3) about this equilibrium point. Then the linearized system is given by:

$$\begin{cases} \dot{X} = (-a + \frac{1}{b} - k_{11} + y_1)X + x_1 Y + Z \\ \dot{Y} = -(b + k_{12})Y - 2x_1 X \\ \dot{Z} = -X - (c + k_{13})Z \end{cases} \tag{4}$$

where $(x_1,\ y_1,\ z_1) = (0,\ 0,\ 0)$, that is;

$$\begin{cases} \dot{X} = (-a + \frac{1}{b} - k_{11})X + Z \\ \dot{Y} = -(b + k_{12})Y \\ \dot{Z} = -X - (c + k_{13})Z \end{cases} \tag{5}$$

To prove the asymptotic stability we use the direct method of Lyapunov. Define the Lyapunov function for system (5) by:

$$V(X,\ Y,\ Z) = \frac{1}{2}(X^2 + Y^2 + Z^2) \tag{6}$$

The function Vsatisfied:

i $V(0,\ 0,\ 0) = 0$

ii $V(X,\ Y,\ Z) > 0$ for $X,\ Y$ and Z in the neighbourhood of the origin.

So, $V(X,\ Y,\ Z)$ is positive definite. Also, we have:

$$\frac{dV}{dt} = -\{(a - \frac{1}{b} + k_{11})X^2 + (b + k_{12})Y^2 + (c + k_{13})Z^2\} \tag{7}$$

therefore, the derivative $\frac{dV}{dt} \leq 0$ if,

$$k_{11} \geq \frac{1}{b} - a, \quad k_{12} \geq -b, \quad k_{13} \geq -c \tag{8}$$

i.e. dV/dt is negative definite under condition (8). We deduce the following lemma,

Lemma 4.1. *The equilibrium solution $(x_1,\ y_1,\ z_1)$ of the controlled system (3) is asymptotically stable such that the feedback control gain K satisfy: $k_{11} \geq \frac{1}{b} - a$ and $k_{12} = k_{13} = 0$.*

4.2. *Second*

we introduce the conventional feedback control to guide the chaotic trajectory $(x(t),\ y(t),\ z(t))$ to the second unstable equilibrium point $(x_2,\ y_2,\ z_2) = (\sqrt{1 - ab - b/c},\ a - \frac{1}{b} + \frac{1}{c},\ -\frac{1}{c}\sqrt{1 - ab - b/c})$

$$\begin{cases} \dot{x} = (-a + \dfrac{1}{b})x + xy + z - k_{21}(x - x_2) \\ \dot{y} = -by - x^2 - k_{22}(y - y_2) \\ \dot{z} = -x - cz - k_{23}(z - z_2) \end{cases} \tag{9}$$

The controlled system (9) has one equilibrium point (x_2, y_2, z_2). We linearize (9) about this equilibrium point. Then the linearized system is given by:

$$\begin{cases} \dot{X} = (-a + \dfrac{1}{b} - k_{21} + y_2)X + x_2 Y + Z \\ \dot{Y} = -(b + k_{22})Y - 2x_2 X \\ \dot{Z} = -X - (c + k_{23})Z \end{cases} \tag{10}$$

where $(x_2, y_2, z_2) = (\sqrt{1 - ab - b/c}, a - \frac{1}{b} + \frac{1}{c}, -\frac{1}{c}\sqrt{1 - ab - b/c})$, that is;

$$\begin{cases} \dot{X} = (\dfrac{1}{c} - k_{21})X + (\sqrt{1 - ab - b/c})Y + Z \\ \dot{Y} = -(b + k_{22})Y - 2(\sqrt{1 - ab - b/c})X \\ \dot{Z} = -X - (c + k_{23})Z \end{cases} \tag{11}$$

To prove the asymptotic stability we use the direct method of Lyapunov. Define the Lyapunov function for system(10) by:

$$V(X,\ Y,\ Z) = \frac{1}{2}(X^2 + Y^2 + Z^2) \tag{12}$$

The function V satisfied:

i $V(0,0,0)=0$

ii $V(X,Y,Z)>0$ for X,Y and Z in the neighbourhood of the origin.

So, $V(X,Y,Z)$ is positive definite. Also, we have:

$$\frac{dV}{dt}=-\{2(k_{21}-\frac{1}{c})X^2+(b+k_{22})Y^2+2(c+k_{23})Z^2\} \tag{13}$$

therefore, the derivative $\frac{dV}{dt}\leq 0$ if,

$$k_{21}\geq\frac{1}{c},\quad k_{22}\geq -b,\quad k_{23}\geq -c \tag{14}$$

i.e. $\frac{dV}{dt}$ is negative definite under condition (14). We deduce the following lemma,

LEMMA 4.2. *The equilibrium solution* $(x_2,\ y_2,\ z_2)$ *of the controlled system (9) is asymptotically stable such that the feedback control gain* K *has the simple choice* $k_{21}\geq\frac{1}{c}$ *and* $k_{22}=k_{23}=0$.

4.3. *Third*

we introduce the conventional feedback control to guide the chaotic trajectory (x(t),y(t),z(t)) to the third unstable equilibrium point$(x_3,\ y_3,\ z_3)=(-\sqrt{1-ab-b/c},\ a-\frac{1}{b}+\frac{1}{c},\ \frac{1}{c}\sqrt{1-ab-b/c})$

$$\begin{cases}\dot{x}=(-a+\dfrac{1}{b})x+xy+z-k_{31}(x-x_3)\\ \dot{y}=-by-x^2-k_{32}(y-y_3)\\ \dot{z}=-x-cz-k_{33}(z-z_3)\end{cases} \tag{15}$$

The controlled system (14) has one equilibrium point (x_3,y_3,z_3). We linearize (14) about this equilibrium point. Then the linearized system is given by:

$$\begin{cases}\dot{X}=(-a+\dfrac{1}{b}-k_{31}+y_3)X+x_3Y+Z\\ \dot{Y}=-(b+k_{32})Y-2x_3X\\ \dot{Z}=-X-(c+k_{33})Z\end{cases} \tag{16}$$

where $(x_3,y_3,z_3)=(-\sqrt{1-ab-b/c},a-\frac{1}{b}+\frac{1}{c},\frac{1}{c}\sqrt{1-ab-b/c})$, that is;

$$\begin{cases}\dot{X}=(\dfrac{1}{c}-k_{31})X-(\sqrt{1-ab-b/c})Y+Z\\ \dot{Y}=-(b+k_{32})Y+2(\sqrt{1-ab-b/c})X\\ \dot{Z}=-X-(c+k_{33})Z\end{cases} \tag{17}$$

To prove the asymptotic stability we use the direct method of Lyapunov. Define the Lyapunov function for system(16) by:

$$V(X,\ Y,\ Z)=\frac{1}{2}(X^2+Y^2+Z^2) \tag{18}$$

The function V satisfied:

i $V(0,0,0)=0$

ii $V(X,Y,Z)>0$ for X,Y and Z in the neighbourhood of the origin.

So, $V(X, Y, Z)$ is positive definite. Also, we have:

$$\frac{dV}{dt} = -\{2(k_{31} - \frac{1}{c})X^2 + (b + k_{32})Y^2 + 2(c + k_{33})Z^2\} \tag{19}$$

therefore, the derivative $\frac{dV}{dt} \leq 0$ if,

$$k_{31} \geq \frac{1}{c}, \quad k_{32} \geq -b, \quad k_{33} \geq -c \tag{20}$$

i.e. $\frac{dV}{dt}$ is negative definite under condition (20). We deduce the following lemma,

LEMMA 4.3. *The equilibrium solution $(x_3,\ y_3,\ z_3)$ of the controlled system (15) is asymptotically stable such that the feedback control gain K has the simple choice $k_{31} \geq \frac{1}{c}$ and $k_{32} = k_{33} = 0$.*

5. THE SCHEME OF GENERALIZED FUNCTION PROJECTIVE SYNCHRONIZATION OF CHAOTIC SYSTEMS

The chaotic (master and slave) systems can be given in the following form:

$$\dot{X} = F(X) \tag{21}$$

$$\dot{Y} = G(Y) + U(X, Y, t) \tag{22}$$

Where $X = (x_1, x_2, \ldots, x_n)^T, Y = (y_1, y_2, \ldots, y_n)^T \in \mathbf{R}^n$ are state vectors of the system (20) and (21), respectively; $F, G : \mathbf{R}^n \to \mathbf{R}^n$ are two continuous vector functions and $U : (\mathbf{R}^n, \mathbf{R}^n, \mathbf{R}^n) \to \mathbf{R}^n$ is a controller which will be designed later.

DEFINITION 5.1. *For the master system (20) and the slave system (21), there is said to be generalized function projective synchronization (GFPS) if there exists a vector function $U(X, Y, t)$such that; $\lim_{t\to+\infty} \|Y - \Lambda(X)X\| = 0$ where $\Lambda(X) = diag\{h_1(X), h_2(X), \ldots, h_n(X)\}$ where $h_i(X)$ are continuous functions, $\|.\|$ represents a vector norm induced by the matrix norm.*

REMARK 1. *We define $e = Y - \Lambda(X)X$ which is called the error vector between systems (20) and (21) for GFPS, where $e = (e_1, e_2, \ldots, e_n)^T$,and*
$e_i = Y_i - h_i(X)X_i, (i = 1, 2, \ldots, n)$

REMARK 2.

If $\Lambda = \sigma\mathbf{I}, \sigma \in \mathbf{R}$, the GFPS problem will be reduced to projective synchronization, where $\mathbf{I}$ is an $n \times n$ identity matrix. In particular if $\sigma = 1and - 1$ the problem is further simplified to complete synchronization and antiphase synchronization, respectively. And if $\Lambda = diag\{a_1, a_2, \ldots, a_n\}$,the modified projective synchronization will appear.

We will study the FPS of novel system with known parameters and determine controller function for the FPS of the derive and response systems. Our aim is to design a controller and make the response system trace the drive system and become ultimately. The Noval system as a drive system is given as below;

$$\begin{cases} \dot{x}_1 &= (-a + \frac{1}{b})x_1 + x_1y_1 + z_1 \\ \dot{y}_1 &= -by_1 - x_1^2 \\ \dot{z}_1 &= -x_1 - cz_1 \end{cases} \tag{23}$$

the Noval system as the response system is also given by;

$$\begin{cases} \dot{x}_2 &= (-a + \frac{1}{b})x_2 + x_2y_2 + z_2 + u_1 \\ \dot{y}_2 &= -by_2 - x_2^2 + u_2 \\ \dot{z}_2 &= -x_2 - cz_2 + u_3 \end{cases} \tag{24}$$

According to the FPS scheme presented in the previous section, without loss of generality,we choose the scaling function matrix
$\Lambda(X) = diag\{d_{11}x_1 + d_{12}, d_{21}y_1 + d_{22}, d_{31}z_1 + d_{32}\}$ where $d_{ij}(i = 1, 2, 3; j = 1, 2)$ are constant numbers. The error vector can be defined as

$$\begin{cases} e_x &= x_2 - (d_{11}x_1 + d_{12})x_1 \\ e_y &= y_2 - (d_{21}y_1 + d_{22})y_1 \\ e_z &= z_2 - (d_{31}z_1 + d_{32})z_1 \end{cases} \tag{25}$$

The error dynamical system between (23) and (24) is;

$$\begin{cases} \dot{e}_x &= (-a + \frac{1}{b})e_x + x_2y_2 + z_2 - d_{11}(-a + \frac{1}{b})x_1^2 - 2d_{11}x_1z_1 \\ & -2d_{11}x_1^2y_1 - d_{12}x_1y_1 - d_{12}z_1 + u_1 \\ \dot{e}_y &= -be_y - x_2^2 + d_{21}by_1^2 + 2d_{21}y_1x_1^2 + d_{22}x_1^2 + u_2 \\ \dot{e}_z &= -ce_z - x_2 + d_{31}cz_1^2 + 2d_{31}z_1x_1 + d_{32}x_1 + u_3 \end{cases} \tag{26}$$

we can get the controller

$$\begin{cases} u_1 &= \frac{-2}{b}e_x - x_2y_2 - z_2 + d_{11}(-a + \frac{1}{b})x_1^2 + 2d_{11}x_1z_1 + 2d_{11}x_1^2y_1 + d_{12}x_1y_1 + d_{12}z_1 \\ u_2 &= x_2^2 - d_{21}by_1^2 - 2d_{21}y_1x_1^2 - d_{22}x_1^2 \\ u_3 &= x_2 - d_{31}cz_1^2 - 2d_{31}z_1x_1 - d_{32}x_1 \end{cases} \tag{27}$$

then the error dynamical system is described by

$$\begin{cases} \dot{e}_x &= -(a + \frac{1}{b})e_x \\ \dot{e}_y &= -be_y \\ \dot{e}_z &= -ce_z \end{cases} \tag{28}$$

for this choice, the closed loop system (28) has three negative eigenvalues $-(a + \frac{1}{b}), -b, -c$ which implies that the error state e_x, e_y and e_z converge to zero as time t tends to infinity.
Hence the FPS between the identical Noval chaotic system is achieved.

5.1. Numerical Results

In this section, some numerical simulation results are presented to verify the previous analytical results where $a = 2, b = 0.1, c = 1$. Figure 2: shows the convergence of the trajectory of the controlled system to the unstable equilibrium point $(x_1,\ y_1,\ z_1) = (0,\ 0,\ 0)$ of the uncontrolled system (1). Figure 3: shows the convergence of the trajectory of the controlled system to the unstable equilibrium point $(x_2,\ y_2,\ z_2) = (\sqrt{1 - ab - b/c},\ a - \frac{1}{b} + \frac{1}{c},\ -\frac{1}{c}\sqrt{1 - ab - b/c})$ of the uncontrolled system (1). Figure 4: shows the convergence of the trajectory of thec ontrolled system to the unstable equilibrium point $(x_3,\ y_3,\ z_3) = (-\sqrt{1 - ab - b/c},\ a - \frac{1}{b} + \frac{1}{c},\ \frac{1}{c}\sqrt{1 - ab - b/c})$ of the uncontrolled system (1).

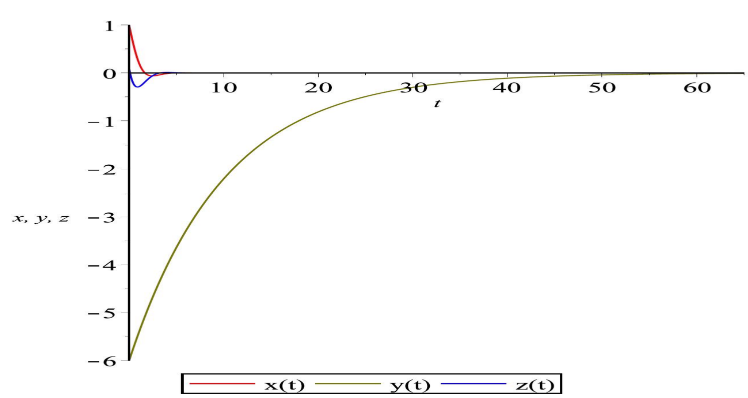


Figure 2: The time responses for the states of the controlled Noval system to a fixed point (x_1, y_1, z_1).

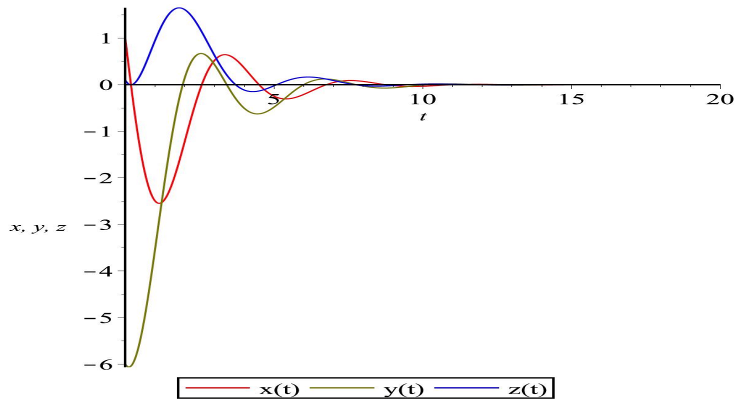


Figure 3: The time responses for the states of the controlled Noval system to a fixed point (x_2, y_2, z_2).

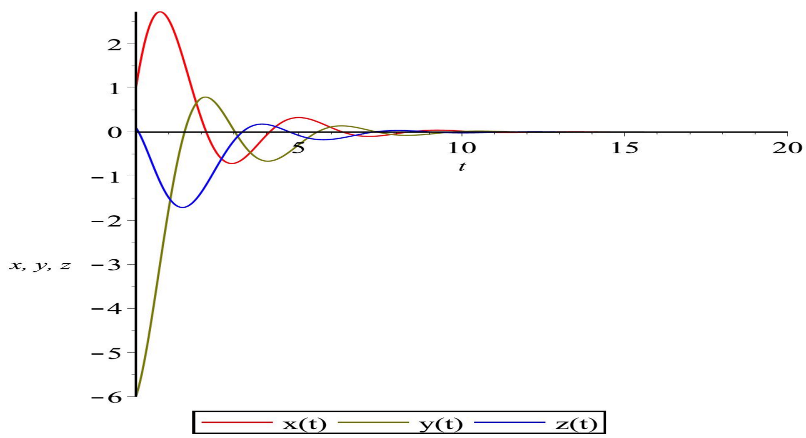


Figure 4: The time responses for the states of the controlled Noval system to a fixed point (x_3, y_3, z_3).

The initial values of the drive system and response system are taken as:

$$(x_1(0), y_1(0), z_1(0))^T = (1, -6, 0.1)^T, (x_2(0), y_2(0), z_2(0))^T = (10, 12, -3)^T.$$

We choose the scaling function factors as:

$$h_1 = x_1 + 2, h_2 = -2y_1 - 2 \; and \; h_3 = z_1 - 2.$$

Figure 5: show the FPS between two identical Noval systems. When the scaling factors are simplified as $h_i = 1 \, (i = 1, 2, 3)$, the complete synchronization between two identical Noval systems are shown in Figure 6. Furthermore, when the scaling factors are simplified as $h_i = -1 \, (i = 1, 2, 3)$, the anti synchronization between two identical Noval systems are shown in Figure 7. Finally, when the scaling factors are simplified as $h_1 = 1.5, h_2 = 2 \; and \; h_3 = 2.5$, the modified projective synchronization (MPS) between two identical Noval

systems are shown in Figure 8.

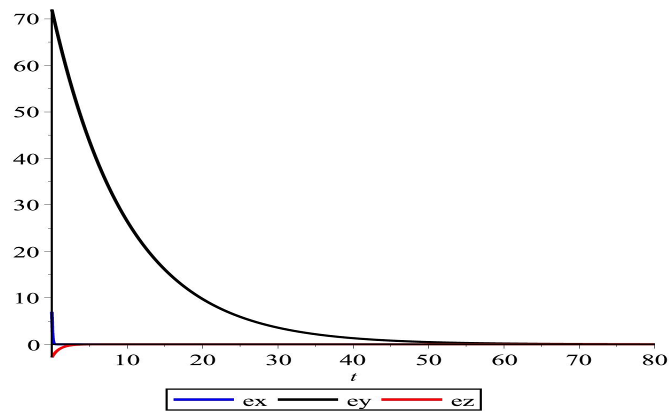


Figure 5: The behaviour of the trajectories e_x, e_y *and* e_z of the error system tends to zero for FPS.

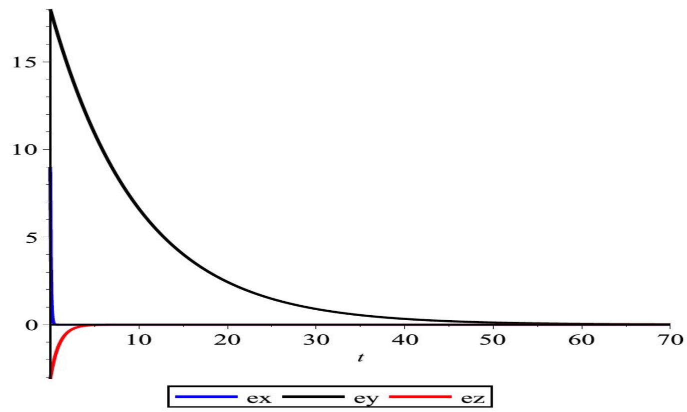


Figure 6: The behaviour of the trajectories e_x, e_y *and* e_z of the error system tends to zero for complete synchronization

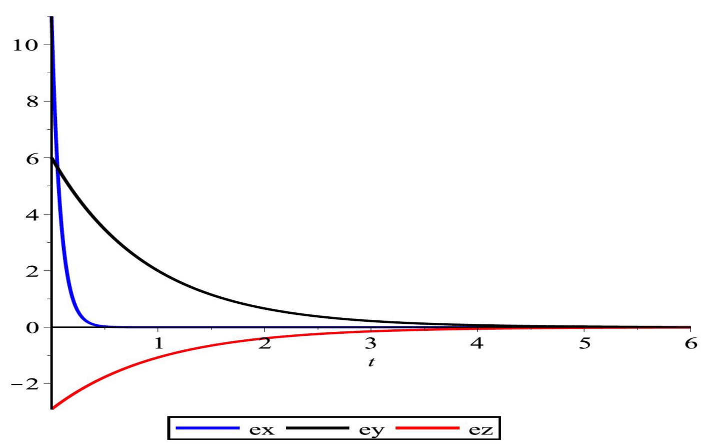


Figure 7: The behaviour of the trajectories e_x, e_y *and* e_z of the error system tends to zero for anti synchronization

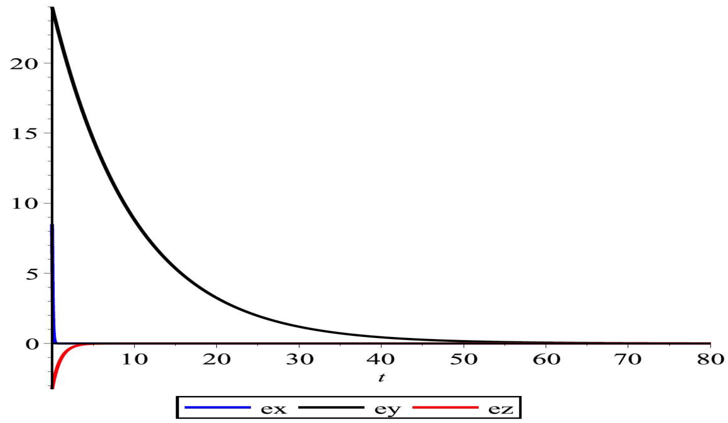


Figure 8: The behaviour of the trajectories e_x, e_y *and* e_z of the error system tends to zero for MPS.

6. CONCLUSION

The paper has studied the noval chaotic dynamical system, including some basic dynamical properties, Lyapunov exponents, Lyapunov dimension. A feedback control has been proposed to the noval chaotic dynamical system. The controlling conditions are derived from the Lyapunov direct method. The function projective synchronization has been used to synchronize two identical chaotic systems with known parameters. By the Lyapunov stability theory, the sufficient condition of the function projective synchronization has been obtained. Finally, numerical simulations are provided to verify the effectiveness of the results obtained.

Acknowledgments

This article was funded by the Deanship of Scientific Research (DSR), King Abdulaziz University, Jeddah. The authors, therefore, acknowledge with thanks DSR technical and financial support.

References

[1] E.Ott, C. Grebogi, J A. Yorke, Controlling Chaos, Phys Rev Lett (64), (1999), 1179-1184.

[2] G. Chen, Chaos on Some Controllability Conditions for Chaotic Dynamics Control, Chaos Solitons Fractals, 8(9), (1997), 1461-1470.

[3] C. Hwang , J. Hsheh , and R. Lin, A Linear Continuous Feedback Control of Chua's Circuit, Chaos Solitons Fractals 8(9), 1997, 1507-1515.

[4] G. Chen and X. Dong, On feedback control of chaotic dynamic systems, Int. J. Bifurcation and Chaos 2, (1995), 407-411.

[5] K. Pyragas, Continuous control of chaos by self-controlling feedback, Physics letters A 170, pp. 421-428, (1992).

[6] A. Hegazi, H. N. Agiza, and M. M. El-Dessoky. Controlling chaotic behaviour for spin generator and Rossler dynamical systems with feedback control, Chaos Solitons Fractals, 12, (2001), 631-658.

[7] Sara Dadras, and Hamid Momeni, Control of a fractional-order economical system via sliding mode, Physica A (389), (2010), 2434-2442.

[8] H. N. Agiza, On The Analysis of Stability, Bifurcation, Chaos and Chaos Control of Kopel map, Chaos Solitons Fractals 10(11), (1999), 1909-1916.

[9] C. Hwang, J. Hsheh and R. Lin, A linear continuous feedback control of Chua's circuit, Chaos Solitons Fractals, 8(9), (1997), 1507-1515.

[10] Seyed Mehdi Abedi Pahnehkolaei, Alireza Alfi, J. A. Tenreiro Machado, Chaos suppression in fractional systems using adaptive fractional state feedback control, Chaos Solitons Fractals, 103, (2017), 488-503.

[11] V. Bala Shunmuga Jothi, S. Selvaraj, V. Chinnathambi, S. Rajasekar, Bifurcations and chaos in two-coupled periodically driven four-well Duffing-van der Pol oscillators, Chinese J. Phys., 55(5), (2017), 1849-1856.

[12] Anuraj Singh, Sunita Gakkhar, Controlling chaos in a food chain model, Math. Comput. Simulation, 115, (2015), 24-36.

[13] L. M. Pecora, T. L. Carroll, Synchronization in chaotic systems, Phys. Rev. Lett.64 (8), (1990), 821-824.

[14] T. L. Carroll, L. M. Perora, Synchronizing a chaotic systems, IEEE Trans, Circuits Systems 38, (1991), 453-456.

[15] G. Chen, Control and Synchronization of Chaos, a Bibliography, Dept. of Elect. Eng., Univ. Houston, TX, (1997).

[16] Yongguang Yua, and Han-Xiong Li. Adaptive generalized function projective synchronization of uncertain chaotic systems, Nonlinear Analysis: Real World Applications, Vol.11, (2010), 2456-2464.

[17] E. M. Elabbasy,and M. M. El-Dessoky, Adaptive Coupled Synchronization of Coupled Chaotic Dynamical Systems, Applied Sciences Research , (2), (2007), 88-102.

[18] Na Cai, Yuanwei Jing, and Siying Zhang, Modi…ed projective synchronization of chaotic systems with disturbances via active sliding mode control, Commun Nonlinear Sci Numer Simulat., (15), (2010), 1613-1620.

[19] Guo-Hui Li, Generalized Projective Synchronization between Lorenz System and Chen's System, Chaos,Solitons and Fractals (32), 2007, 1454-1458.

[20] Guo-Hui Li. Modi…ed Projective Synchronization of Chaotic System, Chaos Solitons Fractals (32), (2007), 1786-1790.

[21] Johannes Petereit, Arkady Pikovsky, Chaos synchronization by nonlinear coupling, Commun. Nonlinear Sci. Numer. Simul., 44, (2017), 344-351.

[22] K. Vishal, Saurabh K. Agrawal, On the dynamics, existence of chaos, control and synchronization of a novel complex chaotic system, Chinese J. Phys., 55(2), (2017), 519-532.

[23] M. M. El-Dessoky, M. T. Yassen and E. Salah, Adaptive Modi…ed Function Projective Synchronization between two di¤erent Hyperchaotic Dynamical Systems, Math. Probl. Eng. , Vol., 2012, (2012), Article ID 810626, 16 pages, doi:10.1155/2012/810626.

[24] N. F. Rulkov, M. M. Sushchik, L. S. Tsimring, and Henry D. I. Abarbanel, Generalized Synchronization of Chaos in Directionally Coupled Chaotic Systems, Phys. Rev. E, (51),1995, 980- 994.

[25] Yong Chen, X. Li. Function Projective Synchronization between Two Identical Chaotic Systems, Int. J. Mod. Phys. C(18), 2007, 883-888.

[26] M. M. El-Dessoky, and M. T. Yassen, Adaptive feedback control for chaos control and synchronization for new chaotic dynamical system, Math. Probl. Eng. , Vol. 2012, (2012), Article ID 347210, 12 pages, doi:10.1155/2012/347210.

[27] Guo-Hui Li, Generalized Synchronization of Chaos Based on Suitable Separation, Chaos Solitons Fractals, (39), (2009), 2056-2062.

[28] M. M. El-Dessoky, E. O. Alzahrany, and N. A. Almohammadi. Function Projective Synchronization for Four Scroll Attractor by Nonlinear Control, Appl. Math. Sci., Vol.11(26), (2017), 1247-1259.

[29] Er-Wei Bai and Karl E. Lonngren, Sequential synchronization of two Lorenz system using active control, Chaos Solitons Fractals, 11(1), (2000), 1041-1044.

[30] Shao FuWang, and Da-zhuan Xu, The dynamic analysis of a chaotic system, Adv. Mech. Eng. , Vol. 9(3), 2017, 1-6.

UMBRAL CALCULUS APPROACH TO r-STIRLING NUMBERS OF THE SECOND KIND AND r-BELL POLYNOMIALS

TAEKYUN KIM[1], DAE SAN KIM[2], HYUCK-IN KWON[3], AND JONGKYUM KWON[4,*]

ABSTRACT. In this paper, we would like to use umbral calculus in order to derive some properties, recurrence relations and identities related to r-Stirling numbers of second kind and r-Bell polynomials. In particular, we will express the r-Bell polynomials as linear combinations of many well-known families of special polynomials.

1. INTRODUCTION

The Stirling numbers $S_2(n,k)$ of the second kind counts the number of partitions of the set $[n]=\{1,2,\cdots,n\}$ into k nonempty disjoint subsets.

The $S_2(n,k),(n,k\geq 0)$ are given by the recurrence relation

$$S_2(n,k)=kS_2(n-1,k)+S_2(n-1,k-1),\ (n,k\geq 1), \tag{1.1}$$

with the initial conditions

$$S_2(n,0)=\delta_{0n},S_2(0,k)=\delta_{0k}. \tag{1.2}$$

They are also given by

$$x^n=\sum_{k=0}^{n}S_2(n,k)(x)_k, \tag{1.3}$$

with $(x)_0=1,(x)_k=x(x-1)\cdots(x-k+1)$, for $k\geq 1$, and by

$$\frac{1}{k!}(e^t-1)^k=\sum_{n=k}^{\infty}S_2(n,k)\frac{t^n}{n!}. \tag{1.4}$$

More explicitly, they are given by

$$\begin{aligned} S_2(n,k)&=\frac{1}{k!}\sum_{j=0}^{k}\binom{k}{j}(-1)^{k-j}j^n \\ &=\frac{1}{k!}\triangle^k 0^n,\ (n\geq k), \end{aligned} \tag{1.5}$$

where $\triangle^k 0^n=\triangle^k x^n|_{x=0}$, and $\triangle f(x)=f(x+1)-f(x)$ is the forward difference operator. For these well known facts, one may refer to [3,4].

2010 *Mathematics Subject Classification.* 05A19, 05A40, 11B73, 11B83.

Key words and phrases. r-Stirling numbers of the second kind, r-Bell polynomials, umbral calculus.

* corresponding author.

1

Let r be any positive integer. These 'classical' Stirling numbers $S_2(n,k)$ of the second kind were generalized to the r-Stirling numbers $S_{2,r}(n,k)$ of the second kind (see, [1,2,7]). The $S_{2,r}(n,k)$ enumerates the number of partitions of the set $[n]=\{1,2,\cdots,n\}$ into k nonempty disjoint subsets in such a way that $1,2,\cdots,r$ are in distinct subsets.

They are given by the recurrence relation

$$S_{2,r}(n,k)=kS_{2,r}(n-1,k)+S_{2,r}(n-1,k-1),\ (n>r), \tag{1.6}$$

with the initial conditions

$$S_{2,r}(n,k)=0,(n<r);S_{2,r}(n,k)=\delta_{kr},(n=r). \tag{1.7}$$

The $S_{2,r}(n,k)$ are also given by

$$(x+r)^n=\sum_{k=0}^{n}S_{2,r}(n+r,k+r)(x)_k, \tag{1.8}$$

and by

$$\frac{1}{k!}e^{rt}(e^t-1)^k=\sum_{n=k}^{\infty}S_{2,r}(n+r,k+r)\frac{t^n}{n!}. \tag{1.9}$$

Analogously to the classical case, they are explicitly given by

$$\begin{aligned}S_{2,r}(n+r,k+r)&=\frac{1}{k!}\sum_{j=0}^{k}\binom{k}{j}(-1)^{k-j}(r+j)^n\\&=\frac{1}{k!}\triangle^k r^n,\ (n\geq k),\end{aligned} \tag{1.10}$$

where $\triangle^k r^n=\triangle^k x^n|_{x=r}$.

For more details about r-Stirling numbers of the second kind, one may refer to [1,2,7].

The Bell polynomials $Bel_n(x)$ (also called exponential or Touchard polynomials) are defined by

$$e^{x(e^t-1)}=\sum_{n=0}^{\infty}Bel_n(x)\frac{t^n}{n!},\ (see\ [3,4,8,9]). \tag{1.11}$$

Then it is immediate to see that

$$Bel_n(x)=\sum_{k=0}^{n}S_2(n,k)x^k. \tag{1.12}$$

For $x=1$, $Bel_n=Bel_n(1)=\sum_{k=0}^{n}S_2(n,k)$ are called Bell numbers so that

$$e^{e^t-1}=\sum_{n=0}^{\infty}Bel_n\frac{t^n}{n!}. \tag{1.13}$$

Further, the Bell polynomial is given by Dobinski's formula

$$Bel_n(x)=e^{-x}\sum_{k=0}^{\infty}\frac{k^n}{k!}x^k. \tag{1.14}$$

On the other hand, the r-Bell polynomials $Bel_{n,r}(x)$ are defined by

$$e^{rt}e^{x(e^t-1)} = \sum_{n=0}^{\infty} Bel_{n,r}(x)\frac{t^n}{n!}, \ (see\ [5]). \tag{1.15}$$

Then it is easy to see that

$$Bel_{n,r}(x) = \sum_{k=0}^{n} S_{2,r}(n+r, k+r)x^k. \tag{1.16}$$

Moreover, they satisfy the generalized Dobinski's formula

$$Bel_{n,r}(x) = e^{-x}\sum_{k=0}^{\infty}\frac{(k+r)^n}{k!}x^k. \tag{1.17}$$

When $x = 1$, $Bel_{n,r} = Bel_{n,r}(1) = \sum_{k=0}^{n} S_{2,r}(n+r, k+r)$ are called r-Bell numbers so that

$$e^{e^t-1+rt} = \sum_{n=0}^{\infty} Bel_{n,r}\frac{t^n}{n!}. \tag{1.18}$$

We note here, in passing, that r-Bell numbers were called in another name, namely extended Bell numbers,(see [6]).

In this paper, we would like to use umbral calculus in order to derive some properties, recurrence relations and identities related to r-Stirling numbers of the second kind and r-Bell polynomials. In particular, we will express the r-Bell polynomials as linear combinations of many well-known families of special polynomials.

2. **Review on umbral calculus**

Here we will go over some of the basic facts about umbral calculus. For a complete treatment, the reader may refer to [4].
Let $\mathcal{F}$ be the algebra of all formal power series in the single variable t with the coefficients in the field $\mathbb{C}$ of complex numbers:

$$\mathcal{F} = \left\{ f(t) = \sum_{k=0}^{\infty} a_k \frac{t^k}{k!} \ \middle| \ a_k \in \mathbb{C} \right\}. \tag{2.1}$$

Let $\mathbb{P} = \mathbb{C}[x]$ denote the ring of polynomials in x with the coefficients in $\mathbb{C}$, and let $\mathbb{P}^*$ be the vector space of all linear functionals on $\mathbb{P}$. For $L \in \mathbb{P}^*$, $p(x) \in \mathbb{P}$, $< L \,|\, p(x) >$ denotes the action of the linear functional L on $p(x)$. For $f(t) = \sum_{k=0}^{\infty} a_k \frac{t^k}{k!} \in \mathcal{F}$, the linear functional $< f(t) \,|\, \cdot >$ on $\mathbb{P}$ is defined by

$$< f(t) \,|\, x^n >= a_n, \ \ (n \geq 0). \tag{2.2}$$

For $L \in \mathbb{P}^*$, let $f_L(t) = \sum_{k=0}^{\infty} \left\langle L|x^k \right\rangle \frac{t^k}{k!} \in \mathcal{F}$. Then we evidently have $\left\langle f_L(t)|x^n \right\rangle = \left\langle L|x^n \right\rangle$, and the map $L \to f_L(t)$ is a vector space isomorphism from $\mathbb{P}^*$ to $\mathcal{F}$. Thus $\mathcal{F}$ may be viewed as the vector space of all linear functionals on $\mathbb{P}$ as well as the algebra of formal power series in t. So an element $f(t) \in \mathcal{F}$ will be thought of as both a formal power series and a linear functional on $\mathbb{P}$. $\mathcal{F}$ is called the umbral algebra, the study of which is the umbral calculus.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 27, NO.1, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC

The order $o(f(t))$ of $0 \neq f(t) \in \mathcal{F}$ is the smallest integer k such that the coefficients of t^k does not vanish. In particular, for $0 \neq f(t) \in \mathcal{F}$, it is called an invertible series if $o(f(t)) = 0$ and a delta series if $o(f(t)) = 1$.

Let $f(t), g(t) \in \mathcal{F}$, with $o(g(t)) = 0$, $o(f(t)) = 1$. Then there exists a unique sequence of polynomials $S_n(x)$ ($\deg S_n(x) = n$) such that $\Big\langle g(t)f(t)^k | S_n(x) \Big\rangle = n!\delta_{n,k}$, for $n, k \geq 0$. Such a sequence is called the Sheffer sequence for the Sheffer pair $(g(t), f(t))$, which is concisely denoted by $S_n(x) \sim (g(t), f(t))$.

It is known that $S_n(x) \sim (g(t), f(t))$ if and only if

$$\frac{1}{g(\bar{f}(t))} e^{x\bar{f}(t)} = \sum_{n=0}^{\infty} S_n(x) \frac{t^n}{n!}, \tag{2.3}$$

where $\bar{f}(t)$ is the compositional inverse of $f(t)$ satisfying $f(\bar{f}(t)) = \bar{f}(f(t)) = t$.

Let $p_n(x) \sim (1, f(t)), q_n(x) \sim (1, l(t))$. Then the transfer formula says that

$$q_n(x) = x \left(\frac{f(t)}{l(t)} \right)^n x^{-1} p_n(x), \ (n \geq 1). \tag{2.4}$$

Let $S_n(x) \sim (g(t)), f(t))$. Then we have the Sheffer identity:

$$S_n(x + y) = \sum_{k=0}^{n} \binom{n}{k} S_k(x) p_{n-k}(y), \tag{2.5}$$

where $p_n(x) = g(t)S_n(x) \sim (1, f(t))$.

The derivative of $S_n(x)$ is given by

$$\frac{d}{dx} S_n(x) = \sum_{k=0}^{n-1} \binom{n}{k} \Big\langle \bar{f}(t) | x^{n-k} \Big\rangle S_k(x), \ (n \geq 1). \tag{2.6}$$

Also, we have the recurrence formula:

$$S_{n+1}(x) = \left(x - \frac{g'(t)}{g(t)} \right) \frac{1}{f'(t)} S_n(x). \tag{2.7}$$

Assume that $S_n(x) \sim (g(t), f(t))$, $r_n(x) \sim (h(t), l(t))$. Then

$$S_n(x) = \sum_{k=0}^{n} C_{n,k} r_k(x), \tag{2.8}$$

where

$$C_{n,k} = \frac{1}{k!} \Big\langle \frac{h(\bar{f}(t))}{g(\bar{f}(t))} l(\bar{f}(t))^k | x^n \Big\rangle. \tag{2.9}$$

Finally, we also need the following: for any $h(t) \in \mathcal{F}$, $p(x) \in \mathbb{P}$,

$$\Big\langle h(t) | xp(x) \Big\rangle = \Big\langle \partial_t h(t) | p(x) \Big\rangle. \tag{2.10}$$

3. Main Results

As we can see from (1.15) and (2.3), we see that

$$Bel_{n,r}(x) \sim \left(\frac{1}{(1+t)^r}, \log((1+t) \right) = (g(t), f(t)). \tag{3.1}$$

Let $n \geq 1$. Then, using (2.10), we have

$$\begin{aligned} Bel_{n,r}(y) &= \Big\langle \sum_{m=0}^{\infty} Bel_{m,r}(y) \frac{t^m}{m!} | x^n \Big\rangle \\ &= \Big\langle e^{rt} e^{y(e^t-1)} | x^n \Big\rangle \\ &= \Big\langle \partial_t (e^{rt} e^{y(e^t-1)}) | x^{n-1} \Big\rangle \\ &= \Big\langle r e^{rt} e^{y(e^t-1)} + e^{rt} e^{y(e^t-1)} y e^t | x^{n-1} \Big\rangle \\ &= r \Big\langle e^{rt} e^{y(e^t-1)} | x^{n-1} \Big\rangle + y \Big\langle e^{(r+1)t} e^{y(e^t-1)} | x^{n-1} \Big\rangle \\ &= r Bel_{n-1,r}(y) + y Bel_{n-1,r+1}(y). \end{aligned} \tag{3.2}$$

Thus we obtain the following recurrence relation for r-Bell polynomials.

Theorem 3.1. *For all integers $n \geq 1$, we have the recurrence relation.*

$$Bel_{n,r}(x) = r Bel_{n-1,r}(x) + x Bel_{n-1,r+1}(x), \ (n \geq 1).$$

From (2.6), we have

$$\begin{aligned} \frac{d}{dx} Bel_{n,r}(x) &= \sum_{k=0}^{n-1} \binom{n}{k} \Big\langle e^t - 1 | x^{n-k} \Big\rangle Bel_{k,r}(x) \\ &= \sum_{k=0}^{n-1} \binom{n}{k} (1 - \delta_{n,k}) Bel_{k,r}(x) \\ &= \sum_{k=0}^{n-1} \binom{n}{k} Bel_{k,r}(x), \ (n \geq 1). \end{aligned} \tag{3.3}$$

Using (2.7), we obtain

$$\begin{aligned} Bel_{n+1,r}(x) &= (x + r \frac{1}{1+t})(1+t) Bel_{n,r}(x) \\ &= x(1+t) Bel_{n,r}(x) + r Bel_{n,r}(x) \\ &= x Bel_{n,r}(x) + x \frac{d}{dx} Bel_{n,r}(x) + r Bel_{n,r}(x), \end{aligned} \tag{3.4}$$

from which it follows that

$$\begin{aligned} &\frac{d}{dx} Bel_{n,r}(x) \\ =& \frac{Bel_{n+1,r}(x)}{x} - \frac{r Bel_{n,r}(x)}{x} - Bel_{n,r}(x). \end{aligned} \tag{3.5}$$

This agrees with the result in [2].

Noting that $p_n(x) = g(t)Bel_{n,r}(x) \sim (1, \log(1+t))$, we have $p_n(x) = Bel_n(x)$. Hence from (2.5), we get the following Sheffer identity

$$Bel_{n,r}(x+y) = \sum_{k=0}^{n} \binom{n}{k} Bel_{k,r}(x) Bel_{n-k}(y). \tag{3.6}$$

$$\begin{aligned} Bel_{n,r}(y) &= \left\langle e^{rt} e^{y(e^t-1)} | x^n \right\rangle \\ &= \left\langle e^{rt} | e^{y(e^t-1)} x^n \right\rangle \\ &= \left\langle e^{rt} | \sum_{m=0}^{\infty} Bel_m(y) \frac{t^m}{m!} x^n \right\rangle \\ &= \sum_{m=0}^{n} \binom{n}{m} Bel_m(y) \left\langle e^{rt} | x^{n-m} \right\rangle \\ &= \sum_{m=0}^{n} \binom{n}{m} Bel_m(y) r^{n-m}. \end{aligned} \tag{3.7}$$

Hence we get

$$Bel_{n,r}(x) = \sum_{m=0}^{n} \binom{n}{m} r^{n-m} Bel_m(x). \tag{3.8}$$

Here we apply the transfer formula in (2.4) to $x^n \sim (1,t)$, $\frac{1}{(1+t)^r} Bel_{n,r}(x) \sim (1, \log((1+t))$.

For $n \geq 1$, we have

$$\begin{aligned} \frac{1}{(1+t)^r} Bel_{n,r}(x) &= x \left(\frac{t}{\log(1+t)} \right)^n x^{-1} x^n \\ &= x \sum_{k=0}^{\infty} b_k^{(n)} \frac{t^k}{k!} x^{n-1} \\ &= \sum_{k=0}^{n-1} \binom{n-1}{k} b_k^{(n)} x^{n-k}. \end{aligned} \tag{3.9}$$

Here $b_k^{(n)}$ are the Bernoulli numbers of the second kind of order n defined by

$$\left(\frac{t}{\log(1+t)} \right)^n = \sum_{k=0}^{\infty} b_k^{(n)} \frac{t^k}{k!}. \tag{3.10}$$

Here, as is well known, $b_k^{(n)} = B_k^{(k-n+1)}(1)$, with $B_k^{(n)}(x)$ denoting the Bernoulli polynomials of order n. Thus we obtain

$$\begin{aligned} Bel_{n,r}(x) &= \sum_{k=0}^{n-1} \binom{n-1}{k} b_k^{(n)} (1+t)^r x^{n-k} \\ &= \sum_{k=0}^{n-1} \binom{n-1}{k} b_k^{(n)} \sum_{l=0}^{r} \binom{r}{l} t^l x^{n-k} \\ &= \sum_{k=0}^{n-1} \sum_{l=0}^{r} \binom{n-1}{k} \binom{r}{l} (n-k)_l b_k^{(n)} x^{n-k-l}. \end{aligned} \tag{3.11}$$

As $\frac{1}{(1+t)^r} Bel_{n,r}(x) = Bel_n(x) = \sum_{j=0}^{n} S_2(n,j)x^j$, we can proceed as follows.

$$\begin{aligned} Bel_{n,r}(x) &= (1+t)^r Bel_n(x) \\ &= \sum_{k=0}^{\infty} \binom{r}{k} t^k Bel_n(x) \\ &= \sum_{k=0}^{n} \binom{r}{k} t^k \sum_{j=0}^{n} S_2(n,j)x^j \\ &= \sum_{k=0}^{n} \binom{r}{k} \sum_{j=k}^{n} S_2(n,j)(j)_k x^{j-k} \\ &= \sum_{k=0}^{n} \binom{r}{k} \sum_{l=0}^{n-k} S_2(n,k+l)(k+l)_k x^l \\ &= \sum_{l=0}^{n} \left(\sum_{k=0}^{n-l} \binom{r}{k} (k+l)_k S_2(n,k+l) \right) x^l. \end{aligned} \tag{3.12}$$

Also, from $Bel_{n,r}(x) = (1+t)^r Bel_n(x)$,

$$(1+t)^s Bel_{n,r}(x) = Bel_{n,r+s}(x), \ (s \geq 0). \tag{3.13}$$

In particular, for $s = 1$, we have

$$Bel_{n,r+1}(x) = Bel_{n,r}(x) + \frac{d}{dx} Bel_{n,r}(x). \tag{3.14}$$

Hence in addition to (3.3) and (3.4) we obtain another expression for the derivative of $Bel_{n,r}(x)$, namely

$$\frac{d}{dx} Bel_{n,r}(x) = Bel_{n,r+1}(x) - Bel_{n,r}(x). \tag{3.15}$$

Combining this with (3.3), we get

$$Bel_{n,r+1}(x) = \sum_{k=0}^{n} \binom{n}{k} Bel_{k,r}(x). \tag{3.16}$$

We are now going to summarize the results obtained so far as the following three theorems. Theorem 2 follows from (3.3), (3.5) and (3.15), Theorem 3 from (3.6), (3.8) and (3.16), and Theorem 4 from (3.11) and (3.12).

Theorem 3.2. *For all integers $n \geq 1$, the derivative of r-Bell polynomials can be given as follows:*

$$\begin{aligned}\frac{d}{dx}Bel_{n,r}(x) &= \sum_{k=0}^{n-1}\binom{n}{k}Bel_{k,r}(x)\\ &= \frac{Bel_{n+1,r}(x)}{x} - \frac{rBel_{n,r}(x)}{x} - Bel_{n,r}(x)\\ &= Bel_{n,r+1}(x) - Bel_{n,r}(x).\end{aligned}$$

Theorem 3.3. *For all integers $n \geq 0$, the following identities hold true.*

$$\begin{aligned}Bel_{n,r}(x+y) &= \sum_{k=0}^{n}\binom{n}{k}Bel_{k,r}(x)Bel_{n-k}(y),\\ Bel_{n,r}(x) &= \sum_{m=0}^{n}\binom{n}{m}r^{n-m}Bel_m(x),\\ Bel_{n,r+1}(x) &= \sum_{k=0}^{n}\binom{n}{k}Bel_{k,r}(x).\end{aligned}$$

Theorem 3.4. *For all integers $n \geq 0$, we have the following expressions of r-Bell polynomials.*

$$\begin{aligned}Bel_{n,r}(x) &= \sum_{k=0}^{n-1}\sum_{l=0}^{r}\binom{n-1}{k}\binom{r}{l}(n-k)_l b_k^{(n)} x^{n-k-l}\\ &= \sum_{l=0}^{n}\left(\sum_{k=0}^{n-l}\binom{r}{k}(k+l)_k S_2(n,k+l)\right)x^l,\end{aligned}$$

where $b_k^{(n)}$ are the Bernoulli numbers of the second kind of order n given by (3.10).

From now on, we will apply the formula (2.9) in order to express $Bel_{n,r}(x)$ as linear combinations of well-known families of special polynomials. For this, let us remind you of the fact in (3.1), namely

$$Bel_{n,r}(x) \sim \left(\frac{1}{(1+t)^r}, \log(1+t)\right). \tag{3.17}$$

Noting that the Bernoulli polynomial $B_n(x)$ is Sheffer for $\left(\frac{e^t-1}{t}, t\right)$, we write $Bel_{n,r}(x) = \sum_{k=0}^{n} C_{n,k}B_k(x)$.Then

$$\begin{aligned}C_{n,k} &= \left\langle \frac{e^{e^t-1}-1}{e^t-1} \Big| \frac{1}{k!}e^{rt}(e^t-1)^k x^n \right\rangle\\ &= \left\langle \frac{e^{e^t-1}-1}{e^t-1} \Big| \sum_{l=k}^{\infty} S_{2,r}(l+r,k+r)\frac{t^l}{l!}x^n \right\rangle\\ &= \sum_{l=k}^{n}\binom{n}{l}S_{2,r}(l+r,k+r)\left\langle \frac{e^{e^t-1}-1}{e^t-1} \Big| x^{n-l}\right\rangle.\end{aligned} \tag{3.18}$$

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 27, NO.1, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC

Here we observe that

$$\begin{aligned}
&\Big\langle \frac{e^{e^t-1}-1}{e^t-1} | x^{n-l} \Big\rangle \\
&= \Big\langle \frac{e^{e^t-1}-1}{t} | \frac{t}{e^t-1} x^{n-l} \Big\rangle \\
&= \Big\langle \frac{e^{e^t-1}-1}{t} | \sum_{m=0}^{\infty} B_m \frac{t^m}{m!} x^{n-l} \\
&= \sum_{m=0}^{n-l} \binom{n-l}{m} B_m \Big\langle \frac{e^{e^t-1}-1}{t} | x^{n-l-m} \Big\rangle \\
&= \sum_{m=0}^{n-l} \frac{1}{n-l-m+1} \binom{n-l}{m} B_m \Big\langle e^{e^t-1}-1 | x^{n-l-m+1} \Big\rangle \\
&= \sum_{m=0}^{n-l} \frac{1}{n-l-m+1} \binom{n-l}{m} B_m Bel_{n-l-m+1}.
\end{aligned} \tag{3.19}$$

Thus we see that

$$\begin{aligned}
C_{n,k} =& \frac{1}{n+1} \sum_{l=k}^{n} \sum_{m=0}^{n-l} \binom{n+1}{l} \binom{n-l+1}{m} S_{2,r}(l+r,k+r) \\
& \times B_m Bel_{n-l-m+1}.
\end{aligned} \tag{3.20}$$

Finally, we obtain

$$\begin{aligned}
Bel_{n,r}(x) =& \frac{1}{n+1} \sum_{k=0}^{n} \Big(\sum_{l=k}^{n} \sum_{m=0}^{n-l} \binom{n+1}{l} \binom{n-l+1}{m} S_{2,r}(l+r,k+r) \\
& \times B_m Bel_{n-l-m+1} \Big) B_k(x).
\end{aligned} \tag{3.21}$$

Let $Bel_{n,r}(x) = \sum_{k=0}^{n} C_{n,k} E_k(x)$. Here $E_n(x)$ are the Euler polynomials with $E_n(x) \sim (\frac{e^t+1}{2}, t)$. Then

$$\begin{aligned}
C_{n,k} &= \frac{1}{2} \Big\langle e^{e^t-1}+1 | \frac{1}{k!} e^{rt}(e^t-1)^k x^n \Big\rangle \\
&= \frac{1}{2} \sum_{l=k}^{n} \binom{n}{l} S_{2,r}(l+r,k+r) \Big\langle e^{e^t-1}+1 | x^{n-l} \Big\rangle \\
&= \frac{1}{2} \sum_{l=k}^{n} \binom{n}{l} S_{2,r}(l+r,k+r)(Bel_{n-l}+\delta_{n,l}).
\end{aligned} \tag{3.22}$$

Hence we get

$$Bel_{n,r}(x) = \frac{1}{2} \sum_{k=0}^{n} \left(\sum_{l=k}^{n} \binom{n}{l} S_{2,r}(l+r,k+r)(Bel_{n-l}+\delta n, l) \right) E_k(x). \tag{3.23}$$

We summarize the expressions of $Bel_{n,r}(x)$ in (3.21) and (3.23) as a theorem.

Theorem 3.5. *For all integers $n \geq 0$, we have the following expressions.*

$$\begin{aligned} Bel_{n,r}(x) &= \frac{1}{n+1}\sum_{k=0}^{n}(\sum_{l=k}^{n}\sum_{m=0}^{n-l}\binom{n+1}{l}\binom{n-l+1}{m}S_{2,r}(l+r,k+r) \\ &\times B_m Bel_{n-l-m+1})B_k(x) \\ &= \frac{1}{2}\sum_{k=0}^{n}\left(\sum_{l=k}^{n}\binom{n}{l}S_{2,r}(l+r,k+r)(Bel_{n-l}+\delta n,l)\right)E_k(x). \end{aligned}$$

Write $Bel_{n,r}(x) = \sum_{k=0}^{n} C_{n,k}(x)_k$, where $(x)_n$ are the falling factorials with $(x)_n \sim (1, e^t - 1)$. Then

$$\begin{aligned} C_{n,k} &= \left\langle e^{rt}|\frac{1}{k!}(e^{e^t-1}-1)^k x^n\right\rangle \\ &= \left\langle e^{rt}|\sum_{l=k}^{\infty}S_2(l,k)\frac{1}{l!}(e^t-1)^l x^n\right\rangle \\ &= \sum_{l=k}^{n}S_2(l,k)\left\langle e^{rt}|\sum_{m=l}^{\infty}S_2(m,l)\frac{t^m}{m!}x^n\right\rangle \\ &= \sum_{l=k}^{n}S_2(l,k)\sum_{m=l}^{n}\binom{n}{m}S_2(m,l)\left\langle e^{rt}|x^{n-m}\right\rangle \\ &= \sum_{l=k}^{n}\sum_{m=l}^{n}\binom{n}{m}S_2(l,k)S_2(m,l)r^{n-m}. \end{aligned} \tag{3.24}$$

Thus we have

$$Bel_{n,r}(x) = \sum_{k=0}^{n}\left(\sum_{l=k}^{n}\sum_{m=l}^{n}\binom{n}{m}S_2(l,k)S_2(m,l)r^{n-m}\right)(x)_k. \tag{3.25}$$

As in (3.24), we let $Bel_{n,r}(x) = \sum_{k=0}^{n} C_{n,k}(x)_k$. But here we compute the coefficients $C_{n,k}$ in a way different from (3.24). Then

$$\begin{aligned} C_{n,k} &= \frac{1}{k!}\left\langle e^{rt}|(e^{e^t-1}-1)^k x^n\right\rangle \\ &= \frac{1}{k!}\left\langle e^{rt}|\sum_{l=0}^{k}\binom{k}{l}(-1)^{k-l}e^{l(e^t-1)}x^n\right\rangle \\ &= \frac{1}{k!}\sum_{l=0}^{k}\binom{k}{l}(-1)^{k-l}\left\langle e^{rt}|\sum_{m=0}^{\infty}Bel_m(l)\frac{t^m}{m!}x^n\right\rangle \\ &= \frac{1}{k!}\sum_{l=0}^{k}\binom{k}{l}(-1)^{k-l}\sum_{m=0}^{n}\binom{n}{m}Bel_m(l)r^{n-m} \\ &= \frac{1}{k!}\sum_{l=0}^{k}\sum_{m=0}^{n}(-1)^{k-l}\binom{k}{l}\binom{n}{m}r^{n-m}Bel_m(l). \end{aligned} \tag{3.26}$$

Hence we obtain

$$Bel_{n,r}(x) = \sum_{k=0}^{n}\left(\sum_{l=0}^{k}\sum_{m=0}^{n}\frac{(-1)^{k-l}}{k!}\binom{k}{l}\binom{n}{m}r^{n-m}Bel_m(l)\right)(x)_k. \tag{3.27}$$

Combining (3.25) and (3.27), we get the following theorem.

Theorem 3.6. *For all integers $n \geq 0$, we have the following expressions.*

$$\begin{aligned} Bel_{n,r}(x) &= \sum_{k=0}^{n} \left(\sum_{l=k}^{n} \sum_{m=l}^{n} \binom{n}{m} S_2(l,k) S_2(m,l) r^{n-m} \right) (x)_k \\ &= \sum_{k=0}^{n} \left(\sum_{l=0}^{k} \sum_{m=0}^{n} \frac{(-1)^{k-l}}{k!} \binom{k}{l} \binom{n}{m} r^{n-m} Bel_m(l) \right) (x)_k. \end{aligned}$$

We recall here that the Abel polynomial $A_n(x;a)(a \neq 0)$ is the associated sequence for te^{at}, namely $A_n(x;a) \sim (1, te^{at})$. Let $Bel_{n,r}(x) = \sum_{k=0}^{n} C_{n,k} A_k(x;a)$. Then

$$\begin{aligned} C_{n,k} &= \left\langle e^{rt} e^{ak(e^t-1)} \middle| \frac{1}{k!}(e^t-1)^k x^n \right\rangle \\ &= \left\langle e^{rt} e^{ak(e^t-1)} \middle| \sum_{l=k}^{\infty} S_2(l,k) \frac{t^l}{l!} x^n \right\rangle \\ &= \sum_{l=k}^{n} \binom{n}{l} S_2(l,k) \left\langle e^{rt} \middle| e^{ak(e^t-1)} x^{n-l} \right\rangle \\ &= \sum_{l=k}^{n} \binom{n}{l} S_2(l,k) \left\langle e^{rt} \middle| \sum_{m=0}^{\infty} Bel_m(ak) \frac{t^m}{m!} x^{n-l} \right\rangle \\ &= \sum_{l=k}^{n} \binom{n}{l} S_2(l,k) \sum_{m=0}^{n-l} \binom{n-l}{m} Bel_m(ak) r^{n-l-m} \\ &= \sum_{l=k}^{n} \sum_{m=0}^{n-l} \binom{n}{l} \binom{n-l}{m} S_2(l,k) r^{n-l-m} Bel_m(ak). \end{aligned} \tag{3.28}$$

Thus we have the following result.

Theorem 3.7. *For all integers $n \geq 0$, we have the following expression.*

$$Bel_{n,r}(x) = \sum_{k=0}^{n} \left(\sum_{l=k}^{n} \sum_{m=0}^{n-l} \binom{n}{l} \binom{n-l}{m} S_2(l,k) r^{n-l-m} Bel_m(ak) \right) A_k(x;a),$$

where $A_n(x;a)$ are the Abel polynomials.

The ordered Bell polynomials $Ob_n(x)$ are the Appell polynomial with $Ob_n(x) \sim (2-e^t, t)$. Write $Bel_{n,r}(x) = \sum_{k=0}^{n} C_{n,k} Ob_k(x)$. Then

$$\begin{aligned} C_{n,k} &= \left\langle 2 - e^{e^t-1} \middle| \frac{1}{k!} e^{rt} (e^t-1)^k x^n \right\rangle \\ &= \sum_{l=k}^{n} \binom{n}{l} S_{2,r}(l+r, k+r) \left\langle 2 - e^{e^t-1} \middle| x^{n-l} \right\rangle \\ &= \sum_{l=k}^{n} \binom{n}{l} S_{2,r}(l+r, k+r) \big(2\delta_{n,l} - Bel_{n-l} \big). \end{aligned} \tag{3.29}$$

Hence we obtain the following theorem.

Theorem 3.8. *For all integers $n \geq 0$, we have the following expression.*

$$Bel_{n,r}(x) = \sum_{k=0}^{n} \left(\sum_{l=k}^{n} \binom{n}{l} S_{2,r}(l+r, k+r)\big(2\delta_{n,l} - Bel_{n-l}\big) \right) Ob_k(x),$$

where $Ob_n(x)$ are the ordered Bell polynomials.

In (3.29), we saw that the ordered Bell polynomials $Ob_m(x)$ are given by generating function

$$\frac{1}{2-e^t} e^{xt} = \sum_{m=0}^{\infty} Ob_m(x) \frac{t^m}{m!}. \tag{3.30}$$

More generally, the ordered Bell polynomials $Ob_m^{(\alpha)}(x)$ of order α are defined by

$$\left(\frac{1}{2-e^t} \right)^{\alpha} e^{xt} = \sum_{m=0}^{\infty} Ob_m^{(\alpha)}(x) \frac{t^m}{m!}. \tag{3.31}$$

For $x = 0$, $Ob_m^{(\alpha)} = Ob_m^{(\alpha)}(0)$ are called the ordered Bell numbers of order α and given by

$$\left(\frac{1}{2-e^t} \right)^{\alpha} = \sum_{m=0}^{\infty} Ob_m^{(\alpha)} \frac{t^m}{m!}. \tag{3.32}$$

Let $Bel_{n,r}(x) = \sum_{k=0}^{n} C_{n,k} L_k^{(\alpha)}(x)$. Here $L_n^{(\alpha)}(x)$ are the Laguerre polynomials of order α with $L_n^{(\alpha)}(x) \sim \big((1-t)^{-\alpha-1}, \frac{t}{t-1}\big)$. Then

$$\begin{aligned}
C_{n,k} &= \frac{1}{k!} \left\langle \big(2-e^t\big)^{-\alpha-1} e^{rt} \left(\frac{e^t-1}{e^t-2} \right)^k \middle| x^n \right\rangle \\
&= (-1)^k \left\langle \big(2-e^t\big)^{-(k+\alpha+1)} \middle| \frac{1}{k!} e^{rt} \big(e^t-1\big)^k x^n \right\rangle \\
&= (-1)^k \sum_{l=k}^{n} \binom{n}{l} S_{2,r}(l+r, k+r) \left\langle \big(2-e^t\big)^{-(k+\alpha+1)} \middle| x^{n-l} \right\rangle \\
&= (-1)^k \sum_{l=k}^{n} \binom{n}{l} S_{2,r}(l+r, k+r) Ob_{n-l}^{(k+\alpha+1)}.
\end{aligned} \tag{3.33}$$

Then we have the following theorem.

Theorem 3.9. *For all integers $n \geq 0$, we have the following expression.*

$$\begin{aligned}
Bel_{n,r}(x) = \sum_{k=0}^{n} \Bigg(&\sum_{l=k}^{n} (-1)^k \binom{n}{l} S_{2,r}(l+r, k+r) \\
&\times Ob_{n-l}^{(k+\alpha+1)} \Bigg) L_k^{(\alpha)}(x),
\end{aligned}$$

where $Ob_n^{(\alpha)}(x)$ and $L_n^{(\alpha)}(x)$ are the higher-order ordered Bell polynomials and the Laguerre polynomials of order α, respectively.

Let $Bel_{n,r}(x)=\sum_{k=0}^{n}C_{n,k}D_k(x)$, where $D_n(x)$ are the Daehee polynomials with $D_n(x)\sim\big(\frac{e^t-1}{t},e^t-1\big)$. Then

$$\begin{aligned}
C_{n,k}&=\frac{1}{k!}\Big\langle\frac{t}{e^t-1}e^{rt}\frac{e^{e^t-1}-1}{t}\big(e^{e^t-1}-1\big)^k\Big|x^n\Big\rangle\\
&=\frac{1}{k!}\frac{1}{n+1}\Big\langle\frac{t}{e^t-1}e^{rt}(e^{e^t-1}-1)\big(e^{e^t-1}-1\big)^k\Big|x^{n+1}\Big\rangle\\
&=\frac{k+1}{n+1}\Big\langle\frac{t}{e^t-1}e^{rt}\Big|\frac{1}{(k+1)!}\big(e^{e^t-1}-1\big)^{k+1}x^{n+1}\Big\rangle\\
&=\frac{k+1}{n+1}\Big\langle\frac{t}{e^t-1}e^{rt}\Big|\sum_{l=k+1}^{\infty}S_2(l,k+1)\frac{1}{l!}\big(e^t-1\big)^lx^{n+1}\Big\rangle\\
&=\frac{k+1}{n+1}\sum_{l=k+1}^{n+1}S_2(l,k+1)\Big\langle\frac{t}{e^t-1}e^{rt}\Big|\sum_{m=l}^{\infty}S_2(m,l)\frac{t^m}{m!}x^{n+1}\Big\rangle\\
&=\frac{k+1}{n+1}\sum_{l=k+1}^{n+1}S_2(l,k+1)\sum_{m=l}^{n+1}\binom{n+1}{m}S_2(m,l)\Big\langle\frac{t}{e^t-1}e^{rt}\Big|x^{n+1-m}\Big\rangle\\
&=\sum_{l=k+1}^{n+1}\sum_{m=l}^{n+1}\frac{k+1}{n+1}\binom{n+1}{m}S_2(l,k+1)S_2(m,l)B_{n+1-m}(r).
\end{aligned}\tag{3.34}$$

Thus we have the following theorem.

Theorem 3.10. *For all integers $n\geq 0$, we have the following expression.*

$$Bel_{n,r}(x)=\sum_{k=0}^{n}\left(\sum_{l=k+1}^{n+1}\sum_{m=l}^{n+1}\frac{k+1}{n+1}\binom{n+1}{m}S_2(l,k+1)S_2(m,l)B_{n+1-m}(r)\right)D_k(x),$$

where $D_n(x)$ are the Daehee polynomials.

Write $Bel_{n,r}(x)=\sum_{k=0}^{n}C_{n,k}H_k^{(\nu)}(x)$. Here $H_n^{(\nu)}(x)$ are the Hermite polynomials with $H_n^{(\nu)}(x)\sim(e^{\frac{\nu t^2}{2}},t)$. Then

$$\begin{aligned}
C_{n,k}&=\Big\langle e^{\frac{\nu(e^t-1)^2}{2}}\Big|\frac{1}{k!}e^{rt}\big(e^t-1\big)^kx^n\Big\rangle\\
&=\sum_{l=k}^{n}\binom{n}{l}S_{2,r}(l+r,k+r)\Big\langle e^{\frac{\nu(e^t-1)^2}{2}}\Big|x^{n-l}\Big\rangle\\
&=\sum_{l=k}^{n}\binom{n}{l}S_{2,r}(l+r,k+r)\Big\langle\sum_{m=0}^{\infty}\frac{\nu^m}{m!}\frac{(e^t-1)^{2m}}{2^m}\Big|x^{n-l}\Big\rangle\\
&=\sum_{l=k}^{n}\binom{n}{l}S_{2,r}(l+r,k+r)\sum_{m=0}^{[\frac{n-l}{2}]}\frac{(2m)!\nu^m}{m!2^m}\Big\langle\frac{1}{(2m)!}(e^t-1)^{2m}\Big|x^{n-l}\Big\rangle\\
&=\sum_{l=k}^{n}\binom{n}{l}S_{2,r}(l+r,k+r)\sum_{m=0}^{[\frac{n-l}{2}]}\frac{(2m)!}{m!}(\frac{\nu}{2})^m\Big\langle\sum_{i=2m}^{\infty}S_2(i,2m)\frac{t^i}{i!}\Big|x^{n-l}\Big\rangle\\
&=\sum_{l=k}^{n}\sum_{m=0}^{[\frac{n-l}{2}]}\binom{n}{l}\frac{(2m)!}{m!}(\frac{\nu}{2})^mS_{2,r}(l+r,k+r)S_2(n-l,2m).
\end{aligned}\tag{3.35}$$

Hence we obtain the following result.

Theorem 3.11. *For all integers $n \geq 0$, we have the following expression.*

$$Bel_{n,r}(x) = \sum_{k=0}^{n} \left(\sum_{l=0}^{n} \sum_{m=0}^{[\frac{n-l}{2}]} \binom{n}{l} \frac{(2m)!}{m!} (\frac{\nu}{2})^m S_{2,r}(l+r,k+r) \right.$$
$$\left. \times S_2(n-l,2m) \right) H_k^{(\nu)}(x),$$

where $H_n^{(\nu)}(x)$ are the Hermite polynomials.

Let $Bel_{n,r}(x) = \sum_{k=0}^{n} C_{n,k} p_k(x)$. Here $p_n(x) = x^n y_{n-1}(\frac{1}{x}) \sim (1, t - \frac{1}{2}t^2)$, where $y_n(x) = \sum_{k=0}^{n} \frac{(n+k)!}{(n-k)!k!} \left(\frac{x}{2}\right)^k$ are called Bessel polynomials and satisfy the differential equation

$$x^2 y'' + (2x+2)y' + n(n+1)y = 0. \tag{3.36}$$

$$\begin{aligned} C_{n,k} &= \left(-\frac{1}{2}\right)^k \left\langle \left(e^t - 3\right)^k \middle| \frac{1}{k!} e^{rt} \left(e^t - 1\right)^k x^n \right\rangle \\ &= \left(-\frac{1}{2}\right)^k \sum_{l=k}^{n} \binom{n}{l} S_{2,r}(l+r,k+r) \left\langle \left(e^t - 3\right)^k \middle| x^{n-l} \right\rangle \\ &= \left(-\frac{1}{2}\right)^k \sum_{l=k}^{n} \binom{n}{l} S_{2,r}(l+r,k+r) \left\langle \sum_{m=0}^{k} \binom{k}{m} (-3)^{k-m} e^{mt} \middle| x^{n-l} \right\rangle \\ &= \left(-\frac{1}{2}\right)^k \sum_{l=k}^{n} \binom{n}{l} S_{2,r}(l+r,k+r) \sum_{m=0}^{k} \binom{k}{m} (-3)^{k-m} m^{n-l} \\ &= \left(\frac{3}{2}\right)^k \sum_{l=k}^{n} \sum_{m=0}^{k} \binom{n}{l} \binom{k}{m} (-\frac{1}{3})^m m^{n-l} S_{2,r}(l+r,k+r). \end{aligned} \tag{3.37}$$

Hence we have the following result.

Theorem 3.12. *For all integers $n \geq 0$, we have the following expression.*

$$Bel_{n,r}(x) = \sum_{k=0}^{n} \left(\sum_{l=k}^{n} \sum_{m=0}^{k} \binom{n}{l} \binom{k}{m} \left(\frac{3}{2}\right)^k (-\frac{1}{3})^m m^{n-l} \right.$$
$$\left. \times S_{2,r}(l+r,k+r) \right) p_k(x),$$

where $p_n(x) = x^n y_{n-1}(\frac{1}{x})$, with $y_n(x)$ the Bessel polynomials.

Let $Bel_{n,r}(x) = \sum_{k=0}^{n} C_{n,k} b_k(x)$, where $b_n(x)$ are the Bernoulli polynomials of the second kind with $b_n(x) \sim \left(\frac{t}{e^t-1}, e^t - 1\right)$.

$$\begin{aligned}
C_{n,k} &= \left\langle \frac{e^t-1}{e^{e^t-1}-1} e^{rt} \middle| \frac{1}{k!}(e^{e^t-1}-1)^k x^n \right\rangle \\
&= \left\langle \frac{e^t-1}{e^{e^t-1}-1} e^{rt} \middle| \sum_{l=k}^{\infty} S_2(l,k) \frac{1}{l!}(e^t-1)^l x^n \right\rangle \\
&= \sum_{l=k}^{n} S_2(l,k) \left\langle \frac{e^t-1}{e^{e^t-1}-1} e^{rt} \middle| \sum_{m=l}^{\infty} S_2(m,l) \frac{t^m}{m!} x^n \right\rangle \\
&= \sum_{l=k}^{n} S_2(l,k) \sum_{m=l}^{n} \binom{n}{m} S_2(m,l) \left\langle e^{rt} \middle| \frac{e^t-1}{e^{e^t-1}-1} x^{n-m} \right\rangle \\
&= \sum_{l=k}^{n} S_2(l,k) \sum_{m=l}^{n} \binom{n}{m} S_2(m,l) \left\langle e^{rt} \middle| \sum_{i=0}^{\infty} B_i \frac{1}{i!}(e^t-1)^i x^{n-m} \right\rangle \qquad (3.38) \\
&= \sum_{l=k}^{n} \sum_{m=l}^{n} \binom{n}{m} S_2(l,k) S_2(m,l) \sum_{i=0}^{n-m} B_i \left\langle e^{rt} \middle| \sum_{j=i}^{\infty} S_2(j,i) \frac{t^j}{j!} x^{n-m} \right\rangle \\
&= \sum_{l=k}^{n} \sum_{m=l}^{n} \binom{n}{m} S_2(l,k) S_2(m,l) \sum_{i=0}^{n-m} B_i \sum_{j=i}^{n-m} \binom{n-m}{j} S_2(j,i) r^{n-m-j} \\
&= \sum_{l=k}^{n} \sum_{m=l}^{n} \sum_{i=0}^{n-m} \sum_{j=i}^{n-m} \binom{n}{m} \binom{n-m}{j} S_2(l,k) S_2(m,l) S_2(j,i) r^{n-m-j} B_i.
\end{aligned}$$

Thus we get the final result of this paper.

Theorem 3.13. *For all integers $n \geq 0$, we have the following expression.*

$$\begin{aligned}
Bel_{n,r}(x) = \sum_{k=0}^{n} \Bigg(& \sum_{l=k}^{n} \sum_{m=l}^{n} \sum_{i=0}^{n-m} \sum_{j=i}^{n-m} \binom{n}{m} \binom{n-m}{j} \\
& \times S_2(l,k) S_2(m,l) S_2(j,i) r^{n-m-j} B_i \Bigg) b_k(x),
\end{aligned}$$

where $b_n(x)$ are the Bernoulli polynomials of the second kind.

References

1. A.Z. Broder, *The r-Stirling numbers,* Discrete Math., **49** (1984), 241-259.
2. I. Mezö, *On the maximum of r-Stirmilg numbers,* Adv. Appl. Math., **41** (2008), 293-306.
3. J. Quaintance, H. W. Gould, *Combinatorial identities for Stirling numbers. The unpublished notes of H. W. Gould. With a foreword by George E. Andrews.* World Scientific Publishing Co. Pte. Ltd., Singapore, 2016. xv+260 pp. ISBN: 978-981-4725-26-2
4. S. Roman,*The umbral calculus, Pure and Applied Mathematics, Vol.111,* Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York, 1984.
5. M. Mihoubi, H. Belbachir, *Linear recurrences for r-Bell polynomials,* J. Integer Seq., **17** (2014), Article 14.10.6.
6. T. Kim, D. S. Kim, *Extended Stirling polynomials of the second kind and Bell polynomials,* Preprint.
7. D. S. Kim, T. Kim, *Identities involving r-Stirling numbers,* J. Comput. Anal. Appl., **17** (2014), no. 4, 674-680.

8. D. S. Kim, T. Kim, *Some identities of Bell polynomials,* Sci. China Math., **58** (2015), no. 10, 2095-2104.
9. T. Kim, D. S. Kim, *On λ-Bell polynomials associated with umbral calculus,* Russ. J. Math. Phys., **24** (2017), no. 1, 69-78.

[1] Department of Mathematics, College of Science, Tianjin Polytechnic University, Tianjin 300160, China, Department of Mathematics, Kwangwoon University, Seoul, 139-701, Republic of Korea
E-mail address: tkkim@kw.ac.kr

[2] Department of Mathematics, Sogang University, Seoul, 121-742, Republic of Korea
E-mail address: dskim@sogang.ac.kr

[3] Department of Mathematics, Kwangwoon University, Seoul, 139-701, Republic of Korea
E-mail address: sura@kw.ac.kr

[4,*] Department of Mathematics Education and RINS, Gyeongsang National University, Jinju, Gyeongsangnamdo, 52828, Republic of Korea
E-mail address: mathkjk26@gnu.ac.kr

TABLE OF CONTENTS, JOURNAL OF COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 27, NO. 1, 2019

TABLE OF CONTENTS, JOURNAL OF COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 27, NO. 1, 2019

(continued)

Volume 27, Number 2 August 2019
ISSN:1521-1398 PRINT,1572-9206 ONLINE

Journal of Computational Analysis and Applications

EUDOXUS PRESS,LLC

Journal of Computational Analysis and Applications
ISSNno.'s:1521-1398 PRINT,1572-9206 ONLINE

SCOPE OF THE JOURNAL

An international publication of Eudoxus Press, LLC
(fifteen times annually)

Editor in Chief: George Anastassiou
Department of Mathematical Sciences,
University of Memphis, Memphis, TN 38152-3240, U.S.A
ganastss@memphis.edu
http://www.msci.memphis.edu/~ganastss/jocaaa

The main purpose of "J.Computational Analysis and Applications" is to publish high quality research articles from all subareas of Computational Mathematical Analysis and its many potential applications and connections to other areas of Mathematical Sciences. Any paper whose approach and proofs are computational,using methods from Mathematical Analysis in the broadest sense is suitable and welcome for consideration in our journal, except from Applied Numerical Analysis articles. Also plain word articles without formulas and proofs are excluded. The list of possibly connected mathematical areas with this publication includes, but is not restricted to: Applied Analysis, Applied Functional Analysis, Approximation Theory, Asymptotic Analysis, Difference Equations, Differential Equations, Partial Differential Equations, Fourier Analysis, Fractals, Fuzzy Sets, Harmonic Analysis, Inequalities, Integral Equations, Measure Theory, Moment Theory, Neural Networks, Numerical Functional Analysis, Potential Theory, Probability Theory, Real and Complex Analysis, Signal Analysis, Special Functions, Splines, Stochastic Analysis, Stochastic Processes, Summability, Tomography, Wavelets, any combination of the above, e.t.c.

"J.Computational Analysis and Applications" is a peer-reviewed Journal. See the instructions for preparation and submission of articles to JoCAAA. Assistant to the Editor: Dr.Razvan Mezei,mezei_razvan@yahoo.com, Madison,WI,USA.

Journal of Computational Analysis and Applications(JoCAAA) is published by **EUDOXUS PRESS,LLC**,1424 Beaver Trail Drive,Cordova,TN38016,USA,anastassioug@yahoo.com http://www.eudoxuspress.com. **Annual Subscription Prices**:For USA and Canada,Institutional:Print $800, Electronic OPEN ACCESS. Individual:Print $400. For any other part of the world add $160 more(handling and postages) to the above prices for Print. No credit card payments.

JoCAAA is reviewed and abstracted by AMS Mathematical Reviews,MATHSCI,and Zentralblaat MATH.

Editorial Board
Associate Editors of Journal of Computational Analysis and Applications

Dartmouth College
8000 Cummings Hall,
Hanover, NH 03755-8000
603-646-3843 (X 3546 Secr.)
e-mail:george.cybenko@dartmouth.edu
Approximation Theory and Neural
Networks

Sever S. Dragomir
School of Computer Science and
Mathematics, Victoria University,
PO Box 14428,
Melbourne City,
MC 8001, AUSTRALIA
Tel. +61 3 9688 4437
Fax +61 3 9688 4050
sever.dragomir@vu.edu.au
Inequalities, Functional Analysis,
Numerical Analysis, Approximations,
Information Theory, Stochastics.

Oktay Duman
TOBB University of Economics and
Technology,
Department of Mathematics, TR-
06530,
Ankara, Turkey,
oduman@etu.edu.tr
Classical Approximation Theory,
Summability Theory, Statistical
Convergence and its Applications

Saber N. Elaydi
Department Of Mathematics
Trinity University
715 Stadium Dr.
San Antonio, TX 78212-7200
210-736-8246
e-mail: selaydi@trinity.edu
Ordinary Differential Equations,
Difference Equations

J .A. Goldstein
Department of Mathematical Sciences
The University of Memphis
Memphis, TN 38152
901-678-3130
jgoldste@memphis.edu
Partial Differential Equations,
Semigroups of Operators

H. H. Gonska
Department of Mathematics
University of Duisburg
Duisburg, D-47048
Germany
011-49-203-379-3542
e-mail: heiner.gonska@uni-due.de
Approximation Theory, Computer
Aided Geometric Design

John R. Graef
Department of Mathematics
University of Tennessee at
Chattanooga
Chattanooga, TN 37304 USA
John-Graef@utc.edu
Ordinary and functional
differential equations, difference
equations, impulsive systems,
differential inclusions, dynamic
equations on time scales, control
theory and their applications

Weimin Han
Department of Mathematics
University of Iowa
Iowa City, IA 52242-1419
319-335-0770
e-mail: whan@math.uiowa.edu
Numerical analysis, Finite element
method, Numerical PDE, Variational
inequalities, Computational
mechanics

Tian-Xiao He
Department of Mathematics and
Computer Science
P.O. Box 2900, Illinois Wesleyan
University
Bloomington, IL 61702-2900, USA
Tel (309)556-3089
Fax (309)556-3864
the@iwu.edu
Approximations, Wavelet,
Integration Theory, Numerical
Analysis, Analytic Combinatorics

Margareta Heilmann
Faculty of Mathematics and Natural
Sciences, University of Wuppertal
Gaußstraße 20
D-42119 Wuppertal, Germany,
heilmann@math.uni-wuppertal.de
Approximation Theory (Positive
Linear Operators)

Xing-Biao Hu
Institute of Computational
Mathematics
AMSS, Chinese Academy of Sciences
Beijing, 100190, CHINA
hxb@lsec.cc.ac.cn

Computational Mathematics

Jong Kyu Kim
Department of Mathematics
Kyungnam University
Masan Kyungnam,631-701,Korea
Tel 82-(55)-249-2211
Fax 82-(55)-243-8609
jongkyuk@kyungnam.ac.kr
Nonlinear Functional Analysis, Variational Inequalities, Nonlinear Ergodic Theory, ODE, PDE, Functional Equations.

Robert Kozma
Department of Mathematical Sciences
The University of Memphis
Memphis, TN 38152, USA
rkozma@memphis.edu
Neural Networks, Reproducing Kernel Hilbert Spaces,
Neural Percolation Theory

Mustafa Kulenovic
Department of Mathematics
University of Rhode Island
Kingston, RI 02881,USA
kulenm@math.uri.edu
Differential and Difference Equations

Irena Lasiecka
Department of Mathematical Sciences
University of Memphis
Memphis, TN 38152
PDE, Control Theory, Functional Analysis, lasiecka@memphis.edu

Burkhard Lenze
Fachbereich Informatik
Fachhochschule Dortmund
University of Applied Sciences
Postfach 105018
D-44047 Dortmund, Germany
e-mail: lenze@fh-dortmund.de
Real Networks, Fourier Analysis, Approximation Theory

Hrushikesh N. Mhaskar
Department Of Mathematics
California State University
Los Angeles, CA 90032
626-914-7002
e-mail: hmhaska@gmail.com
Orthogonal Polynomials, Approximation Theory, Splines, Wavelets, Neural Networks

Ram N. Mohapatra
Department of Mathematics
University of Central Florida
Orlando, FL 32816-1364
tel.407-823-5080
ram.mohapatra@ucf.edu
Real and Complex Analysis, Approximation Th., Fourier Analysis, Fuzzy Sets and Systems

Gaston M. N'Guerekata
Department of Mathematics
Morgan State University
Baltimore, MD 21251, USA
tel: 1-443-885-4373
Fax 1-443-885-8216
Gaston.N'Guerekata@morgan.edu
nguerekata@aol.com
Nonlinear Evolution Equations, Abstract Harmonic Analysis, Fractional Differential Equations, Almost Periodicity & Almost Automorphy

M.Zuhair Nashed
Department Of Mathematics
University of Central Florida
PO Box 161364
Orlando, FL 32816-1364
e-mail: znashed@mail.ucf.edu
Inverse and Ill-Posed problems, Numerical Functional Analysis, Integral Equations, Optimization, Signal Analysis

Mubenga N. Nkashama
Department OF Mathematics
University of Alabama at Birmingham
Birmingham, AL 35294-1170
205-934-2154
e-mail: nkashama@math.uab.edu
Ordinary Differential Equations, Partial Differential Equations

Vassilis Papanicolaou
Department of Mathematics
National Technical University of Athens
Zografou campus, 157 80
Athens, Greece
tel:: +30(210) 772 1722
Fax +30(210) 772 1775
papanico@math.ntua.gr
Partial Differential Equations, Probability

Instructions to Contributors
Journal of Computational Analysis and Applications

An international publication of Eudoxus Press, LLC, of TN.

Editor in Chief: George Anastassiou
Department of Mathematical Sciences
University of Memphis
Memphis, TN 38152-3240, U.S.A.

1. Manuscripts files in Latex and PDF and in English, should be submitted via email to the Editor-in-Chief:

Prof.George A. Anastassiou
Department of Mathematical Sciences
The University of Memphis
Memphis,TN 38152, USA.
Tel. 901.678.3144
e-mail: ganastss@memphis.edu

Authors may want to recommend an associate editor the most related to the submission to possibly handle it.

Also authors may want to submit a list of six possible referees, to be used in case we cannot find related referees by ourselves.

2. Manuscripts should be typed using any of TEX,LaTEX,AMS-TEX,or AMS-LaTEX and according to EUDOXUS PRESS, LLC. LATEX STYLE FILE. (Click HERE to save a copy of the style file.)They should be carefully prepared in all respects. Submitted articles should be brightly typed (not dot-matrix), double spaced, in ten point type size and in 8(1/2)x11 inch area per page. Manuscripts should have generous margins on all sides and should not exceed 24 pages.

3. Submission is a representation that the manuscript has not been published previously in this or any other similar form and is not currently under consideration for publication elsewhere. A statement transferring from the authors(or their employers,if they hold the copyright) to Eudoxus Press, LLC, will be required before the manuscript can be accepted for publication.The Editor-in-Chief will supply the necessary forms for this transfer.Such a written transfer of copyright,which previously was assumed to be implicit in the act of submitting a manuscript,is necessary under the U.S.Copyright Law in order for the publisher to carry through the dissemination of research results and reviews as widely and effective as possible.

4. The paper starts with the title of the article, author's name(s) (no titles or degrees), author's affiliation(s) and e-mail addresses. The affiliation should comprise the department, institution (usually university or company), city, state (and/or nation) and mail code.

The following items, 5 and 6, should be on page no. 1 of the paper.

5. An abstract is to be provided, preferably no longer than 150 words.

6. A list of 5 key words is to be provided directly below the abstract. Key words should express the precise content of the manuscript, as they are used for indexing purposes.

The main body of the paper should begin on page no. 1, if possible.

7. All sections should be numbered with Arabic numerals (such as: 1. INTRODUCTION) .
Subsections should be identified with section and subsection numbers (such as 6.1. Second-Value Subheading).
If applicable, an independent single-number system (one for each category) should be used to label all theorems, lemmas, propositions, corollaries, definitions, remarks, examples, etc. The label (such as Lemma 7) should be typed with paragraph indentation, followed by a period and the lemma itself.

8. Mathematical notation must be typeset. Equations should be numbered consecutively with Arabic numerals in parentheses placed flush right, and should be thusly referred to in the text [such as Eqs.(2) and (5)]. The running title must be placed at the top of even numbered pages and the first author's name, et al., must be placed at the top of the odd numbed pages.

9. Illustrations (photographs, drawings, diagrams, and charts) are to be numbered in one consecutive series of Arabic numerals. The captions for illustrations should be typed double space. All illustrations, charts, tables, etc., must be embedded in the body of the manuscript in proper, final, print position. In particular, manuscript, source, and PDF file version must be at camera ready stage for publication or they cannot be considered.

Tables are to be numbered (with Roman numerals) and referred to by number in the text. Center the title above the table, and type explanatory footnotes (indicated by superscript lowercase letters) below the table.

10. List references alphabetically at the end of the paper and number them consecutively. Each must be cited in the text by the appropriate Arabic numeral in square brackets on the baseline.
References should include (in the following order):
initials of first and middle name, last name of author(s)
title of article,

name of publication, volume number, inclusive pages, and year of publication.

Authors should follow these examples:

Journal Article

1. H.H.Gonska,Degree of simultaneous approximation of bivariate functions by Gordon operators, (journal name in italics) *J. Approx. Theory*, 62,170-191(1990).

Book

2. G.G.Lorentz, (title of book in italics) *Bernstein Polynomials* (2nd ed.), Chelsea,New York,1986.

Contribution to a Book

3. M.K.Khan, Approximation properties of beta operators,in(title of book in italics) *Progress in Approximation Theory* (P.Nevai and A.Pinkus,eds.), Academic Press, New York,1991,pp.483-495.

11. All acknowledgements (including those for a grant and financial support) should occur in one paragraph that directly precedes the References section.

12. Footnotes should be avoided. When their use is absolutely necessary, footnotes should be numbered consecutively using Arabic numerals and should be typed at the bottom of the page to which they refer. Place a line above the footnote, so that it is set off from the text. Use the appropriate superscript numeral for citation in the text.

13. After each revision is made please again submit via email Latex and PDF files of the revised manuscript, including the final one.

14. Effective 1 Nov. 2009 for current journal page charges, contact the Editor in Chief. Upon acceptance of the paper an invoice will be sent to the contact author. The fee payment will be due one month from the invoice date. The article will proceed to publication only after the fee is paid. The charges are to be sent, by money order or certified check, in US dollars, payable to Eudoxus Press, LLC, to the address shown on the Eudoxus homepage.

No galleys will be sent and the contact author will receive one (1) electronic copy of the journal issue in which the article appears.

15. This journal will consider for publication only papers that contain proofs for their listed results.

An iterative algorithm of poles assignment for LDP systems *

Lingling Lv †, Zhe Zhang ‡, Lei Zhang §, Xianxing Liu ¶

Abstract

The problem of poles assignment and robust poles assignment in linear discrete-time periodic (LDP) system via periodic state feedback is discussed in this paper. Based on a numerical solution to the periodic Sylvester matrix equation, an iterative algorithm of computing the periodic feedback gain can be obtained. By optimizing the free parameter matrix in the proposed algorithm, according to robustness principle, an algorithm on the minimum norm and robust poles assignment for the LDP systems is presented. Two numerical examples are worked out to illustrate the effect of the proposed approaches.

Keywords: Linear discrete-time periodic (LDP) systems; poles assignment; robustness.

1 Introduction

Linear discrete-time periodic (LDP) systems are important bridges connecting time-varying systems and time-invariant systems. In fact, Many natural and engineering phenomena can be reduced to a composite of periodic systems thus applications of periodic systems would be found in different field, where periodic controllers could be used to dealing with the problem in which time-invariant controllers is helpless(for example, [1–3]). Moreover, another major role of the periodic controller is to improve the performance of the closed-loop system, which has also been extensively studied(one can see [4, 5] and references therein). Therefore, researches on LDP systems have attracted more and more attentions.

Since poles assignment techniques to modify the dynamic response of linear systems are the most studied problems among modern control theory, the above mentioned advantages of periodic systems and periodic controllers provide sufficient impetus for the researchers to carry out the study of poles assignment for periodic systems (see [6–9] and literatures therein). Due to the constraints of the constant controller in the periodic system, it is advocated in [6] that linear periodic output feedback is adequate to assign poles of a linear periodic discrete-time system. By utilizing a computational method on Sylvester equation, [7] proposes a complete parametric approach for pole assignment via periodic output feedback, in which parameter existed in the feedback gain could be used to accomplish some properties of plant system, robustness for instance. Using gradient search methods on the defined cost function, a computational approach is proposed in [8] to solve the minimum norm and robust pole assignment problem for linear periodic discrete-time system. Based on the proposed algorithm for parametric pole assignment problem, [9] considers the robust and minimum norm pole assignment problem and an explicit algorithm is proposed.

In this paper, the problem of poles assignment and robust poles assignment in LDP systems via state feedback is considered. Based on an iterative algorithm proposed in [13] for periodic Sylvester matrix equation, an algorithm on the problem of poles assignment in periodic linear discrete-time system with periodic state

*This work is supported by the Programs of National Natural Science Foundation of China (Nos. 11501200, U1604148, 61402149), Innovative Talents of Higher Learning Institutions of Henan (No. 17HASTIT023), China Postdoctoral Science Foundation (No. 2016M592285).

†1. College of Environment and Planning, Henan University, Kaifeng, 475004, P. R. China. 2. Institute of Electric power, North China University of Water Resources and Electric Power, Zhengzhou 450011, P. R. China. Email: lingling_lv@163.com (Lingling Lv).

‡Institute of electric power, North China University of Water Resources and Electric Power, Zhengzhou 450011, P. R. China. Email: zhe_Zhang5218@163.com (Zhe Zhang)

§Computer and Information Engineering College, Henan University, Kaifeng 475004, P. R. China. Email: zhanglei@henu.edu.cn (Lei Zhang).

¶Computer and Information Engineering College, Henan University, Kaifeng 475004, P. R. China. Email: liuxianxing@henu.edu.cn (Xianxing Liu). Corresponding author.

feedback is presented. The algorithm can realize accurate configuration of the closed-loop poles and obtain the numerical solution of the control gain. After solving the basic poles assignment problem, it is tempting to think: can we improve this algorithm to achieve the robustness of the system? The answer is positive. By optimizing the free parameter matrix in the proposed algorithm, this paper presents an algorithm on the minimum norm and robust poles assignment for the periodic linear discrete-time system. This algorithm can significantly improve the robust performance of closed-loop system. Two numerical examples are worked out to illustrate the effect of the proposed approaches.

Here, we give descriptions of some symbols which will be encountered in the rest of this paper. $\mathrm{tr}(A)$ means the trace of matrix A. Norm $\|A\|$ is a Frobenius norm of matrix A. $\Lambda(A)$ means the eigenvalue set of matrix A and Φ_{A_k} denotes the monodromy matrix $A_{K-1}A_{K-2}\cdots A_0$.

2 Main Discussions

2.1 Poles Assignment with Periodic State Feedback

Consider the completely reachable LDP systems as:

$$q_{k+1} = A_k q_k + B_k u_k, \tag{1}$$

where state matrix $A_k \in \mathbb{R}^{n\times n}$ and input matrix $B_k \in \mathbb{R}^{n\times r}$ are K-periodic. Based on the periodic feedback law in the form of

$$u_k = F_k q_k, \tag{2}$$

where F_k is the K-periodic control gain, the closed-loop system can be obtained as

$$q_{k+1} = A_{c,k} q_k, \tag{3}$$

where $A_{c,k}$ denotes $(A_k + B_k F_k)$. Then the problem of poles assignment for periodic discrete-time linear system by control law (2) can be represented as

Problem 1 *Consider the completely reachable periodic discrete-time linear system (1), seek the periodic state feedback gain $F_k \in \mathbb{R}^{m\times n}, k \in \overline{0, K-1}$, such that the poles of corresponding periodic closed-loop system (3) are set to the predetermined position $\Gamma = \{\lambda_1, \cdots, \lambda_n\}$, where Γ should be symmetrical about the real axis.*

In the following, we will first present a new poles assignment algorithm via periodic state feedback, then give strict mathematical argument to show the correctness of the proposed algorithm.

Algorithm 1 *(Poles assignment with periodic state feedback)*

1. *Choose the appropriate K-periodic matrices $\widetilde{A}_k \in \mathbb{R}^{n\times n}, k \in \overline{0, K-1}$, satisfying $\Lambda(\Phi_{\widetilde{A}_k}) = \Gamma$. Further, choose $G_k \in \mathbb{R}^{r\times n}, k \in \overline{0, K-1}$ such that periodic matrix pairs $(\widetilde{A}_k, G_k)$ are completely observable and $\Lambda(\Phi_{\widetilde{A}_k}) \cap \Lambda(\Phi_{A_k}) = 0$;*
2. *Set tolerance ε, for arbitrary initial matrix $X_k(0) \in \mathbb{R}^{n\times n}$, $k \in \overline{0, K-1}$, calculate*

$$\begin{aligned} Q_k(0) &= B_k G_k + A_k X_k(0) - X_{k+1}(0)\widetilde{A}_k; \\ R_k(0) &= -A_k^{\mathrm{T}} Q_k(0) + Q_{k-1}(0)\widetilde{A}_{k-1}^{\mathrm{T}}; \\ P_k(0) &= -R_k(0); \\ j &:= 0; \end{aligned}$$

3. While $\|R_k(j)\| \leq \varepsilon, k \in \overline{0, K-1}$*, calculate*

$$\begin{aligned}
\alpha(j) &= \frac{\sum_{k=0}^{K-1} \operatorname{tr}\left[P_k^{\mathrm{T}}(j) R_k(j)\right]}{\sum_{k=0}^{K-1} \left\| -A_k P_k(j) + P_{k+1}(j) \widetilde{A}_k \right\|^2}; \\
X_k(j+1) &= X_k(j) + \alpha(j) P_k(j) \in \mathbb{R}^{n \times n}; \\
Q_k(j+1) &= B_k G_k + A_k X_k(j+1) - X_{k+1}(j+1) \widetilde{A}_k \in \mathbb{R}^{n \times n}; \\
R_k(j+1) &= -A_k^{\mathrm{T}} Q_k(j+1) + Q_{k-1}(j+1) \widetilde{A}_{k-1}^{\mathrm{T}}; \\
P_k(j+1) &= -R_k(j+1) + \frac{\sum_{k=0}^{K-1} \|R_k(j+1)\|^2}{\sum_{k=0}^{K-1} \|R_k(j)\|^2} P_k(j) \in \mathbb{R}^{n \times n}; \\
j &= j+1;
\end{aligned}$$

4. Let $X_k^* = X_k(j)$*, calculate the periodic state feedback gain* F_k *by*

$$F_k = G_k (X_k^*)^{-1}, k \in \overline{0, K-1}.$$

To verify the validity of the above algorithm, we would provide several necessary lemmas for the problem under discussion, whose correctness can be easily checked by detail computation or derivation, and their proof is omitted due to space limitations.

Lemma 1 *For* $k \geq 0$*, the following equation holds:*

$$\sum_{k=0}^{T-1} \operatorname{tr}\left[R_k^{\mathrm{T}}(j+1) P_k(j)\right] = 0$$

for all $\{R_k(j)\}$ *and* $\{P_k(j)\}$ *derived from Algorithm 1.*

Lemma 2 *For* $k \geq 0$*, the following equation holds:*

$$\sum_{k=0}^{T-1} \operatorname{tr}\left[R_k^{\mathrm{T}}(j) P_k(j)\right] = -\sum_{j=0}^{T-1} \|R_k(j)\|^2$$

for all $\{R_k(j)\}$ *and* $\{P_k(j)\}$ *generated by Algorithm 1.*

Lemma 3 *For* $k \geq 0$*, the following relation holds:*

$$\sum_{j \geq 0} \frac{\left(\sum_{k=0}^{T-1} \|R_k(j)\|^2\right)^2}{\sum_{k=0}^{K-1} \|P_k(j)\|^2} < \infty.$$

for all $\{R_k(j)\}$ *and* $\{P_k(j)\}$ *generated by Algorithm 1.*

Based on these lemmas, we can further draw the following conclusion.

Theorem 1 *The matrices* $X_k^*, k \in \overline{0, T-1}$ *generated by Algorithm 1 satisfy periodic Sylvester matrix equation*

$$A_k X_k - X_{k+1} \widetilde{A}_k + B_k G_k = 0, k \in \overline{0, K-1}. \tag{4}$$

Proof. To explain matrices $X_k, k \in \overline{0, K-1}$ generated by Algorithm 1 are solutions to equation (10), we first illustrate that this problem is related to the convergence of matrix sequence $\{R_k(j)\}, k \in \overline{0, T-1}$ generated by Algorithm 1.

According to equation (10), construct the following index function:

$$J = \sum_{k=0}^{K-1} \frac{1}{2} \left\| B_k G_k + A_k X_k - X_{k+1} \widetilde{A}_k \right\|^2 . \tag{5}$$

It is easily obtained that for $k \in \overline{0, K-1}$,

$$\frac{\partial J}{\partial X_k} = -A_k^T \left(B_k G_k + A_k X_k - X_{k+1} \widetilde{A}_k \right) + \left(B_{k-1} G_{k-1} + A_{k-1} X_{k-1} - X_k \widetilde{A}_{k-1} \right) \widetilde{A}_{k-1}^T$$

So far, if we can find matrices $X_k^*, k \in \overline{0, K-1}$ such that

$$\left. \frac{\partial J}{\partial X_k} \right|_{X_k = X_k^*} = 0,$$

then matrices $X_k^*, k \in \overline{0, K-1}$ must be the solution to equation (10) in the meaning of least squares. From the formulation of sequence $\{R_k(j)\}, k \in \overline{0, T-1}$ in Algorithm 1, we can see

$$R_k(j) = \left. \frac{\partial J}{\partial X_k} \right|_{X_k = X_k(j)} .$$

That is to say, if matrix sequence $\{R_k(j)\}, k \in \overline{0, T-1}$ can converge to zero, matrices $X_k^*, k \in \overline{0, K-1}$ generated by Algorithm 1 must satisfy periodic matrix equation (10).

In the remaining, we only need proof that, for $k \in \overline{0, K-1}$

$$\lim_{j \to \infty} \| R_k(j) \| = 0.$$

By Lemma 1 and the expressions of $P_k(j+1)$ in Algorithm 1, we have

$$\begin{aligned} \sum_{k=0}^{K-1} \| P_k(j+1) \|^2 &= \sum_{k=0}^{K-1} \left\| -R_k(j+1) + \frac{\sum_{k=0}^{K-1} \| R_k(j+1) \|^2}{\sum_{k=0}^{K-1} \| R_k(j) \|^2} P_k(j) \right\|^2 \\ &= \left(\frac{\sum_{k=0}^{K-1} \| R_k(j+1) \|^2}{\sum_{k=0}^{K-1} \| R_k(j) \|^2} \right)^2 \sum_{k=0}^{K-1} \| P_k(j) \|^2 + \sum_{k=0}^{K-1} \| R_k(j+1) \|^2 . \end{aligned}$$

Let

$$t_j = \frac{\sum_{k=0}^{K-1} \| P_k(j) \|^2}{\left(\sum_{k=0}^{K-1} \| R_k(j) \|^2 \right)^2}.$$

Then the preceding relation can be written as

$$t_{j+1} = t_j + \frac{1}{\sum_{k=0}^{K-1} \| R_k(j+1) \|^2}. \tag{6}$$

equivalently.
We now proceed by contradiction and assume that

$$\lim_{j \to \infty} \sum_{k=0}^{K-1} \| R_k(j) \|^2 \neq 0. \tag{7}$$

This relation implies that there exists a constant $\delta > 0$ such that

$$\sum_{k=0}^{K-1} \| R_k(j) \|^2 \geq \delta$$

for all $j \geq 0$. It follows from (6) and (7) that

$$t_{j+1} \leq t_0 + \frac{j+1}{\delta}.$$

4

And it shows that

$$\frac{1}{t_{j+1}} \geq \frac{\delta}{\delta t_0 + j + 1}.$$

So we have

$$\sum_{j=1}^{\infty} \frac{1}{t_j} \geq \sum_{j=1}^{\infty} \frac{\delta}{\delta t_0 + j + 1} = \infty.$$

However, it follows from Lemma 3 that

$$\sum_{j=1}^{\infty} \frac{1}{t_j} < \infty.$$

This gives a contradiction.

Thus, the correctness of the theorem has been proved. ■

As for the effectiveness of Algorithm 1, we have the following conclusion:

Theorem 2 *Consider completely reachable periodic discrete-time linear system (1), the K-periodic matrix F_k generated from Algorithm 1 is a solution of the problem of poles assignment with periodic state feedback.*

Proof. Notice that the poles of LDP system (1) are the poles of the monodromy matrix Φ_{A_k}. According to algorithm 1, $\Phi_{\widetilde{A_k}}$ possesses the desired pole set Γ. To assign the poles of the closed-loop system (3) to set Γ, we just need find n-order invertible matrices $X_k, k \in \overline{0, K-1}$, such that

$$X_{k+1}^{-1} A_{ck} X_k = \widetilde{A}_k, \tag{8}$$

namely

$$X_{k+1}^{-1}(A_k + B_k F_k) X_k = \widetilde{A}_k, \tag{9}$$

Pre-multiplying the above equation by matrix X_{k+1} gives

$$A_k X_k - X_{k+1}\widetilde{A}_k + B_k F_k X_k = 0, k \in \overline{0, K-1},$$

Let

$$G_k = F_k X_k,$$

then Problem 1 is converted to the problem of solving the periodic Sylvester matrix equation in the form of

$$A_k X_k - X_{k+1}\widetilde{A}_k + B_k G_k = 0, k \in \overline{0, K-1}. \tag{10}$$

The step 2-3 in Algorithm 1 involve the solution of this matrix equation, and its correctness has been proved in [13]. By solving the solution matrix X_k, the periodic feedback gain can be obtained as

$$F_k = G_k X_k^{-1}, k \in \overline{0, K-1}. \tag{11}$$

That is, the periodic feedback gain F_k derived from (11) is a solution to Problem 1. ■

Remark 1 *For the periodic matrix $\widetilde{A}_k$, it should satisfy $\Lambda(\Phi_{\widetilde{A}}) = \Gamma$. This requirement can be achieved by letting F_0 be the real Jordan canonical form of the desired pole set and $F_k, k \in \overline{1, K-1}$ be unit matrices of corresponding dimension.*

Remark 2 *If system (1) is completely reachable and $\Lambda(\Phi_{\widetilde{A}}) \cap \Lambda(\Phi_A) = 0$, then X_k will be invertible naturally. That's why the algorithm requires condition $\Lambda(\Phi_{\widetilde{A}}) \cap \Lambda(\Phi_A) = 0$.*

2.2 Robust Consideration

In the previous subsection, the iterative algorithm can provide infinite numerical solutions for the pole assignment problem via periodic state feedback by choose different parameter matrix G_k. Therefore, by adding some additional conditions to the feedback gain matrix $F_k, k \in \overline{0, K-1}$ and transforming matrix $X_k, k \in \overline{0, K-1}$, the free parameter matrix G_k can be used to achieve the robustness of the system. In general, the small feedback gain is robust. Because small gain means small control signals, that is beneficial to reduce noise amplification. At the same time, in the sense of poles assignment, the closed-loop poles to be configured should be not as sensitive as possible to disturbances in the system matrix. Thus, the following robust and minimum norm pole assignment problem via periodic state feedback is proposed.

Problem 2 *Consider the completely reachable linear periodic discrete-time system (1), seek the K-periodic state feedback gain $F_k \in \mathbb{R}^{m\times n}$, such that*

1. *the poles of corresponding periodic closed-loop system are set to the predetermined position $\Gamma = \{\lambda_1, \cdots, \lambda_n\}$;*
2. *The periodic feedback gain is as small as possible and the closed-loop poles are not as sensitive as possible to disturbances in the system matrix.*

In order to solve Problem 2, the index function in [8] is introduced as follows:

$$J(G_k) = \gamma \frac{1}{2} \sum_{k=0}^{K-1} \kappa_{\mathrm{F}}^2(X_k) + (1-\gamma)\frac{1}{2} \sum_{k=0}^{K-1} \|F_k\|^2, \tag{12}$$

where $0 \leq \gamma \leq 1$ is a weighting factor. It is noted that when $\gamma = 0$, $J(G_k)$ degenerates into the index function of the minimum norm problem; when $\gamma = 1$, $J(G_k)$ becomes a purely objective function to solve the robust problem. Obviously, the weight γ leads to the combination of these two problems. [8] gives explicit analytical expressions for the index function J and its gradient. So it's easy to minimize $J(G_k)$ by using any gradient-based search method. Therefore, we can present an algorithm for the problem of periodic robust and minimum norm poles assignment.

Algorithm 2 *(Robust and minimum norm poles assignment)*

1. *Choose the appropriate K-periodic matrices $\widetilde{A}_k \in \mathbb{R}^{n\times n}$ satisfying $\Lambda(\Phi_{\widetilde{A}_k}) = \Gamma$, and initialize $G_k \in \mathbb{R}^{r\times n}$ such that periodic matrix pairs $(\widetilde{A}_k, G_k)$ are completely observable and $\Lambda(\Phi_{\widetilde{A}_k}) \cap \Lambda(\Phi_{A_k}) = 0$;*
2. *Set tolerance ε, for arbitrary initial matrix $X_k(0) \in \mathbb{R}^{n\times n}$, $k \in \overline{0, K-1}$, calculate*

$$\begin{aligned} Q_k(0) &= B_k G_k + A_k X_k(0) - X_{k+1}(0)\widetilde{A}_k; \\ R_k(0) &= -A_k^{\mathrm{T}} Q_k(0) + Q_{k-1}(0)\widetilde{A}_{k-1}^{\mathrm{T}}; \\ P_k(0) &= -R_k(0); \\ j &:= 0; \end{aligned}$$

3. *While $\|R_k(j)\| \leq \varepsilon, k \in \overline{0, K-1}$, calculate*

$$\begin{aligned} \alpha(j) &= \frac{\sum_{k=0}^{K-1} \mathrm{tr}\left[P_k^{\mathrm{T}}(j) R_k(j)\right]}{\sum_{k=0}^{K-1} \left\| -A_k P_k(j) + P_{k+1}(j)\widetilde{A}_k \right\|^2}; \\ X_k(j+1) &= X_k(j) + \alpha(j) P_k(j) \in \mathbb{R}^{n\times n}; \\ Q_k(j+1) &= B_k G_k + A_k X_k(j+1) - X_{k+1}(j+1)\widetilde{A}_k \in \mathbb{R}^{n\times n}; \\ R_k(j+1) &= -A_k^{\mathrm{T}} Q_k(j+1) + Q_{k-1}(j+1)\widetilde{A}_{k-1}^{\mathrm{T}}; \\ P_k(j+1) &= -R_k(j+1) + \frac{\sum_{k=0}^{K-1} \|R_k(j+1)\|^2}{\sum_{k=0}^{K-1} \|R_k(j)\|^2} P_k(j) \in \mathbb{R}^{n\times n}; \\ j &= j+1; \end{aligned}$$

4. *Based on gradient-based search methods and the index (12), choosing the appropriate weighting factor γ, solve the optimization problem*
$$\text{Minimize } J(G_k).$$
Denote the optimal decision matrix by $G_{opt,k}$;

5. *Substituting $G_{opt,k}$ into step 2-3 gives optimization solution $X_{opt,k}(j)$;*

6. *Let $X_{opt,k} = X_{opt,k}(j)$, calculate the robust and minimum norm periodic state feedback gain $F_{opt,k}$ by*
$$F_{opt,k} = G_{opt,k} X_{opt,k}^{-1}, k \in \overline{0, K-1}.$$

3 Numerical examples

Example 1 *Consider the completely reachable system described by*

$$q(t+1) = A(t)q(t) + B(t)u(t)$$

with

$$A_0 = \begin{bmatrix} 0 & e & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & e & 0 & 0 \\ 0 & 0 & 0 & e^{-1} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, A_1 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & e & 0 & 0 \\ 0 & 1-e^{-1} & 0 & e^{-1} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$
$$B_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ e-1 & 0 \\ 0 & 1-e^{-1} \\ 1 & 0 \end{bmatrix}, B_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ e-1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Find 2-periodic control law $u(t) = F(t)q(t)$ such that the poles of the periodic close-loop system are assigned at $\Gamma = \{0.5 \pm 0.5i, 0.7 \pm 0.7i, -0.6\}$. Specially, let

$$G(t) = \begin{bmatrix} e & 0 & 2 & 0 & 1 \\ 0.5 & -e^{-1} & 0 & 1 & 2 \end{bmatrix}, t = 0, 1$$
$$\widetilde{A}(t) = \begin{cases} \begin{bmatrix} 0.5 & 0.5 & 0 & 0 & 0 \\ -0.5 & 0.5 & 0 & 0 & 0 \\ 0 & 0 & 0.7 & 0.7 & 0 \\ 0 & 0 & -0.7 & 0.7 & 0 \\ 0 & 0 & 0 & 0 & 0.6 \end{bmatrix}, t = 0 \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, t = 1 \end{cases}$$

The proposed Algorithm 1 applied to the example gives the following 2-periodic feedback gain:

$$F(0) = \begin{bmatrix} 2.8249 & -0.4278 & -2.6334 & 2.3210 & 0.4035 \\ 1.1033 & 0.2796 & -0.8349 & 1.4695 & 0.2045 \end{bmatrix},$$
$$F(1) = \begin{bmatrix} -0.2648 & -1.0196 & -0.7015 & -0.2593 & -0.0573 \\ 1.0698 & -1.7859 & 1.4382 & -0.7656 & -0.2827 \end{bmatrix}.$$

What can be verified is that the poles assignment is valid.

Example 2 *This example is borrowed from [12]. The desired close-loop eigenvalues set is* $\Gamma = \{0.5, 0.6, 0.7, -0.6, -0.7\}$. *Arbitrarily assigning the parameter matrix* G_k *as*

$$G(t) = \begin{bmatrix} 0.3 & 0.5 & 2.1 & 0 & 1.1 \\ 0.6 & 1.1 & 0.7 & 1.2 & 0.2 \end{bmatrix}, t = 0, 1$$

gives a group of feedback gains as follows:

$$F_{\text{rand}}(0) = \begin{bmatrix} 1.0000 & -0.0000 & 0.0000 & 0.0000 & 0.0000 \\ 36.9007 & -19.7886 & 93.1374 & 19.1142 & -9.4571 \end{bmatrix},$$
$$F_{\text{rand}}(1) = \begin{bmatrix} -0.0045 & 0.0419 & -1.3397 & -0.0351 & 0.0476 \\ -0.8356 & 0.1582 & 1.9971 & 0.4532 & -0.5408 \end{bmatrix}.$$

Applying Algorithm 2 with $\gamma = 0.5$ *gives the following robust feedback gains:*

$$F_{\text{robu}}(0) = \begin{bmatrix} 1.0000 & 0.0000 & 0.0000 & -0.0000 & -0.0000 \\ -0.0289 & -2.6601 & -0.0603 & 2.9199 & 0.0054 \end{bmatrix},$$
$$F_{\text{robu}}(1) = \begin{bmatrix} -0.0332 & 0.0005 & -1.2358 & -0.0004 & 0.0200 \\ 0.0042 & -0.8145 & -0.0068 & 1.0742 & 0.0029 \end{bmatrix}.$$

Let the close-loop system matrices be perturbed by $\Delta_k \in \mathbb{R}^{n\times n}, k = 0, 1$, *which are random perturbations satisfying* $\|\Delta_k\| = 1, k = 0, 1$. *Then the close-loop system with perturbations can be represented as:*

$$A_{ck} + \mu\Delta_k, k = 0, 1,$$

where $\mu > 0$ *is a factor representing the disturbance level. According to [14], the following index can be adopted to measure the robustness of the corresponding close-loop system:*

$$d_\mu(\Delta_k) = \max_{1\le i\le 5}\{|\lambda_i\{(A_{c1} + \mu\Delta_1)(A_{c0} + \mu\Delta_0)\}|\},$$

where $\lambda_i\{A\}$ *denotes the i-th eigenvalue of matrix A. 3,000 randomized trials were performed at* μ *equal to 0.002, 0.003 and 0.005, respectively. The worst and the average value of* $d_\mu(\Delta_k)$ *corresponding to* F_{robu} *and* F_{rand} *respectively are listed in Table 1. Polar plots of the trials are depicted in Fig.1, where the left hand side refers to* F_{robu} *and the right hand side refers to* F_{rand}. *As we can see, in the presence of disturbances, the robust periodic feedback gain* F_{robu} *always performs better than* F_{rand}.

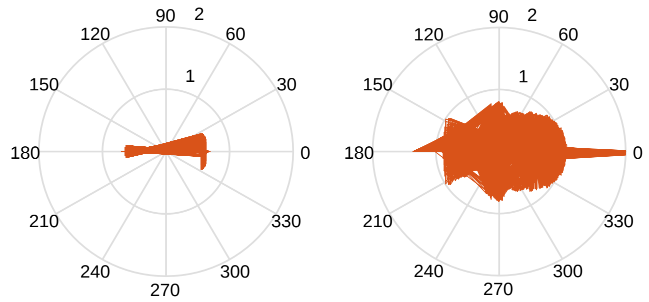
Table 1: Comparison between K_{robu} and K_{rand}

μ	μ=0.002		μ=0.003		μ=0.005	
d_μ	F_{robu}	F_{rand}	F_{robu}	F_{rand}	F_{robu}	F_{rand}
Worst	1.0237	3.3798	1.0197	4.7927	1.1561	10.9309
Mean	0.7262	1.3667	0.7244	1.5881	0.9022	2.5102

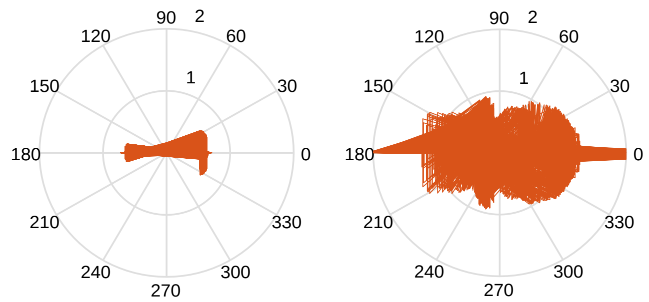
In terms of minimum norm, we compute the robust periodic feedback gains by minimize the index $J(G_k)$ *at* γ *equal to 0,0.5 and 1 respectively and the feedback norm* $\|F_0\|$, $\|F_1\|$ *together with* $\|F\| = \sqrt{\|F_0\|^2 + \|F_1\|^2}$. *The results can be see in Table 2.*

Table 2: Comparison between K_{robu} and K_{rand}

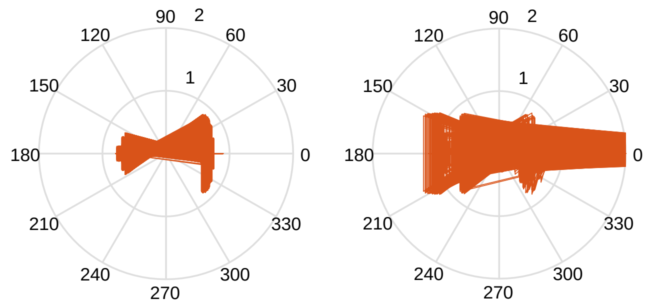
Factor	$\|F_0\|$	$\|F_1\|$	$\|F\|$
$\gamma = 0$	2.2230	2.2549	3.1665
$\gamma = 0.5$	4.0751	1.8292	4.4668
$\gamma = 1$	4.0727	1.8289	4.4645



(a) Perturbed eigenvalues of the close-loop system with $\mu = 0.002$



(b) Perturbed eigenvalues of the close-loop system with $\mu = 0.003$



(c) Perturbed eigenvalues of the close-loop system with $\mu = 0.005$

Figure 1: Perturbed eigenvalues of the close-loop system with different disturbance levels

4 Conclusions

Poles assignment with periodic state feedback and periodic robust and minimum norm poles assignment are discussed in this paper. Through mathematical derivation, the poles assignment problem is transformed into the solution to the periodic Sylvester matrix equation. Based on the recent method of solving the equation, an algorithm for solving the problem of poles assignment is presented. In this algorithm, the parameter matrix G_k can be used for further discussion on robustness. By analyzing the theory of robustness and the minimum norm, an index function of matrix G_k is adopted. Based on the gradient search algorithm, the optimization decision matrix is finally given, and the robust and minimum norm gain is thus obtained. Two examples demonstrate the effectiveness of the proposed approaches.

References

[1] P Khargonekar, K Poolla, A Tannenbaum. Robust control of linear time-invariant plants using periodic compensation. IEEE Transactions on Automatic Control, 1985, 30(11):1088-1096.

[2] E Carlos. De Souza, A. Trofino. An LMI approach to stabilization of linear discrete-time periodic systems. International Journal of Control, 2000, 73(8):696-703.

[3] C Farges, D Peaucelle, D Arzelier, et al. Robust H2 performance analysis and synthesis of linear polytopic discrete-time periodic systems via LMIs. Systems & Control Letters, 2007, 56(2):159-166.

[4] S Longhi, R Zulli. A robust periodic pole assignment algorithm. Automatic Control IEEE Transactions on, 1995, 40(5):890-894.

[5] J Lavaei, S Sojoudi, Aghdam A G. Pole Assignment With Improved Control Performance by Means of Periodic Feedback. IEEE Transactions on Automatic Control, 2007, 55(1):248-252.

[6] D Aeyels, J L Willems. Pole assignment for linear periodic systems by memoryless output feedback. IEEE Transactions on Automatic Control, 1995, 40(4):735-739.

[7] L L Lv, G R Duan, B Zhou. Parametric Pole Assignment for Discrete-time Linear Periodic Systems via Output Feedback. Acta Automatica Sinica, 2010, 36(36):113-120.

[8] A Varga. Robust and minimum norm pole assignment with periodic state feedback. Automatic Control IEEE Transactions on, 2000, 45(5):1017-1022.

[9] L L Lv, G Duan, B Zhou. Parametric pole assignment and robust pole assignment for discrete-time linear periodic systems. SIAM Journal on Control and Optimization, 2010, 48(6): 3975-3996.

[10] L H Keel, J A Fleming, S P Bhattacharyya. Minimum Norm Pole Assignment via Sylvester's Equation. Linear Algebra & Its Role in Systems Theory, 1985, 47:265-272.

[11] G R Duan. Solutions of the equation AV+ BW= VF and their application to eigenstructure assignment in linear systems. IEEE Transactions on Automatic Control, 1993, 38(2): 276-280.

[12] S. Longhi, R. Zulli. A note on robust pole assignment for periodic systems. IEEE Transactions on Automatic Control, 1996, 41(10):1493-1497.

[13] L. Lv, Z. Zhang. Finite Iterative Solutions to Periodic Sylvester Matrix Equations. Journal of the Franklin Institute, 2017, 354(5):2358-2370.

[14] L. James, H. K. TSO, N. K. Tsing. Robust deadbeat regulation. International Journal of Control, 1997, 67(4):587-602.

C^*-ALGEBRA-VALUED MODULAR METRIC SPACES AND RELATED FIXED POINT RESULTS

BAHMAN MOEINI[1], ARSLAN HOJAT ANSARI[2], CHOONKIL PARK[3] AND DONG YUN SHIN[4]

ABSTRACT. In this paper, a concept of C^*-algebra-valued modular metric space is introduced which is a generalization of a modular metric space of Chistyakov (Folia Math. **14** (2008), 3-25). Next, some common fixed point theorems are proved for generalized contraction type mappings on such spaces. Also, to support of our results an application is provided for existence and uniqueness of solution for a system of integral equations.

1. INTRODUCTION

One of the main directions in obtaining possible generalizations of fixed point results is introduced in new types of spaces. The notion of modular spaces, as a generalization of metric spaces, was introduced by Nakano [18] and was intensively developed by Koshi and Shimogaki [12], Yamamuro [23] and others. Also, the theory of fixed points in the content of modular spaces was investigated by Khamsi *et al.* [11] and many authors generalized these results [1, 2, 9, 10, 15, 22].

In 2008, Chistyakov [3] introduced the notion of modular metric spaces generated by F-modular and developed the theory of this space. In 2010, Chistyakov [4] defined the notion of modular on an arbitrary set and developed the theory of metric spaces generated by modular which are t called the modular metric spaces. Recently, Mongkolkeha *et al.* [16, 17] have introduced some notions and established some fixed point results in modular metric spaces.

In [14], Ma *et al.* introduced the concept of C^*-algebra-valued metric spaces. The main idea consists in using the set of all positive elements of a unital C^*-algebra instead of the set of real numbers. They showed that if $(X, \mathbb{A}, d)$ is a complete C^*-algebra-valued metric space and $T : X \to X$ is a contractive mapping, i.e., there exists an $a \in \mathbb{A}$ with $\|a\| < 1$ such that

$$d(Tx, Ty) \preceq a^* d(x, y)a, \quad (\forall x, y \in X).$$

Then T has a unique fixed point in X. This line of research was continued in [7, 8, 13, 21, 24], where several other fixed point results were obtained in the framework of C^*-algebra valued metric, as well as (more general) C^*-algebra-valued b-metric spaces. Recently, Shateri [20] introduced the concept of C^*-algebra-valued modular space which is a generalization of a modular space and next proved some fixed point theorems for self-mappings with contractive or expansive conditions on such spaces.

In this paper, new type of modular metric space is introduced and by using some ideas of [19] some common fixed point results are proved for self-mappings with contractive

[0]Corresponding authors: baak@hanyang.ac.kr (Choonkil Park), dyshin@uos.ac.kr (Dong Yun Shin)

2010 *Mathematics Subject Classification.* Primary 47H10; 54H25; 46L05.

Key words and phrases. modular metric space, C^*-algebra-valued modular metric space, common fixed point, occasionally weakly compatible, integral equation.

conditions on such spaces. Also, some examples to elaborate and illustrate our results are constructed. Finally, as application, existence and uniqueness of solution for a type of system of nonlinear integral equations is established.

2. Basic notions

Let X be a nonempty set, $\lambda \in (0,\infty)$ and due to the disparity of the arguments, function $\omega : (0,\infty) \times X \times X \to [0,\infty]$ will be written as $\omega_\lambda(x,y) = \omega(\lambda,x,y)$ for all $\lambda > 0$ and $x,y \in X$.

Definition 2.1. [3] Let X be a nonempty set. A function $\omega : (0,\infty)\times X\times X \to [0,\infty]$ is said to be a modular metric on X if it satisfies the following three axioms:

(i) given $x,y \in X$, $\omega_\lambda(x,y) = 0$ for all $\lambda > 0$ if and only if $x = y$;
(ii) $\omega_\lambda(x,y) = \omega_\lambda(y,x)$ for all $\lambda > 0$ and $x,y \in X$;
(iii) $\omega_{\lambda+\mu}(x,y) \le \omega_\lambda(x,z) + \omega_\mu(z,y)$ for all $\lambda > 0$ and $x,y,z \in X$.

Then (X,ω) is called a modular metric space.

Recall that a Banach algebra $\mathbb{A}$ (over the field $\mathbb{C}$ of complex numbers) is said to be a C^*-algebra if there is an involution $*$ in $\mathbb{A}$ (i.e., a mapping $* : \mathbb{A} \to \mathbb{A}$ satisfying $a^{**} = a$ for each $a \in \mathbb{A}$) such that, for all $a,b \in \mathbb{A}$ and $\lambda,\mu \in \mathbb{C}$, the following holds:

(i) $(\lambda a + \mu b)^* = \bar{\lambda}a^* + \bar{\mu}b^*$;
(ii) $(ab)^* = b^*a^*$;
(iii) $\|a^*a\| = \|a\|^2$.

Note that, from (iii), it follows that $\|a\| = \|a^*\|$ for each $a \in \mathbb{A}$. Moreover, the pair $(\mathbb{A},*)$ is called a unital $*$-algebra if $\mathbb{A}$ contains the unit element $1_\mathbb{A}$. A positive element of $\mathbb{A}$ is an element $a \in \mathbb{A}$ such that $a^* = a$ and its spectrum $\sigma(a) \subset \mathbb{R}_+$, where $\sigma(a) = \{\lambda \in \mathbb{R} : \lambda 1_\mathbb{A} - a \text{ is noninvertible}\}$. The set of all positive elements will be denoted by $\mathbb{A}_+$. Such elements allow us to define a partial ordering '$\succeq$' on the elements of $\mathbb{A}$. That is,

$$b \succeq a \quad \text{if and only if} \quad b - a \in \mathbb{A}_+.$$

If $a \in \mathbb{A}$ is positive, then we write $a \succeq \theta$, where θ is the zero element of $\mathbb{A}$. Each positive element a of a C^*-algebra $\mathbb{A}$ has a unique positive square root. From now on, by $\mathbb{A}$ we mean a unital C^*-algebra with unit element $1_\mathbb{A}$. Further, $\partial_+ = \{a \in \mathbb{A} : a \succeq \theta\}$ and $(a^*a)^{\frac{1}{2}} = |a|$.

Lemma 2.2. *[5] Suppose that $\mathbb{A}$ is a unital C^*-algebra with a unit $1_\mathbb{A}$.*

(1) *For any $x \in \mathbb{A}_+$, we have $x \preceq 1_\mathbb{A} \Leftrightarrow \|x\| \le 1$.*
(2) *If $a \in \mathbb{A}_+$ with $\|a\| < \frac{1}{2}$, then $1_\mathbb{A} - a$ is invertible and $\|a(1_\mathbb{A} - a)^{-1}\| < 1$.*
(3) *Suppose that $a,b \in \mathbb{A}$ with $a,b \succeq \theta$ and $ab = ba$. Then $ab \succeq \theta$.*
(4) *By $\mathbb{A}'$ we denote the set $\{a \in \mathbb{A} : ab = ba, \forall b \in \mathbb{A}\}$. Let $a \in \mathbb{A}'$. If $b,c \in \mathbb{A}$ with $b \succeq c \succeq \theta$ and $1_\mathbb{A} - a \in \mathbb{A}'$ is an invertible operator, then*
$$(1_\mathbb{A} - a)^{-1}b \succeq (1_\mathbb{A} - a)^{-1}c.$$

Notice that in a C^*-algebra, if $\theta \preceq a,b$, one cannot conclude that $\theta \preceq ab$. For example, consider the C^*-algebra $\mathbb{M}_2(\mathbb{C})$ and set $a = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}$, $b = \begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix}$. Then $ab = \begin{pmatrix} -1 & 2 \\ -4 & 8 \end{pmatrix}$. Clearly $a,b \in \mathbb{M}_2(\mathbb{C})_+$, while ab is not.

In the following we begin to introduce and study a new type of modular metric space that is called a C^*-algebra-valued modular metric space.

Definition 2.3. Let X be a nonempty set. A function $\omega : (0,\infty)\times X\times X \to \mathbb{A}$ is said to be a C^*-algebra-valued modular metric (briefly, C^*.m.m) on X if it satisfies the following three axioms:

(i) given $x, y \in X$, $\omega_\lambda(x,y) = \theta$ for all $\lambda > 0$ if and only if $x = y$;
(ii) $\omega_\lambda(x,y) = \omega_\lambda(y,x)$ for all $\lambda > 0$ and $x, y \in X$;
(iii) $\omega_{\lambda+\mu}(x,y) \preceq \omega_\lambda(x,z) + \omega_\mu(z,y)$ for all $\lambda, \mu > 0$ and $x, y, z \in X$.

The truple $(X, \mathbb{A}, \omega)$ is called a C^*.m.m space.

If instead of (i), we have the condition
(i') $\omega_\lambda(x,x) = \theta$ for all $\lambda > 0$ and $x \in X$, then ω is said to be a C^*-algebra-valued pseudo modular metric (briefly, C^*.p.m.m) on X and if ω satisfies (i'), (iii) and
(i'') given $x, y \in X$, if there exists a number $\lambda > 0$, possibly depending on x and y, such that $\omega_\lambda(x,y) = \theta$, then $x = y$, then ω is called a C^*-algebra-valued strict modular metric (briefly, C^*.s.m.m) on X.

A C^*.m.m (or C^*.p.m.m, C^*.s.m.m) ω on X is said to be convex if, instead of (iii), we replace the following condition:

(iv) $\omega_{\lambda+\mu}(x,y) \preceq \frac{\lambda}{\lambda+\mu}\omega_\lambda(x,z) + \frac{\mu}{\lambda+\mu}\omega_\mu(z,y)$ for all $\lambda, \mu > 0$ and $x, y, z \in X$.

Clearly, if ω is a C^*.s.m.m, then ω is a C^*.m.m, which in turn implies that ω is a C^*.p.m.m on X, and similar implications hold for convex ω. The essential property of a C^*.m.m ω on a set X is as follows: given $x, y \in X$, the function $0 < \lambda \to \omega_\lambda(x,y) \in \mathbb{A}$ is non increasing on $(0,\infty)$. In fact, if $0 < \mu < \lambda$, then we have

$$\omega_\lambda(x,y) \preceq \omega_{\lambda-\mu}(x,x) + \omega_\mu(x,y) = \omega_\mu(x,y). \tag{2.1}$$

It follows that at each point $\lambda > 0$ the right limit $\omega_{\lambda+0}(x,y) := \lim_{\varepsilon\to+0}\omega_{\lambda+\varepsilon}(x,y)$ and the left limit $\omega_{\lambda-0}(x,y) := \lim_{\varepsilon\to+0}\omega_{\lambda-\varepsilon}(x,y)$ exist in $\mathbb{A}$ and the following two inequalities hold:

$$\omega_{\lambda+0}(x,y) \preceq \omega_\lambda(x,y) \preceq \omega_{\lambda-0}(x,y).$$

It can be checked that if $x_0 \in X$, then the set

$$X_\omega = \{x \in X : \lim_{\lambda\to\infty} \omega_\lambda(x,x_0) = \theta\}$$

is a C^*-algebra-valued metric space, called a C^*-algebra-valued modular space, where $d^0_\omega : X_\omega \times X_\omega \to \mathbb{A}$ is given by

$$d^0_\omega = \inf\{\lambda > 0 : \|\omega_\lambda(x,y)\| \le \lambda\} \text{ for all } x, y \in X_\omega.$$

Moreover, if ω is convex, then the set X_ω is equal to

$$X^*_\omega = \{x \in X : \exists\ \lambda = \lambda(x) > 0 \text{ such that } \|\omega_\lambda(x,x_0)\| < \infty\}$$

and $d^*_\omega : X^*_\omega \times X^*_\omega \to \mathbb{A}$ is given by

$$d^*_\omega = \inf\{\lambda > 0 : \|\omega_\lambda(x,y)\| \le 1\} \text{ for all } x, y \in X^*_\omega.$$

It is easy to see that if X is a real linear space, $\rho : X \to \mathbb{A}$ and

$$\omega_\lambda(x,y) = \rho(\frac{x-y}{\lambda}) \text{ for all } \lambda > 0 \text{ and } x, y \in X, \tag{2.2}$$

then ρ is a C^*-algebra valued modular (convex C^*-algebra-valued modular) on X if and only if ω is C^*.m.m (convex C^*.m.m, respectively) on X. On the other hand, assume that ω satisfies the following two conditions:

(i) $\omega_\lambda(\mu x, 0) = \omega_{\frac{\lambda}{\mu}}(x,0)$ for all $\lambda, \mu > 0$ and $x \in X$;
(ii) $\omega_\lambda(x+z, y+z) = \omega_\lambda(x,y)$ for all $\lambda > 0$ and $x, y, z \in X$.

If we set $\rho(x) = \omega_1(x, 0)$ with (2.2), where $x \in X$, then $X_\rho = X_\omega$ is a linear subspace of X and the functional $\|x\|_\rho = d^0_\omega(x, 0),\ x \in X_\rho$ is an F-norm on X_ρ. If ω is convex, then $X^*_\rho \equiv X^*_\omega = X_\rho$ is a linear subspace of X and the functional $\|x\|_\rho = d^*_\omega(x, 0),\ x \in X^*_\rho$, is a norm on X^*_ρ.

Similar assertions hold if we replace C^*.m.m by C^*.p.m.m. If ω is C^*.m.m in X, then the set X_ω is a C^*.m.m space.

By the idea of property in C^*-algebra-valued metric spaces and C^*-algebra-valued modular spaces, we define the following:

Definition 2.4. Let X_ω be a C^*.m.m space.

(1) The sequence $(x_n)_{n\in\mathbb{N}}$ in X_ω is said to be ω-convergent to $x \in X_\omega$ with respect to $\mathbb{A}$ if
$\omega_\lambda(x_n, x) \to \theta$ as $n \to \infty$ for all $\lambda > 0$.

(2) The sequence $(x_n)_{n\in\mathbb{N}}$ in X_ω is said to be ω-Cauchy with respect to $\mathbb{A}$ if
$\omega_\lambda(x_m, x_n) \to \theta$ as $m, n \to \infty$ for all $\lambda > 0$.

(3) A subset C of X_ω is said to be ω-closed with respect to $\mathbb{A}$ if the limit of the ω-convergent sequence of C always belongs to C.

(4) X_ω is said to be ω-complete if any ω-Cauchy sequence with respect to $\mathbb{A}$ is ω-convergent.

(5) A subset C of X_ω is said to be ω-bounded with respect to $\mathbb{A}$ if for all $\lambda > 0$
$\delta_\omega(C) = \sup\{\|\omega_\lambda(x, y)\|;\ x, y \in C\} < \infty$.

Definition 2.5. Let X_ω be a C^*.m.m space. Let f, g be self-mappings of X_ω. A point x in X_ω is called a coincidence point of f and g if and only if $fx = gx$. We shall call $w = fx = gx$ a point of coincidence of f and g.

Definition 2.6. Let X_ω be a C^*.m.m space. Two self-mappings f and g of X_ω are said to be weakly compatible if they commute at coincidence points.

Definition 2.7. Let X_ω be a C^*.m.m space. Two self-mappings f and g of X_ω are occasionally weakly compatible (owc) if and only if there is a point x in X_ω which is a coincidence point of f and g at which f and g commute.

Lemma 2.8. *[6] Let X_ω be a C^*.m.m space and f, g owc self-mappings of X_ω. If f and g have a unique point of coincidence, $w = fx = gx$, then w is a unique common fixed point of f and g.*

3. Main results

Theorem 3.1. *Let X_ω be a C^*.m.m space and $I, J, R, S, T, U : X_\omega \to X_\omega$ be self-mappings of X_ω such that the pairs (SR, I) and (TU, J) are occasionally weakly compatible. Suppose there exist $a, b, c \in \mathbb{A}$ with $0 < \|a\|^2 + \|b\|^2 + \|c\|^2 < 1$ such that the following assertion for all $x, y \in X_\omega$ and $\lambda > 0$ hold:*

(3.1.1) $\omega_\lambda(SRx, TUy) \preceq a^*\omega_\lambda(Ix, Jy)a + b^*\omega_\lambda(SRx, Jy)b + c^*\omega_{2\lambda}(TUy, Ix)c;$

(3.1.2) $\|\omega_\lambda(SRx, TUy)\| < \infty.$

Then SR, TU, I and J have a common fixed point in X_ω. Furthermore, if the pairs $(S, R), (S, I), (R, I), (T, J), (T, U), (U, J)$ are commuting pairs of mappings, then I, J, R, S, T and U have a unique common fixed point in X_ω.

Proof. Since the pair (SR, I) and (TU, J) are occasionally weakly compatible, there exist $u, v \in X_\omega : SRu = Iu$ and $TUv = Jv$. Moreover, $SR(Iu) = I(SRu)$ and

$TU(Jv) = J(TUv)$. Now we can assert that $SRu = TUv$. If not then by (3.1.1)

$$\begin{aligned} \omega_\lambda(SRu, TUv) &\preceq a^*\omega_\lambda(Iu, Jv)a + b^*\omega_\lambda(SRu, Jv)b + c^*\omega_{2\lambda}(TUv, Iu)c \\ &= a^*\omega_\lambda(Iu, Jv)a + b^*\omega_\lambda(Iu, Jv)b + c^*\omega_{2\lambda}(Jv, Iu)c \\ &= a^*\omega_\lambda(Iu, Jv)a + b^*\omega_\lambda(Iu, Jv)b + c^*\omega_{2\lambda}(Iu, Jv)c. \end{aligned} \tag{3.1}$$

By definition of C^*.m.m space and (2.1) and (3.1), we have

$$\begin{aligned} \omega_\lambda(SRu, TUv) &\preceq a^*\omega_\lambda(Iu, Jv)a + b^*\omega_\lambda(Iu, Jv)b + c^*(\omega_\lambda(Iu, Iu) + \omega_\lambda(Iu, Jv))c \\ &= a^*\omega_\lambda(Iu, Jv)a + b^*\omega_\lambda(Iu, Jv)b + c^*\omega_\lambda(Iu, Jv)c \\ &= a^*(\omega_\lambda(Iu, Jv))^{\frac{1}{2}}(\omega_\lambda(Iu, Jv))^{\frac{1}{2}}a + b^*(\omega_\lambda(Iu, Jv))^{\frac{1}{2}}(\omega_\lambda(Iu, Jv))^{\frac{1}{2}}b \\ &+c^*(\omega_\lambda(Iu, Jv))^{\frac{1}{2}}(\omega_\lambda(Iu, Jv))^{\frac{1}{2}}c \\ &= (a(\omega_\lambda(Iu, Jv))^{\frac{1}{2}})^*(a(\omega_\lambda(Iu, Jv))^{\frac{1}{2}}) \\ &+(b(\omega_\lambda(Iu, Jv))^{\frac{1}{2}})^*(b(\omega_\lambda(Iu, Jv))^{\frac{1}{2}}) \\ &+(c(\omega_\lambda(Iu, Jv))^{\frac{1}{2}})^*(c(\omega_\lambda(Iu, Jv))^{\frac{1}{2}}) \\ &= |a(\omega_\lambda(Iu, Jv))^{\frac{1}{2}}|^2 + |b(\omega_\lambda(Iu, Jv))^{\frac{1}{2}}|^2 + |c(\omega_\lambda(Iu, Jv))^{\frac{1}{2}}|^2 \\ &\preceq \|a(\omega_\lambda(Iu, Jv))^{\frac{1}{2}}\|^2 1_{\mathbb{A}} + \|b(\omega_\lambda(Iu, Jv))^{\frac{1}{2}}\|^2 1_{\mathbb{A}} + \|c(\omega_\lambda(Iu, Jv))^{\frac{1}{2}}\|^2 1_{\mathbb{A}}. \end{aligned}$$

Thus

$$\begin{aligned} \|\omega_\lambda(SRu, TUv)\| &\leq \|\omega_\lambda(Iu, Jv)\|(\|a\|^2 + \|b\|^2 + \|c\|^2) \\ &< \|\omega_\lambda(Iu, Jv)\|. \end{aligned}$$

So $\|\omega_\lambda(Iu, Jv)\| < \|\omega_\lambda(Iu, Jv)\|$, which is a contradiction. Hence $SRu = TUv$ and thus

$$SRu = Iu = TUv = Jv.$$

Moreover, assume that there is another point z such that $SRz = Iz$. Using (3.1.1),

$$\begin{aligned} \omega_\lambda(SRz, TUv) &\preceq a^*\omega_\lambda(Iz, Jv)a + b^*\omega_\lambda(SRz, Jv)b + c^*\omega_{2\lambda}(TUv, Iz)c \\ &= a^*\omega_\lambda(SRz, TUv)a + b^*\omega_\lambda(SRz, TUv)b + c^*\omega_{2\lambda}(SRz, TUv)c. \end{aligned} \tag{3.2}$$

By a similar way, $\|\omega_\lambda(SRz, TUv)\| < \|\omega_\lambda(SRz, TUv)\|(\|a\|^2 + \|b\|^2 + \|c\|^2)$, which is a contradiction. Hence we get

$$SRu = Iu = TUv = Jv. \tag{3.3}$$

Thus from (3.2) and (3.3), it follows that $SRu = SRz$. Hence $w = SRu = Iu$, for some $w \in X_\omega$, is the unique point of coincidence of SR and I. Then by Lemma 2.8, w is a unique common fixed point of SR and I. So $SRw = Iw = w$.

Similarly, there is another common fixed point $w' \in X_\omega : TUw' = Jw' = w'$.

For the uniqueness, suppose $w \neq w'$. Then by (3.1.1), we have

$$\begin{aligned} \omega_\lambda(SRw, TUw') &= \omega_\lambda(w, w') \\ &\preceq a^*\omega_\lambda(Iw, Jw'^*\omega_\lambda(SRw, Jw'^*\omega_{2\lambda}(TUw, Iw')c \\ &= a^*\omega_\lambda(w, w'^*\omega_\lambda(w, w'^*\omega_{2\lambda}(w, w')c. \end{aligned}$$

Thus $\|\omega_\lambda(w, w')\| < \|\omega_\lambda(w, w'^2 + \|b\|^2 + \|c\|^2)$, which is a contradiction. Hence $w = w'$. So w is a unique common fixed point of SR, TU, I and J.

Furthermore, if $(S, R),\ (S, I),\ (R, I),\ (T, J),\ (T, U),\ (U, J)$ are commuting pairs, then

$$\begin{aligned} Sw &= S(SRw) = S(RS)w = SR(Sw) \\ Sw &= S(Iw) = S(RS)w = I(Sw) \\ Rw &= R(SRw) = RS(Rw) = SR(Rw) \\ Rw &= R(Iw) = (Rw), \end{aligned}$$

which show that Sw and Rw is a common fixed point of (SR, I), which gives $SRw = Sw = Rw = Iw = w$.

Similarly, we have $TUw = Tw = Uw = Jw = w$. Hence w is a unique common fixed point of $S,\ R,\ I,\ J,\ T,\ U$. □

Corollary 3.2. *Let X_ω be a C^*.m.m space and $I, J, S, T : X_\omega \to X_\omega$ be self-mappings of X_ω such that the pairs (S, I) and (T, J) are occasionally weakly compatible. Suppose there exist $a, b, c \in \mathbb{A}$ with $0 < \|a\|^2 + \|b\|^2 + \|c\|^2 < 1$ such that the following assertions for all $x, y \in X_\omega$ and $\lambda > 0$ hold:*

(3.2.1) $\omega_\lambda(Sx, Ty) \preceq a^*\omega_\lambda(Ix, Jy)a + b^*\omega_\lambda(Sx, Jy)b + c^*\omega_{2\lambda}(Ty, Ix)c$;
(3.2.2) $\|\omega_\lambda(Sx, Ty)\| < \infty$.

Then S, T, I and J have a unique common fixed point in X_ω.

Proof. If we put $R = U := I_{X_\omega}$ where I_{X_ω} is an identity mapping on X_ω, then the result follows from Theorem 3.1. □

Corollary 3.3. *Let X_ω be a C^*.m.m space and $S, T : X_\omega \to X_\omega$ be self-mappings of X_ω such that S and T are occasionally weakly compatible. Suppose there exist $a, b, c \in \mathbb{A}$ with $0 < \|a\|^2 + \|b\|^2 + \|c\|^2 < 1$ such that the following assertions for all $x, y \in X_\omega$ and $\lambda > 0$ hold:*

(3.3.1) $\omega_\lambda(Tx, Ty) \preceq a^*\omega_\lambda(Sx, Sy)a + b^*\omega_\lambda(Tx, Sy)b + c^*\omega_{2\lambda}(Ty, Sx)c$;
(3.3.2) $\|\omega_\lambda(Tx, Ty)\| < \infty$.

Then S and T have a unique common fixed point in X_ω.

Proof. If we put $I = J := S$ and $S := T$ in (3.2.1) and (3.2.2), then the result follows from Theorem 3.1. □

Corollary 3.4. *Let X_ω be a C^*.m.m space and $S, T : X_\omega \to X_\omega$ be self-mappings of X_ω such that S and T are occasionally weakly compatible. Suppose there exists $a \in \mathbb{A}$ with $0 < \|a\| < 1$ such that the following assertions for all $x, y \in X_\omega$ and $\lambda > 0$ hold:*

(3.4.1) $\omega_\lambda(Tx, Ty) \preceq a^*\omega_\lambda(Sx, Sy)a$;
(3.4.2) $\|\omega_\lambda(Tx, Ty)\| < \infty$.

Then S and T have a unique common fixed point in X_ω.

Proof. If we put $b = c := 0_{\mathbb{A}}$ in (3.3.1), then the result follows from Corollary 3.3. □

4. Examples

In this section we provide some nontrivial examples in favour of our results.

Example 4.1. Let $X = \mathbb{R}$ and consider $\mathbb{A} = M_2(\mathbb{R})$ of all 2×2 matrices with the usual operation of addition, scalar multiplication and matrix multiplication. Define a norm on $\mathbb{A}$ by $\|A\| = \left(\sum_{i,j=1}^2 |a_{ij}|^2\right)^{\frac{1}{2}}$ and $* : \mathbb{A} \to \mathbb{A}$, given by $A^* = A$ for all $A \in \mathbb{A}$, defines an involution on $\mathbb{A}$. Thus $\mathbb{A}$ becomes a C^*-algebra. For

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \in \mathbb{A} = M_2(\mathbb{R}),$$

we denote $A \preceq B$ if and only if $(a_{ij} - b_{ij}) \le 0$ for all $i, j = 1, 2$.
Define $\omega : (0, \infty) \times X \times X \to \mathbb{A}$ by

$$\omega_\lambda(x, y) = \begin{pmatrix} |\frac{x-y}{\lambda}| & 0 \\ 0 & |\frac{x-y}{\lambda}| \end{pmatrix}$$

for all $x, y \in X$ and $\lambda > 0$. It is easy to check that ω satisfies all the conditions of Definition 2.3. So $(X, \mathbb{A}, \omega)$ is a C^*.m.m space.

Example 4.2. Let $X = \{\frac{1}{c^n} : n = 1, 2, \cdots\}$ where $0 < c < 1$ and $\mathbb{A} = M_2(\mathbb{R})$. Define $\omega : (0,\infty) \times X \times X \to \mathbb{A}$ by

$$\omega_\lambda(x,y) = \begin{pmatrix} \|\frac{x-y}{\lambda}\| & 0 \\ 0 & \alpha\|\frac{x-y}{\lambda}\| \end{pmatrix}$$

for all $x, y \in X$, $\alpha \geq 0$ and $\lambda > 0$. Then it is easy to check that ω is a C^*.m.m. space.

Example 4.3. Let $X = L^\infty(E)$ and $H = L^2(E)$, where E is a Lebesgue measurable set. By $B(H)$ we denote the set of bounded linear operators on the Hilbert space H. Clearly, $B(H)$ is a C^*-algebra with the usual operator norm.
Define $\omega : (0,\infty) \times X \times X \to B(H)$ by

$$\omega_\lambda(f,g) = \pi_{|\frac{f-g}{\lambda}|}, \quad (\forall f, g \in X).$$

Here $\pi_h : H \to H$ is the multiplication operator defined by

$$\pi_h(\phi) = h \cdot \phi$$

for $\phi \in H$. Then ω is a C^*.m.m and $(X_\omega, B(H), \omega)$ is an ω-complete C^*.m.m space. It suffices to verify the completeness of X_ω. For this, let $\{f_n\}$ be an ω-Cauchy sequence with respect to $B(H)$, that is, for an arbitrary $\varepsilon > 0$, there is $N \in \mathbb{N}$ such that for all $m, n \geq N$,

$$\|\omega_\lambda(f_m, f_n)\| = \|\pi_{|\frac{f_m - f_n}{\lambda}|}\| = \|\frac{f_m - f_n}{\lambda}\|_\infty \leq \varepsilon.$$

So $\{f_n\}$ is a Cauchy sequence in Banach space X. Hence there are a function $f \in X$ and $N_1 \in \mathbb{N}$ such that

$$\|\frac{f_n - f}{\lambda}\|_\infty \leq \varepsilon \ \ (n \geq N_1),$$

which implies that

$$\|\omega_\lambda(f_n, f)\| = \|\pi_{|\frac{f_n - f}{\lambda}|}\| = \|\frac{f_n - f}{\lambda}\|_\infty \leq \varepsilon, \quad (n \geq N_1).$$

Consequently, the sequence $\{f_n\}$ is an ω-convergent sequence in X_ω and so X_ω is an ω-complete C^*.m.m space.

Example 4.4. Let $(X, \mathbb{A}, \omega)$ be C^*.m.m space defined as in Example 4.1. Define $S, T, I, J : X_\omega \to X_\omega$ by

$$Sx = Tx = 1, \quad Jx = 2 - x, \quad Ix = \begin{cases} \frac{x}{2} & if \ x \in (-\infty, 1), \\ 1 & if \ x = 1, \\ 0 & if \ x \in (1, \infty) \end{cases}$$

for all $x, y \in X_\omega = \mathbb{R}$ and $\lambda > 0$. Then we have

$0 = \left\| \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\| = \|\omega_\lambda(Sx, Ty)\| < \infty.$

For all $a, b, c \in \mathbb{A}$ with $0 < \|a\|^2 + \|b\|^2 + \|c\|^2 < 1$, we get

$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \omega_\lambda(Sx, Ty) \preceq a^*\omega_\lambda(Ix, Jy)a + b^*\omega_\lambda(Sx, Jy)b + c^*\omega_{2\lambda}(Ty, Ix)c$ for all $x, y \in X_\omega$ and $\lambda > 0$. Also clearly, the pairs (S, I) and (T, J) are occasionally weakly compatible. So all the conditions of Corollary 3.2 are satisfied and $x = 1$ is a unique common fixed point of S, T, I and J.

5. Application

Remind that if for $\lambda > 0$ and $x, y \in L^\infty(E)$, we define $\omega : (0,\infty)\times L^\infty(E)\times L^\infty(E) \to B(H)$ by

$$\omega_\lambda(x, y) = \pi_{|\frac{x-y}{\lambda}|},$$

where $\pi_h : H \to H$ is defined as in Example 4.3, then $(L^\infty(E)_\omega, B(H), \omega)$ is an ω-complete C^*.m.m space.

Let E be a Lebesgue measurable set, $X = L^\infty(E)$ and $H = L^2(E)$ be the Hilbert space. Consider the following system of nonlinear integral equations:

$$x(t) = w(t) + k_i(t, x(t)) + \mu \int_E n(t, s)h_j(s, x(s))ds \tag{5.1}$$

for all $t \in E$, where $w \in L^\infty(E)_\omega$ is known, $k_i(t, x(t)),\ n(t, s),\ h_j(s, x(s)),\ i, j = 1, 2$ and $i \neq j$ are real or complex valued functions that are measurable both in t and s on E and μ is a real or complex number, and assume the following conditions:

(a) $sup_{s\in E} \int_E |n(t, s)|dt = M_1 < +\infty$,

(b) $k_i(s, x(s)) \in L^\infty(E)_\omega$ for all $x \in L^\infty(E)_\omega$, and there exists $L_1 > 1$ such that for all $s \in E$,

$$|k_1(s, x(s)) - k_2(s, y(s))| \geq L_1|x(s) - y(s)| \text{ for all } x, y \in L^\infty(E)_\omega,$$

(c) $h_i(s, x(s)) \in L^\infty(E)_\omega$ for all $x \in L^\infty(E)_\omega$, and there exists $L_2 > 0$ such that for all $s \in E$,

$$|h_1(s, x(s)) - h_2(s, y(s))| \leq L_2|x(s) - y(s)| \text{ for all } x, y \in L^\infty(E)_\omega,$$

(d) there exists $x(t) \in L^\infty(E)_\omega$ such that

$$x(t) - w(t) - \mu \int_E n(t, s)h_1(s, x(s))ds = k_1(t, x(t)),$$

which implies

$$\begin{aligned}&k_1(t, x(t)) - w(t) - \mu \textstyle\int_E n(t, s)h_1(s, k_1(s, x(s)))ds\\ &= k_1(t, x(t) - w(t) - \mu \textstyle\int_E n(t, s)h_1(s, x(s))ds),\end{aligned}$$

(e) there exists $y(t) \in L^\infty(E)_\omega$ such that

$$y(t) - w(t) - \mu \int_E n(t, s)h_2(s, y(s))ds = k_2(t, y(t)),$$

which implies

$$\begin{aligned}&k_2(t, y(t)) - w(t) - \mu \textstyle\int_E n(t, s)h_i(s, k_2(s, y(s)))ds\\ &= k_2(t, y(t) - w(t) - \mu \textstyle\int_E n(t, s)h_2(s, y(s))ds).\end{aligned}$$

Theorem 5.1. *With the assumptions (a)-(e), the system of nonlinear integral equations (5.1) has a unique solution x^* in $L^\infty(E)_\omega$ for each real or complex number μ with $\frac{1+|\mu|L_2M_1}{L_1} < 1$.*

Proof. Define

$$Sx(t) = x(t) - w(t) - \mu \int_E n(t, s)h_1(s, x(s))ds,$$

$$Tx(t) = x(t) - w(t) - \mu \int_E n(t, s)h_2(s, x(s))ds,$$

$$Ix(t) = k_1(t, x(t)),\ \ Jx(t) = k_2(t, x(t)).$$

Set $a = \sqrt{\frac{1+|\mu|M_1L_2}{L_1}} \cdot 1_{B(H)}$, $b = c = 0_{B(H)}$. Then $a \in B(H)_+$ and $0 < \|a\|^2 + \|b\|^2 + \|c\|^2 = \frac{1+|\mu|M_1L_2}{L_1} < 1$.

For any $h \in H$, we have

$$\|\omega_\lambda(Sx, Ty)\| = \sup_{\|h\|=1}(\pi_{|\frac{Sx-Ty}{\lambda}|}h, h)$$

$$= \sup_{\|h\|=1} \int_E \left[\frac{1}{\lambda}\Big|(x-y) + \mu \int_E n(t,s)(h_2(s,y(s) - h_1(s,x(s))ds\Big|\right] h(t)\overline{h(t)}dt$$

$$\leq \sup_{\|h\|=1} \int_E \left[\frac{1}{\lambda}\Big|(x-y) + \mu \int_E n(t,s)(h_2(s,y(s) - h_1(s,x(s))ds\Big|\right] |h(t)|^2 dt$$

$$\leq \frac{1}{\lambda} \sup_{\|h\|=1} \int_E |h(t)|^2 dt \Big[\|x-y\|_\infty + |\mu|M_1L_2\|x-y\|_\infty\Big]$$

$$\leq \left(\frac{1+|\mu|M_1L_2}{\lambda}\right)\|x-y\|_\infty$$

$$\leq \left(\frac{1+|\mu|M_1L_2}{L_1}\right)\left\|\frac{k_1(t,x(t)) - k_2(t,y(t))}{\lambda}\right\|_\infty$$

$$= \left(\frac{1+|\mu|M_1L_2}{L_1}\right)\|\omega_\lambda(Ix, Jy)\|$$

$$= \|a\|^2\|\omega_\lambda(Ix, Jy)\|.$$

Then

$$\|\omega_\lambda(Sx, Ty)\| \leq \|a\|^2\|\omega_\lambda(Ix, Jy)\| + \|b\|^2\|\omega_\lambda(Sx, Jy)\| + \|c\|^2\|\omega_{2\lambda}(Ty, Ix)\|$$

for all $x, y \in L^\infty(E)_\omega$ and $\lambda > 0$. Also by conditions (d) and (e) the pairs (S, I) and (T, J) are occasionally weakly compatible. Therefore, by Corollary 3.2, there exists a unique common fixed point $x^* \in L^\infty(E)_\omega$ such that $x^* = Sx^* = Tx^* = Ix^* = Jx^*$, which proves the existence of unique solution of (5.1) in $L^\infty(E)_\omega$. This completes the proof. □

References

[1] G.A. Anastassiou, I.K. Argyros, Approximating fixed points with applications in fractional calculus, *J. Comput. Anal. Appl.* **21** (2016), 1225–1242.

[2] A. Batool, T. Kamran, S. Jang, C. Park, Generalized φ-weak contractive fuzzy mappings and related fixed point results on complete metric space, *J. Comput. Anal. Appl.* **21** (2016), 729–737.

[3] V.V. Chistyakov, Modular metric spaces generated by F-modulars, *Folia Math.* **14** (2008), 3–25.

[4] V.V. Chistyakov, Modular metric spaces I: basic concepts, *Nonlinear Anal.* **72** (2010), 1–14.

[5] R. Douglas, Banach Algebra Techniques in Operator Theory, *Springer, Berlin*, 1998.

[6] G. Jungck and B.E. Rhoades, Fixed point theorems for occasionally weakly compatible mappings, *Fixed Point Theory*, **7** (2006), 287–296.

[7] Z. Kadelburg and S. Radenović, Fixed point results in C^*-algebra-valued metric spaces are direct consequences of their standard metric counterparts, *Fixed Point Theory Appl.* **2016**, 2016:53.

[8] T. Kamran, M. Postolache, A. Ghiura, S. Batul and R. Ali, The Banach contraction principle in C^*-algebra-valued b-metric spaces with application, *Fixed Point Theory Appl.* **2016**, 2016:10.

[9] M.A. Khamsi, A convexity property in modular function spaces, *Math. Jpn.* **44** (1996), 269–279.

[10] M.A. Khamsi, Quasicontraction mapping in modular spaces without Δ_2-condition, *Fixed Point Theory Appl.* **2008**, Article ID 916187 (2008).

[11] M.A. Khamsi, W.M. Kozlowski and S. Reich, Fixed point theory in modular function spaces, *Nonlinear Anal.* **14** (1990), 935–953.

[12] S. Koshi and T. Shimogaki, On F-norms of quasi-modular spaces, *J. Fac. Sci. Hokkaido Univ. Ser. I*, **15** (1961), 202–218.
[13] Z. Ma and L. Jiang, C^*-Algebra-valued b-metric spaces and related fixed point theorems, *Fixed Point Theory Appl.* **2015**, 2015:222.
[14] Z. Ma, L. Jiang and H. Sun, C^*-Algebra-valued metric spaces and related fixed point theorems, *Fixed Point Theory Appl.* **2014**, 2014:206.
[15] C. Mongkolkeha and P. Kumam, Common fixed points for generalized weak contraction mappings in modular spaces, *Sci. Math. Jpn.* **75** (2012), 69–79.
[16] C. Mongkolkeha, W. Sintunavarat and P. Kumam, Fixed point theorems for contraction mappings in modular metric spaces, *Fixed Point Theory Appl.* **2011**, 2011:93.
[17] C. Mongkolkeha, W. Sintunavarat and P. Kumam, Fixed point theorems for contraction mappings in modular metric spaces, *Fixed Point Theory Appl.* **2012**, 2012:103.
[18] H. Nakano, Modulared: Semi-Ordered Linear Spaces, *In Tokyo Math. Book Ser. Vol. 1, Maruzen Co., Tokyo*, 1950.
[19] A. Parya, P. Pathak, V.H. Badshah and N. Gupta, Common fixed point theorems for generalized contraction mappings in modular metric spaces, *Adv. Inequal. Appl.* **2017**, 2017:9.
[20] T.L. Shateri, C^*-algebra-valued modular spaces and fixed point theorems, *J. Fixed Point Theory Appl.* **19** (2017), 1551–1560.
[21] D. Shehwar and T. Kamran, C^*-Valued G-contraction and fixed points, *J. Inequal. Appl.* **2015**, 2015:304.
[22] X. Wang and Y. Chen, Fixed points of asymptotic pointwise nonexpansive mappings in modular spaces, *Appl. Math.* **2012**, Article ID 319394 (2012).
[23] S. Yamamuro, On conjugate spaces of Nakano spaces, *Trans. Amer. Math. Soc.* **90** (1959), 291–311.
[24] A. Zada, S. Saifullah and Z. Ma, Common fixed point theorems for G-contraction in C^*-algebra-valued metric spaces, *Int. J. Anal. Appl.* **11** (2016), 23–27.

[1]Department of Mathematics, Hidaj Branch, Islamic Azad University, Hidaj, Iran
E-mail address: moeini145523@gmail.com

[2]Department of Mathematics, Karaj Branch, Islamic Azad University, Karaj, Iran
E-mail address: analsisamirmath2@gmail.com

[3]Research Institute for Natural Sciences, Hanyang University, Seoul 04763, Korea
E-mail address: baak@hanyang.ac.kr

[4]Department of Mathematics, University of Seoul, Seoul 02504, Korea
E-mail address: dyshin@uos.ac.kr

Strong Convergence Theorems and Applications of a New Viscosity Rule for Nonexpansive Mappings

Waqas Nazeer[1], Mobeen Munir[1], Sayed Fakhar Abbas Naqvi[2], Chahn Yong Jung[3,*] and Shin Min Kang[4,5,*]

[1]Division of Science and Technology, University of Education, Lahore 54000, Pakistan
e-mails: nazeer.waqas@ue.edu.pk (W.N); mmunir@ue.edu.pk (M.M)

[2]Department of Mathematics, Lahore Leads University, Lahore 54810, Pakistan
e-mail: fabbas27@gmail.com

[3]Department of Business Administration, Gyeongsang National University, Jinju 52828, Korea
e-mail: bb5734@gnu.ac.kr

[4]Center for General Education, China Medical University, Taichung 40402, Taiwan

[5]Department of Mathematics and RINS, Gyeongsang National University, Jinju 52828, Korea
e-mail: smkang@gnu.ac.kr

Abstract

We introduced new viscosity rule for nonexpansive mappings in Hilbert Spaces. The strong convergence theorem of the new rule is proved under certain assumptions imposed on the sequence of parameters. Moreover, applications of proposed viscosity rule are also given.

2010 Mathematics Subject Classification: 47H09

Key words and phrases: viscosity rule, Hilbert space, nonexpansive mapping, variational inequality

1 Introduction

In this paper, we shall take H as a real Hilbert space, $\langle\cdot,\cdot\rangle$ as inner product, $\|\cdot\|$ as the induced norm, and C as a nonempty closed subset of H.

Definition 1.1. Let $T : H \to H$ be a mapping. T is called *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in H.$$

Definition 1.2. A mapping $f : H \to H$ is called a *contraction* if for all $x, y \in H$ and $\theta \in [0, 1)$

$$\|fx - fy\| \leq \theta\|x - y\|.$$

[*] Corresponding authors

Definition 1.3. $P_c : H \to C$ is called a *metric projection* if for every $x \in H$ there exists a unique nearest point in C, denoted by $P_c x$, such that

$$\|x - P_c x\| \leq \|x - y\|, \quad \forall y \in C.$$

The following theorem gives the condition for a projection mapping to be nonexpansive.

Theorem 1.4. *Let C be a nonempty closed convex subset of the real Hilbert space H and $P_c : H \to H$ a metric projection. Then*
(*a*) $\|P_c x - P_c y\|^2 \leq \langle x - y, P_c x - P_c y\rangle$ *for all* $x, y \in H$.
(*b*) P_c *is a nonexpansive mapping, that is,* $\|x - P_c x\| \leq \|x - y\|$ *for all* $y \in C$.
(*c*) $\langle x - P_c x, y - P_c x\rangle \leq 0$ *for all* $x \in H$ *and* $y \in C$.

In order to verify the weak convergence of an algorithm to a fixed point of a nonexpansive mapping we need the demiclosedness principle:

Theorem 1.5. (The demiclosedness principle) ([2]) *Let C be a nonempty closed convex subset of the real Hilbert space H and $T : C \to C$ such that $x_n \rightharpoonup x^* \in C$ and $(I - T)x_n \to 0$. Then $x^* = Tx^*$. (Here $\to$ and $\rightharpoonup$ denote strong and weak convergence, respectively).*

Moreover, the following result gives the conditions for the convergence of a nonnegative real sequence.

Theorem 1.6. ([9]) *Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that $a_{n+1} \leq (1 - \gamma_n)a_n + \delta_n, \forall n \geq 0$, where $\{\gamma_n\}$ is a sequence in $(0, 1)$ and $\{\delta_n\}$ is a sequence with*
(1) $\sum_{n=0}^{\infty} \gamma_n = \infty$,
(2) $\lim_{n\to\infty} \sup \frac{\delta_n}{\gamma_n} \leq 0$ *or* $\sum_{n=0}^{\infty} |\delta_n| < \infty$.
Then $a_n \to 0$ as $n \to \infty$.

The following strong convergence theorem, which is also called the *viscosity approximation method*, for nonexpansive mappings in real Hilbert spaces is given by Moudafi [8] in 2000.

Theorem 1.7. *Let C be a noneempty closed convex subset of the real Hilbert space H. Let T be a nonexpansive mapping of C into itself such that $F(T) := \{x \in H : T(x) = x\}$ is nonempty. Let f be a contraction of C into itself. Consider the sequence*

$$x_{n+1} = \frac{\epsilon_n}{1 + \epsilon_n} f(x_n) + \frac{1}{1 + \epsilon_n} T(x_n), \quad n \geq 0,$$

where the sequence $\{\epsilon_n\} \in (0, 1)$ satisfies
(1) $\lim_{n\to\infty} \epsilon_n = 0$,
(2) $\sum_{n=0}^{\infty} \epsilon_n = \infty$,
(3) $\lim_{n\to\infty} |\frac{1}{\epsilon_{n+1}} - \frac{1}{\epsilon_n}| = 0$.
Then $\{x_n\}$ converges strongly to a fixed point x^* of the nonexpansive mapping T, which is also the unique solution of the variational inequality

$$\langle (I - f)x, y - x\rangle \geq 0, \quad \forall \in F(T).$$

In 2015, Xu et al. [9] applied the viscosity method on the midpoint rule for nonexpansive mappings and give the generalized viscosity implicit rule:

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)T\left(\frac{x_n + x_{n+1}}{2}\right), \quad \forall n \geq 0.$$

This, using contraction, regularizes the implicit midpoint rule for nonexpansive mappings. They also proved that the sequence generated by the generalized viscosity implicit rule converges strongly to a fixed point of T. Ke and Ma [6], motivated and inspired by the idea of Xu et al. [9], proposed two generalized viscosity implicit rules:

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)T\left(s_n x_n + (1-s_n)x_{n+1}\right)$$

and

$$x_{n+1} = \alpha_n x_n + \beta f(x_n) + \gamma_n T(s_n x_n + (1-s_n)x_{n+1}).$$

In [3], Jung et al. presented the following viscosity rule

$$\begin{cases} x_{n+1} = T(y_n), \\ y_n = \alpha_n(w_n) + \beta_n f(w_n) + \gamma_n T(w_n), \\ w_n = \frac{x_n + x_{n+1}}{2}. \end{cases}$$

In [7], Kwun et al. proved the strong convergence of the following viscosity rule

$$\begin{cases} x_{n+1} = T(y_n), \\ y_n = \alpha_n(x_n) + \beta_n f(x_n) + \gamma_n T\left(\frac{x_n + x_{n+1}}{2}\right). \end{cases}$$

Our contribution in this direction is the following new viscosity rule

$$x_{n+1} = \alpha_n\left(\frac{x_n + x_{n+1}}{2}\right) + \beta_n f\left(\frac{x_n + x_{n+1}}{2}\right) + \gamma_n T\left(\frac{x_n + x_{n+1}}{2}\right). \tag{1.1}$$

2 New viscosity rule

Theorem 2.1. *Let C be a nonempty closed convex subset of the real Hilbert space H. Let $T : C \to C$ be a nonexpansive mapping with $F(T) \neq \emptyset$ and $f : C \to C$ a contraction with coefficient $\theta \in [0, 1)$. Pick any $x_0 \in C$, let $\{x_n\}$ be a sequence generated by the new viscosity rule* (1.1)*, where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $(0, 1)$ satisfying the following conditions:*

(i) $\alpha_n + \beta_n + \gamma_n = 1$,

(ii) $\lim_{n\to\infty} \alpha_n = 0 = \lim_{n\to\infty} \beta_n$ *and* $\lim_{n\to\infty} \gamma_n \to 1$,

(iii) $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$,

(iv) $\sum_{n=0}^{\infty} |\beta_{n+1} - \beta_n| < \infty$.

Then $\{x_n\}$ converges strongly to a fixed point x^* of the nonexpansive mapping T, which is also the unique solution of the variational inequality $\langle (I-f)x, y-x\rangle \geq 0$, $\forall y \in F(T)$.

In other words, x^* is the unique fixed point of the contraction $P_{F(T)}f$, that is, $P_{F(T)}f(x^*) = x^*$.

Proof. This proof is divided into five steps.

STEP 1. ($\{x_n\}$ is bounded)

Taking an arbitrary point p of $F(T)$, we have

$$\begin{aligned}
&\|x_{n+1}-p\| \\
&= \left\|\alpha_n\left(\frac{x_n+x_{n+1}}{2}\right)+\beta_n f\left(\frac{x_n+x_{n+1}}{2}\right)+\gamma_n T\left(\frac{x_n+x_{n+1}}{2}\right)-p\right\| \\
&= \left\|\alpha_n\left(\frac{x_n+x_{n+1}}{2}\right)-\alpha_n p+\beta_n f\left(\frac{x_n+x_{n+1}}{2}\right)-\beta_n p\right. \\
&\quad \left. +\gamma_n T\left(\frac{x_n+x_{n+1}}{2}\right)+(\alpha_n+\beta_n-1)p\right\| \\
&\le \alpha_n\left\|\left(\frac{x_n+x_{n+1}}{2}\right)-p\right\|+\beta_n\left\|f\left(\frac{x_n+x_{n+1}}{2}\right)-p\right\|+\gamma_n\left\|T\left(\frac{x_n+x_{n+1}}{2}\right)-p\right\| \\
&\le \frac{\alpha_n}{2}\|x_n-p\|+\frac{\alpha_n}{2}\|x_{n+1}-p\|+\beta_n\left\|f\left(\frac{x_n+x_{n+1}}{2}\right)-f(p)\right\| \\
&\quad +\beta_n\|f(p)-p\|+\gamma_n\left\|\frac{x_n+x_{n+1}}{2}-p\right\| \\
&\le \frac{\alpha_n}{2}\|x_n-p\|+\frac{\alpha_n}{2}\|x_{n+1}-p\|+\theta\beta\left\|\frac{x_n+x_{n+1}}{2}-p\right\|+\beta\|f(p)-p\| \\
&\quad +\gamma_n\left[\frac{1}{2}\|x_n-p\|+\frac{1}{2}\|x_{n+1}-p\|\right] \\
&= \left(\frac{\alpha_n+\gamma_n+\theta\beta_n}{2}\right)\|x_n-p\|+\left(\frac{\alpha_n+\gamma_n+\theta\beta_n}{2}\right)\|x_{n+1}-p\| \\
&\quad +\frac{\gamma_n}{2}\|x_{n+1}-p\|+\beta_n\|f(p)-p\| \\
&= \left(\frac{1-\beta_n+\theta\beta_n}{2}\right)\|x_n-p\|+\left(\frac{1-\beta_n+\theta\beta_n}{2}\right)\|x_{n+1}-p\| \\
&\quad +\frac{\gamma_n}{2}\|x_{n+1}-p\|+\beta_n\|f(p)-p\|.
\end{aligned}$$

It follows that

$$\left(1-\frac{1-\beta_n+\theta\beta_n}{2}\right)\|x_{n+1}-p\| \le \left(\frac{1-\beta_n+\theta\beta_n}{2}\right)\|x_n-p\|+\beta_n\|f(p)-p\|$$

implies

$$(1+\beta_n(1-\theta))\|x_{n+1}-p\| \le (1-\beta_n(1-\theta))\|x_n-p\|+2\beta_n\|f(p)-p\|. \tag{2.1}$$

Since $\beta_n,\theta\in(0,1)$, $1-\beta_n(1-\theta)\ge 0$. Moreover, by (2.1) and $\alpha_n+\beta_n+\gamma_n=1$ we get

$$\begin{aligned}
&\|x_{n+1}-p\| \\
&\le \frac{1-\beta_n(1-\theta)}{1+\beta_n(1-\theta)}\|x_n-p\|+\frac{2\beta_n}{1+\beta_n(1-\theta)}\|f(p)-p\| \\
&\le \left[1-\frac{2\beta_n(1-\theta)}{1+\beta_n(1-\theta)}\right]\|x_n-p\|+\frac{2\beta_n(1-\theta)}{1+\beta_n(1-\theta)}\left(\frac{1}{1-\theta}\|f(p)-p\|\right).
\end{aligned}$$

Thus, we have

$$\|x_{n+1}-p\| \leq \max\left\{\|x_n-p\|, \frac{1}{1-\theta}\|f(p)-p\|\right\}.$$

By induction we obtain

$$\|x_{n+1}-p\| \leq \max\left\{\|x_0-p\|, \frac{1}{1-\theta}\|f(p)-p\|\right\}.$$

Hence, we concluded that $\{x_n\}$ is bounded. Consequently, $\{f(\frac{x_n+x_{n+1}}{2})\}$ and $\{T(\frac{x_n+x_{n+1}}{2})\}$ are bounded.

STEP 2. ($\lim_{n\to\infty}\|x_{n+1}-x_n\|=0$)

$$\begin{aligned}
&\|x_{n+1}-x_n\| \\
&= \bigg\|\alpha_n\left(\frac{x_n+x_{n+1}}{2}\right)+\beta_n f\left(\frac{x_n+x_{n+1}}{2}\right)+\gamma_n T\left(\frac{x_n+x_{n+1}}{2}\right) \\
&\quad - \left[\alpha_{n-1}\left(\frac{x_n+x_{n-1}}{2}\right)+\beta_{n-1} f\left(\frac{x_n+x_{n-1}}{2}\right)+\gamma_{n-1} T\left(\frac{x_{n-1}+x_n}{2}\right)\right]\bigg\| \\
&= \bigg\|\frac{\alpha_n}{2}(x_{n+1}-x_n)+\frac{\alpha_n}{2}(x_n-x_{n-1})+\frac{1}{2}(\alpha_n-\alpha_{n-1})x_n+\frac{1}{2}(\alpha_n-\alpha_{n-1})x_{n-1} \\
&\quad + \beta_n\left(f\left(\frac{x_n+x_{n+1}}{2}\right)-f\left(\frac{x_n+x_{n-1}}{2}\right)\right)+(\beta_n-\beta_{n-1})f\left(\frac{x_n+x_{n-1}}{2}\right) \\
&\quad + \gamma_n\left[T\left(\frac{x_{n+1}+x_n}{2}\right)-T\left(\frac{x_{n-1}+x_n}{2}\right)\right]+(\gamma_n-\gamma_{n-1})T\left(\frac{x_{n-1}+x_n}{2}\right)\bigg\| \\
&= \bigg\|\frac{\alpha_n}{2}(x_{n+1}-x_n)+\frac{\alpha_n}{2}(x_n-x_{n-1})+\frac{1}{2}(\alpha_n-\alpha_{n-1})(x_n+x_{n-1}) \\
&\quad + \beta_n\left(f\left(\frac{x_n+x_{n+1}}{2}\right)-f\left(\frac{x_n+x_{n-1}}{2}\right)\right)+(\beta_n-\beta_{n-1})f\left(\frac{x_n+x_{n-1}}{2}\right) \\
&\quad + \gamma_n\left[T\left(\frac{x_{n+1}+x_n}{2}\right)-T\left(\frac{x_{n-1}+x_n}{2}\right)\right] \\
&\quad - \Big[(\alpha_n-\alpha_{n-1})+(\beta_n-\beta_{n-1})\Big]T\left(\frac{x_{n-1}+x_n}{2}\right)\bigg\| \\
&\leq \frac{\alpha_n}{2}\|x_{n+1}-x_n\|+\frac{\alpha_n}{2}\|x_n-x_{n-1}\| \\
&\quad + \frac{1}{2}|\alpha_n-\alpha_{n-1}|\left\|x_{n-1}+x_n-2T\left(\frac{x_{n-1}+x_n}{2}\right)\right\| \\
&\quad + \beta_n\left\|f\left(\frac{x_n+x_{n+1}}{2}\right)-f\left(\frac{x_n+x_{n-1}}{2}\right)\right\| \\
&\quad + |\beta_n-\beta_{n-1}|\left\|f\left(\frac{x_n+x_{n-1}}{2}\right)-T\left(\frac{x_n+x_{n-1}}{2}\right)\right\| \\
&\quad + \gamma_n\left\|T\left(\frac{x_{n+1}+x_n}{2}\right)-T\left(\frac{x_{n-1}+x_n}{2}\right)\right\| \\
&\leq \frac{\alpha_n}{2}\|x_{n+1}-x_n\|+\frac{\alpha_n}{2}\|x_n-x_{n-1}\|+\left(\frac{1}{2}|\alpha_n-\alpha_{n-1}|+|\beta_n-\beta_{n-1}|\right)M \\
&\quad + \theta\beta_n\left\|\frac{x_{n+1}+x_n}{2}-\frac{x_n-x_{n-1}}{2}\right\|+\gamma_n\left\|\frac{x_{n+1}+x_n}{2}-\frac{x_n-x_{n-1}}{2}\right\|
\end{aligned}$$

$$
\begin{aligned}
&= \frac{\alpha_n}{2}\|x_{n+1}-x_n\| + \frac{\alpha_n}{2}\|x_n-x_{n-1}\| + \left(\frac{1}{2}|\alpha_n-\alpha_{n-1}| + |\beta_n-\beta_{n-1}|\right)M \\
&\quad + \frac{\theta\beta_n}{2}\|x_{n+1}-x_n\| + \frac{\theta\beta_n}{2}\|x_n-x_{n-1}\| + \frac{\gamma_n}{2}\|x_{n+1}-x_n\| + \frac{\gamma_n}{2}\|x_n-x_{n-}\| \\
&= \frac{\alpha_n+\theta\beta_n+\gamma_n}{2}\|x_{n+1}-x_n\| + \frac{\alpha_n+\theta\beta_n+\gamma_n}{2}\|x_n-x_{n-1}\| \\
&\quad + \left(\frac{1}{2}|\alpha_n-\alpha_{n-1}| + |\beta_n-\beta_{n-1}|\right)M,
\end{aligned}
$$

where $M > 0$ is a constant such that

$$
\begin{aligned}
M \geq \max\Bigg\{ &\sup_{n\geq 0}\left\|x_n + x_{n-1} - 2T\left(\frac{x_{n-1}+x_n}{2}\right)\right\|, \\
&\sup_{n\geq 0}\left\|f\left(\frac{x_n+x_{n-1}}{2}\right) - T\left(\frac{x_n+x_{n-1}}{2}\right)\right\|\Bigg\}.
\end{aligned}
$$

It gives

$$
\begin{aligned}
&\left(1-\frac{\alpha_n+\theta\beta_n+\gamma_n}{2}\right)\|x_{n+1}-x_n\| \\
&\leq \frac{\alpha_n+\theta\beta_n+\gamma_n}{2}\|x_n-x_{n-1}\| + \left(\frac{1}{2}|\alpha_n-\alpha_{n-1}| + |\beta_n-\beta_{n-1}|\right)M
\end{aligned}
$$

implies

$$
\begin{aligned}
&\left(1-\frac{1-\beta_n+\theta\beta_n}{2}\right)\|x_{n+1}-x_n\| \\
&\leq \frac{1-\beta_n+\theta\beta_n}{2}\|x_n-x_{n-1}\| + \left(\frac{1}{2}|\alpha_n-\alpha_{n-1}| + |\beta_n-\beta_{n-1}|\right)M
\end{aligned}
$$

implies

$$
\begin{aligned}
(1+\beta_n(1-\theta))\|x_{n+1}-x_n\| &\leq (1-\beta_n(1-\theta))\|x_n-x_{n-1}\| \\
&\quad + (|\alpha_n-\alpha_{n-1}| + 2|\beta_n-\beta_{n-1}|)M.
\end{aligned}
$$

Thus we have

$$
\begin{aligned}
\|x_{n+1}-x_n\| &\leq \left(\frac{1-\beta_n(1-\theta)}{1+\beta_n(1-\theta)}\right)\|x_n-x_{n-1}\| \\
&\quad + \frac{M}{1+\beta_n(1-\theta)}(|\alpha_n-\alpha_{n-1}| - 2|\beta_n-\beta_{n-1}|).
\end{aligned}
$$

Since $\theta, \beta_n \in (0,1)$, $1+\beta_n(1-\theta) \geq 1$ and hence

$$
\frac{1-\beta_n(1-\theta)}{1+\beta_n(1-\theta)} \leq 1-\beta_n(1-\theta).
$$

Thus

$$
\begin{aligned}
\|x_{n+1}-x_n\| &\leq \Big[1-\beta_n(1-\theta)\Big]\|x_n-x_{n-1}\| \\
&\quad + \frac{M}{1+\beta_n(1-\theta)}(|\alpha_n-\alpha_{n-1}| - 2|\beta_n-\beta_{n-1}|).
\end{aligned}
$$

Since

$$\sum_{n=0}^{\infty}\beta_n=\infty,\quad \sum_{n=0}^{\infty}|\alpha_{n+1}-\alpha_n|<\infty,\quad \text{and}\quad \sum_{n=0}^{\infty}|\beta_{n+1}-\beta_n|<\infty,$$

by Theorem 1.6, we have $\|x_{n+1}-x_n\|\to 0$ as $n\to\infty$.

STEP 3. ($\|x_n-Tx_n\|\to 0$ as $\to\infty$)

Consider

$$\begin{aligned}
&\|x_n-Tx_n\|\\
&=\left\|x_n-x_{n+1}+x_{n+1}-T\left(\frac{x_n+x_{n+1}}{2}\right)+T\left(\frac{x_n+x_{n+1}}{2}\right)-Tx_n\right\|\\
&\le\|x_n-x_{n+1}\|+\left\|x_{n+1}-T\left(\frac{x_n+x_{n+1}}{2}\right)\right\|+\left\|T\left(\frac{x_n+x_{n+1}}{2}\right)-Tx_n\right\|\\
&\le\|x_n-x_{n+1}\|+\left\|\alpha_n\left(\frac{x_n+x_{n+1}}{2}\right)+\beta_n f\left(\frac{x_n+x_{n+1}}{2}\right)\right.\\
&\qquad\left.+\gamma_n T\left(\frac{x_n+x_{n+1}}{2}\right)-T\left(\frac{x_n+x_{n+1}}{2}\right)\right\|+\left\|\frac{x_n+x_{n+1}}{2}-x_n\right\|\\
&=\|x_n-x_{n+1}\|+\left\|\frac{\alpha_n}{2}(x_n+x_{n+1})+\beta_n f\left(\frac{x_n+x_{n+1}}{2}\right)\right.\\
&\qquad\left.-(1-\gamma_n)T\left(\frac{x_n+x_{n+1}}{2}\right)\right\|+\frac{1}{2}\|x_{n+1}-x_n\|\\
&\le\frac{3}{2}\|x_n-x_{n+1}\|+\left\|\frac{\alpha_n}{2}(x_n+x_{n+1})+\beta_n f\left(\frac{x_n+x_{n+1}}{2}\right)\right.\\
&\qquad\left.-(\alpha_n+\beta_n)T\left(\frac{x_n+x_{n+1}}{2}\right)\right\|\\
&\le\frac{3}{2}\|x_n-x_{n+1}\|+\frac{\alpha_n}{2}\left\|x_n+x_{n+1}-2T\left(\frac{x_n+x_{n+1}}{2}\right)\right\|\\
&\quad+\beta_n\left\|f\left(\frac{x_n+x_{n+1}}{2}\right)-T\left(\frac{x_n+x_{n+1}}{2}\right)\right\|\\
&\le\frac{3}{2}\|x_{n+1}-x_n\|+\left(\frac{\alpha_n}{2}+\beta_n\right)M.
\end{aligned}$$

Then by $\lim_{n\to\infty}\|x_{n+1}-x_n\|=0$ and $\lim_{n\to\infty}\gamma_n=1$, we get

$$\|x_n-Tx_n\|\to 0.$$

STEP 4. ($\lim_{n\to\infty}\sup\langle x^*-f(x^*),x^*-x_n\rangle\le 0$, where $x^*=P_{F(T)}f(x^*)$)

Indeed, we take a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ which converges weakly to a fixed point p of T. Without loss of generality, we may assume that $\{x_{n_i}\}\rightharpoonup p$. From $\lim_{n\to\infty}\|x_n-Tx_n\|=0$ and Theorem 1.5 we have $p=Tp$. This, together with the property of the metric projection, implies that

$$\begin{aligned}
\lim_{n\to\infty}\sup\langle x^*-f(x^*),x^*-x_n\rangle&=\lim_{n\to\infty}\sup\langle x^*-f(x^*),x^*-x_{n_i}\rangle\\
&=\langle x^*-f(x^*),x^*-p\rangle\\
&\le 0.
\end{aligned}$$

STEP 5. ($x_n \to x^*$ as $n \to \infty$)
Now we again take $x^* \in F(T)$ as the unique fixed point of the contraction $P_{F(T)}f$. Consider

$$
\begin{aligned}
&\|x_{n+1} - x^*\|^2 \\
&= \left\|\alpha_n\left(\frac{x_n + x_{n+1}}{2}\right) + \beta_n f\left(\frac{x_n + x_{n+1}}{2}\right) + \gamma_n T\left(\frac{x_n + x_{n+1}}{2}\right) - x^*\right\|^2 \\
&= \left\|\alpha_n\left(\frac{x_n + x_{n+1}}{2}\right) - \alpha_n x^* + \beta_n f\left(\frac{x_n + x_{n+1}}{2}\right) - \beta_n x^*\right. \\
&\quad \left. + \gamma_n T\left(\frac{x_n + x_{n+1}}{2}\right) + (\alpha_n + \beta_n - 1)x^*\right\|^2 \\
&= \alpha_n^2\left\|\left(\frac{x_n + x_{n+1}}{2}\right) - x^*\right\|^2 + \beta_n^2\left\|f\left(\frac{x_n + x_{n+1}}{2}\right) - x^*\right\|^2 \\
&\quad + \gamma_n^2\left\|T\left(\frac{x_n + x_{n+1}}{2}\right) - x^*\right\|^2 \\
&\quad + 2\alpha_n\beta_n\left\langle \frac{x_n + x_{n+1}}{2} - x^*, f\left(\frac{x_n + x_{n+1}}{2}\right) - x^*\right\rangle \\
&\quad + 2\alpha_n\gamma_n\left\langle \frac{x_n + x_{n+1}}{2} - x^*, T\left(\frac{x_n + x_{n+1}}{2}\right) - x^*\right\rangle \\
&\quad + 2\beta_n\gamma_n\left\langle f\left(\frac{x_n + x_{n+1}}{2}\right) - x^*, T\left(\frac{x_n + x_{n+1}}{2}\right) - x^*\right\rangle \\
&\le \alpha_n^2\left\|\left(\frac{x_n + x_{n+1}}{2}\right) - x^*\right\|^2 + \beta_n^2\left\|f\left(\frac{x_n + x_{n+1}}{2}\right) - x^*\right\|^2 + \gamma_n^2\left\|\frac{x_{n+1} + x_n}{2} - x^*\right\|^2 \\
&\quad + 2\alpha_n\beta_n\left\langle \left(\frac{x_n + x_{n+1}}{2}\right) - x^*, f\left(\frac{x_n + x_{n+1}}{2}\right) - x^*\right\rangle \\
&\quad + 2\alpha_n\gamma_n\|x_n - x^*\|\left\|T\left(\frac{x_n + x_{n+1}}{2}\right) - x^*\right\| \\
&\quad + 2\beta_n\gamma_n\left\langle f\left(\frac{x_n + x_{n+1}}{2}\right) - f(x^*), T\left(\frac{x_n + x_{n+1}}{2}\right) - x^*\right\rangle \\
&\quad + 2\beta_n\gamma_n\left\langle f(x^*) - x^*, T\left(\frac{x_n + x_{n+1}}{2}\right) - x^*\right\rangle \\
&\le (\alpha_n^2 + \gamma_n^2)\left\|\frac{x_n + x_{n+1}}{2} - x^*\right\|^2 + 2\alpha_n\gamma_n\left\|\frac{x_{n+1} + x_n}{2} - x^*\right\|^2 \\
&\quad + 2\beta_n\gamma_n\left\|f\left(\frac{x_n + x_{n+1}}{2}\right) - f(x^*)\right\|\left\|\frac{x_{n+1} + x_n}{2} - x^*\right\| + K_n \\
&\le (\alpha_n + \gamma_n)^2\left\|\frac{x_{n+1} + x_n}{2} - x^*\right\|^2 + 2\theta\beta_n\gamma_n\left\|\frac{x_{n+1} + x_n}{2} - x^*\right\|^2 + K_n \\
&\le \Big((\alpha_n + \gamma_n)^2 + 2\theta\beta_n\gamma_n\Big)\left\|\frac{x_{n+1} + x_n}{2} - x^*\right\|^2 + K_n \\
&\le \Big((1 - \beta_n)^2 + 2\theta\beta_n\gamma_n\Big)\left\|\frac{x_{n+1} + x_n}{2} - x^*\right\|^2 + K_n,
\end{aligned}
$$

where

$$\begin{aligned}K_n = &\beta_n^2\left\| f\left(\frac{x_{n+1}+x_n}{2}\right) - x^*\right\|^2 \\ &+ 2\alpha_n\beta_n\left\langle \left(\frac{x_{n+1}+x_n}{2}\right) - x^*, f\left(\frac{x_{n+1}+x_n}{2}\right) - x^*\right\rangle \\ &+ 2\beta_n\gamma_n\left\langle f(x^*) - x^*, T\left(\frac{x_{n+1}+x_n}{2}\right) - x^*\right\rangle.\end{aligned}$$

It follows that

$$\left[(1-\beta_n)^2 + 2\theta\beta_n\gamma_n\right]\left\|\frac{x_{n+1}+x_n}{2} - x^*\right\|^2 \geq \|x_{n+1}-x_n\|^2 - K_n$$

implies

$$\sqrt{(1-\beta_n)^2 + 2\theta\beta_n\gamma_n}\left\|\frac{x_{n+1}+x_n}{2} - x^*\right\| \geq \sqrt{\|x_{n+1}-x_n\|^2 - K_n}$$

implies

$$\frac{1}{2}\sqrt{(1-\beta_n)^2 + 2\theta\beta_n\gamma_n}(\|x_{n+1}-x^*\| + \|x_n - x^*\|) \geq \sqrt{\|x_{n+1}-x_n\|^2 - K_n}$$

implies

$$\begin{aligned}&\frac{1}{4}((1-\beta_n)^2 + 2\theta\beta_n\gamma_n)(\|x_{n+1}-x^*\|^2 + \|x_n - x^*\|^2 \\ &\quad + 2\|x_{n+1}-x^*\|\|x_n-x^*\|) \\ &\geq \|x_{n+1}-x_n\|^2 - K_n\end{aligned}$$

implies

$$\begin{aligned}&\frac{1}{4}((1-\beta_n)^2 + 2\theta\beta_n\gamma_n)(\|x_{n+1}-x^*\|^2 + \|x_n - x^*\|^2 \\ &\quad + (\|x_{n+1}-x^*\|^2 + \|x_n-x^*\|^2)) \\ &\geq \|x_{n+1}-x_n\|^2 - K_n\end{aligned}$$

implies

$$\begin{aligned}&\left[1 - \frac{1}{2}((1-\beta_n)^2 + 2\theta\beta_n\gamma_n)\right]\|x_{n+1}-x^*\|^2 \\ &\leq \left[\frac{1}{2}((1-\beta_n)^2 + 2\theta\beta_n\gamma_n)\right]\|x_n - x^*\|^2 + K_n.\end{aligned}$$

Thus we have

$$\begin{aligned}&\|x_{n+1}-x^*\|^2 \\ &\leq \frac{\frac{1}{2}((1-\beta_n)^2 + 2\theta\beta_n\gamma_n)}{1-\frac{1}{2}((1-\beta_n)^2 + 2\theta\beta_n\gamma_n)}\|x_n - x^*\|^2 + \frac{K_n}{1-\frac{1}{2}((1-\beta_n)^2 + 2\theta\beta_n\gamma_n)} \\ &= \frac{1-\frac{1}{2}((1-\beta_n)^2 + 2\theta\beta_n\gamma_n) - 1 + ((1-\beta_n)^2 + 2\theta\beta_n\gamma_n)}{1-\frac{1}{2}((1-\beta_n)^2 + 2\theta\beta_n\gamma_n)}\|x_n - x^*\|^2 \\ &\quad + \frac{K_n}{1-\frac{1}{2}((1-\beta_n)^2 + 2\theta\beta_n\gamma_n)} \\ &= \left[1 - \frac{1-((1-\beta_n)^2 + 2\theta\beta_n\gamma_n)}{1-\frac{1}{2}((1-\beta_n)^2 + 2\theta\beta_n\gamma_n)}\right]\|x_n - x^*\|^2 + \frac{K_n}{1-\frac{1}{2}((1-\beta_n)^2 + 2\theta\beta_n\gamma_n)}.\end{aligned}$$

Note that

$$0 < 1 - \frac{1}{2}((1-\beta_n)^2 + 2\theta\beta_n\gamma_n) < 1$$

implies

$$\frac{1 - ((1-\beta_n)^2 + 2\theta\beta_n\gamma_n)}{1 - \frac{1}{2}((1-\beta_n)^2 + 2\theta\beta_n\gamma_n)} \geq 1 - ((1-\beta_n)^2 + 2\theta\beta_n\gamma_n).$$

Thus we have

$$\begin{aligned}
&\|x_{n+1} - x^*\|^2 \\
&\leq [1 - (1 - ((1-\beta_n)^2 + 2\theta\beta_n\gamma_n))]\|x_n - x^*\|^2 + \frac{K_n}{1 - \frac{1}{2}((1-\beta_n)^2 + 2\theta\beta_n\gamma_n)} \\
&= [(1-\beta_n)^2 - 2\theta\beta_n\gamma_n]\|x_n - x^*\|^2 + \frac{K_n}{1 - \frac{1}{2}((1-\beta_n)^2 + 2\theta\beta_n\gamma_n)} \\
&\leq (1-\beta_n)^2\|x_n - x^*\|^2 + \frac{K_n}{1 - \frac{1}{2}((1-\beta_n)^2 + 2\theta\beta_n\gamma_n)}.
\end{aligned}$$

Since $0 < 1 - \beta_n < 1$, this give $(1-\beta_n)^2 < (1-\beta_n)$ and

$$\|x_{n+1} - x^*\|^2 \leq (1-\beta_n)\|x_n - x^*\|^2 + \frac{K_n}{1 - \frac{1}{2}((1-\beta_n)^2 + 2\theta\beta_n\gamma_n)}. \tag{2.2}$$

By $\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \beta_n = 0$ and $\lim_{n\to\infty} \gamma_n = 1$ we have

$$\begin{aligned}
&\lim_{n\to\infty} \frac{K_n}{1 - \frac{1}{2}((1-\beta_n)^2 + 2\theta\beta_n\gamma_n)} \\
&= \lim_{n\to\infty} \Bigg(\frac{\beta_n^2\|f\big(\frac{x_{n+1}+x_n}{2}\big) - x^*\|^2 + 2\alpha_n\beta_n\big\langle\big(\frac{x_{n+1}+x_n}{2}\big) - x^*, f\big(\frac{x_{n+1}+x_n}{2}\big) - x^*\big\rangle}{1 - \frac{1}{2}((1-\beta_n)^2 + 2\theta\beta_n\gamma_n)} \\
&\qquad + \frac{2\beta_n\gamma_n\langle f(x^*) - x^*, T\big(\frac{x_{n+1}+x_n}{2}\big) - x^*\rangle}{1 - \frac{1}{2}((1-\beta_n)^2 + 2\theta\beta_n\gamma_n)} \Bigg) \\
&\leq 0.
\end{aligned} \tag{2.3}$$

From (2.2), (2.3), and Theorem 1.6 we have $\lim_{n\to\infty} \|x_{n+1} - x^*\|^2 = 0$, which implies that $x_n \to x^*$ as $n \to \infty$. This completes the proof. □

3 Applications

The scheme can be used to solve problems of system of variational inequalities and constrained convex minimization. Moreover, it can be applied to find a fixed point in K-mappings.

3.1 A more general system of variational inequalities

Let C be a nonempty closed convex subset of the real Hilbert Space H and $\{A_i\}_{i=1}^N : C \to H$ be a family of mappings. In [1] Cai and Bu considered the problem of finding

$(x_1^*, x_2^*, \ldots, x_N^*) \in C \times C \times \cdots \times C$ such that

$$\begin{cases} \langle \lambda_N A_N x_N^* + x_1^* - x_N^*, x - x_1^* \rangle \geq 0, \\ \langle \lambda_{N-1} A_{N-1} x_{N-1}^* + x_N^* - x_{N-1}^*, x - x_N^* \rangle \geq 0, \\ \quad \vdots \\ \langle \lambda_2 A_2 x_2^* + x_3^* - x_2^*, x - x_3^* \rangle \geq 0, \\ \langle \lambda_1 A_1 x_1^* + x_2^* - x_1^*, x - x_2^* \rangle \geq 0, \quad \forall x \in C. \end{cases} \tag{3.1}$$

The equation (3.1) can be written as

$$\begin{cases} \langle x_1^* - (I - \lambda_N A_N) x_N^*, x - x_1^* \rangle \geq 0, \\ \langle x_N^* - (I - \lambda_{N-1} A_{N-1}) x_{N-1}^*, x - x_N^* \rangle \geq 0, \\ \quad \vdots \\ \langle x_3^* - (I - \lambda_2 A_2) x_2^*, x - x_3^* \rangle \geq 0, \\ \langle x_2^* - (I - \lambda_1 A_1) x_1^*, x - x_2^* \rangle \geq 0, \end{cases}$$

which is a more general system of variational inequalities in Hilbert spaces with $\lambda_i > 0$ for all $i \in \{1, 2, 3, \ldots, N\}$. Moreover, we have some useful results:

Lemma 3.1. ([1]) *Let C be a nonempty closed convex subset of the real Hilbert spaces H. For $i \in \{1, 2, 3, \cdots, N\}$, let $A_i : C \to H$ be δ_i-inverse strongly monotone for some positive real number δ_i, namely,*

$$\langle A_i x - A_i y, x - y \rangle \geq \delta_i \|A_i x - A_i y\|^2, \forall x, y \in C$$

Let $G : C \to C$ be a mapping defined by

$$\begin{aligned} G(x) = {} & P_C(I - \lambda_N A_N) P_C(I - \lambda_{N-1} A_{N-1}) \cdots \\ & P_C(I - \lambda_2 A_2) P_C(I - \lambda_1 A_1) x, \quad \forall x \in C. \end{aligned} \tag{3.2}$$

If $0 < \lambda_i \leq 2\delta_i$ for all $i \in \{1, 2, 3, \cdots, N\}$, then G is nonexpansive.

Lemma 3.2. ([5]) *Let C be a nonempty closed convex subject of the real Hilbert Spaces H. Let $A_i : C \to H$ be a nonlinear mapping,where $i \in \{1, 2, 3, ..., N\}$. For given $x_i^* \in C$, $i \in \{1, 2, 3, ..., N\}$, $(x_1^*, x_2^*, x_3^*, ..., x_N^*)$ is a solution of the problem (3.1) if and only if*

$$\begin{aligned} x_1^* &= P_C(I - \lambda_N A_N) x_N^*, x_i^* \\ &= P_C(I - \lambda_{i-1} A_{i-1}) x_{i-1}^*, \quad i = 2, 3, 4, ..., N, \end{aligned}$$

that is,

$$\begin{aligned} x_1^* = {} & P_C(I - \lambda_N A_N) P_C(I - \lambda_{N-1} A_{N-1}) \cdots \\ & P_C(I - \lambda_2 A_2) P_C(I - \lambda_1 A_1) x_1^*, \quad \forall x \in C. \end{aligned}$$

From Lemma 3.2, we know that $x_1^* = G(x_1^*)$, that is, x_1^* is a fixed point of the mapping G, where G is defined by (3.2). Moreover, if we find the fixed point x_1^*, it is easy to get the other points by (3.3). Applying Theorem 2.1 we get the result.

Theorem 3.3. *Let C be a nonempty closed convex subject of the real Hilbert spaces H. For $i \in \{1, 2, 3, ..., N\}$, let $A_i : C \to H$ be δ_i-inverse-strongly monotone for some positive real number δ_i with $F(G) \neq \emptyset$, where $G : C \to C$ is defined by*

$$G(x) = P_C(I - \lambda_N A_N)P_C(I - \lambda_{N-1}A_{N-1}) \cdots$$
$$P_C(I - \lambda_2 A_2)P_C(I - \lambda_1 A_1)x, \quad \forall x \in C.$$

Let $f : C \to C$ be a contraction with coefficient $\theta \in [0, 1)$. Pick any $x_0 \in C$, let $\{x_n\}$ be a sequence generated by

$$x_{n+1} = \alpha_n\left(\frac{x_n + x_{n+1}}{2}\right) + \beta_n f\left(\frac{x_n + x_{n+1}}{2}\right) + \gamma_n G\left(\frac{x_n + x_{n+1}}{2}\right),$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $(0, 1)$ satisfying the conditions (i)-(iv)*.*

Then $\{x_n\}$ converges strongly to a fixed point x^* of the nonexpansive mapping G, which is also the unique solution of the variational inequality $\langle (I - f)x, y - x\rangle \geq 0$, $\forall y \in F(T)$.

In other words, x^* is the unique fixed point of the contraction $P_{F(G)}f$, that is, $P_{F(G)}f(x^*) = x^*$.

3.2 The constrained convex minimization problem

Now, we consider the following constrained convex minimization problem;

$$\min_{x \in C} \phi(x), \tag{3.4}$$

where $\phi : C \to R$ is a real-valued convex function and assumes that the problem (3.4)is consistent. Let Ω denote its solution set. For the minimization problem (3.4), if ϕ is (Fréchet)differentiable, then we have the following lemma.

Lemma 3.4. (Optimality Condition) ([5]) *A necessary condition of optimality for a point $x^* \in C$ to be a solution of the minimization problem* (3.4) *is that x^* solves the variational inequality*

$$\langle \nabla\phi(x^*), x - x^*\rangle \geq 0, \quad \forall x \in C. \tag{3.5}$$

Equivalently, $x^* \in C$ solves the fixed point equation

$$x^* = P_C\Big(x^* - \lambda\nabla\phi(x^*)\Big)$$

for every constant $\lambda > 0$. If, in a addition ϕ is convex, then the optimality condition (3.5) *is also sufficient.*

It is well known that the mapping $P_C(I - \lambda A)$ is nonexpansive when the mapping A is δ-inverse-strongly monotone and $0 < \lambda < 2\delta$. We therefore have the following result.

Theorem 3.5. *Let C be a nonempty closed convex subset of the real Hilbert Space H. For the minimization problem* (3.4)*, assume that ϕ is* (*Fréchet*) *differentiable and the gradient $\nabla\phi$ is a δ-inverse-strongly monotone mapping for some positive real number δ. Let $f : C \to C$ be a contraction with coefficient $\theta \in [0, 1)$. Pick any $x_0 \in C$. Let $\{x_n\}$ be a sequence generated by*

$$x_{n+1} = \alpha_n\left(\frac{x_n + x_{n+1}}{2}\right) + \beta_n f\left(\frac{x_n + x_{n+1}}{2}\right) + \gamma_n P_C(I - \lambda\nabla\phi)\left(\frac{x_n + x_{n+1}}{2}\right),$$

where $\{\alpha_n\}$, $\{\beta_n\}$ *and* $\{\gamma_n\}$ *are sequences in* $(0,1)$ *satisfying the conditions* (i)-(iv).

Then $\{x_n\}$ *converges strongly to a solution* x^* *of the minimization problem* (3.4), *which is also the unique solution of the variational inequality* $\langle (I-f)x, y-x\rangle \geq 0$, $\forall y \in \Omega$.

In other words, x^* *is the unique fixed point of the contraction* $P_\Omega f$, *that is,* $P_\Omega f(x^*) = x^*$.

3.3 K-mapping

Kangtunyakarn and Suantai [4] in 2009 gave K-mapping generated by $T_1, T_2, T_3, ..., T_N$ and $\lambda_1, \lambda_2, \lambda_3, ..., \lambda_N$ as follows.

Definition 3.6. ([4]) Let C be a nonempty convex subset of real Banach Space. Let $\{T_i\}_{i=1}^N$ be a family of mappings of C into itself and let $\lambda_1, \lambda_2, \lambda_3, ..., \lambda_N$ be real numbers such that $0 \leq \lambda_i \leq 1$ for every $i = 1, 2, 3, ..., N$. We define a mapping $K : C \to C$ as follows;

$$\begin{cases} U_1 = \lambda_1 T_1 + (1-\lambda_1)I, \\ U_2 = \lambda_2 T_2 U_1 + (1-\lambda_2)U_1, \\ \quad\vdots \\ U_{N-1} = \lambda_{N-1}T_{N-1}U_{N-2} + (1-\lambda_{N-1})U_{N-2}, \\ U_N = \lambda_N T_N U_{N-1} + (1-\lambda_N)U_{N-1}. \end{cases}$$

Such a mapping is called a K-mapping generated by $T_1, T_2, T_3, ..., T_N$ and $\lambda_1, \lambda_2, \lambda_3, ..., \lambda_N$.

In 2014, Kangtunyakarn and Suwannaut [10] established the following result for K-mapping generated by $T_1, T_2, T_3, ..., T_N$ and $\lambda_1, \lambda_2, \lambda_3, ..., \lambda_N$.

Lemma 3.7. ([10]) *Let* C *be a nonempty closed convex subset of the real Hilbert space* H. *For* $i = 1, 2, 3, ..., N$, *let* $\{T_i\}_{i=1}^N$ *be a finite family of* K_i*-strictly pseudo-contractive mapping of* C *into itself with* $K_i \leq \omega_i$ *and* $\bigcap_{i=1}^N F(T_i) \neq \emptyset$, *namely, there exist constants* $K_i \in [0,1)$ *such that*

$$\|T_i x - T_i y\|^2 \leq \|x-y\|^2 + K_i\|(I-T_i)x - (I-T_i)y\|^2, \quad \forall x, y \in C.$$

Let $\lambda_1, \lambda_2, \lambda_3, ..., \lambda_N$ *be real numbers with* $0 < \lambda_i < \omega_2, \forall i = 1, 2, 3, ..., N$ *and* $\omega_1 + \omega_2 < 1$. *Let* K *be the K-mapping generated by* $T_1, T_2, T_3, ..., T_N$ *and* $\lambda_1, \lambda_2, \lambda_3, ..., \lambda_N$. *Then the following properties hold:*

(*a*) $F(K) = \bigcap_{i=1}^N F(T_i)$.

(*b*) K *is a nonexpansive mapping.*

On the bases of above lemma, we have the following results.

Theorem 3.8. *Let* C *be a nonempty closed convex subset of the real Hilbert space* H. *For* $i = 1, 2, 3, ..., N$, *let* $\{T_i\}_{i=1}^N$ *be a finite family of* K_i*-strictly pseudo-contractive mapping of* C *into itself with* $K_i \leq \omega_i$ *and* $\bigcap_{i=1}^N F(T_i) \neq \emptyset$. *Let* $\lambda_1, \lambda_2, \lambda_3, ..., \lambda_N$ *be real numbers with* $0 < \lambda_i < \omega_2, \forall i = 1, 2, 3, ..., N$ *and* $\omega_1 + \omega_2 < 1$. *Let* K *be the* K*-mapping generated by* $T_1, T_2, T_3, ..., T_N$ *and* $\lambda_1, \lambda_2, \lambda_3, ..., \lambda_N$. *Let* $f : C \to C$ *be a contraction with coefficient* $\theta \in [0,1)$. *Pick any* $x_0 \in C$, *let* $\{x_n\}$ *be sequence generated by*

$$x_{n+1} = \alpha_n\left(\frac{x_n + x_{n+1}}{2}\right) + \beta_n f\left(\frac{x_n + x_{n+1}}{2}\right) + \gamma_n K\left(\frac{x_n + x_{n+1}}{2}\right),$$

where $\{\alpha_n\}$, $\{\beta_n\}$ *and* $\{\gamma_n\}$ *are sequences in* $(0,1)$ *satisfying the conditions* (i)-(iv).

Then $\{x_n\}$ *converges strongly to a fixed point* x^* *of the mappings* $\{T_i\}_{i=1}^N$, *which is also the unique solution of the variational inequality* $\langle (I-f)x, y-x\rangle$, $\forall y \in F(K) = \bigcap_{i=1}^N F(T_i)$.

In other words, x^* *is the unique fixed point of the contraction* $P_{\bigcap_{i=1}^N F(T_i)} f$, *that is,* $P_{\bigcap_{i=1}^N F(T_i)} f(x^*) = x^*$.

References

[1] G. Cai and S. Q. Bu, Hybrid algorithm for generalized mixed equilibrium problems and variational inequality problems and fixed point problems, *Comput. Math. Appl.*, **62** (2011), 4772–4782.

[2] K. Goebel and W. Kirk, Topics in Metric Fixed Point Theory, Cambridge Studies in Advanced Mathematics, vol. 28. Cambridge University Press, Cambridge, 1990.

[3] C. Y. Jung, W. Nazeer, S. F. A. Naqvi and S. M. Kang, An implicit viscosity technique of nonexpansive mappings in Hilbert spaces, *Int. J. Pure Appl. Math.*, **108** (2016), 635–650.

[4] A. Kangtunyakarn and S. Suantai, A new mapping for finding common solutions of equilibrium problems and fixed point problems of finite family of nonexpansive mappings, *Nonlinear Anal.*, **71** (2009), 4448–4460.

[5] Y. F. Ke and C. F. Ma, A new relaxed extragradient-like algorithm for approaching common solutions of generalized mixed equilibrium problems, a more general system of variational inequalities and a fixed point problem, *Fixed point Theory Appl.*, **126** (2013), 21 pages.

[6] Y. F. Ke and C. F. Ma, The generalized viscosity implicit rules of nonexpansive mappings in Hilbert spaces, *Fixed point Theory Appl.*, **190** (2015), 21 pages.

[7] Y. C. Kwun, W. Nazeer, S. F. A. Naqvi and S. M. Kang, Viscosity approximation methods of nonexpansive mappings in Hilbert spaces and applications, *Int. J. Pure Appl. Math.*, **108** (2016), 929–944.

[8] A. Moudafi, Viscosity approximation methods for fixed points problems, *J. Math. Anal. Appl.*, **241** (2000), 46–55.

[9] H. K. Xu, M. A. Alghamdi and N. Shahzad., The viscosity technique for the implicit mid point rule of nonexpansive mappings in Hilbert spaces, *Fixed point Theory Appl.*, **41** (2015), 12 pages.

[10] S. Suwannaut and A. Kangtunyakarn, Strong convergence theorem for the modified generalized equilibrium problem and fixed point problem of strictly pseudo-contractive mappings, *Fixed Point Theory Appl.*, **86** (2014), 31 pages.

GENERALIZED STABILITY OF CUBIC FUNCTIONAL EQUATIONS WITH AN AUTOMORPHISM ON A QUASI-β NORMED SPACE

DONGSEUNG KANG[1] AND HOEWOON B. KIM[2]

[1]Mathematics Education, Dankook University, 152, Jukjeon, Suji, Yongin, Gyeonggi, 16890, Korea

E-mail address: dskang@dankook.ac.kr

[2]Department of Mathematics, Oregon State University, Corvallis, Oregon 97331

E-mail address: kimho@math.oregonstate.edu

Abstract. We introduce a generalized cubic functional equation with an automorphism and investigate the generalized stability of the cubic functions as solutions to the generalized cubic functional equation on a quasi-β Banach space by the fixed point of the alternative method.

Keywords: Hyers-Ulam Stability, Cubic functional equations, Quasi-β normed space, Fixed Point, Functional equations

1. Introduction

In a talk before the Mathematics Club of the University of Wisconsin in 1940, a Polish-American mathematician, S. M. Ulam [25] proposed the stability problem of the linear functional equation $f(x+y) = f(x) + f(y)$ where any solution $f(x)$ of this equation is called a linear function.

To make the statement of the problem precise, *let G_1 be a group and G_2 a metric group with the metric $d(\cdot,\cdot)$. Then given $\epsilon > 0$, does there exist a $\delta > 0$ such that if a function $f : G_1 \longrightarrow G_2$ satisfies the inequality $d(f(xy), f(x)f(y)) < \delta$ for all $x, y \in G_1$, then there is a homomorphism $F : G_1 \longrightarrow G_2$ with $d(f(x), F(x)) < \epsilon$ for all $x \in G_1$?*. In other words, the question would be generalized as "Under what conditions a mathematical object satisfying a certain property approximately must

2000 *Mathematics Subject Classification.* 39B52.

Correspondence: Hoewoon B. Kim, kimho@math.oregonstate.edu.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 27, NO.2, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC

be close to an object satisfying the property exactly?".

In 1941, the first, affirmative, and partial solution to Ulam's question was provided by D. H. Hyers [10]. In his celebrated theorem Hyers explicitly constructed the linear function (or additive function) in Banach spaces directly from a given approximate function satisfying the well-known weak Hyers inequality with a positive constant. The Hyers stability result was first generalized in the stability of additive mappings by Aoki [1] allowing the Cauchy difference to become unbounded. In 1978 Th. M. Rassias [16] also provided a generalization of Hyers' theorem with the possibly unbounded Cauchy difference for linear mappings. For the last decades, stability problems of various functional equations, not only linear case, have been extensively investigated and generalized by many mathematicians (see [4, 7, 9, 11, 17, 20, 21]).

The functional equation

$$f(x+y)+f(x-y)=2f(x)+2f(y) \tag{1.1}$$

is called a quadratic functional equation and every solution of this functional equation is said to be a quadratic function or mapping (e.g. $f(x)=cx^2$). The Hyers-Ulam stability problem for the quadratic functional equation was first studied by Skof [23] in a normed space as the domain of a quadracitc mapping of the equation. Cholewa [6] noticed that the results of Skof still hold in abelian groups. In [7] Czerwik obtained the Hyers-Ulam-Rassias stability (or generalized Hyers-Ulam stability) of the quadratic functional equation. See [2, 15, 27] for more results on the equation (1.1). Also the quadratic equation (1.1) was generalized by Stetkær in [24] introducing an involution σ of an abelian group G, i.e., an automorphism $\sigma : G \to G$ with $\sigma^2 = I$ (I denotes the identity) and considering the following functional equation

$$f(x+y)+f(x+\sigma(y))=2f(x)+2f(y) \tag{1.2}$$

for all $x, y \in G$. As we already notice the equation (1.1) corresponds to the equation (1.2) with $\sigma=-I$.

Jun and Kim [11] considered the following cubic functional equation

$$f(2x+y)+f(2x-y)=2f(x+y)+2f(x-y)+12f(x) \tag{1.3}$$

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 27, NO.2, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC

since it should be easy to see that a function $f(x) = cx^3$ is a solution of the equation (1.3) as the quadratic equation case. In a year they [12] proved the generalized Hyers-Ulam stability of a different version of a cubic functional equation

$$f(x+2y) + f(x-2y) + 6f(x) = f(x+y) + 4f(x-y). \tag{1.4}$$

Since then the stability of cubic functional equations has been investigated by a number of authors (see [5, 14] for details). In particular, Najati [14] investigated the following generalized cubic functional equation

$$f(sx+y) + f(sx-y) = sf(x+y) + sf(x-y) + 2(s^3-s)f(x) \tag{1.5}$$

for a positive integer $s \geq 2$.

As we might notice there are various definitions for the stability of the cubic functional equations and here we consider the following definition of a generalized cubic functional equation

$$\begin{aligned} &f(ax+y) - f(x+ay) + a(a-1)f(x-y) \\ &= (a-1)(a+1)^2 f(x) - (a-1)(a+1)^2 f(y) \end{aligned} \tag{1.6}$$

for all $a \in \mathbb{Z}$ $(a \neq 0, \pm 1)$ and generalized the equation (1.6) with the involution σ of a linear space X when $a = 2$;

$$f(2x+y) - f(x+2y) + 2f(x+\sigma(y)) - 9f(x) + 9f(y) = 0. \tag{1.7}$$

In this paper we will study the generalized Hyers-Ulam stability problem of the equation (1.7).

In order to give our results in Section 3 it is convenient to state the definition of a generalized metric on a set X and a result on a fixed point theorem of the alternative by Diaz and Margolis [8].

Let X be a set. A function $d : X \times X \longrightarrow [0, \infty]$ is called a generalized metric on X if d satisfies

(1) $d(x,y) = 0$ if and only if $x = y$;
(2) $d(x,y) = d(y,x)$ for all $x, y \in X$;
(3) $d(x,z) \leq d(x,y) + d(y,z)$ for all $x, y, z \in X$.

Theorem 1.1. *Let (X, d) be a complete generalized metric space and let $J : X \longrightarrow X$ be a strictly contractive mapping with Lipschitz constant $0 < L < 1$. Then for*

each element $x \in X$, either $d(J^n x, J^{n+1}x) = \infty$ for all nonnegative integers n or there exists a positive n_0 such that

(1) *$d(J^n x, J^{n+1}x) < \infty$ for all $n \geq n_0$;*
(2) *the sequence $\{J^n x\}$ converges to a fixed point y^* of J;*
(3) *y^* is the unique fixed point of J in the set $Y = \{y \in X | d(J^{n_0}x, y) < \infty\}$;*
(4) *$d(y, y^*) \leq (1/(1-L))d(y, Jy)$ for all $y \in Y$.*

In 2009, Rassias and Kim [18] investigated the Hyers-Ulam stability of Cauchy and Jensen type additive mappings in quasi-β-normed spaces with the following definition of a quasi-β-norm:

Definition 1.2. Let β be a real number with $0 < \beta \leq 1$ and $\mathbb{K}$ be either $\mathbb{R}$ or $\mathbb{C}$. Let X be a linear space over a field $\mathbb{K}$. A *quasi-β-norm* $||\cdot||$ is a real-valued function on X satisfying the following properties:

(1) $||x|| \geq 0$ for all $x \in X$ and $||x|| = 0$ if and only if $x = 0$
(2) $||\lambda x|| = |\lambda|^\beta ||x||$ for all $\lambda \in \mathbb{K}$ and all $x \in X$
(3) There is a constant $K \geq 1$ such that $||x + y|| \leq K(||x|| + ||y||)$ for all $x, y \in X$.

The pair $(X, ||\cdot||)$ is called a *quasi-β-normed space* if $||\cdot||$ is a quasi-β-norm on X. A smallest possible constant K is called the modulus of concavity of $||\cdot||$. A quasi-β-Banach space is a complete quasi-β-normed space. A quasi-β-norm $||\cdot||$ is called a (β, p)-norm $(0 < p \leq 1)$ if the property (3) of the Definition 1.2 takes the form $||x+y||^p \leq ||x||^p + ||y||^p$ for all $x, y \in X$. In this case, a quasi-β-Banach space is referred to as a (β, p)-Banach space; see [3, 18, 19] for datails.

In this paper, using the Fixed Point method we prove the generalized Hyers-Ulam stability of the generalized cubic functional equation (1.7) in a quasi-β-normed linear space we just defined above (Definition 1.2). In Section 2 we establish the general solution of the cubic functional equation (1.7) applying the symmetric n-additive mappings for the cubic functional equation (1.7) that will be explained in detail in the Section. Finally, we obtain, in Section 3, the generalized Hyers-Ulam stability of the generalized cubic functional equation (1.7) with the Fixed Point theorem of the Alternative.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 27, NO.2, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC

2. The General Solution with $\sigma = -I$

In this section we study the general solution of the cubic functional equation (1.7) with $\sigma = -I$ by introducing and applying n-additive symmetric mappings and their properties that are found in [22, 26]. Before we proceed, let us give some basic backgrounds of n-additive symmetric mappings. Let X and Y be real vector spaces and n a positive integer. A function $A_n : X^n \longrightarrow Y$ is called *n-additive* if it is additive in each of its variables. A function $A_n : X^n \longrightarrow Y$ is said to be *symmetric* if $A_n(x_1, x_2, \cdots, x_n) = A_n(x_{\sigma(1)}, x_{\sigma(2)}, \cdots, x_{\sigma(n)})$ for every permutation $\{\sigma(1), \sigma(2), \cdots, \sigma(n)\}$ of $\{1, 2, \cdots, n\}$. If $A_n(x_1, x_2, \cdots, x_n)$ is an n-additive symmetric map, then $A^n(x)$ will denote the diagonal $A_n(x, x, \cdots, x)$ and $A^n(rx) = r^n A^n(x)$ for all $x \in X$ and $r \in \mathbb{Q}$. Such a function $A^n(x)$ will be called a *monomial function of degree n* assuming $A^n(x) \not\equiv 0$. Moreover, the resulting function after substituting $x_1 = x_2 = \cdots = x_s = x$ and $x_{s+1}, x_{s+2}, \cdots = x_n = y$ in $A_n(x_1, x_2, \cdots, x_n)$ will be denoted by $A^{s,n-s}(x, y)$.

Theorem 2.1. *A function $f : X \longrightarrow Y$ is a solution of the functional equation (1.7) with $\sigma = -I$ if and only if f is of the form $f(x) = A^3(x)$ for all $x \in X$, where $A^3(x)$ is the diagonal of the 3-additive symmetric mapping $A_3 : X^3 \longrightarrow Y$.*

Proof. Assume that f satisfies the functional equation (1.7). Taking $x = y = 0$ in the equation (1.7) it's not hard to have $f(0) = 0$ since $\sigma(0) = 0$. Substituting $y = 0$ in (1.7) also gives

$$f(2x) - f(x) + 2f(x) - 9f(x) = 0,$$

that is,

$$f(2x) = 2^3 f(x) \tag{2.1}$$

for all $x \in X$. Similarly, when $x = 0$ in the equation (1.7) we have

$$2f(y) + 2f(\sigma(y)) = 0,$$

i.e.,

$$f(y) + f(-y)) = 0 \tag{2.2}$$

for all $y \in X$ since $\sigma(y) = -y$. This observation leads us to $f(-y) = -f(y)$ for all $y \in X$ and hence f is an odd function. Rewriting the equation (1.7) as

$$f(x) - \frac{1}{9}f(2x+y) + \frac{1}{9}f(x+2y) - \frac{2}{9}f(x-y) - f(y) = 0 \tag{2.3}$$

and applying Theorems 3.5 and 3.6 in [26] we express f as

$$f(x) = A^3(x) + A^2(x) + A^1(x) + A^0 \tag{2.4}$$

where A^0 is an arbitrary element in Y and $A^i(x)$ is the diagonal of the i-additive symmetric mapping $A_i : X^i \longrightarrow Y$ for $i = 1, 2, 3$. Since f is odd and $f(0) = 0$ it follows that

$$f(x) = A^3(x) + A^1(x)$$

for all $x \in X$. By the property (2.1) of f and $A^n(rx) = r^n A^n(x)$ for all $x \in X$ and $r \in \mathbb{Q}$ we should obtain $A^1(x) = 0$ for all $x \in X$. Therefore we conclude that $f(x) = A^3(x)$ for all $x \in X$.
Conversely, let us assume that $f(x) = A^3(x)$ for all $x \in X$, where $A^3(x)$ is the diagonal of a 3-additive symmetric mapping $A_3 : X^3 \longrightarrow Y$. Noting that

$$A^3(qx + ry) = q^3A^3(x) + 3q^2rA^{2,1}(x, y) + 3qr^2A^{1,2}(x, y) + r^3A^3(y)$$

and calculating simple computation for the equation (1.7) with $\sigma = -I$ in term of $A^3(x)$, we show that the function f satisfies the cubic equation (1.7) with $\sigma = -I$, which completes the proof. $\square$

3. General Hyers-Ulam Stability in a Quasi-β Banach Space: A Fixed Point Theorem of the Alternative Approach

In this section we will investigate the generalized Hyers-Ulam stability of the cubic functional equation (1.7) which is introduced earlier in previous sections

$$f(2x + y) - f(x + 2y) + 2f(x + \sigma(y)) - 9f(x) + 9f(y) = 0.$$

for all $x, y \in X$ by the approach of the fixed point of the alternative. As we used the notations in the previous sections we assume that X is a vector space and $(Y, ||\cdot||)$ is a quasi-β-Banach space in this section. A set $\mathbb{R}_+$ denotes the set of all nonnegative real numbers.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 27, NO.2, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC

Theorem 3.1. *Suppose that a function $\phi : X^2 \longrightarrow \mathbb{R}_+$ is given and there exists a constant L with $0 < L < 1$ such that*

$$\phi(2x, 2y) \leq 2L\phi(x, y) \quad \text{and} \quad \phi(x + \sigma(x), y + \sigma(y)) \leq 2L\phi(x, y) \tag{3.1}$$

for all $x, y \in X$. Furthermore, let $f : X \longrightarrow Y$ be a mapping such that $f(0) = 0$ and

$$||f(2x + y) - f(x + 2y) + 2f(x + \sigma(y)) - 9f(x) + 9f(y)|| \leq \phi(x, y) \tag{3.2}$$

for all $x, y \in X$ where σ is an automorphism on X with $\sigma^2 = I$ where I is the identity.
Then there exists the unique generalized cubic function $C : X \longrightarrow Y$ defined by $C(x) := lim_{n\to\infty} \left(\frac{1}{2^{3n}}\right)(f(2^n x) + (2^n - 1)f(2^{n-1}x + 2^{n-1}\sigma(x)))$ such that

$$||f(x) - C(x)|| \leq \left(\frac{1 + L}{2^3(1 - L)}\right)\Phi(x) \tag{3.3}$$

for all $x \in X$ where $\Phi(x) = \max\{\phi(x, 0), \phi(0, x)\}$ for all $x \in X$.

Proof. First, we put $y = 0$ in the inequality (3.2) to obtain

$$||f(2x) - 2^3 f(x)|| \leq \phi(x, 0) \tag{3.4}$$

for $x \in X$ since $\sigma(0) = 0$. Similarly we substitute $x = 0$ into the inequality (3.2) again to have

$$||10f(y) - f(2y) + 2f(\sigma(y))|| \leq \phi(0, y) \tag{3.5}$$

for all $y \in X$. Combining the two inequalities (3.4) and (3.5) we note that

$$\begin{aligned} ||2f(x) + 2f(\sigma(x))|| &= ||10f(x) - f(2x) + 2f(\sigma(x)) + f(2x) - 2^3 f(x)|| \\ &\leq \phi(x, 0) + \phi(0, x) \end{aligned}$$

and hence we conclude that

$$||f(x) + f(\sigma(x))|| \leq \frac{1}{2}(\phi(x, 0) + \phi(0, x)) \tag{3.6}$$

Then we let $x = x + \sigma(x)$ in the above inequality (3.6) and we are able to get

$$||f(x + \sigma(x))|| \leq \frac{1}{4}(\phi(x + \sigma(x), 0) + \phi(0, x + \sigma(x))) \leq \frac{L}{2}(\phi(x, 0) + \phi(0, x)) \tag{3.7}$$

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 27, NO.2, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC

We also define a function $T(f)$ from X to Y by $T(f)(x) = \frac{1}{2^3}(f(2x)+f(x+\sigma(x)))$ and we then consider the following estimation

$$\begin{aligned}
||T(f)(x) - f(x)|| &= \left\|\frac{1}{2^3}(f(2x) + f(x+\sigma(x))) - f(x)\right\| \\
&= \left\|\frac{1}{2^3}(f(2x) - 2^3 f(x)) + \frac{1}{2^3} f(x+\sigma(x))\right\| \\
&\le \frac{1}{2^3}\phi(x,0) + \frac{1}{2^3}\left(\frac{L}{2}\right)(\phi(x,0) + \phi(0,x)) \\
&\le \frac{1}{2^3}(1+L)\Phi(x)
\end{aligned} \tag{3.8}$$

This idea enables us to define a sequence $\{T^n(f)\}$ in Y for each $x \in X$ by

$$T^n(f)(x) = \frac{1}{2^{3n}}(f(2^n x) + (2^n - 1) f(2^{n-1}x + 2^{n-1}\sigma(x)))$$

for a nonnegative integer n with $T^0(f) = f$ and we claim that it should be a Cauchy sequence in Y. In order to show this we use the inequalities (3.4), (3.7), and (3.8) to compute the following estimations;

(3.9)

$$\begin{aligned}
&||T^n(f)(x) - T^{n-1}(f)(x)|| = ||\frac{1}{2^{3n}}(f(2^n x) + (2^n - 1) f(2^{n-1}x + 2^{n-1}\sigma(x))) \\
&- \frac{1}{2^{3(n-1)}}(f(2^{n-1}x) + (2^{n-1} - 1) f(2^{n-2}x + 2^{n-2}\sigma(x)))|| \\
&= ||\frac{1}{2^{3n}}(f(2^n x) + f(2^{n-1}x + 2^{n-1}\sigma(x)) + (2^n - 2) f(2^{n-1}x + 2^{n-1}\sigma(x))) \\
&- \frac{1}{2^{3(n-1)}}(f(2^{n-1}x) + (2^{n-1} - 1) f(2^{n-2}x + 2^{n-2}\sigma(x)))|| \\
&= ||\frac{1}{2^{3n}}(f(2^n x) + f(2^{n-1}x + 2^{n-1}\sigma(x)) - 2^3 f(2^{n-1}x)) \\
&+ \frac{1}{2^{3n}}((2^n - 2) f(2^{n-1}x + 2^{n-1}\sigma(x)) - 2^2(2^n - 2) f(2^{n-2}x + 2^{n-2}\sigma(x)))|| \\
&= ||\frac{1}{2^{3n}}(f(2^n x) + f(2^{n-1}x + 2^{n-1}\sigma(x)) - 2^3 f(2^{n-1}x)) \\
&+ \frac{1}{2}\left(\frac{2^n - 2}{2^{3n}}\right)(2f(2^{n-1}x + 2^{n-1}\sigma(x)) - 2^3 f(2^{n-2}x + 2^{n-2}\sigma(x)))|| \\
&\le \frac{1}{2^{3n}}(\phi(2^{n-1}x, 0) + \frac{L}{2}(\phi(2^{n-1}x, 0) + \phi(0, 2^{n-1}x))) \\
&+ \left(\frac{1}{2}\left(\frac{2^n - 2}{2^{3n}}\right)\right)\left(\phi(2^{n-2}x + 2^{n-2}\sigma(x), 0) + \frac{L}{2}(\phi(2^{n-2}x + 2^{n-2}\sigma(x), 0) + \phi(0, 2^{n-2}x + 2^{n-2}\sigma(x)))\right) \\
&\le \frac{(2L)^{n-1}}{2^{3n}}(1+L)\Phi(x) + \frac{2^{n-1} - 1}{2^{3n}}(2L)^{n-1}(1+L)\Phi(x) = \frac{1}{2^3}(1+L)\left(\frac{L}{2}\right)^{n-1}\Phi(x)
\end{aligned}$$

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 27, NO.2, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC

for all $x \in X$ and all nonnegative integer n. Hence we note that

$$\|T^n(f)(x) - T^m(f)(x)\| \leq \frac{1+L}{2^3} \sum_{j=m}^{n-1} \left(\frac{L}{2}\right)^j \Phi(x) \tag{3.10}$$

for all $x \in X$ and $n > m \in \mathbb{N}$.

With this result in mind we consider the set $\Omega = \{g | g : X \longrightarrow Y, g(0) = 0\}$ and then define a generalized metric d on Ω as follows:

$$d(g,h) = \inf\{\lambda \in [0,\infty] : \|g(x) - h(x)\| \leq \lambda\Phi(x) \text{ for all } x \in X\}$$

with $\inf \emptyset = \infty$. Then (S,d) is a complete generalized metric space; see Lemma 2.1 in [13]. Now we define a mapping $T : \Omega \longrightarrow \Omega$ by

$$T(g)(x) = \frac{1}{2^3}(g(2x) + g(x + \sigma(x))) \tag{3.11}$$

for all $x \in X$. We, then, will show that T is strictly contractive on Ω.

Given $g, h \in \Omega$, let $\lambda \in [0,\infty]$ be a constant with $d(g,h) \leq \lambda$. Then we have $\|g(x) - h(x)\| \leq \lambda\Phi(x)$ for all $x \in X$.

By the equation (3.1) we have

$$\begin{aligned}
&\|T(g)(x) - T(h)(x)\| = \frac{1}{2^3}\|g(2x) - h(2x) + g(x+\sigma(x)) - h(x+\sigma(x))\| \\
&\leq \frac{1}{2^3}\|g(2x) - h(2x)\| + \frac{1}{2^3}\|g(x+\sigma(x)) - h(x+\sigma(x))\| \\
&\leq \frac{\lambda}{2^3}\Phi(2x) + \frac{\lambda}{2^3}\Phi(x+\sigma(x)) \leq \frac{1}{2}L\lambda \leq L\lambda
\end{aligned}$$

for all $x \in G$, which implies

$$d(T(g), T(h)) \leq L\lambda.$$

Therefore we may conclude that

$$d(T(g), T(h)) \leq Ld(g,h)$$

for any $g, h \in \Omega$. Since L is a constant with $0 < L < 1$, T is strictly contractive as claimed.

Also the inequality (3.8) implies that

$$d(T(f), f) \leq \frac{1}{2^3}(1+L) < \infty. \tag{3.12}$$

By the Alternative of Fixed Point as we introduced in the Introduction Section, there exists a mapping $C : X \longrightarrow Y$ which is a fixed point of T such that $d(T^n(f), C) \to 0$ as $n \to \infty$, that is,

$$C(x) = \lim_{n\to\infty} T^n(f)(x)$$

for all $x \in X$. Then we will show that C is cubic and it would not be hard if we recall the approximation inequality (3.2) for f where we let $x = 2^n x$, $y = 2^n y$ and $x = 2^{n-1}(x + \sigma(x))$, $y = 2^{n-1}(y + \sigma(y))$, respectably, as follows;

$$\begin{aligned}
&\|C(2x + y) - C(x + 2y) + 2C(x + \sigma(y)) - 9C(x) + 9C(y)\| \\
&\leq \lim_{n\to\infty} \frac{1}{2^{3n}}\phi(2^n x, 2^n y) + \lim_{n\to\infty} \frac{2^n - 1}{2^{3n}}\phi(2^{n-1}(x + \sigma(x)), 2^{n-1}(y + \sigma(y))) \\
&\leq \lim_{n\to\infty} \frac{(2L)^n}{2^{3n}}\phi(x, y) + \lim_{n\to\infty} \frac{(2^n - 1)(2L)^n}{2^{3n}}\phi(x, y) \\
&= \lim_{n\to\infty} \left(\frac{L}{2}\right)^n \phi(x, y) = 0
\end{aligned}$$

for all $x, y \in X$, which implies that C is cubic.

By the Alternative of Fixed Point theorem and the inequality (3.12) we get

$$d(f, C) \leq \frac{1}{1 - L} d(f, T(f)) \leq \frac{1 + L}{2^3(1 - L)}.$$

Hence the inequality (3.3) is true for all $x \in X$.

By the uniqueness of the fixed point of T, the cubic function C should be unique, which completes the proof. □

Let us give the classical Cauchy difference type stability of the generalized cubic functional equation (1.7) when $\sigma = -I$ from Theorem 3.1 as we see the following Corollary.

Corollary 3.2. *Let $\epsilon \geq 0$, $0 < p < \frac{1}{\beta}$ be a real number. Suppose $f : X \longrightarrow Y$ is a function satisfying $f(0) = 0$ and*

$$||f(2x + y) - f(x + 2y) + 2f(x - y) - 9f(x) + 9f(y)|| \leq \epsilon(||x||^p + ||y||^p)$$

for all $x, y \in X$. Then there exists the unique cubic function $C : X \longrightarrow Y$ defined by

$$C(x) = lim_{n\to\infty} \left(\frac{1}{2^{3n}}\right) f(2^n x)$$

satisfying

$$||f(x) - C(x)|| \leq \left(\frac{\epsilon(1+L)}{2^3(1-L)} \right) ||x||^p \tag{3.13}$$

for all $x \in X$.

Proof. This proof follows from Theorem 3.1 by taking $\phi(x, y) = \epsilon(||x||^p + ||y||^p)$ for all $x, y \in X$ with $L = |2|^{p\beta-1}$.

□

References

[1] T. Aoki, On the stability of the linear transformation in Banach spaces, *Journal of the Mathematical Society of Japan*, vol. 2, pp. 64-66, (1950)

[2] B. Belaid, E. Elhoucien, and R. Ahmed, Hyers-Ulam Stability of the Generalized Quadratic Functional Equation in Amenable Semigroups, *Journal of inequalities in pure and applied mathematics*, Vol. 8 (2007), Issue 2, Article 56, 18 pp.

[3] Y. Benyamini and J. Lindenstrauss, *Geometric nonlinear functional analysis*, vol. 1, Colloq Publ, vol. 48, Amer Math Soc, Providence, (2000)

[4] JH Bae and WG Park, On the generalized Hyers-Ulam-Rassias stability in Banach modules over a C^*-algebra, *Journal of Mathematical Analysis and Applications*, vol. 294, no. 1, pp. 196-205, (2004)

[5] IS Chang, KW Jun, and YS Jung The modified Hyers-Ulam-Rassias stability of a cubic type functional equation, *Math Inequal Appl.* 8(4), 675-683 (2005)

[6] PW Cholewa, Remarks on the stability of functional equations, *Aequ Math.* 27, 7686 (1984). doi:10.1007/BF02192660

[7] St. Czerwik, On the stability of the quadratic mapping in normed spaces, *Abhandlungen aus dem Mathematischen Seminar der Universit́at Hamburg*, vol. 62, pp. 59-64, (1992)

[8] J. B. Diaz and B. Margolis, A fixed point theorem of the alternative, for contractions on a generalized complete metric space, *Bulletin of the American Mathematical Society*, vol. 74, pp. 305-309, (1968)

[9] Z. Gajda, On stability of additive mappings, *International Journal of Mathematics and Mathematical Sciences*, vol. 14, no. 3, pp. 431-434, (1991)

[10] D. H. Hyers, On the stability of the linear functional equation, *Proceedings of the National Academy of Sciences of the United States of America*, vol. 27, pp. 222-224, (1941)

[11] KW Jun and HM Kim, The generalized Hyers-Ulam-Rassias stability of a cubic functional equation, *J. Math. Anal. Appl.* 274 (2002) 867-878

[12] KW Jun and HM Kim, On the Hyers-Ulam-Rassias stability of a general cubic functional equation, *Math Inequal Appl.* 6(2), 289-302 (2003)

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 27, NO.2, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC

[13] D. Mihett and V. Radu, On the stability of the additive Cauchy functional equation in random normed spaces, *J Math Anal Appl* 343(1) (2008), 567572

[14] A. Najati, The generalized Hyers-Ulam-Rassias stability of a cubic functional equation, *Turk J Math.* 31, 395-408 (2007)

[15] C. Park, Generalized Hyers-Ulam Stability of Quadratic Functional Equations: A Fixed Point Approach, *Fixed Point Theory and Applications*, Vol. (2008), Article ID 493751, 9 pages

[16] Th. M. Rassias, On the stability of the linear mapping in Banach spaces, *Proceedings of the American Mathematical Society*, vol. 72, no. 2, pp. 297300, (1978)

[17] Th. M. Rassias, On the stability of functional equations in Banach spaces, *Journal of Mathematical Analysis and Applications*, vol. 251, no. 1, pp. 264-284, (2000)

[18] J.M. Rassias and HM. Kim, Generalized, Hyers-Ulam stability for general additive functional equations in quasi-β-normed spaces, *J. Math. Anal. Appl.* 356 (2009), 302309

[19] S. Rolewicz, *Metric Linear Spaces*, Reidel/PWN-Polish Sci. Publ, Dordrecht, (1984)

[20] Th. M. Rassias and P. Semrl, On the Hyers-Ulam stability of linear mappings, *Journal of Mathematical Analysis and Applications*, vol. 173, no. 2, pp. 325-338, (1993)

[21] Th. M. Rassias and K. Shibata, Variational problem of some quadratic functionals in complex analysis, *Journal of Mathematical Analysis and Applications*, vol. 228, no. 1, pp. 234-253, (1998)

[22] P.K. Sahoo, A generalized cubic functional equation, *Acta Math Sinica* 21(5) (2005), 11591166

[23] F. Skof, Proprietá locali e approssimazione di operatori, *Rend Sem Mat Fis Milano.* 53, 113129 (1983). doi:10.1007/BF02924890

[24] H. Stetkær, Functional equations on abelian groups with involution, *Aequationes Math.*, 54 (1997), 144172.

[25] S. M. Ulam, *Problems in Morden Mathematics*, Wiley, New York (1960)

[26] T.Z. Xu, J.M. Rassias and W.X. Xu, A generalized mixed quadratic-quartic functional equation, *Bull, Malaysian Math Scien Soc* 35(3) (2012), 633649

[27] Dilian Yang, The stability of the quadratic functional equation on amenable groups, *Journal of Math. Anal. Appl.* 291 (2004) 666-672

[1]Mathematics Education, Dankook University, 152, Jukjeon, Suji, Yongin, Gyeonggi, 16890, Korea

E-mail address: dskang@dankook.ac.kr

[2]Department of Mathematics, Oregon State University, Corvallis, Oregon 97331

E-mail address: kimho@math.oregonstate.edu

Two quotient BI-algebras induced by fuzzy normal subalgebras and fuzzy congruence relations

Yinhua Cui[1] **and Sun Shin Ahn**[2,*]

[1] *Department of Mathematics, Yanbian University, Yanji 133002, P. R. China*

[2] *Department of Mathematics Education, Dongguk University, Seoul 04620, Korea*

Abstract. In this paper, we discuss two quotient BI-algebras induced by fuzzy normal subalgebras and induced by fuzzy congruence relations, which are useful in the study of the structural theory of fuzzy quotient BI-algebras.

1. Introduction

Zadeh [14] introduced the notion of a fuzzy subset A of a set X as a function from X into $[0,1]$. Rosenfeld [11] applied this concept to the theory of groupoids and groups. Liu [7] introduced and studied the notion of the fuzzy ideals of a ring. Mukherjee and Sen [9] defined and examined the fuzzy prime ideals of a ring. The concept of fuzzy ideals was applied to several algebras: BN-algebras [2], BL-algebras [8], semirings [5] and semigroups [3]. Recently, Song et al. [13] discussed positive implicative superior ideals induced by superior mappings in BCK-algebras.

Saeid et al. [12] introduced a new algebra, called a BI-algebra, which is a generalization of a (dual) implication algebra, and they discussed ideals and congruence relations. Ahn et al. [1] introduced the notion of normal subalgebras in BI-algebras, and studied its analytic construction.

In this paper, we discuss two quotient BI-algebras induced by fuzzy normal subalgebras and induced by fuzzy congruence relations, which are useful in the study of the structural theory of fuzzy quotient BI-algebras.

2. Preliminaries

We recall some definitions and results discussed in [12].

An algebra $(X,*,0)$ of type (2, 0) is called a *BI-algebra* [12] if

(B1) $x*x=0$ for all $x\in X$,

(B2) $x*(y*x)=x$ for all $x,y\in X$.

We introduced a relation "$\leq$" on a BI-algebra X by $x\leq y$ if and only if $x*y=0$. We note that the relation "$\leq$" is not a partial order, since it is only reflexive. A non-empty subset S of a BI-algebra X is said to be a *subalgebra* of X if it is closed under the operation "$*$". Since $x*x=0$ for all $x\in X$, it is clear that $0\in S$.

---

[0] **2010 Mathematics Subject Classification**: 08A72.

[0] **Keywords**: BI-algebra; fuzzy (normal) subalgebra; fuzzy congruence relation.

* Correspondence: Tel: +82 2 2260 3410, Fax: +82 2 2266 3409 (S. S. Ahn).

[0] **E-mail**: cuiyh@ybu.edu.cn (Y. Cui); sunshine@dongguk.edu (S. S. Ahn).

Definition 2.1. Let $(X, *, 0)$ be a BI-algebra and let I be a non-empty subset of X. Then I is called an *ideal* [12] of X if

(I1) $0 \in I$,

(I2) $x * y \in I$ and $y \in I$ imply $x \in I$

for any $x, y \in X$. Obviously, $\{0\}$ and X are ideals of X. We call $\{0\}$ and X a *zero ideal* and a *trivial ideal*, respectively. An ideal I is said to be *proper* if $I \neq X$.

Example 2.2. Let $X := \{0, a, b, c\}$ be a BI-algebra [12] with the following table:

$*$	0	a	b	c
0	0	0	0	0
a	a	0	a	b
b	b	b	0	b
c	c	b	c	0

Then it is easy to check that $I_1 := \{0, a, c\}$ is an ideal of X, but $I_2 := \{0, a, b\}$ is not an ideal of X, since $c * a = b \in I_2$ and $a \in I_2$, but $c \notin I_2$.

Proposition 2.3. [12] *Let I be an ideal of a BI-algebra X. If $y \in I$ and $x \leq y$, then $x \in I$.*

Proposition 2.4. [12] *Let X be a BI-algebra. Then*

(i) $x * 0 = x$,

(ii) $0 * x = 0$,

(iii) $x * y = (x * y) * y$,

(iv) *if $y * x = x$, then $X = \{0\}$,*

(v) *if $x * (y * z) = y * (x * z)$, then $X = \{0\}$,*

(vi) *if $x * y = z$, then $z * y = z$ and $y * z = y$,*

(vii) *if $(x * y) * (z * u) = (x * z) * (y * u)$, then $X = \{0\}$,*

for all $x, y, z, u \in X$.

Definition 2.5. A non-empty subset N of a BI-algebra X is said to be *normal* (or a *normal subalgebra*) [1] if $(x * a) * (y * b) \in N$ for any $x * y, a * b \in N$.

Definition 2.6. A BI-algebra X is called a BI_1*-algebra* [1] if $x * y = 0 = y * x$ implies $x = y$ for all $x, y \in X$.

3. Quotient BI-algebras induced by fuzzy normal subalgebras

Definition 3.1. A fuzzy set μ in a BI-algebra X is called a *fuzzy subalgebra* of X if for any $x, y \in X$,

(F0) $\mu(x * y) \geq \min\{\mu(x), \mu(y)\}$.

Example 3.2. Let $X := \{0, a, b, c\}$ be a BI-algebra [12] with the following table:

$*$	0	a	b	c
0	0	0	0	0
a	a	0	0	0
b	b	0	0	b
c	c	0	c	0

Define a fuzzy set $\mu : X \to [0,1]$ by $\mu(0) > \mu(a) = \mu(b) > \mu(c)$. Then μ is a fuzzy subalgebra of X.

Proposition 3.3. Let μ be a fuzzy subalgebra of a BI-algebra X. Then $\mu(0) \geq \mu(x)$ for all $x \in X$.

Proof. By (B1), we have $x * x = 0$ for all $x \in X$. Using (F0), $\mu(0) = \mu(x * x) \geq \min\{\mu(x), \mu(x)\} = \mu(x)$ for all $x \in X$. □

We denote a notation $\prod^n x * x$ by $\prod^n x * x := \underbrace{x * (x * (x * (\cdots * (x * x}_{\text{n}})) \cdots)$ for any natural number n.

Proposition 3.4. *Let μ be a fuzzy subalgebra of a BI-algebra X and let $n \in \mathbb{N}$. Then*

(i) $\mu(\prod^n x * x) \geq \mu(x)$ *whenever n is odd,*
(ii) $\mu(\prod^n x * x) = \mu(x)$ *whenever n is even.*

Proof. Let $x \in X$ and n be an odd natural number. Then $n = 2k - 1$ for some positive integer k. Then $\mu(\prod^{2(k+1)-1} x * x) = \mu(\prod^{2k+1} x * x) = \mu(\prod^{2k-1} x * (x * (x * x))) = \mu(\prod^{2k-1} x * x) \geq \mu(x)$ which proves (i). Similarly we can prove the second part, but we omit it. □

Definition 3.5. A fuzzy set μ in a BI-algebra X is said to be *fuzzy normal* if it satisfies the inequality

$$\mu((x * a) * (y * b)) \geq \min\{\mu(x * y), \mu(a * b)\}$$

for all $a, b, x, y \in$X.

Example 3.6. Let $X := \{0, 1, 2, 3\}$ be a BI-algebra [1] set with the following table:

$*$	0	1	2	3
0	0	0	0	0
1	1	0	1	1
2	2	2	0	2
3	3	3	3	0

Define a fuzzy set $\mu : X \to [0,1]$ by $\mu(0) > \mu(1) > \mu(2) = \mu(3)$. Then it easy to see that μ is fuzzy normal of X.

Theorem 3.7 *Every fuzzy normal set μ in a BI-algebra X is a fuzzy subalgebra of X.*

Proof. Let $x, y \in X$. Since μ is fuzzy normal, we have $\mu(x * y) = \mu((x * y) * (0 * 0)) \geq \min\{\mu(x * 0), \mu(y * 0)\} = \min\{\mu(x), \mu(y)\}$, which shows that μ is a fuzzy subalgebra of X. □

The converse of Theorem 3.7 may not be true in general.

Example 3.8. Consider a BI-algebra $X = \{0, a, b, c\}$ and a fuzzy set μ as in Example 3.2. Then μ is a fuzzy subalgebra of X, but not fuzzy normal, since $\mu((c * b) * (c * c)) = \mu(c) \ngeq \mu(b) = \min\{\mu(c * c), \mu(b * c)\}$.

Definition 3.9. A fuzzy set μ in a BI-algebra X is called a *fuzzy normal subalgebra* of X if it is both a fuzzy subalgebra and a fuzzy normal subset of X.

Example 3.10. Consider a BI-algebra $X = \{0, 1, 2, 3\}$ as in Example 3.6. Define a fuzzy set $\nu : X \to [0, 1]$ by

$$\nu(x) := \begin{cases} 0.7 & \text{if } x \in \{0, 1\}, \\ 0.3 & \text{if } x \in \{2, 3\}. \end{cases}$$

It is easy to show that ν is a fuzzy normal subalgebra of X.

Proposition 3.11. *If a fuzzy set μ in a BI-algebra X is fuzzy normal, then $\mu(x * y) = \mu(y * x)$ for all $x, y \in X$.*

Proof. Let $x, y \in X$. Using Proposition 3.3, we have $\mu(x*y) = \mu((x*y)*(x*x)) \geq \min\{\mu(x*x), \mu(y*x)\} = \mu(y*x)$. Interchanging x with y, we obtain $\mu(y * x) \geq \mu(x * y)$, which proves the proposition. □

Theorem 3.12. *Let μ be a fuzzy normal BI-algebra X. Then the set*

$$X_\mu := \{x \in X | \mu(x) = \mu(0)\}$$

is a normal subalgebra of X.

Proof. Let $a, b, x, y \in X$ be such that $x * y \in X_\mu$ and $a * b \in X_\mu$. Then $\mu(x * y) = \mu(0) = \mu(a * b)$. Since μ is fuzzy normal, we have $\mu((x * a) * (y * b)) \geq \min\{\mu(x * y), \mu(a * b)\} = \mu(0)$. It follows from Proposition 3.3 that $\mu((x * a) * (y * b)) = \mu(0)$. Hence $(x * a) * (y * b) \in X_\mu$. This completes the proof. □

Theorem 3.13. *The intersection of a family of fuzzy normal subalgebras of a BI-algebra X is also a fuzzy normal subalgebra of X.*

Proof. Let $\{\mu_\alpha | \alpha \in \Lambda\}$ be a family of fuzzy normal subalgebras and let $a, b, x, y \in X$. Then

$$\begin{aligned} \cap_{\alpha\in\Lambda}\mu_\alpha((x * a) * (y * b)) &= \inf_{\alpha\in\Lambda} \mu_\alpha((x * a) * (y * b)) \\ &\geq \inf_{\alpha\in\Lambda} \{\min\{\mu_\alpha(x * y), \mu_\alpha(a * b)\}\} \\ &= \min\{\inf_{\alpha\in\Lambda} \mu_\alpha(x * y), \inf_{\alpha\in\Lambda} \mu_\alpha(a * b)\} \\ &= \min\{\cap_{\alpha\in\Lambda}\mu_\alpha(x * y), \cap_{\alpha\in\Lambda}\mu_\alpha(a * b)\} \end{aligned}$$

which shows that $\cap_{\alpha\in\Lambda}\mu_\alpha$ is fuzzy normal of X. By Proposition 3.7, we know that $\cap_{\alpha\in\Lambda}\mu_\alpha$ is a fuzzy normal subalgebra of X. □

Suppose that μ is a fuzzy normal subalgebra of a BI-algebra X. Define a binary relation " $\sim^\mu$ " on X by putting $x \sim^\mu y$ if and only if $\mu(x * y) = \mu(0)$ for any $x, y \in X$.

Lemma 3.14. *The relation $\sim^\mu$ is an equivalence relation on a BI-algebra X.*

Proof. Using (B1), $\mu(x * x) = \mu(0)$ and so $x \sim^\mu x$, which means $\sim^\mu$ is reflexive. Suppose that $x \sim^\mu y$ for any $x, y \in X$. Then $\mu(0) = \mu(x * y)$. By Proposition 3.11, $\mu(y * x) = \mu(0)$. So $y \sim^\mu x$, which means $\sim^\mu$ is symmetric.

Suppose that $x \sim^\mu y$ and $y \sim^\mu z$ for any $x, y, z \in X$. Then $\mu(x * y) = \mu(0)$, $\mu(y * z) = \mu(0) = \mu(z * y)$ and

$$\begin{aligned}\mu(x * z) =& \mu((x * z) * 0) = \mu((x * z) * (y * y))\\ \geq& \min\{\mu(x * y), \mu(z * y)\}\\ =& \min\{\mu(0), \mu(0)\} = \mu(0).\end{aligned}$$

Also since $\mu(0) \geq \mu(x)$ for all $x \in X$, $\mu(0) \geq \mu(x * z)$ and so $\mu(x * z) = \mu(0)$. Hence $x \sim^\mu z$. Therefore $\sim^\mu$ is an equivalence relation on a BI-algebra X. □

Lemma 3.15. *For all x, y, z in a BI-algebra X, $x \sim^\mu y$ implies $x * z \sim^\mu y * z$ and $z * x \sim^\mu z * y$.*

Proof. Let $x \sim^\mu y$. Then $\mu(x * y) = \mu(0)$. Since $x * x = 0$ and $\mu(0) \geq \mu(x)$ for all $x \in X$, we have

$$\begin{aligned}\mu((x * z) * (y * z)) \geq& \min\{\mu(x * y), \mu(z * z)\}\\ =& \min\{\mu(0), \mu(0)\} = \mu(0).\end{aligned}$$

Since $\mu(0) \geq \mu(x)$ for all $x \in X$, $\mu(0) \geq \mu((x * z) * (y * z))$. Therefore $\mu(0) = \mu((x * z) * (y * z))$, so $x * z \sim^\mu y * z$. By a similar way, we can prove that $z * x \sim^\mu z * y$. The proof is complete. □

Lemma 3.16. *Let X be a BI-algebra. For any $x, y, z, w \in X$, $x \sim^\mu y$ and $z \sim^\mu w$ imply $x * z \sim^\mu y * w$.*

Proof. Let $x \sim^\mu y$ and $z \sim^\mu w$ for any $x, y, z, w \in X$. Then $\mu(x * y) = \mu(0)$ and $\mu(z * w) = \mu(0)$. Hence $\mu((x * z) * (y * w)) \geq \min\{\mu(x * y), \mu(z * w)\} = \min\{\mu(0), \mu(0)\} = \mu(0)$. Since $\mu(0) \geq \mu(x)$ for all $x \in X$, $\mu(0) \geq \mu((x * z) * (y * w))$. Thus $\mu(0) = \mu((x * z) * (y * w))$, so $x * z \sim^\mu y * w$. The proof is complete. □

The above Lemmas 3.14, 3.15 and 3.16 give the following theorem.

Theorem 3.17. *The relation " $\sim^\mu$ " is a congruence relation on a BI-algebra X.*

Denote by μ_x the equivalence class containing x, and let X/μ be the set of all equivalence classes with respect to $\sim^\mu$, that is, $\mu_x = \{y \in X | y \sim^\mu x\}$ and $X/\mu = \{\mu_x | x \in X\}$. Now we define a binary operation " $*$ " in X/μ by putting $\mu_x * \mu_y := \mu_{x*y}$. Theorem 3.17 guarantees that this operation is well defined.

Theorem 3.18. *Let μ be a fuzzy normal subalgebra in a BI_1-algebra X. Then $(X/\mu, *, \mu_0)$ is a BI_1-algebra.*

Proof. Let $\mu_x, \mu_y, \mu_z \in X/\mu$. Then $\mu_x * \mu_x = \mu_{x*x} = \mu_0$ and $\mu_x = \mu_{x*(y*x)} = \mu_x * \mu_{y*x} = \mu_x * (\mu_y * \mu_x)$. If $\mu_x * \mu_y = \mu_0$ and $\mu_y * \mu_x = \mu_0$, then $\mu_{x*y} = \mu_0 = \mu_{y*x}$ and so $x * y = 0 = y * x$. Hence $x = y$ and therefore $\mu_x = \mu_y$. Thus $(X/\mu, *, \mu_0)$ is a BI_1-algebra. □

Corollary 3.19. *Let μ be a fuzzy normal subalgebra in a BI-algebra. Then $(X/\mu; *, \mu_0)$ is a BI-algebra.*

This algebra X/μ is called the *quotient BI-algebra* of a BI-algebra X induced by a fuzzy normal subalgebra μ.

If μ is a fuzzy normal subalgebra in a BI-algebra X, then the set $X_\mu := \{x \in X | \mu(x) = \mu(0)\}$ is a normal subalgebra of X.

Theorem 3.20. *Let μ be a fuzzy normal subalgebra of a BI-algebra X. The mapping $\gamma : X \to X/\mu$, given by $\gamma(x) = \mu_x$, is a surjective homomorphism, and $ker\gamma = \{x \in X | \gamma(x) = \mu_0\} = X_\mu$.*

Proof. Let $\mu_x \in X/\mu$. Then there exists an element $x_0 \in \mu_x$, so $x_0 \in X$ such that $\gamma(x_0) = \mu_x$. Hence γ is surjective. For any $x, y \in X$, $\gamma(x * y) = \mu_{x*y} = \mu_x * \mu_y = \gamma(x) * \gamma(y)$. Thus γ is a homomorphism. And $ker\gamma = \{x \in X|\gamma(x) = \mu_0\} = \{x \in X|x \sim^{\mu} 0\} = \{x \in X|\mu(x) = \mu(0)\} = X_\mu$. □

Let X, Y be BI-algebras. If we define $(x_1, y_1) * (x_2, y_2) := (x_1 * x_2, y_1 * y_2)$ in $X \times Y$, then $(X \times Y, *, (0, 0))$ becomes a BI-algebra, and we call it a *product BI-algebra*.

Theorem 3.21. *Let μ (resp., ν) be a fuzzy normal subalgebra in a BI-algebra X (resp., Y). If we define $(\mu\times\nu)(x, y) := \min\{\mu(x), \nu(x)\}$ in $X\times Y$ for $x \in X, y \in Y$, then $\mu\times\nu$ is also a fuzzy normal subalgebra in $X\times Y$.*

Proof. Let μ (resp., ν) be a fuzzy normal subalgebra in X (resp., Y). Then

$$\begin{aligned}(\mu \times \nu)((x_1, y_1)*(x_2, y_2)) &= (\mu \times \nu)(x_1 * x_2, y_1 * y_2)\\ &= \min\{\mu(x_1 * x_2), \nu(y_1 * y_2)\}\\ &\geq \min\{\min\{\mu(x_1), \mu(x_2)\}, \min\{\nu(y_1), \nu(y_2)\}\}\\ &= \min\{\min\{\mu(x_1), \nu(y_1)\}, \min\{\mu(x_2), \nu(y_2)\}\}\\ &= \min\{(\mu \times \nu)(x_1, y_1), (\mu \times \nu)(x_2, y_2)\}\end{aligned}$$

for any $(x_1, y_1), (x_2, y_2) \in X \times Y$. Hence $\mu \times \nu$ is a fuzzy subalgebra of $X \times Y$. And

$$\begin{aligned}&(\mu \times \nu)(((x_1, y_1) * (a_1, b_1)) * ((x_2, y_2) * (a_2, b_2)))\\ &=(\mu \times \nu)((x_1 * a_1, y_1 * b_1) * (x_2 * a_2, y_2 * b_2))\\ &=(\mu \times \nu)((x_1 * a_1) * (x_2 * a_2), (y_1 * b_1) * (y_2 * b_2))\\ &= \min\{\mu((x_1 * a_1) * (x_2 * a_2)), \nu((y_1 * b_1) * (y_2 * b_2))\}\\ &\geq \min\{\min\{\mu(x_1 * x_2), \mu(a_1 * a_2)\}, \min\{\nu(y_1 * y_2), \nu(b_1 * b_2)\}\}\\ &= \min\{\min\{\mu(x_1 * x_2), \nu(y_1 * y_2)\}, \min\{\mu(a_1 * a_2), \nu(b_1 * b_2)\}\}\\ &= \min\{(\mu \times \nu)((x_1 * x_2), (y_1 * y_2)), (\mu \times \nu)((a_1 * a_2), (b_1 * b_2))\}\\ &= \min\{(\mu \times \nu)((x_1, y_1) * (x_2, y_2)), (\mu \times \nu)((a_1, b_1) * (a_2, b_2))\}.\end{aligned}$$

So $\mu \times \nu$ is fuzzy normal. Therefore $\mu \times \nu$ is also a fuzzy normal subalgebra of $X \times Y$. □

Proposition 3.22. *Let μ be a fuzzy normal subalgebra of a BI-algebra X. If J is a normal subalgebra of X, then J/μ is a normal subalgebra of X/μ.*

Proof. Let μ be a fuzzy normal subalgebra of X and J be a normal subalgebra of X. Then for any $x, y \in J$, $x * y \in J$. Let $\mu_x, \mu_y \in J/\mu$. Then $\mu_x * \mu_y = \mu_{x*y} \in J/\mu$. So $J/\mu = \{\mu_x|x \in J\}$ is a subalgebra of X/μ. For any $x * y, a * b \in J$, $(x * a) * (y * b) \in J$. For any $\mu_x * \mu_y, \mu_a * \mu_b \in J/\mu$, we have

$$\begin{aligned}(\mu_x * \mu_a) * (\mu_y * \mu_b) &=\mu_{x*a} * \mu_{y*b}\\ &=\mu_{(x*a)*(y*b)} \in J/\mu.\end{aligned}$$

Hence J/μ is a normal subalgebra of X/μ. □

Theorem 3.23. *If J^* is a normal subalgebra of X/μ, then there exists a normal subalgebra $J = \cup\{x \in X|\mu_x \in J^*\}$ in X such that $J/\mu = J^*$.*

Proof. Since J^* is a normal subalgebra of X/μ, we have $\mu_x * \mu_y = \mu_{x*y} \in J^*$ for any $\mu_x, \mu_y \in J^*$. Hence $x * y \in J$ for any $x, y \in J$. And $\mu_{x*a} * \mu_{y*b} = \mu_{(x*a)*(y*b)} \in J^*$ for any $\mu_{x*y}, \mu_{a*b} \in J^*$. Therefore $(x * a) * (y * b) \in J$ for any $x * y, a * b \in J$. Thus J is a normal subalgebra of X. By Theorem 3.20,

$$\begin{aligned} J/\mu =& \{\mu_j | j \in J\} \\ =& \{\mu_j | \exists \mu_x \in J^* \text{ such that } j \sim^{\mu} x\} \\ =& \{\mu_j | \exists \mu_x \in J^* \text{ such that } \mu_x = \mu_j\} \\ =& \{\mu_j | \mu_j \in J^*\} = J^*. \end{aligned}$$

This completes the proof. □

4. Quotient BI-algebras induced by fuzzy congruence relations

Definition 4.1. [10] A binary operation θ from $X \times X \to [0, 1]$ is a *fuzzy equivalence relation* on X if for all $x, y, z, u \in X$

(FC1) $\theta(x, x) = \sup\{\theta(y, z) | y, z \in X\} = \theta(0, 0)$,

(FC2) $\theta(x, y) = \theta(y, x)$,

(FC3) $\theta(x, z) \geq \min\{\theta(x, y), \theta(y, z)\}$.

Moreover, if it satisfies

(FC4) $\theta(x * u, y * u) \geq \theta(x, y), \theta(u * x, u * y) \geq \theta(x, y)$

for all $x, y, u \in X$, we say that θ is a *fuzzy congruence relation* on $(X, *, 0)$.

Let $FCo(X)$ be the set of all fuzzy congruence relations on a BI-algebra X.

Lemma 4.2. *If θ satisfies the condition (FC2) $\sim$ (FC4) above, then (FC1) is equivalent to $\theta(0, 0) \geq \theta(x, y)$ for all $x, y \in X$.*

Proof. Suppose that $\theta(0, 0) = \theta(x, x)$. By (FC2) and (FC3), we have $\theta(0, 0) = \theta(x, x) \geq \min\{\theta(x, y), \theta(y, x)\} = \theta(x, y)$ for all $x, y \in X$.

Conversely, assume that $\theta(0, 0) \geq \theta(x, y)$ for all $x, y \in X$. It follows from (FC4) that $\theta(0, 0) \leq \theta(x * 0, x * 0) = \theta(x, x)$ By assumption, we have $\theta(0, 0) = \theta(x, x)$. Hence (FC1) holds. □

Proposition 4.3. *Let θ be a fuzzy congruence relation on a BI-algebra X. Then $\theta(x, y) = \theta(x * y, 0)$ for all $x, y \in X$.*

Proof. By (FC4) and Lemma 4.2, we have $\min\{\theta(x, y), \theta(y, y)\} = \min\{\theta(x, y), \theta(0, 0)\} = \theta(x, y) \leq \theta(x * y, y * y) = \theta(x * y, 0)$ for all $x, y \in X$. On the other hand, $\theta(x * y, 0) = \theta(x * y, x * x) \geq \theta(y, x)$. Hence $\theta(x, y) = \theta(x * y, 0)$ □

For every element $x \in X$, we define $\theta_x := \{y \in X | \theta(x, y) = \theta(0, 0)\}$ of X and $X/\theta := \{\theta_x | x \in X\}$. Define an operation "$\bullet$" on the set X/θ by

$$\theta_x \bullet \theta_y := \theta_{x*y}.$$

This operation is well defined. In fact, if $\theta_x = \theta_{x'}$ and $\theta_y = \theta_{y'}$, then we have $\theta(x, x') = \theta(y, y') = \theta(0, 0)$. Since $\theta(0, 0) = \min\{\theta(x, x'), \theta(y, y')\} \leq \theta(x * y, x' * y') \leq \theta(0, 0)$, we have $\theta(x * y, x' * y') = \theta(0, 0)$ and so $\theta_{x*y} = \theta_{x'*y'}$. Hence $\bullet$ is well defined.

Theorem 4.4. *If $\theta \in FCo(X)$, where X is a BI-algebra, then $(X/\theta, \bullet, \theta_0)$ is a BI-algebra.*

Proof. Straightforward. □

Proposition 4.5. *Let $f : X \to Y$ be a homomorphism of BI-algebras. If θ is a fuzzy congruence relation of Y, then $\bar{\theta}(x, y) := \theta(f(x), f(y))$ is a fuzzy congruence relation of X.*

Proof. It is obvious that $\bar{\theta}$ is well-defined. Let $x, y, z, u \in X$. Then
(i) $\bar{\theta}(x, x) = \theta(f(x), f(x)) = \theta(0, 0)$.
(ii) $\bar{\theta}(x, y) = \theta(f(x), f(y)) = \theta(f(y), f(x)) = \bar{\theta}(y, x)$.
(iii) It can be shown that $\bar{\theta}(x, z) = \theta(f(x), f(z)) \geq \min\{\theta(f(x), f(y)), \theta(f(y), f(z))\} = \min\{\bar{\theta}(x, y), \bar{\theta}(y, z)\}$.
(iv) It can be shown that $\bar{\theta}(x*u, y*u) = \theta(f(x*u), f(y*u)) = \theta(f(x)*f(u), f(y)*f(u)) \geq \theta(f(x), f(y)) = \bar{\theta}(x, y)$.
By a similar way, we have $\bar{\theta}(u * x, u * y) \geq \bar{\theta}(x, y)$. Thus $\bar{\theta}$ is a fuzzy congruence relation. □

Proposition 4.6. *Let θ be a fuzzy congruence relation of a BI-algebra X. Then the mapping $\gamma : X \to X/\theta$, given by $\gamma(x) := \theta_x$, is a surjective homomorphism.*

Proof. Let $\theta_x \in X/\theta$. Then there exists an element $x_0 \in \theta_x$ such that $\gamma(x_0) = \theta_x$. Hence γ is surjective. For any $x, y \in X$, $\gamma(x * y) = \theta_{x*y} = \theta_x \bullet \theta_y = \gamma(x) \bullet \gamma(y)$. Thus γ is a homomorphism. □

Theorem 4.7. *Let $f : (X, *, 0_X) \to (Y, *, 0_Y)$ be an epimorphism of BI_1-algebras and let θ be a fuzzy congruence relation of Y. If $\bar{\theta} = \theta \circ f$, then the quotient algebra $X/\bar{\theta} := (X/(\theta \circ f), \bullet_X, \bar{\theta}_{0_X})$ is isomorphic to the quotient algebra $Y/\theta := (Y/\theta, \bullet_Y, \theta_{0_Y})$.*

Proof. By Theorem 4.4 and Proposition 4.5, $X/(\theta \circ f) := (X/(\theta \circ f), \bullet_X, \bar{\theta}_{0_X})$ is a BI-algebra and $Y/\theta := (Y/\theta, \bullet_Y, \theta_{0_Y})$ is a BI-algebra. Define a map

$$\eta : X/(\theta \circ f) \to Y/\theta,\ (\theta \circ f)_x \mapsto \theta_{f(x)}$$

for all $x \in X$. Then the function η is well-defined. In fact, assume that $(\theta \circ f)_x = (\theta \circ f)_y$ for all $x, y \in X$. Then we have $\theta(f(x) *_Y f(y)) = \theta(f(x *_X y)) = (\theta \circ f)(x *_X y) = (\theta \circ f)(0_X) = \theta(f(0_X)) = \theta(0_Y)$ and $\theta(f(y) *_Y f(x)) = \theta(f(y *_X x)) = (\theta \circ f)(y *_X x) = (\theta \circ f)(0_X) = \theta(f(0_X)) = \theta(0_Y)$. Hence $\theta_{f(x)} = \theta_{f(y)}$.

For any $(\theta \circ f)_x, (\theta \circ f)_y \in X/(\theta \circ f)$, we have $\eta((\theta \circ f)_x \bullet_X (\theta \circ f)_y) = \eta((\theta \circ f)_{x*y}) = \theta_{f(x*_X y)} = \theta_{f(x)*_Y f(y)} = \theta_{f(x)} \bullet \theta_{f(y)} = \eta((\theta \circ f_x)) \bullet_Y \eta((\theta \circ f)_y)$. Therefore η is a homomorphism.

Let $\theta_a \in Y/\theta$. Then there exists $x \in X$ such that $f(x) = a$, since f is surjective. Hence $\eta((\theta \circ f)_x) = \theta_{f(x)} = \theta_a$ and so η is surjective.

Let $x, y \in X$ be such that $\theta_{f(x)} = \theta_{f(y)}$. Then we have $(\theta \circ f)(x *_X y) = \theta(f(x *_X y)) = \theta(f(x) *_Y f(y)) = \theta(0_Y) = \theta(f(0_X)) = (\theta \circ f)(0_X)$ and $(\theta \circ f)(y *_X x) = \theta(f(y *_X x)) = \theta(f(y) *_Y f(x)) = \theta(0_Y) = \theta(f(0_X)) = (\theta \circ f)(0_X)$. It follows that $(\theta \circ f)_x = (\theta \circ f)_y$. Thus η is injective. This completes the proof. □

The homomorphism $\pi : X \to X/\theta$, $x \to \theta_x$, is called the *natural homomorphism* of X onto X/θ. In Theorem 4.7, if we define natural homomorphisms $\pi_X : X \to X/\theta \circ f$ and $\pi_Y : Y \to Y/\theta$, then it is easy to show that $\eta \circ \pi_X = \pi_Y \circ f$, i.e., the following diagram commutes.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\pi_X"'] & Y \arrow[d, "\pi_Y"'] \\ X/(\theta \circ f) \arrow[r, "\eta"] & Y/\theta. \end{tikzcd}$$

The fuzzy subset θ_x of a BI-algebra X, which is defined by $\theta_x(y) := \theta(x, y)$, is called the fuzzy congruence class containing x and X/θ is the set of all fuzzy congruences classes θ_x.

Proposition 4.8. *Let θ be a fuzzy congruence relation in a BI-algebra X. Then θ_0 is a fuzzy ideal of X.*

Proof. Let $x, y \in X$. Then $\theta_0(0) = \theta(0, 0) \geq \theta(x, y)$ by Lemma 4.2. Put $y := 0$ in above inequality. Then $\theta_0(0) \geq \theta(x, 0) = \theta_0(x)$. By (FC3), (FC2) and Proposition 5.3, we have $\theta_0(y) = \theta(0, y) \geq \min\{\theta(0, x), \theta(x, y)\} = \min\{\theta(x, 0), \theta(x * y, 0)\} = \min\{\theta_0(x), \theta_0(x * y)\}$. Thus θ_0 is a fuzzy ideal of X. □

References

[1] S. S. Ahn, J. M. Ko and A. B. Saeid, *Normal subalgebras of BI-algebras and its analytic constructions*, (submitted).

[2] G. Dymek and A. Walendziak, *(Fuzzy) ideals of BN-algebras*, The Scientific World Journal **2015** (2015), Article ID 925049.

[3] M. Khan, F. Feng and M. N. A. Khan, *On minimal fuzzy ideals of semigroups*, Journal of Mathematics **2013** (2013), Article ID 475190.

[4] H. S. Kim, C. B. Kim and K. S. So, *Radical structures of fuzzy polynomial ideals in a ring*, Discrete Dynamics in Nature and Society **2016** (2016), Article ID: 782178.

[5] H. V. Kumbhojar, *Proper fuzzification of prime ideals of a hemiring*, Advances in fuzzy systems **2012** (2012), Article ID: 801650.

[6] T. Kuraoka and N. Kuroki, *On fuzzy quotient rings induced by fuzzy ideals*, Fuzzy Sets and Systems **47** (1992), 381-386.

[7] W. J. Liu, *Fuzzy invariant subgroups and fuzzy ideals*, Fuzzy sets and Systems **8** (1982), 133-139.

[8] B. L. Meng and X. L. Xin, *On fuzzy ideals of BL-algebras*, The Scientific World Journal **2014** (2014), Article ID 757382.

[9] T. K. Mukherjee and M. K. Sen, *On fuzzy ideals of a ring 1*, Fuzzy Sets and Systems **21** (1987), 99-104.

[10] V. Murali, *Fuzzy congruences relations*, Fuzzy Sets and Systems **30** (1989), 155-163.

[11] A. Rosenfeld, *Fuzzy groups*, J. Math. Anal. Appl. **35** (1971), 512-517.

[12] A. B. Saeid, H. S. Kim and A. Rezaei, *On BI-algebras*, An. Şt. Univ. Ovidius Constanţa **25** (2017), 177-194.

[13] S. Z. Song, Y. B. Jun and H. S. Kim, *Characterizations of positive implicative superior ideals induced by superior mappings*, J. Computational Anal. and Appl. **25** (2018), 634-643.

[14] L. A. Zadeh, *Fuzzy sets*, Inform. and Control **8** (1965), 338-353.

General quadratic functional equations in quasi-β-normed spaces: solution, superstability and stability

Shahrokh Farhadabadi[1*], Choonkil Park[2*] and Sungsik Yun[3*]

[1]Young Researchers and Elite Club, Parand Brunch, Islamic Azad University, Parand, Iran
[2]Research Institute for Natural Sciences Hanyang University, Seoul 04763, Korea
[3]Department of Financial Mathematics, Hanshin University, Gyeonggi-do 18101, Korea
e-mail: shahrokh_math@yahoo.com; baak@hanyang.ac.kr; ssyun@hs.ac.kr

Abstract. Let $f : \mathcal{X} \to \mathcal{Y}$ be a mapping where $\mathcal{X}$ is a quasi-α-normed space and $\mathcal{Y}$ is a quasi-β-normed space. The following quadratic functional equation

$$\sum_{i=1}^{n} f\Big(\sum_{\substack{j=1\\ j\neq i}}^{n} x_j + \frac{2-n}{2}x_i\Big) = \frac{n^2}{4}\sum_{i=1}^{n} f(x_i), \qquad (n \geq 3) \tag{0.1}$$

is introduced and solved by giving its general solution.

Moreover, we prove the Hyers-Ulam stability of the functional equation (0.1) by using a direct method.

1. Introduction and preliminaries

Studying functional equations by focusing on their approximate and exact solutions conduces to one of the most substantial significant study brunches in functional equations, what we call *"the theory of stability of functional equations"*. This theory specifically analyzes relationships between approximate and exact solutions of functional equations. Actually a functional equation is considered to be *stable* if one can find an exact solution for any approximate solution of that certain functional equation. Another related and close term in this area is *superstability*, which has a similar nature and concept to the stability problem. As a matter of fact, superstability for a given functional equation occurs when any approximate solution is an exact solution too. In such this situation the functional equation is called *superstable*.

In 1940, the most preliminary form of stability problems was proposed by Ulam [35]. He gave a talk and asked the following: "when and under what conditions does an exact solution of a functional equation near an approximately solution of that exist?"

In 1941, this question that today is considered as the source of the stability theory, was formulated and solved by Hyers [13] for the Cauchy's functional equation in Banach spaces. Then the result of Hyers was generalized by Aoki [1] for additive mappings and by Rassias [24] for linear mappings by considering an unbounded Cauchy difference. In 1994, Găvruţa [12] provided a further generalization of Rassias' theorem in which he replaced the unbounded Cauchy difference by a general control function for the existence of a unique linear mapping. For more epochal information and various aspects about the stability of functional equations theory, we refer the reader to the monographs [10, 11, 14, 15, 18, 20, 25, 27, 29, 30, 31, 32, 33], which also include many interesting results concerning the stability of different functional equations in many various spaces.

Now we present some brief explanations about the functional equation (0.1) and also generally about quadratic functional equations. Consider the functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y) \tag{1.1}$$

---

[0]2010 Mathematics Subject Classification: 39B52, 39B72, 46Bxx, 39Bxx.

[0]Keywords: Hyers-Ulam stability; functional equation; quadratic functional equation; superstability; direct method.

*Corresponding authors.

which is called *the classic quadratic functional equation.* Obviously, the function $f(x) = cx^2$ is its solution and so it is called quadratic. There are some other different types of quadratic functional equations. For examples, the following n-dimensional quadratic functional equations

$$\sum_{k=2}^{n}\Big[\sum_{i_1=2}^{k}\sum_{i_2=i_1+1}^{k+1}\cdots\sum_{i_{n-k+1}=i_{n-k}+1}^{n} f\Big(\sum_{\substack{i=1\\ i\neq i_1,\cdots,i_{n-k+1}}}^{n} x_i-\sum_{r=1}^{n-k+1}x_{i_r}\Big)\Big] +f\Big(\sum_{i=1}^{n}x_i\Big)=2^{n-1}\sum_{i=1}^{n}f(x_i),$$

$$f\Big(\sum_{i=1}^{n}x_i\Big)+\sum_{1\le i<j\le n}f(x_i-x_j)=n\sum_{i=1}^{n}f(x_i)$$

have been introduced in [9] and [3], respectively. These n-dimensional versions are generalized forms of (1.1), but each in a different way.

In this paper, we introduce another n-dimensional version as follows:

$$\sum_{i=1}^{n} f\Big(\sum_{\substack{j=1\\ j\neq i}}^{n} x_j+\frac{2-n}{2}x_i\Big)=\frac{n^2}{4}\sum_{i=1}^{n}f(x_i) \qquad (n\ge 3), \tag{1.2}$$

in which the simplest case (for $n = 3$) is the functional equation

$$f\left(x+y-\frac{z}{2}\right)+f\left(x+z-\frac{y}{2}\right)+f\left(y+z-\frac{x}{2}\right)=\frac{9}{4}\Big[f(x)+f(y)+f(z)\Big]. \tag{1.3}$$

Note that (1.2), for each fixed integer $n \ge 3$, is symmetric with respect to any permutation of the variables.

In the next section, we will show that (1.2) is equivalent to (1.1). Nevertheless (1.2) is not a generalization of (1.1), rather in fact it is a generalized form of (1.3).

The stability problem for the classic quadratic functional equation was first proved by F. Skof [34] and then generalized by Cholewa [6], Czerwik [7, 8] and others [2, 4, 22, 23, 25, 26]. Many stability problems for some other versions can be found in [3, 5, 16, 17, 19, 21].

Now we give briefly some useful definitions, preliminary and fundamental results of quasi-β-normed spaces. Throughout this paper β will be a fixed real number with $0 < \beta \le 1$ and $\mathbb{K}$ denotes either $\mathbb{R}$ or $\mathbb{C}$.

Definition 1.1. ([28]) Let $\mathcal{X}$ be a linear space over $\mathbb{K}$. A *quasi-β-norm* $\|\cdot\|$ is a real-valued function on $\mathcal{X}$ satisfying the following conditions:

($\mathcal{C}_1$) $\|x\| \ge 0$ for all $x \in \mathcal{X}$ and $\|x\| = 0$ if only if $x = 0$;

($\mathcal{C}_2$) $\|\lambda \cdot x\| = |\lambda|^\beta \cdot \|x\|$ for all $\lambda \in \mathbb{K}$ and all $x \in \mathcal{X}$;

($\mathcal{C}_3$) There is a constant $\mathcal{K} \ge 1$ such that $\|x + y\| \le \mathcal{K}(\|x\| + \|y\|)$ for all $x, y \in \mathcal{X}$.

The pair $(\mathcal{X}, \|\cdot\|)$ is called a *quasi-β-normed space* and the smallest possible $\mathcal{K}$ is called the modulus of concavity of $\|\cdot\|$.

A complete quasi-β-normed space is a *quasi-β-Banach space.*

Definition 1.2. ([28]) Let $0 < p \le 1$ be a real number. A quasi-β-normed space $(\mathcal{X}, \|\cdot\|)$ is called a (β, p)-*normed space* if

$$\|x+y\|^p \le \|x\|^p + \|y\|^p$$

for all $x, y \in \mathcal{X}$. In this case, a quasi-β-Banach space is called a (β, p)-Banach space.

2. The general solution

In this section, we give the general solution of the functional equation (0.1) by proving the fact that it is equivalent to the functional equation (1.1), which implies that it is quadratic too.

First, we prove a useful lemma.

Lemma 2.1. *Let $\mathcal{X}$ and $\mathcal{Y}$ be linear spaces. If a mapping $f : \mathcal{X} \to \mathcal{Y}$ satisfies the functional equation*

$$f(x+y) + f(x-y) = 2f(x) + 2f(y)$$

for all $x, y \in \mathcal{X}$, then

$$f(rx) = r^2 f(x)$$

for all $x \in \mathcal{X}$ and all $r \in \mathbb{Q}$.

Proof. Let $n \geq 2$ be a natural number. Replacing (x, y) by $(0, 0), (0, -x)$ and $(x, (n-1)x)$ respectively, we get $f(0) = 0, f(x) = f(-x)$ and

$$f(nx) = 2f(x) + 2f((n-1)x) - f((n-2)x)$$

for all $x \in \mathcal{X}$. This simply implies that

$$\begin{aligned} f(-2x) = f(2x) &= 2f(x) + 2f(x) - f(0) = 4f(x), \\ f(-3x) = f(3x) &= 2f(x) + 2f(2x) - f(x) = 9f(x), \\ f(-4x) = f(4x) &= 2f(x) + 2f(3x) - f(2x) = 16f(x), \\ &\vdots \\ f(kx) &= k^2 f(x) \end{aligned}$$

for all $x \in \mathcal{X}$ and all $k \in \mathbb{Z}$. Putting $\frac{x}{k}$ instead of x in the above line, we obtain

$$f\Big(\frac{x}{k}\Big) = \frac{1}{k^2} f(x)$$

for all $x \in \mathcal{X}$ and all $k \in \mathbb{Z}$. Therefore, we can conclude for any $r \in \mathbb{Q}$ that

$$f(rx) = f\Big(\frac{m}{n}x\Big) = m^2 f\Big(\frac{x}{n}\Big) = \frac{m^2}{n^2} f(x) = r^2 f(x)$$

for all $x \in \mathcal{X}$ and all $r \in \mathbb{Q}$, which ends the proof. □

Theorem 2.2. *Let $\mathcal{X}$ and $\mathcal{Y}$ be linear spaces and let $n \geq 3$ be a fixed positive integer. A mapping $f : \mathcal{X} \to \mathcal{Y}$ satisfies the functional equation*

$$\sum_{i=1}^{n} f\Big(\sum_{\substack{j=1 \\ j \neq i}}^{n} x_j + \frac{2-n}{2} x_i\Big) = \frac{n^2}{4} \sum_{i=1}^{n} f(x_i) \tag{2.1}$$

for all $x_1, x_2, \cdots, x_n \in \mathcal{X}$ if and only if f satisfies the functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y) \tag{2.2}$$

for all $x, y \in \mathcal{X}$.

Proof. Sufficiency. For the *'only if'* part of the proof, suppose that $f : \mathcal{X} \to \mathcal{Y}$ satisfies (2.1), we will show that f satisfies the classic quadratic functional equation (2.2).

First we except the cases $n = 3, 4$ and investigate them separately. In the case $n = 3$, (2.1) is in the form

$$f\left(x + y - \frac{z}{2}\right) + f\left(x + z - \frac{y}{2}\right) + f\left(y + z - \frac{x}{2}\right) = \frac{9}{4}\Big[f(x) + f(y) + f(z)\Big] \tag{2.3}$$

for all $x, y, z \in \mathcal{X}$. Replacing (x, y, z) in (2.3), by $(0, 0, 0)$, $(x, 0, 0)$, $(x, x, 0)$, $(\frac{2}{3}x, \frac{2}{3}x, \frac{2}{3}x)$ and $\left(\frac{2}{3}x, -\frac{2}{3}x, y\right)$, respectively, we obtain $f(0) = 0$ and

$$f\Big(\frac{-1}{2}x\Big) = \frac{1}{4} f(x), \tag{2.4}$$

$$f\Big(\frac{1}{2}x\Big) = \frac{9}{4} f(x) - \frac{1}{2} f(2x), \tag{2.5}$$

$$f\Big(\frac{2}{3}x\Big) = \frac{4}{9} f(x), \tag{2.6}$$

$$f\Big(\frac{-1}{2}y\Big) + f(x+y) + f(y-x) = \frac{9}{4}\Big[f\Big(\frac{2}{3}x\Big) + f\Big(\frac{-2}{3}x\Big) + f(y)\Big] \tag{2.7}$$

for all $x, y \in \mathcal{X}$, respectively. By using (2.4) and (2.6), we can rewrite (2.7) as

$$f(x+y) + f(y-x) = f(x) + f(-x) + 2f(y) \tag{2.8}$$

for all $x, y \in \mathcal{X}$. Letting $x = y$ in (2.8), we see that

$$f(2x) = 3f(x) + f(-x) \tag{2.9}$$

for all $x \in \mathcal{X}$. From (2.4), (2.5) and (2.9), it follows that

$$\frac{1}{4}f(-x) = f\Big(\frac{1}{2}x\Big) = \frac{9}{4}f(x) - \frac{1}{2}f(2x) = \frac{9}{4}f(x) - \frac{3}{2}f(x) - \frac{1}{2}f(-x)$$

for all $x \in \mathcal{X}$, which implies that $f(x) = f(-x)$ for all $x \in \mathcal{X}$. So (2.8) can be rewritten as

$$f(x+y) + f(x-y) = 2f(x) + 2f(y)$$

for all $x, y \in \mathcal{X}$, which is exactly (2.2).

Now the case $n = 4$. In this case we have the functional equation

$$\begin{aligned} & f(x+y+z-w) + f(x+y+w-z) + f(x+z+w-y) \\ & \quad + f(y+z+w-x) = 4\big[f(x) + f(y) + f(z) + f(w)\big] \end{aligned} \tag{2.10}$$

for all $x, y, z, w \in \mathcal{X}$. Replacing (x, y, z, w) in (2.10), by $(0,0,0,0)$, $(x,0,0,0)$, $(\frac{1}{2}x, \frac{1}{2}x, 0, 0)$ and $\Big(\frac{1}{2}x, -\frac{1}{2}x, y, 0\Big)$, respectively, we get $f(0) = 0$, $f(x) = f(-x)$, $f(\frac{1}{2}x) = \frac{1}{4}f(x)$ and

$$f(-y) + f(y) + f(x+y) + f(y-x) = 4\Big[f\Big(\frac{1}{2}x\Big) + f\Big(\frac{-1}{2}x\Big) + f(y)\Big]$$

for all $x, y \in \mathcal{X}$, respectively, which can easily be simplified to (2.2).

Now we assume that $n \geq 5$.

Replacing the variables in (2.1), by $(0, \cdots, 0)$, $(x, 0, \cdots, 0)$, $(x, x, 0, \cdots, 0)$, $(-x, \cdots, -x, 0, 0)$ and $(\frac{2}{n}x, \cdots, \frac{2}{n}x)$, respectively, we have $f(0) = 0$ and

$$\begin{aligned} f\Big(\frac{2-n}{2}x\Big) &= \frac{(n-2)^2}{4}f(x), & (2.11)\\ f\Big(\frac{4-n}{2}x\Big) &= \frac{n^2}{4}f(x) + \frac{2-n}{2}f(2x), & (2.12)\\ f\Big(\frac{4-n}{2}x\Big) &= \frac{n^2}{4}f(-x) + \frac{2}{2-n}f\Big((2-n)x\Big), & (2.13)\\ f\Big(\frac{2}{n}x\Big) &= \frac{4}{n^2}f(x) & (2.14) \end{aligned}$$

for all $x \in \mathcal{X}$, respectively. Note that $n \geq 5$ might be either an odd or an even number. In the cases of oddness and evenness, if we respectively put

$$\begin{aligned} \Big(x_1, \cdots, x_{\frac{n-1}{2}}, x_{\frac{n+1}{2}}, \cdots, x_{n-1}, x_n\Big) &= \Big(\frac{2}{n}x, \cdots, \frac{2}{n}x, \frac{-2}{n}x, \cdots, \frac{-2}{n}x, y\Big), \\ \Big(x_1, \cdots, x_{\frac{n-2}{2}}, x_{\frac{n}{2}}, \cdots, x_{n-2}, x_{n-1}, x_n\Big) &= \Big(\frac{2}{n}x, \cdots, \frac{2}{n}x, \frac{-2}{n}x, \cdots, \frac{-2}{n}x, y, 0\Big) \end{aligned}$$

in (2.1), then we get

$$\begin{aligned} f\Big(\frac{2-n}{2}y\Big) + \frac{n-1}{2}\Big[f(x+y) + f(y-x)\Big] &= \frac{n^2(n-1)}{8}\Big[f\Big(\frac{2}{n}x\Big) + f\Big(\frac{-2}{n}x\Big)\Big] + \frac{n^2}{4}f(y), \\ f\Big(\frac{2-n}{2}y\Big) + \frac{n-2}{2}\Big[f(x+y) + f(y-x)\Big] &= \frac{n^2(n-2)}{8}\Big[f\Big(\frac{2}{n}x\Big) + f\Big(\frac{-2}{n}x\Big)\Big] + \frac{n^2-4}{4}f(y) \end{aligned}$$

for all $x, y \in \mathcal{X}$, which both by using (2.11) and (2.14), are easily simplified to (2.8). From (2.8), we obtain (2.9) again. By using (2.9), we can rewrite (2.12) as

$$f\Big(\frac{4-n}{2}x\Big) = \frac{n^2-6n+12}{4}f(x) + \frac{2-n}{2}f(-x) \tag{2.15}$$

for all $x \in \mathcal{X}$. Subtracting (2.13) from (2.15), we get

$$f\Big((2-n)x\Big) = \frac{(2-n)(n^2-6n+12)}{8}f(x) + \frac{(n-2)(n^2+2n-4)}{8}f(-x)$$

for all $x \in \mathcal{X}$. Putting $\frac{x}{2}$ instead of x, and then applying (2.11), we obtain

$$\begin{aligned} \frac{(n-2)^2}{4}f(x) &= \frac{(2-n)(n^2-6n+12)}{8}f\Big(\frac{x}{2}\Big) + \frac{(n-2)(n^2+2n-4)}{8}f\Big(\frac{-x}{2}\Big), \\ f(x) &= \frac{-n^2+6n-12}{2(n-2)}f\Big(\frac{x}{2}\Big) + \frac{n^2+2n-4}{2(n-2)}f\Big(\frac{-x}{2}\Big) \end{aligned} \tag{2.16}$$

for all $x \in \mathcal{X}$. On the other hand from (2.9), we have

$$f(x) = 3f\Big(\frac{x}{2}\Big) + f\Big(\frac{-x}{2}\Big) \tag{2.17}$$

for all $x \in \mathcal{X}$. Comparing (2.16) and (2.17), we conclude that $f(x) = f(-x)$ for all $x \in \mathcal{X}$, which simply transforms the form of (2.8) to (2.2).

Necessity. For the *'if'* part of the proof, suppose that $f : \mathcal{X} \to \mathcal{Y}$ satisfies the functional equation (2.2). We show that f satisfies (2.1) too.

First, we prove the following

$$\begin{aligned} f\Big(x_2+\cdots+x_n+\frac{2-n}{2}x_1\Big) &= \frac{n}{2}f(x_2+\cdots+x_n) + \Big(\frac{2-n}{2}\Big)f(x_1+\cdots+x_n) \\ &+ \frac{n}{2}\Big(\frac{n}{2}-1\Big)f(x_1) \end{aligned} \tag{2.18}$$

for all $x_1,\cdots,x_n \in \mathcal{X}$ and any fixed integer $n \geq 3$.

Let $k \in \mathbb{N}$. Replacing (x,y) in (2.2) by (x, kx_1) respectively, we get

$$f(x-x_1) = 2f(x)+2f(x_1)-f(x+x_1), \tag{2.19}$$

$$f(x-2x_1) = 2f(x)+2f(2x_1)-f(x+2x_1), \tag{2.20}$$

$$f(x-3x_1) = 2f(x)+2f(3x_1)-f(x+3x_1), \tag{2.21}$$

$$f(x-4x_1) = 2f(x)+2f(4x_1)-f(x+4x_1) \tag{2.22}$$

$$\vdots$$

for all $x, x_1, \in \mathcal{X}$. Replacing (x,y) in (2.2) by (x_1+x, kx_1), respectively, we get

$$\begin{aligned} f(x+2x_1) = f(x_1+x+x_1) &= 2f(x_1+x)+2f(x_1)-f(x), \\ f(x+3x_1) = f(x_1+x+2x_1) &= 2f(x_1+x)+2f(2x_1)-f(x-x_1), \\ f(x+4x_1) = f(x_1+x+3x_1) &= 2f(x_1+x)+2f(3x_1)-f(x-2x_1), \\ &\vdots \end{aligned}$$

for all $x, x_1, \in \mathcal{X}$. Continuous process of the above equations (2.20), (2.21), $\cdots$ and Lemma 2.1 generally lead to

$$f(x-kx_1) = (k+1)f(x)+k(k+1)f(x_1)-kf(x+x_1) \tag{2.23}$$

for all $x, x_1 \in \mathcal{X}$, and all $k \in \mathbb{N}$. Replacing (x,y) in (2.2), by $\big(x-\frac{k}{2}x_1, \frac{k}{2}x_1\big)$, using (2.23) and Lemma 2.1, we obtain

$$\begin{aligned} f\Big(x-\frac{k}{2}x_1\Big) &= \frac{1}{2}f(x)+\frac{1}{2}f(x-kx_1)-f\Big(\frac{k}{2}x_1\Big) \\ &= \frac{k+2}{2}f(x)+\frac{k^2+2k}{4}f(x_1)-\frac{k}{2}f(x+x_1) \end{aligned} \tag{2.24}$$

for all $x, x_1 \in \mathcal{X}$, and all $k \in \mathbb{N}$.

Now (2.23) and (2.24) imply that (2.18) holds. The first one proves it for any fixed even integer $n_e \geq 4$, and the second one proves it for any fixed odd integer $n_o \geq 3$. It is done simply by putting $x = x_2 + \cdots + x_n$ in both (2.23) and (2.24) and by $k = \frac{n_e-2}{2}$, $k = n_o - 2$ in (2.23) and (2.24), respectively.

It follows from (2.18) that

$$\begin{matrix} f\Big(x_2 + \cdots + x_n + \frac{2-n}{2}x_1\Big) \\ + \\ \vdots \\ + \\ f\Big(x_1 + \cdots + x_{n-1} + \frac{2-n}{2}x_n\Big) \end{matrix} = \frac{n}{2}\begin{bmatrix} f(x_2 + \cdots + x_n) \\ + \\ \vdots \\ + \\ f\Big(x_1 + \cdots + x_{n-1}\Big) \end{bmatrix} + \frac{n}{2}\Big(\frac{n}{2} - 1\Big)\begin{bmatrix} f(x_1) \\ + \\ \vdots \\ + \\ f(x_n) \end{bmatrix}$$
$$+ \frac{n}{2}(2-n)\Big[f(x_1 + \cdots + x_n)\Big]$$

for all $x_1, \cdots, x_n \in \mathcal{X}$ and any fixed integer $n \geq 3$. This signifies that in order to get (2.1), it is just necessary to show the following

$$\begin{bmatrix} f(x_2 + \cdots + x_n) \\ + \\ \vdots \\ + \\ f\Big(x_1 + \cdots + x_{n-1}\Big) \end{bmatrix} + (2-n)\Big[f(x_1 + \cdots + x_n)\Big] = \begin{bmatrix} f(x_1) \\ + \\ \vdots \\ + \\ f(x_n) \end{bmatrix} \tag{2.25}$$

for all $x_1, \cdots, x_n \in \mathcal{X}$ and any fixed integer $n \geq 3$.

As it is clear, the proof of (2.25) directly depends on the specific value of $n \geq 3$. Nevertheless, we try to provide a general idea to prove it.

First assume that $n \geq 3$ is an odd number. In this case, by frequently using (2.2), the left hand side of (2.25) will be in the form

$$\frac{1}{2}\begin{bmatrix} f(x_1 + x_2 + 2x_3 + \cdots + 2x_n) \\ + \\ \vdots \\ + \\ f\Big(2x_1 + \cdots + 2x_{n-3} + x_{n-2} + x_{n-1} + 2x_n\Big) \end{bmatrix} + \frac{1}{2}\begin{bmatrix} f(x_1 - x_2) \\ + \\ \vdots \\ + \\ f(x_{n-2} - x_{n-1}) \end{bmatrix}$$
$$+f\Big(x_1 + \cdots + x_{n-1}\Big) + (2-n)\Big[f(x_1 + \cdots + x_n)\Big] \tag{2.26}$$

for all $x_1, \cdots, x_n \in \mathcal{X}$ and any fixed odd number $n \geq 3$. Since we have

$$\frac{1}{2}f(x_1 + x_2 + 2x_3 + \cdots + 2x_n) = f(x_1 + \cdots + x_n) + f(x_3 + \cdots + x_n) - \frac{1}{2}f(x_1 + x_2)$$

for all $x_1, \cdots, x_n \in \mathcal{X}$ from (2.2), (2.26) is simplified to

$$\begin{bmatrix} f(x_3 + \cdots + x_n) \\ + \\ \vdots \\ + \\ f\Big(x_1 + \cdots + x_{n-3} + x_n\Big) \end{bmatrix} + \frac{1}{2}\begin{bmatrix} f(x_1 - x_2) - f(x_1 + x_2) \\ + \\ \vdots \\ + \\ f(x_{n-2} - x_{n-1}) - f(x_{n-2} + x_{n-1}) \end{bmatrix}$$
$$+f(x_1 + \cdots + x_{n-1}) + \Big(\frac{3-n}{2}\Big)\Big[f(x_1 + \cdots + x_n)\Big] \tag{2.27}$$

for all $x_1, \cdots, x_n \in \mathcal{X}$ and any odd fixed integer $n \geq 3$. For the case $n = 3$, (2.27) leads to the right hand side of (2.25), which means that the proof is complete for the case $n = 3$. So we assume that $n \geq 5$ and continue the proof. We come across two cases:

a) $\frac{n-1}{2}$, which is the number of the terms in the first term of (2.27), is an even integer;

b) $\frac{n-1}{2}$ is an odd integer.

In the case *a*), similar to the process in which we obtain (2.27) from (2.25), we get (2.28) from (2.27), as follows:

$$\begin{aligned}
&\begin{bmatrix} f(x_5+\cdots+x_n) \\ + \\ \vdots \\ + \\ f\Big(x_1+\cdots+x_{n-5}+x_n\Big) \end{bmatrix} + \frac{1}{2}\begin{bmatrix} f(x_1-x_2)-f(x_1+x_2) \\ + \\ \vdots \\ + \\ f(x_{n-2}-x_{n-1})-f(x_{n-2}+x_{n-1}) \end{bmatrix} \\
&+\frac{1}{2}\begin{bmatrix} f(x_1+x_2-x_3-x_4)-f(x_1+\cdots+x_4) \\ + \\ \vdots \\ + \\ f(x_{n-4}+x_{n-3}-x_{n-2}-x_{n-1})-f(x_{n-4}+\cdots+x_{n-1}) \end{bmatrix} \\
&+f(x_1+\cdots+x_{n-1})+\Big(\frac{5-n}{4}\Big)\Big[f(x_1+\cdots+x_n)\Big] \qquad (2.28)
\end{aligned}$$

for all $x_1,\cdots,x_n\in\mathcal{X}$ and any fixed $n=5,9,13,\cdots$. In the case $n=5$, (2.28) is in the form

$$f(x_5)+\frac{1}{2}\begin{bmatrix} f(x_1-x_2)-f(x_1+x_2) \\ + \\ f(x_3-x_4)-f(x_3+x_4) \end{bmatrix}+\frac{1}{2}\Big[f(x_1+x_2-x_3-x_4)+f(x_1+\cdots+x_4)\Big]$$

for all $x_1,\cdots,x_5\in\mathcal{X}$, which simply by using (2.2) gives the right hand side of (2.25). By continuing the process we can obtain the result for $n=9,13,\cdots$.

Similarly in the case *b*), we get (2.29) from (2.27), as follows:

$$\begin{aligned}
&\begin{bmatrix} f(x_5+\cdots+x_n) \\ + \\ \vdots \\ + \\ f\Big(x_1+\cdots+x_{n-7}+x_{n-2}+x_{n-1}+x_n\Big) \end{bmatrix} + \frac{1}{2}\begin{bmatrix} f(x_1-x_2)-f(x_1+x_2) \\ + \\ \vdots \\ + \\ f(x_{n-2}-x_{n-1})-f(x_{n-2}+x_{n-1}) \end{bmatrix} \\
&+\frac{1}{2}\begin{bmatrix} f(x_1+x_2-x_3-x_4)-f(x_1+\cdots+x_4) \\ + \\ \vdots \\ + \\ f(x_{n-6}+x_{n-5}-x_{n-4}-x_{n-3})-f(x_{n-6}+\cdots+x_{n-3}) \end{bmatrix} \\
&+f(x_1+\cdots+x_{n-3}+x_n)+f(x_1+\cdots+x_{n-1})+\Big(\frac{3-n}{4}\Big)\Big[f(x_1+\cdots+x_n)\Big] \qquad (2.29)
\end{aligned}$$

for all $x_1,\cdots,x_n\in\mathcal{X}$ and any fixed $n=7,11,15,\cdots$. If we put $n=7$, in (2.29), then we have

$$\begin{aligned}
&f(x_5+x_6+x_7)+\frac{1}{2}\begin{bmatrix} f(x_1-x_2)-f(x_1+x_2) \\ + \\ f(x_3-x_4)-f(x_3+x_4) \\ + \\ f(x_5-x_6)-f(x_5+x_6) \end{bmatrix}+\frac{1}{2}f(x_1+x_2-x_3-x_4) \\
&-\frac{1}{2}f(x_1+\cdots+x_4)+f(x_1+\cdots+x_4+x_7)+f(x_1+\cdots+x_6)-f(x_1+\cdots+x_7)
\end{aligned}$$

for all $x_1,\cdots,x_7\in\mathcal{X}$, which could be simplified to the right hand side of (2.25). For $n=11,15,\cdots$, we should continue the process.

Even cases of n are similar and also easier and so we omit them and the proof is complete. □

3. Superstability of the general quadratic functional equation (0.1)

In this section, we provide a superstability theorem for the functional equation (0.1). In fact, $f : \mathcal{X} \to \mathcal{Y}$ will be put in a normed functional inequality instead of an equality which is obviously considered a harder condition for f, in order to be quadratic. From this point of view, one can say that the obtained result in the previous section will be gotten stronger and improved in this section.

Theorem 3.1. *Let $\mathcal{X}$ and $\mathcal{Y}$ be linear spaces and $\frac{1}{6} < l < 1$ be a fixed real number. If $f : \mathcal{X} \to \mathcal{Y}$ satisfies the functional inequality*

$$\begin{aligned} \Big\| f\Big(y+z-\frac{1}{2}x\Big) + f\Big(x+z-\frac{1}{2}y\Big) + f\Big(x+y-\frac{1}{2}z\Big) & \\ -\frac{9}{4}f(y) - \frac{9}{4}f(z) - l\frac{9}{4}f(x)\Big\| & \leq \Big\|(1-l)\frac{9}{4}f(x)\Big\| \end{aligned} \tag{3.1}$$

for all $x, y, z \in \mathcal{X}$, then f is a quadratic mapping.

Proof. Letting $(x, y, z) = (0, 0, 0)$ in (3.1), we get

$$\frac{18l-3}{4}\big\|f(0)\big\| \leq 0.$$

Since $l > \frac{1}{6}$, $\frac{18l-3}{4} > 0$ and so $f(0) = 0$. Letting $(x, y, z) = (0, x, y)$ in (3.1), we have

$$f\Big(x-\frac{1}{2}y\Big) + f\Big(y-\frac{1}{2}x\Big) + f(x+y) = \frac{9}{4}f(x) + \frac{9}{4}f(y) \tag{3.2}$$

for all $x, y \in \mathcal{X}$. Replacing (x, y) in (3.2) by $(x, 0)$, (x, x) and $(x, 2x)$, respectively, we obtain

$$f\Big(-\frac{1}{2}x\Big) = \frac{1}{4}f(x), \tag{3.3}$$

$$2f\Big(\frac{1}{2}x\Big) + f(2x) = \frac{9}{2}f(x), \tag{3.4}$$

$$f\Big(\frac{3}{2}x\Big) + f(3x) = \frac{9}{4}f(x) + \frac{9}{4}f(2x) \tag{3.5}$$

for all $x \in \mathcal{X}$, respectively. Using (3.3) and (3.4) we get $f(x) = f(-x)$ for all $x \in \mathcal{X}$. So (3.3) is rewritten as $f\big(\frac{1}{2}x\big) = \frac{1}{4}f(x)$ for all $x \in \mathcal{X}$. By using this, (3.5) could be simplified to $f(3x) = 9f(x)$, and so

$$f\Big(\frac{1}{3}x\Big) = \frac{1}{9}f(x) \tag{3.6}$$

for all $x \in \mathcal{X}$. Replacing (x, y) in (3.2) by $(x-y, -y)$ and $(y-x, -x)$, we have

$$\begin{aligned} f\Big(x-\frac{1}{2}y\Big) &= \frac{9}{4}f(x-y) + \frac{9}{4}f(y) - \frac{1}{4}f(x+y) - f(x-2y), \\ f\Big(y-\frac{1}{2}x\Big) &= \frac{9}{4}f(x-y) + \frac{9}{4}f(x) - \frac{1}{4}f(x+y) - f(y-2x) \end{aligned}$$

for all $x, y \in \mathcal{X}$. By putting these two equations in (3.2), we obtain

$$\frac{9}{2}f(x-y) + \frac{1}{2}f(x+y) = f(x-2y) + f(y-2x)$$

for all $x, y \in \mathcal{X}$. Now putting $(x, y) = \big(\frac{u+2v}{-3}, \frac{v+2u}{-3}\big)$ in the previous line, we get

$$\frac{9}{2}f\Big(\frac{u-v}{3}\Big) + \frac{1}{2}f(u+v) = f(u) + f(v)$$

for all $u, v \in \mathcal{X}$, which by (3.6) simply leads to (2.2) as desired. □

Theorem 3.2. *Let $\mathcal{X}$ and $\mathcal{Y}$ be linear spaces and n, k be fixed positive integers with $n \geq 4$ and $1 \leq k \leq n$. If $f : \mathcal{X} \to \mathcal{Y}$ satisfies the functional inequality*

$$\Big\|\sum_{i=1}^{n} f\Big(\sum_{\substack{j=1 \\ j\neq i}}^{n} x_j + \frac{2-n}{2}x_i\Big) - \frac{n^2}{4}\sum_{\substack{i=1 \\ i\neq k}}^{n} f(x_i)\Big\| \leq \Big\|\frac{n^2}{4}f(x_k)\Big\| \tag{3.7}$$

for all $x_1, x_2, \cdots, x_n \in \mathcal{X}$, then f is a quadratic mapping.

Proof. Note that the functional inequality (3.7) is symmetric with respect to each variable. So we can take $k=1$ and only prove this case and then conclude the statement for all cases with $1 \le k \le n$. From now on, assume that $k=1$.

Letting $(x_1, x_2, \cdots, x_n) = (0, 0, \cdots, 0)$ in (3.7), we obtain that

$$\frac{n^3-(2n^2+4n)}{4}\big\|f(0)\big\| \le 0.$$

Since $n^3 > (2n^2+4n)$ for all $n \ge 4$, $f(0)=0$. Letting $x_1 = 0$ (or $x_k = 0$) in (3.7), we get

$$\sum_{i=2}^{n} f\Big(\sum_{\substack{j=2\\ j\neq i}}^{n} x_j + \frac{2-n}{2}x_i\Big) + f\Big(\sum_{i=2}^{n} x_i\Big) = \frac{n^2}{4}\sum_{i=2}^{n} f(x_i) \tag{3.8}$$

for all $x_2, x_3, \cdots, x_n \in \mathcal{X}$.

First we investigate the case $n=4$ separately. In this case, by putting $(x_2,x_3,x_4) = (x,y,z)$ in (3.8), we have

$$\begin{aligned} &f\big(y+z-x\big) + f\big(x+z-y\big) + f\big(x+y-z\big) \\ &\qquad + f\big(x+y+z\big) = 4[f(x)+f(y)+f(z)] \end{aligned}$$

for all $x,y,z \in \mathcal{X}$. Replacing (x,y,z) in the above equation by $(x,0,0)$ and $(x,y,0)$, we obtain $f(x) = f(-x)$ and

$$f(y-x) + f(x-y) + 2f(x+y) = 4f(x) + 4f(y)$$

for all $x,y \in \mathcal{X}$, which simply mean that (2.2) holds.

Now the case $n \ge 5$. Replacing $(x_2, x_3, \cdots, x_n)$ in (3.8) by $= (x, 0, \cdots, 0)$, $(x, x, 0, \cdots, 0)$ and $(x, \cdots, x, 0)$, respectively, we obtain

$$f\Big(\frac{2-n}{2}x\Big) = \frac{(n-2)^2}{4}f(x), \tag{3.9}$$

$$f\Big(\frac{4-n}{2}x\Big) = \frac{n^2}{4}f(x) + \frac{2-n}{2}f(2x), \tag{3.10}$$

$$f\Big(\frac{n-4}{2}x\Big) = \frac{n^2}{4}f(x) + \frac{2}{2-n}f\Big((n-2)x\Big) \tag{3.11}$$

for all $x \in \mathcal{X}$, respectively. In the case of evenness and oddness of $n \ge 5$, we respectively put

$$\begin{aligned} \Big(x_2, \cdots, x_{\frac{n}{2}}, x_{\frac{n+2}{2}}, \cdots, x_{n-1}, x_n\Big) &= \Big(\frac{2}{n}x, \cdots, \frac{2}{n}x, \frac{-2}{n}x, \cdots, \frac{-2}{n}x, y\Big), \\ \Big(x_2, \cdots, x_{\frac{n-1}{2}}, x_{\frac{n+1}{2}}, \cdots, x_{n-2}, x_{n-1}, x_n\Big) &= \Big(\frac{2}{n}x, \cdots, \frac{2}{n}x, \frac{-2}{n}x, \cdots, \frac{-2}{n}x, y, 0\Big) \end{aligned}$$

in (3.8), to get

$$\begin{aligned} f\Big(\frac{2-n}{2}y\Big) + \frac{n-2}{2}\Big[f(x+y)+f(y-x)\Big] &= \frac{n^2(n-2)}{8}\Big[f\Big(\frac{2}{n}x\Big)+f\Big(\frac{-2}{n}x\Big)\Big] + \frac{n^2-4}{4}f(y), \\ f\Big(\frac{2-n}{2}y\Big) + \frac{n-3}{2}\Big[f(x+y)+f(y-x)\Big] &= \frac{n^2(n-3)}{8}\Big[f\Big(\frac{2}{n}x\Big)+f\Big(\frac{-2}{n}x\Big)\Big] + \frac{n^2-8}{4}f(y) \end{aligned}$$

for all $x,y \in \mathcal{X}$, which both by (3.9) are simplified to

$$f(x+y) + f(y-x) = \frac{n^2}{4}\Big[f\Big(\frac{2}{n}x\Big)+f\Big(\frac{-2}{n}x\Big)\Big] + 2f(y) \tag{3.12}$$

for all $x,y \in \mathcal{X}$. Letting $y=0$ in (3.12), we have

$$f(x) + f(-x) = \frac{n^2}{4}\Big[f\Big(\frac{2}{n}x\Big)+f\Big(\frac{-2}{n}x\Big)\Big]$$

for all $x \in \mathcal{X}$. By this, (3.12) is equivalent to

$$f(x+y) + f(y-x) = f(x) + f(-x) + 2f(y) \tag{3.13}$$

for all $x, y \in \mathcal{X}$. Now by (3.10), (3.11), (3.13) and a similar argument used in the last part of the proof of Theorem 3.2, we can obtain $f(x) = f(-x)$ for all $x, y \in \mathcal{X}$. This changes (3.13) to $f(x+y) + f(y-x) = 2f(x) + 2f(y)$ for all $x, y \in \mathcal{X}$, which finally ends the proof. □

4. **Hyers-Ulam stability of the general quadratic functional equation (0.1)**

In this section, we prove the Hyers-Ulam stability of the functional equation (0.1). Throughout this section $\mathcal{X}$ denotes a quasi-α-normed space and $\mathcal{Y}$ a quasi-β-Banach space. For a given mapping $f : \mathcal{X} \to \mathcal{Y}$, we define the difference operator:

$$D_\lambda f(x_1, x_2, \cdots, x_n) := \sum_{i=1}^{n} f\Big(\sum_{\substack{j=1\\ j\neq i}}^{n} \lambda x_j + \frac{2-n}{2}\lambda x_i\Big) - \frac{\lambda^2 n^2}{4}\sum_{i=1}^{n} f(x_i)$$

for all $x_1, x_2, \cdots, x_n \in \mathcal{X}$ and all $\lambda \in \mathbb{R}$.

Theorem 4.1. *Let $\varphi : \mathcal{X}^n \to [0, \infty)$ be a function with $\varphi(0, \cdots, 0) = 0$, where $n \geq 3$ is a fixed integer. Denote by ϕ a function such that*

$$\phi(x_1, x_2, \cdots, x_n) := \sum_{m=0}^{\infty}\Big[\frac{4^{m\beta}}{n^{2m\beta}}\varphi\Big(\frac{n^m}{2^m}x_1, \frac{n^m}{2^m}x_2, \cdots, \frac{n^m}{2^m}x_n\Big)\Big]^p < \infty \tag{4.1}$$

for all $x_1, x_2, \cdots, x_n \in \mathcal{X}$. Suppose that $f : \mathcal{X} \to \mathcal{Y}$ is a mapping satisfying

$$\big\|D_1 f(x_1, x_2, \cdots, x_n)\big\|_{\mathcal{Y}} \leq \varphi(x_1, x_2, \cdots, x_n) \tag{4.2}$$

for all $x_1, x_2, \cdots, x_n \in \mathcal{X}$. Then there exists a unique quadratic mapping $\mathcal{Q} : \mathcal{X} \to \mathcal{Y}$ such that

$$\big\|f(x) - \mathcal{Q}(x)\big\|_{\mathcal{Y}} \leq \frac{4^\beta}{n^{3\beta}}\sqrt[p]{\phi(x, x, \cdots, x)} \tag{4.3}$$

for all $x \in \mathcal{X}$.

Proof. Letting $x_1 = x_2 = \cdots = x_n = 0$ in (4.2), we get $f(0) = 0$.

Letting $x_1 = x_2 = \cdots = x_n = x$ in (4.2), we have

$$\begin{aligned}\Big\|nf\Big(\frac{n}{2}x\Big) - \frac{n^3}{4}f(x)\Big\|_{\mathcal{Y}} &\leq \varphi(x, x, \cdots, x),\\ \Big\|\frac{4}{n^2}f\Big(\frac{n}{2}x\Big) - f(x)\Big\|_{\mathcal{Y}} &\leq \frac{4^\beta}{n^{3\beta}}\varphi(x, x, \cdots, x)\end{aligned}$$

for all $x \in \mathcal{X}$. Replacing x by $(\frac{n}{2})^i x$, we get

$$\Big\|\frac{4}{n^2}f\Big(\frac{n^{i+1}}{2^{i+1}}x\Big) - f\Big(\frac{n^i}{2^i}x\Big)\Big\|_{\mathcal{Y}} \leq \frac{4^\beta}{n^{3\beta}}\varphi\Big(\frac{n^i}{2^i}x, \cdots, \frac{n^i}{2^i}x\Big) \tag{4.4}$$

for all $x \in \mathcal{X}$ and all nonnegative integers i. Assume that m, l are positive integers with $m > l$. From the iterative method and (4.4), it follows that

$$\begin{aligned}\Big\|\frac{4^m}{n^{2m}}f\Big(\frac{n^m}{2^m}x\Big) - \frac{4^l}{n^{2l}}f\Big(\frac{n^l}{2^l}x\Big)\Big\|_{\mathcal{Y}}^p &\leq \sum_{i=l}^{m-1}\Big\|\frac{4^{i+1}}{n^{2i+2}}f\Big(\frac{n^{i+1}}{2^{i+1}}x\Big) - \frac{4^i}{n^{2i}}f\Big(\frac{n^i}{2^i}x\Big)\Big\|_{\mathcal{Y}}^p\\ &= \sum_{i=l}^{m-1}\frac{4^{i\beta p}}{n^{2i\beta p}}\Big\|\frac{4}{n^2}f\Big(\frac{n^{i+1}}{2^{i+1}}x\Big) - f\Big(\frac{n^i}{2^i}x\Big)\Big\|_{\mathcal{Y}}^p\\ &\leq \frac{4^{\beta p}}{n^{3\beta p}}\sum_{i=l}^{m-1}\Big[\frac{4^{i\beta}}{n^{2i\beta}}\varphi\Big(\frac{n^i}{2^i}x, \cdots, \frac{n^i}{2^i}x\Big)\Big]^p\end{aligned} \tag{4.5}$$

for all $x \in \mathcal{X}$, in which by (4.1) the right-hand side tends to zero as $m, l \to \infty$. This clarifies that the sequence $\Big\{\frac{4^m}{n^{2m}}f\left(\frac{n^m}{2^m}x\right)\Big\}$ is Cauchy in the complete space $\mathcal{Y}$ and therefore convergent. So we can define for all $x \in \mathcal{X}$, the mapping $\mathcal{Q} : \mathcal{X} \to \mathcal{Y}$ by

$$\mathcal{Q}(x) := \lim_{m\to\infty}\frac{4^m}{n^{2m}}f\Big(\frac{n^m}{2^m}x\Big).$$

Now letting $l = 0$, passing the limit $m \to \infty$ in (4.5) and then using (4.1), we obtain (4.3), as desired.

Lastly, we prove that $\mathcal{Q}$ is unique. Let $\mathcal{Q}' : \mathcal{X} \to \mathcal{Y}$ be another quadratic mapping satisfying (4.3). Then we have

$$\begin{aligned}
\big\|\mathcal{Q}(x) - \mathcal{Q}'(x)\big\|_{\mathcal{Y}}^p &\leq \frac{4^{m\beta p}}{n^{2m\beta p}}\Big\|\mathcal{Q}\Big(\frac{n^m}{2^m}x\Big) - f\big(\frac{n^m}{2^m}x\big)\Big\|_{\mathcal{Y}}^p + \frac{4^{m\beta p}}{n^{2m\beta p}}\Big\|\mathcal{Q}'\Big(\frac{n^m}{2^m}x\Big) - f\Big(\frac{n^m}{2^m}x\Big)\Big\|_{\mathcal{Y}}^p \\
&\leq 2 \cdot \frac{4^{m\beta p}}{n^{2m\beta p}} \cdot \frac{4^{\beta p}}{n^{3\beta p}}\phi\Big(\frac{n^m}{2^m}x, \cdots, \frac{n^m}{2^m}x\Big) \\
&= 2 \cdot \frac{4^{m\beta p}}{n^{2m\beta p}} \cdot \frac{4^{\beta p}}{n^{3\beta p}}\sum_{s=0}^{\infty}\Big[\frac{4^{s\beta}}{n^{2s\beta}}\varphi\Big(\frac{n^{m+s}}{2^{m+s}}x, \cdots, \frac{n^{m+s}}{2^{m+s}}x\Big)\Big]^p \\
&= 2 \cdot \frac{4^{\beta p}}{n^{3\beta p}}\sum_{s=m}^{\infty}\Big[\frac{4^{s\beta}}{n^{2s\beta}}\varphi\Big(\frac{n^s}{2^s}x, \cdots, \frac{n^s}{2^s}x\Big)\Big]^p
\end{aligned}$$

for all $x \in \mathcal{X}$. Now by (4.1), the right-hand side tends to zero as $m \to \infty$, and therefore $\mathcal{Q} : \mathcal{X} \to \mathcal{Y}$ is unique and the proof is complete. $\square$

Theorem 4.2. *Let $\varphi : \mathcal{X}^n \to [0, \infty)$ be a function with $\varphi(0, \cdots, 0) = 0$, where $n \geq 3$ is a fixed integer. Denote by ϕ a function such that*

$$\phi(x_1, x_2, \cdots, x_n) := \sum_{m=0}^{\infty}\Big[\frac{n^{2m\beta}}{4^{m\beta}}\varphi\Big(\frac{2^{m+1}}{n^{m+1}}x_1, \cdots, \frac{2^{m+1}}{n^{m+1}}x_n\Big)\Big]^p < \infty \tag{4.6}$$

for all $x_1, x_2, \cdots, x_n \in \mathcal{X}$. Suppose that $f : \mathcal{X} \to \mathcal{Y}$ is a mapping satisfying (4.2). Then there exists a unique quadratic mapping $\mathcal{Q} : \mathcal{X} \to \mathcal{Y}$ such that

$$\big\|f(x) - \mathcal{Q}(x)\big\|_{\mathcal{Y}} \leq \frac{1}{n^\beta}\sqrt[p]{\phi(x, x, \cdots, x)} \tag{4.7}$$

for all $x \in \mathcal{X}$.

Proof. Letting $x_1 = x_2 = \cdots = x_n = 0$ in (4.2), we get $f(0) = 0$.

Letting $x_1 = x_2 = \cdots = x_n = \frac{2}{n}x$ in (4.2), we have

$$\begin{aligned}
\Big\|nf(x) - \frac{n^3}{4}f\Big(\frac{2}{n}x\Big)\Big\|_{\mathcal{Y}} &\leq \varphi\Big(\frac{2}{n}x, \frac{2}{n}x, \cdots, \frac{2}{n}x\Big), \\
\Big\|\frac{n^2}{4}f\Big(\frac{2}{n}x\Big) - f(x)\Big\|_{\mathcal{Y}} &\leq \frac{1}{n^\beta}\varphi\Big(\frac{2}{n}x, \frac{2}{n}x, \cdots, \frac{2}{n}x\Big)
\end{aligned}$$

for all $x \in \mathcal{X}$. By the same method used in the previous theorem, we can obtain

$$\Big\|\frac{n^{2m}}{4^m}f\Big(\frac{2^m}{n^m}x\Big) - \frac{n^{2l}}{4^l}f\Big(\frac{2^l}{n^l}x\Big)\Big\|_{\mathcal{Y}}^p \leq \frac{1}{n^{\beta p}}\sum_{i=l}^{m-1}\Big[\frac{n^{2i\beta}}{4^{i\beta}}\varphi\Big(\frac{2^{i+1}}{n^{i+1}}x, \cdots, \frac{2^{i+1}}{n^{i+1}}x\Big)\Big]^p \tag{4.8}$$

for positive integers m, l with $m > l$ and all $x \in \mathcal{X}$, in which by (4.6) the right-hand side tends to zero as $m, l \to \infty$.

Now similar to the pervious theorem, the mapping $\mathcal{Q} : \mathcal{X} \to \mathcal{Y}$ is definable as

$$\mathcal{Q}(x) := \lim_{m\to\infty}\frac{n^{2m}}{4^m}f\Big(\frac{2^m}{n^m}x\Big)$$

for all $x \in \mathcal{X}$, which by letting $l = 0$, passing the limit $m \to \infty$ in (4.8) and then using (4.6), satisfies (4.7).

The proof of the uniqueness of $\mathcal{Q}$ is similar to the previous theorem. $\square$

Corollary 4.3. *Let ϑ be a nonnegative real number and q a positive real number with $q < 2\frac{\beta}{\alpha}$. Let $f : \mathcal{X} \to \mathcal{Y}$ be a mapping satisfying*

$$\big\|D_1f(x_1, x_2, \cdots, x_n)\big\|_{\mathcal{Y}} \leq \vartheta\big(\|x_1\|_{\mathcal{X}}^q + \cdots + \|x_n\|_{\mathcal{X}}^q\big) \tag{4.9}$$

for all $x_1, x_2, \cdots, x_n \in \mathcal{X}$. Then there exists a unique quadratic mapping $\mathcal{Q} : \mathcal{X} \to \mathcal{Y}$ such that

$$\big\|f(x) - \mathcal{Q}(x)\big\|_{\mathcal{Y}} \leq 4^\beta\vartheta\frac{n^{1-\alpha q-\beta}}{\sqrt[p]{n^{p(2\beta-\alpha q)} - 2^{p(2\beta-\alpha q)}}}\|x\|_{\mathcal{X}}^q$$

for all $x \in \mathcal{X}$.

Proof. Defining $\varphi(x_1, \cdots, x_n) := \vartheta\big(\|x_1\|_{\mathcal{X}}^q + \cdots + \|x_n\|_{\mathcal{X}}^q\big)$ and applying Theorem 4.1, we get the result. □

Corollary 4.4. *Let* ϑ *be a nonnegative real number and* q *a positive real number with* $q > 2\frac{\beta}{\alpha}$. *Let* $f : \mathcal{X} \to \mathcal{Y}$ *be a mapping satisfying* (4.9). *Then there exists a unique quadratic mapping* $\mathcal{Q} : \mathcal{X} \to \mathcal{Y}$ *such that*

$$\big\|f(x) - \mathcal{Q}(x)\big\|_{\mathcal{Y}} \le 2^{\alpha q}\vartheta \frac{n^{1-3\beta}}{\sqrt[p]{n^{p(\alpha q-2\beta)} - 2^{p(\alpha q-2\beta)}}}\|x\|_{\mathcal{X}}^q$$

for all $x \in \mathcal{X}$.

Proof. Defining $\varphi(x_1, \cdots, x_n) := \vartheta\big(\|x_1\|_{\mathcal{X}}^q + \cdots + \|x_n\|_{\mathcal{X}}^q\big)$ and applying Theorem 4.2, we get the result. □

Corollary 4.5. *Let* ϑ *be a nonnegative real number and* $q_1, \ldots, q_n$ *positive real numbers with* $q_1 + \cdots + q_n < 2\frac{\beta}{\alpha}$. *Let* $f : \mathcal{X} \to \mathcal{Y}$ *be a mapping satisfying*

$$\big\|D_1 f(x_1, x_2, \cdots, x_n)\big\|_{\mathcal{Y}} \le \vartheta\big(\|x_1\|_{\mathcal{X}}^{q_1} \cdot \|x_2\|_{\mathcal{X}}^{q_2} \ldots \|x_n\|_{\mathcal{X}}^{q_n}\big) \tag{4.10}$$

for all $x_1, x_2, \cdots, x_n \in \mathcal{X}$. *Then there exists a unique quadratic mapping* $\mathcal{Q} : \mathcal{X} \to \mathcal{Y}$ *such that*

$$\big\|f(x) - \mathcal{Q}(x)\big\|_{\mathcal{Y}} \le \frac{4^{\beta}\vartheta n^{-\alpha(q_1+\cdots+q_n)-\beta}}{\sqrt[p]{n^{p(2\beta-\alpha(q_1+\cdots+q_n))} - 2^{p(2\beta-\alpha(q_1+\cdots+q_n))}}}\|x\|_{\mathcal{X}}^{(q_1+\cdots+q_n)}$$

for all $x \in \mathcal{X}$.

Proof. Defining $\varphi(x_1, \cdots, x_n) := \vartheta\big(\|x_1\|_{\mathcal{X}}^{q_1} \cdot \|x_2\|_{\mathcal{X}}^{q_2} \cdots \|x_n\|_{\mathcal{X}}^{q_n}\big)$ and applying Theorem 4.1, we get the result. □

Corollary 4.6. *Let* ϑ *be a nonnegative real number and* $q_1, \cdots, q_n$ *positive real numbers with* $q_1 + \cdots + q_n > 2\frac{\beta}{\alpha}$. *Let* $f : \mathcal{X} \to \mathcal{Y}$ *be a mapping satisfying* (4.10). *Then there exists a unique quadratic mapping* $\mathcal{Q} : \mathcal{X} \to \mathcal{Y}$ *such that*

$$\big\|f(x) - \mathcal{Q}(x)\big\|_{\mathcal{Y}} \le \frac{2^{\alpha(q_1+\cdots+q_n)}\vartheta n^{-3\beta}}{\sqrt[p]{n^{p(\alpha(q_1+\cdots+q_n)-2\beta)} - 2^{p(\alpha(q_1+\cdots+q_n)-2\beta)}}}\|x\|_{\mathcal{X}}^{(q_1+\cdots+q_n)}$$

for all $x \in \mathcal{X}$.

Proof. Defining $\varphi(x_1, \cdots, x_n) := \vartheta\big(\|x_1\|_{\mathcal{X}}^{q_1} \cdot \|x_2\|_{\mathcal{X}}^{q_2} \cdots \|x_n\|_{\mathcal{X}}^{q_n}\big)$ and applying Theorem 4.2, we get the result. □

Note that in Corollary 4.5 and 4.6, we can put $q_1 = \cdots = q_n = q$ and make simpler results.

Acknowledgments

S. Farhadabadi was supported for this research by the code 95023 by the Young Researchers and Elite club affiliated to Islamic Azad University, C. Park was supported by Basic Science Research Program through the National Research Foundation of Korea funded by the Ministry of Education, Science and Technology (NRF-2017R1D1A1B04032937), and S. Yun was supported by Hanshin University Research Grant.

References

[1] T. Aoki, *On the stability of the linear transformation in Banach spaces*, J. Math. Soc. Japan **2** (1950), 64–66.
[2] C. Borelli, G.L. Forti, *On a general Hyers-Ulam stability result*, Int. J. Math. Math. Sci. **18** (1995), 229–236.
[3] J. Bae, K. Jun, *On the generalized Hyers-Ulam-Rassias stability of an n-dimensional quadratic functional equation*, J. Math. Anal. Appl. **258** (2001), 183–193.
[4] J. Bae, W. Park, *On the generalized Hyers-Ulam-Rassias stability in Banach modules over a C^*-algebra*, J. Math. Anal. Appl. **294** (2004), 196–205.
[5] I. Chang, H. Kim, *On the Hyers-Ulam stability of quadratic functional equations*, J. Inequal. Pure Appl. Math. **3** (2002), Art. 33.
[6] P.W. Cholewa, *Remarks on the stability of functional equations*, Aequationes Math. **27** (1984), 76–86.
[7] S. Czerwik, *On the stability of the quadratic mapping in normed spaces*, Abh. Math. Sem. Univ. Hamburg **62** (1992), 59–64.

[8] S. Czerwik, *The stability of the quadratic functional equation, in: Th.M. Rassias, J. Tabor (Eds.), Stability of Mappings of Hyers-Ulam Type*, Hadronic Press, Florida, 1994, pp. 81–91.

[9] M. Eshaghi Gordji, H. Khodaei, G. Kim, *Nearly quadratic mappings over p-adic fields*, Abs. Appl. Anal. **2012** (2012), Article ID 285807.

[10] S. Farhadabadi, J. Lee, C. Park *A new quadratic functional equation version and its stability and superstability*, J. Comput. Anal. Appl. **23** (2017), 544–552.

[11] S. Farhadabadi, C. Park, D. Shin *Superstability of (r, s, t)-Jordan C^*-homomorphisms*, J. Comput. Anal. Appl. **20** (2016), 121–134.

[12] P. Găvruţa, *A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings*, J. Math. Anal. Appl. **184** (1994), 431–436.

[13] D.H. Hyers, *On the stability of the linear functional equation*, Proc. Natl Acad. Sci. U.S.A. **27** (1941), 222–224.

[14] D.H. Hyers, G. Isac, Th.M. Rassias, *Stability of Functional Equations in Several Variables*, Birkhäuser, Basel, 1998.

[15] S. Jang, C. Park, P. Efteghar, S. Farhadabadi, *Stability and superstability of (f_r, f_s)-double derivations in quasi-Banah algebras*, J. Computat. Anal. Anal. **18** (2015), 973–983.

[16] S. Jung, *On the Hyers-Ulam-Rassias stability of a quadratic functional equation*, J. Math. Anal. Appl. **232** (1999), 384–393.

[17] H. Kim, *On the stability problem for a mixed type of quartic and quadratic functional equation*, J. Math. Anal. Appl. **324** (2006), 358–372.

[18] M. Kim, S. Lee, G. A. Anastassiou, C. Park, *Functional equations in matrix normed modules*, J. Comput. Anal. Appl. **17** (2014), 336–342.

[19] Y. Lee, S. Chung, *Stability for quadratic functional equation in the spaces of generalized functions*, J. Math. Anal. Appl. **336** (2007), 101–110.

[20] J.M. Rassias, *On approximation of approximately linear mappings by linear mappings*, J. Funct. Anal. **46** (1982), 126–130.

[21] J.M. Rassias, *Solution of the Ulam stability problem for Euler-Lagrange quadratic mappings*, J. Math. Anal. Appl. **220** (1998), 613–639.

[22] J.M. Rassias, *The Ulam stability problem in approximation of approximately quadratic mappings by quadratic mappings*, J. Inequal. Pure Appl. Math. **5** (2004), Art. 52.

[23] J.M. Rassias, *On the general quadratic functional equation*, Bol. Soc. Mat. Mexicana (3) **11** (2005), 259–268.

[24] Th.M. Rassias, *On the stability of the linear mapping in Banach spaces*, Proc. Amer. Math. Soc. **72** (1978), 297–300.

[25] Th.M. Rassias, *On the modified Hyers-Ulam sequence*, J. Math. Anal. Appl. **158** (1991), 106–113.

[26] Th.M. Rassias, *On the stability of the quadratic functional equation and its applications*, Stud. Univ. Babes-Bolyai. **18** (1998), 89–124.

[27] Th.M. Rassias, *On the stability of functional equations and a problem of Ulam*, Acta Appl. Math. **62** (2000), 23–130.

[28] K. Ravi, J.M. Rassias, R. Kodandan, *Generalized Ulam-Hyers stability of an AQ-functional equation in quasi-beta-normed spaces*, Mathematica Aeterna **1** (2011), 217–236.

[29] K. Ravi, J.M. Rassias, S. Pinelas, S. Suresh, *General solution and stability of quattuordecic functional equation in quasi β-normed spaces*, Adv. Pure Math. **6** (2016), 921–941.

[30] K. Ravi, E. Thandapani, B. V. Senthil Kumar, *Solution and stability of a reciprocal type functional equation in several variables*, J. Nonlinear Sci. Appl. **7** (2014), 18–27.

[31] D. Shin, C. Park, S. Farhadabadi, *On the superstability of ternary Jordan C^*-homomorphisms*, J. Comput. Anal. Anal. **16** (2014), 964–973.

[32] D. Shin, C. Park, S. Farhadabadi, *Stability and superstability of J^*-homomorphisms and J^*-derivations for a generalized Cauchy-Jensen equation*, J. Comput. Anal. Anal. **17** (2014). 125–134.

[33] D. Shin, C. Park, S. Farhadabadi, *Ternary Jordan C^*-homomorphisms and ternary Jordan C^*-derivations for a generalized Cauchy-Jensen functional equation*, J. Comput. Anal. Anal. **17** (2014), 681–690.

[34] F. Skof, *Propriet locali e approssimazione di operatori*, Rend. Sem. Mat. Fis. Milano. **53** (1983), 113–129.

[35] S.M. Ulam, *Problems in Modern Mathematics*, science ed, Wiley, New York, 1964, Chapter VI.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 27, NO.2, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC

On Impulsive Sequential Fractional Differential Equations

N. I. Mahmudov and B. Sami
Eastern Mediterranean University
Gazimagusa, TRNC, Mersin 10
Turkey

Abstract

This paper aims to study the existence of the solutions for an Impulsive Sequential Fractional Differential Equations of order $1 < q \leq 2$ involving separate boundary conditions. Our analysis relies on some fixed point theorems. In addition, an example is provided to illustrate the results of this study.

Keywords
Impulsive sequential fractional differential equations, Caputo fractional derivative, fixed point theorem.

1 Introduction

Fractional differential equations have recently proved to be strong tools in the modeling of many physical phenomena. It gives a great application in nonlinear oscillations of earthquakes, many physical phenomena such as seepage flow in porous media and in fluid dynamic traffic model. Impulsive Fractional Differential Equations (IFDEs), and Sequential Fractional Differential Equations (SFDEs) have attracted the attention of many researchers, see [1]-[21]. To the best of our knowledge, the study of impulsive sequential fractional differential equations (ISFDE) supplemented with separated boundary conditions has yet to be initiated.

In [15] Tian and Bai studied the existence solutions for the following IFDEs with boundary conditions, by using Banach's fixed point theorem and Schauder's fixed point theorem:

$$\begin{cases} {}^cD^q u(t) = f(t, u(t)), & q \in (1, 2], t \in [0.1], t \neq t_k, \\ \Delta u|_{t=t_k} = \mathbf{I}_k(u(t_k)),\ \Delta u'|_{t=t_k} = \bar{\mathbf{I}}_k(u(t_k)), k = 1, 2, ..., p, & k = 1, ..., p, \\ u(0) + u^{'}(0) = 0, u(1) + u^{'}(\xi) = 0, \end{cases}$$

with the Caputo fractional derivative ${}^cD^q$, $f \in [0, 1] \times R \to R$ is a continuous function, $\mathbf{I}_k, \bar{\mathbf{I}}_k : R \to R, 0 = t_0 < t_1 < \cdots < t_k < ... < t_p < t_{p+1} = 1$.

In [16] Wang investigated the existence of the solutions of the problem which is given as follows :

$$\begin{cases} {}^cD^q u(t) = f(t, u(t)), & 1 < q \leq 2, t \in J', \\ \Delta u(t_k) = Q_k(u(t_k)),\ \Delta u'(t_k) = \mathbf{I}_k(u(t_k)), k = 1, ..., p, & k = 1, ..., p, \\ au(0) + bu^{'}(0) = x_0, cu(1) + du^{'}(1) = x_1. \end{cases}$$

Mahmudov and Unul, [17] provided existence of solutions for the following IFDEs of order q with mixed BVP :

$$\begin{cases} {}^cD_0^q u(t) = f(t, u(t)), & 1 < q \leq 2, t \in J', \\ \Delta u(t_k) = \mathbf{I}_k(u(t_k)) = u(t_k^+) - u(t_k^-), \Delta u'(t_k) = J_k(u(t_k)) = u'(t_k^+) - u'(t_k^-), & k = 1, ..., p, \\ u(0) + \mu_1 u^{'}(1) = \sigma_1, x(0) + \mu_2 x^{'}(1) = \sigma_2, \end{cases}$$

with ${}^cD^q$ is the Caputo derivative of order q, and $f \in (J \times R, R), \varphi_k, I_k \in C(R \times R), J = [0, 1], 0 = t_0 < t_1 < \cdots < t_k < ... < t_p < t_{p+1} = 1$. $\Delta u(t_k) = u\left(t_k^+\right) - u\left(t_k^-\right), \Delta u'(t_k) = u^{'}\left(t_k^+\right) - u'\left(t_k^-\right)$.

In [20], Ahmad and Ntouyas obtained new existence results by using standard fixed point theorems.

$$\begin{cases} {}^cD^{\xi}(D+\lambda\ )x(t) \in f_1(t,x(t)), & 0<t<1, n<\xi<n-1, \\ x(0)=0.x^{'}(0)=0, \\ x^{'}(0)=0,...,x^{n-1}(0)=0.x(1)=\alpha x(\sigma), \end{cases}$$

where $F:[0.1]\times R\rightarrow \mathcal{F}(R)$ is a multivalued map, $\mathcal{F}(R)$ is the family of all subsets of R.

Sequential fractional integral-differential were studied, in [20]:

$$\begin{cases} ({}^cD^q+\lambda\ {}^cD^{q-1})x(t)=f(t,x(t),{}^c D^{\beta}x\left(t\right),I^{\gamma}x\left(t\right)), & t\in\left[0,1\right], 2<q\leq 3, 0<\beta,\gamma<1, k>0, \\ x(0)=0, x'(0)=0, \\ \sum_{i=1}^{m}a_ix(\zeta_i)=\lambda\int_0^{\eta}\frac{(\eta-s)^{\delta-1}}{\Gamma(\delta)}x(s)ds, \delta\geq 1, 0<\eta<\zeta<...<\zeta<1, \end{cases}$$

Here $f:[0,1]\times R^3\rightarrow R$ is a given continuous function satisfying some natural conditions.

Alsaedi, et al, [21], used fixed point theorems to develop the existence theory for the following problem:

$$\begin{cases} ({}^cD^q+k\ {}^cD^{q-1})u(t)=f(t,u(t), & 1<q\leq 2, t\in\left[0,T\right], \\ \alpha_1u\left(0\right)+\sum_{i=1}^{m}a_iu\left(\eta_i\right)+\gamma_1u\left(T\right)=\beta_1, \\ \alpha_2u'\left(0\right)+\sum_{i=1}^{m}b_iu'\left(\eta_i\right)+\gamma_2u'\left(T\right)=\beta_2, \alpha_3u'\left(0\right)+\sum_{i=1}^{m}c_iu''\left(\eta_i\right)+\gamma_3u^{''}\left(T\right)=\beta_3. \end{cases}$$

This paper is motivated from some recent papers treating the problem of the existence of solutions for ISFDEs with separated boundary conditions:

$$\begin{cases} ({}^cD^q+\lambda\ {}^cD^{q-1})x(t)=f(t,x(t)), 0<t<T & 1<p\leq 2, \\ \alpha_1x(0)+\beta_1x^{'}(0)=\eta_1, \alpha_2x(T)+\beta_2x^{'}(T)=\eta_2, \\ \Delta x|_{t=t_k}=\varphi_k(x(t_k)),\ \Delta x'|_{t=t_k}=\varphi_k^*(x(t_k)), & k=1,...,p, \end{cases} \tag{1}$$

where ${}^cD^q$ is the Caputo derivative of order $q\in(1,2]$, and $f:[0T]\times R\rightarrow R$, $\beta,q,\eta_1,\alpha_1,\alpha_2,\beta_1,\beta_1,\eta_1,\eta_2\in R,\lambda\in R^+,\varphi_k,\varphi_k^*\in C\left(R,R\right)$, and $\Delta x|_{t=t_k}=x\left(t_k^+\right)-x\left(t_k^-\right)$, $\Delta x'|_{t=t_k}=x^{'}\left(t_k^+\right)-x'\left(t_k^-\right)$. $x\left(t_k^+\right)$ and $x\left(t_k^-\right)$ represent the right and the left hand limits of the function $x(t)$, at $t=t_k^+$,respectively.

The rest of the paper is organized as follows. In Section 2, we recall some basic concepts of fractional calculus and obtain the integral solution for the linear variants of the given problems. Section 3 contains the existence results for problem (1) obtained by applying Leray-Schauder's nonlinear alternative, Banach's contraction mapping principle and Krasnoselskii's fixed point theorem. In Section 4, the main result is illustrated with the aid of an example.

2 Basic materials

The basic concepts of fractional calculus are presented in this section [13].

Denote that, $J=\left[0,T\right], t_0=0, t_{p+1}=T$, $J_0=[0,t_1]$, $J_1=(t_1,t_2]$,..., $J_p=(t_p,T]$, $J'=J\backslash\left\{t_1,...,t_p\right\}, 0=t_0<t_1<\cdots<t_p<t_{p+1}=T$, and insert the spaces:

$$PC(J)=\left\{x:J\rightarrow R\mid x\in C(J'),\text{and } x(t_k^+),x(t_k^-)\text{ exist, and } x(t_k^+)=x(t_k), 1\leq k\leq p\right\},$$

with the norm $\left\|x\right\|_{PC}=\sup_{t\in J}\left|x(t)\right|$ and $PC(J)$ is a Banach space.

Definition 1 *The fractional integral of order $q>0$ of a function $\rho:[0,\infty)\rightarrow R$ is given by*

$$\mathbf{I}_{0^+}^q\rho(x)=\frac{1}{\Gamma(q)}\int_0^x\frac{\rho(r)}{(x-r)^{1-q}}dr,\ x>0\ , q>0,$$

provided the right side is point-wise defined on $[0,\infty)$.

Definition 2 *The Caputo fractional derivative of order $q>0$, of a function $\rho:[0,\infty)\to R$ is defined by*

$$\mathbf{D}_{0^+}^q\rho(x)=\frac{1}{\Gamma(n-q)}\int_0^x(x-r)^{n-q-1}\rho^n dr,$$

whenever the right-hand side is defined on $[0,\infty)$.

Lemma 3 *Let $q>0$, then the differential equation*

$$^cD^q\rho(t)=0,$$

has solutions

$$\rho(x)=a_0+a_1x+a_2x^2+...+a_{n-1}x^{n-1},$$

where $a_i\in R,\ i=0,1,2,3,...,n-1;\ n=[\beta]+1.$

Lemma 4 *The set $F\subset PC\left([0,T],R^n\right)$ is relatively compact if and only if*

(i) F is bounded , that $\|x\|\le C$ for each $x\in F$ and some $C>0$;

(ii) F is quasi-equicontinuous in $[0,T]$. That is, to say that for any $\varepsilon>0$ there exist $\gamma>0$ such that if $x\in F;k\in N;s_1,s_2\in(t_{k-1},t_k]$, and $|s_1-s_2|<\gamma,\left|x(s_{1)}-x(s_2)\right|<\varepsilon.$

Lemma 5 *For $\rho\in PC(J,R)$, the solution of the following ISFDEs*

$$\begin{cases}(^cD^q+\lambda\ {}^cD^{q-1})x(t)=\rho(t),\\ \Delta x|_{t=t_k}=\varphi_k(x(t_k)),\ \Delta x'|_{t=t_k}=\varphi_k^*(x(t_k)),\\ \alpha_1x(0)+\beta_1x^{'}(0)=\eta_1,\alpha_2x(T)+\beta_2x^{'}(T)=\eta_2,\quad k=1,...,p,\end{cases}\tag{2}$$

is given by

$$\begin{aligned}x\left(t\right)=&\int_0^t e^{-\lambda(t-s)}I^{q-1}\rho(s)ds+v_1(t)\int_0^T e^{-\lambda(T-s)}I^{q-1}\rho(s)ds\\&+v_2(t)I^{q-1}\rho(T)+v_3(t)\sum_{j=1}^p\varphi_j(x(t_j))+v_4(t)\sum_{k=1}^p\varphi_j^*(x(t_j)\\&+\sum_{j=1}^p z_{1j}\left(t\right)\varphi_j^*(x(t_j))+\sum_{j=k+1}^p z_{2j}\left(t\right)\varphi_j^*(x(t_j))-\sum_{j=k+1}^p\varphi_j(x(t_j))+z_3\left(t\right),\\&t\in\left[t_k,t_{k+1}\right),\ k=0,1,...,p,\end{aligned}\tag{3}$$

where

$$\begin{aligned}\Delta&=\left(\alpha_1-\lambda\beta_1\right)\alpha_2-\left(\alpha_2e^{-\lambda T}-\lambda\beta_2e^{-\lambda T}\right)\alpha_1\neq0,\\v_1(t)&=\frac{\left(\alpha_1e^{-\lambda t}-\alpha_1+\lambda\beta_1\right)\left(\alpha_2-\lambda\beta_2\right)}{\Delta},\\v_2(t)&=\frac{\left(\alpha_1e^{-\lambda t}-\alpha_1+\lambda\beta_1\right)\beta_2}{\Delta},\\v_3(t)&=\frac{\alpha_1\alpha_2}{\Delta}e^{-\lambda t}-\frac{\alpha_1\left(\alpha_2e^{-\lambda T}-\lambda\beta_2e^{-\lambda T}\right)}{\Delta},\\v_4(t)&=\frac{\alpha_1\alpha_2}{\lambda\Delta}e^{-\lambda t}-\frac{\alpha_1\left(\alpha_2e^{-\lambda T}-\lambda\beta_2e^{-\lambda T}\right)}{\lambda\Delta},\\z_{1,j}\left(t\right)&=-e^{\lambda t_j}e^{-\lambda t}\frac{\alpha_2\left(\alpha_1-\lambda\beta_1\right)}{\lambda\Delta}+e^{\lambda t_j}\frac{\left(\alpha_1-\lambda\beta_1\right)\left(\alpha_2e^{-\lambda T}-\lambda\beta_2e^{-\lambda T}\right)}{\lambda\Delta},\\z_{2,j}\left(t\right)&=e^{-\lambda t}\frac{1}{\lambda}e^{\lambda t_j}-\frac{1}{\lambda},\\z_3\left(t\right)&=\left(e^{-\lambda t}\frac{\alpha_2}{\Delta}-\frac{\left(\alpha_2e^{-\lambda T}-\lambda\beta_2e^{-\lambda T}\right)}{\Delta}\right)\eta_1-\left(e^{-\lambda t}\frac{\alpha_1}{\Delta}-\frac{\left(\alpha_1-\lambda\beta_1\right)}{\Delta}\right)\eta_2.\end{aligned}$$

Proof. Assume that x is a solution of

$$({}^cD^q + \lambda \ {}^cD^{q-1})x(t) = \rho(t),$$

on $(t_k, t_{k+1}]$, $(k = 1, 2 \ldots, p)$. Applying the operator $\mathbf{I}^{q-1}$ operator to both sides of the above equation, we get

$$\begin{aligned}\mathbf{I}^{q-1}({}^cD^q + \lambda \ {}^cD^{q-1})x(t) &= \mathbf{I}^{q-1}\rho(t),\\ (D+\lambda)\, x(t) &= c_0 + \mathbf{I}^{q-1}\rho(t).\end{aligned}$$

This can be expressed as

$$e^{\lambda t}\left((D+\lambda)\, x\,(t)\right) = e^{\lambda t}\left(c_0 + \mathbf{I}^{q-1}\rho(t)\right),$$

Solving the above equation, we see that the general solution of (1) on each interval $(t_k, t_{k+1}]$, $(k = 1, 2 \ldots p)$, can be written as

$$x(t) = e^{-\lambda t}A_k + B_k + \int_0^t e^{-\lambda(t-s)}\mathbf{I}^{q-1}\rho(s)ds, t \in J.$$

Next, solving the obtained linear equation on J_0, we get

$$x(t) = e^{-\lambda t}A_0 + B_0 + \int_0^t e^{-\lambda(t-s)}\mathbf{I}^{q-1}\rho(s)ds, \ t \in J_0, \tag{4}$$

where A_0 and B_0 are arbitrary constants. Taking the derivative to (4), we get

$$x^{'}(t) = -\lambda e^{-\lambda t}A_0 - \lambda\int_0^t e^{-\lambda(t-s)}\mathbf{I}^{q-1}\rho(s)ds + I^{q-1}\rho(t), \ t \in J_0. \tag{5}$$

Now, applying the boundary condition, we have

$$(\alpha_1 - \lambda\beta_1)\, A_0 + \alpha_1 B_0 = \eta_1. \tag{6}$$

In general, for $t \in [t_k, t_{k+1})$, we find

$$\begin{aligned}x(t) &= e^{-\lambda t}A_k + B_k + \int_0^t e^{-\lambda(t-s)}\mathbf{I}^{q-1}\rho(s)ds, \\ x^{'}(t) &= -\lambda e^{-\lambda t}A_k - \lambda\int_0^t e^{-\lambda(t-s)}\mathbf{I}^{q-1}\rho(s)ds + \mathbf{I}^{q-1}\rho(t).\end{aligned} \tag{7}$$

Now, applying the boundary condition at $t_{k+1} = T$, we have

$$\left(\alpha_2 e^{-\lambda T} - \lambda\beta_2 e^{-\lambda T}\right)A_p + \alpha_2 B_p = \eta_2 - (\alpha_2 - \lambda\beta_2)\int_0^T e^{-\lambda(T-s)}\mathbf{I}^{q-1}\rho(s)ds - \beta_2\mathbf{I}^{q-1}\rho(T). \tag{8}$$

From $\Delta x'(t_k) = \varphi_k^*(x(t_k))$, we have

$$\begin{aligned}\varphi_k^*(x(t_k)) &= -\lambda e_k^{-\lambda t_k}A_k + \lambda e_{k-1}^{-\lambda t_k}A_{k-1}, \\ A_k - A_{k-1} &= -\frac{1}{\lambda}e^{\lambda t_k}\varphi_k^*(x(t_k)), \ k = 1, ..., p.\end{aligned} \tag{9}$$

Similarly, from $\Delta x(t_k) = \varphi_k(x(t_k))$, we get

$$\begin{aligned}\varphi_k(x(t_k)) &= e^{-\lambda t_k}A_k - e^{-\lambda t_k}A_{k-1} + B_k - B_{k-1}, \\ B_k - B_{k-1} &= \varphi_k(x(t_k)) + \frac{1}{\lambda}\varphi_k^*(x(t_k)), \ k = 1, ..., p.\end{aligned} \tag{10}$$

Next, it follows from (9) and (10) that

$$A_p - A_k = -\frac{1}{\lambda}\sum_{j=k+1}^{p} e^{\lambda t_j}\varphi_j^*(x(t_j)), \tag{11}$$

$$B_p - B_k = \sum_{j=k+1}^{p}\varphi_j(x(t_j)) + \frac{1}{\lambda}\sum_{j=k+1}^{p}\varphi_j^*(x(t_j)), \quad k = 0,1,...,p-1. \tag{12}$$

It follows that for $k=0$ from $(\alpha_1 - \lambda\beta_1) A_0 + \alpha_1 B_0 = \eta_1$ that

$$(\alpha_1 - \lambda\beta_1) A_p + \alpha_1 B_p = \eta_1 - \frac{1}{\lambda}(\alpha_1 - \lambda\beta_1)\sum_{j=1}^{p} e^{\lambda t_j}\varphi_j^*(x(t_j)) + \alpha_1\sum_{j=1}^{p}\varphi_j(x(t_j)) + \frac{1}{\lambda}\alpha_1\sum_{j=1}^{p}\varphi_j^*(x(t_j)).$$

Solving the last equation together(8), for A_p and B_p, we get

$$\begin{aligned}
A_p &= \left(\frac{\alpha_1(\alpha_2 - \lambda\beta_2)}{\Delta}\right)\int_0^T e^{-\lambda(T-s)}\mathbf{I}^{q-1}\rho(s)ds + \left(\frac{\alpha_1\beta_2}{\Delta}\right) I^{q-1}\rho(T) \\
&+ \left(\frac{\alpha_1\alpha_2}{\Delta}\right)\sum_{j=1}^{p}\varphi_j(x(t_j)) + \left(\frac{\alpha_1\alpha_2}{\lambda\Delta}\right)\sum_{j=1}^{p}\varphi_j^*(x(t_j)) - \left(\frac{\alpha_2(\alpha_1 - \lambda\beta_1)}{\lambda\Delta}\right)\sum_{j=1}^{p} e^{\lambda t_j}\varphi_j^*(x(t_j)) \\
&+ \frac{\alpha_2}{\Delta}\eta_1 - \frac{\alpha_1}{\Delta}\eta_2,
\end{aligned}$$

and

$$\begin{aligned}
B_p = &-\left(\frac{(\alpha_1 - \lambda\beta_1)(\alpha_2 - \lambda\beta_2)}{\Delta}\right) \div \int_0^T e^{-\lambda(T-s)}\mathbf{I}^{q-1}\rho(s)ds - \left(\frac{(\alpha_1 - \lambda\beta_1)\beta_2}{\Delta}\right) I^{q-1}\rho(T) \\
&- \left(\frac{\alpha_1\left(\alpha_2 e^{-\lambda T} - \lambda\beta_2 e^{-\lambda T}\right)}{\Delta}\right)\sum_{j=1}^{p}\varphi_j(x(t_j)) - \left(\frac{\alpha_1\left(\alpha_2 e^{-\lambda T} - \lambda\beta_2 e^{-\lambda T}\right)}{\lambda\Delta}\right)\sum_{j=1}^{p}\varphi_j^*(x(t_j)) \\
&+ \left(\frac{(\alpha_1 - \lambda\beta_1)\left(\alpha_2 e^{-\lambda T} - \lambda\beta_2 e^{-\lambda T}\right)}{\lambda\Delta}\right)\sum_{j=1}^{p} e^{\lambda t_j}\varphi_j^*(x(t_j)) - \left(\frac{\left(\alpha_2 e^{-\lambda T} - \lambda\beta_2 e^{-\lambda T}\right)}{\Delta}\right)\eta_1 + \left(\frac{(\alpha_1 - \lambda\beta_1)}{\Delta}\right)\eta_2,
\end{aligned}$$

where $\Delta = (\alpha_1 - \lambda\beta_1)\alpha_2 - \left(\alpha_2 e^{-\lambda T} - \lambda\beta_2 e^{-\lambda T}\right)\alpha_1 \neq 0$. Now, from the equations (11) and (12) it follows that

$$\begin{aligned}
A_k &= A_p + \frac{1}{\lambda}\sum_{j=k+1}^{p} e^{\lambda t_j}\varphi_j^*(x(t_j)), \\
B_k &= B_p - \sum_{j=k+1}^{p}\varphi_j(x(t_j)) - \frac{1}{\lambda}\sum_{j=k+1}^{p}\varphi_j^*(x(t_j)),\ k = 1,...,p-1.
\end{aligned}$$

So

$$\begin{aligned}
A_k &= \left(\frac{\alpha_1(\alpha_2 - \lambda\beta_2)}{\Delta}\right)\int_0^T e^{-\lambda(T-s)}\mathbf{I}^{q-1}\rho(s)ds + \left(\frac{\alpha_1\beta_2}{\Delta}\right) I^{q-1}\rho(T) \\
&+ \left(\frac{\alpha_1\alpha_2}{\Delta}\right)\sum_{j=1}^{p}\varphi_j(x(t_j)) + \left(\frac{\alpha_1\alpha_2}{\lambda\Delta}\right)\sum_{j=1}^{p}\varphi_j^*(x(t_j)) - \left(\frac{\alpha_2(\alpha_1 - \lambda\beta_1)}{\lambda\Delta}\right)\sum_{j=1}^{p} e^{\lambda t_j}\varphi_j^*(x(t_j)) \\
&+ \left(\frac{\alpha_2}{\Delta}\right)\eta_1 - \left(\frac{\alpha_1}{\Delta}\right)\eta_2 + \left(\frac{1}{\lambda}\right)\sum_{j=k+1}^{p} e^{\lambda t_j}\varphi_j^*(x(t_j)).
\end{aligned}$$

Multiplying the above equation by $e^{-\lambda t}$, we get

$$\begin{aligned}
e^{-\lambda t}A_k &= \left(\frac{e^{-\lambda t}\alpha_1\left(\alpha_2-\lambda\beta_2\right)}{\Delta}\right)\int_0^T e^{-\lambda(T-s)}\mathbf{I}^{q-1}\rho(s)ds + \left(\frac{e^{-\lambda t}\alpha_1\beta_2}{\Delta}\right)I^{q-1}\rho(T) \\
&+ \left(\frac{e^{-\lambda t}\alpha_1\alpha_2}{\Delta}\right)\sum_{j=1}^{p}\psi_j(x(t_j)) + \left(\frac{e^{-\lambda t}\alpha_1\alpha_2}{\lambda\Delta}\right)\sum_{j=1}^{p}\psi_j^*(x(t_j)) - \left(\frac{e^{-\lambda t}\alpha_2\left(\alpha_1-\lambda\beta_1\right)}{\lambda\Delta}\right)\sum_{j=1}^{p}e^{\lambda t_j}\psi_j^*\left(x(t_j)\right) \\
&+ \left(\frac{e^{-\lambda t}\alpha_2}{\Delta}\right)\eta_1 - \left(\frac{e^{-\lambda t}\alpha_1}{\Delta}\right)\eta_2 + \left(\frac{e^{-\lambda t}}{\lambda}\right)\sum_{j=k+1}^{p}e^{\lambda t_j}\psi_j^*(x(t_j)).
\end{aligned}$$

and

$$\begin{aligned}
B_k = &-\left(\frac{\left(\alpha_1-\lambda\beta_1\right)\left(\alpha_2-\lambda\beta_2\right)}{\Delta}\right)\int_0^T e^{-\lambda(T-s)}\mathbf{I}^{q-1}\rho(s)ds - \left(\frac{\left(\alpha_1-\lambda\beta_1\right)\beta_2}{\Delta}\right)\mathbf{I}^{q-1}\rho(T) \\
&- \left(\frac{\alpha_1\left(\alpha_2 e^{-\lambda T}-\lambda\beta_2 e^{-\lambda T}\right)}{\Delta}\right)\sum_{j=1}^{p}\varphi_j(x(t_j)) - \left(\frac{\alpha_1\left(\alpha_2 e^{-\lambda T}-\lambda\beta_2 e^{-\lambda T}\right)}{\lambda\Delta}\right)\sum_{j=1}^{p}\varphi_j^*(x(t_j)) \\
&+ \left(\frac{\left(\alpha_1-\lambda\beta_1\right)\left(\alpha_2 e^{-\lambda T}-\lambda\beta_2 e^{-\lambda T}\right)}{\lambda\Delta}\right)\sum_{j=1}^{p}e^{\lambda t_j}\varphi_j^*(x(t_j)) - \left(\frac{\left(\alpha_2 e^{-\lambda T}-\lambda\beta_2 e^{-\lambda T}\right)}{\Delta}\right)\eta_1 \\
&+ \left(\frac{\left(\alpha_1-\lambda\beta_1\right)}{\Delta}\right)\eta_2 - \sum_{j=k+1}^{p}\varphi_j(x(t_j)) - \left(\frac{1}{\lambda}\right)\sum_{j=k+1}^{p}\varphi_j^*(x(t_j)).
\end{aligned}$$

Combining the last two equations, we get

$$
\begin{aligned}
e^{-\lambda t}A_k + B_k = & \left(\frac{e^{-\lambda t}\alpha_1\left(\alpha_2-\lambda\beta_2\right)}{\Delta}\right)\int_0^T e^{-\lambda(T-s)}\mathbf{I}^{q-1}\rho(s)ds + \left(\frac{e^{-\lambda t}\alpha_1\beta_2}{\Delta}\right)I^{q-1}\rho(T) \\
& + \left(\frac{e^{-\lambda t}\alpha_1\alpha_2}{\Delta}\right)\sum_{j=1}^{p}\psi_j(x(t_j)) + \left(\frac{e^{-\lambda t}\alpha_1\alpha_2}{\lambda\Delta}\right)\sum_{j=1}^{p}\psi_j^*(x(t_j)) \\
& - \left(\frac{e^{-\lambda t}\alpha_2\left(\alpha_1-\lambda\beta_1\right)}{\lambda\Delta}\right)\sum_{j=1}^{p}e^{\lambda t_j}\psi_j^*\left(x(t_j)\right) \\
& + \left(\frac{e^{-\lambda t}\alpha_2}{\Delta}\right)\eta_1 - \left(\frac{e^{-\lambda t}\alpha_1}{\Delta}\right)\eta_2 + \left(\frac{e^{-\lambda t}}{\lambda}\right)\sum_{j=k+1}^{p}e^{\lambda t_j}\psi_j^*(x(t_j)) \\
& - \left(\frac{\left(\alpha_1-\lambda\beta_1\right)\left(\alpha_2-\lambda\beta_2\right)}{\Delta}\right)\int_0^T e^{-\lambda(T-s)}\mathbf{I}^{q-1}\rho(s)ds \\
& - \left(\frac{\left(\alpha_1-\lambda\beta_1\right)\beta_2}{\Delta}\right)\mathbf{I}^{q-1}\rho(T) - \left(\frac{\alpha_1\left(\alpha_2e^{-\lambda T}-\lambda\beta_2e^{-\lambda T}\right)}{\Delta}\right)\sum_{j=1}^{p}\varphi_j(x(t_j)) \\
& - \left(\frac{\alpha_1\left(\alpha_2e^{-\lambda T}-\lambda\beta_2e^{-\lambda T}\right)}{\lambda\Delta}\right)\sum_{j=1}^{p}\varphi_j^*(x(t_j)) \\
& + \left(\frac{\left(\alpha_1-\lambda\beta_1\right)\left(\alpha_2e^{-\lambda T}-\lambda\beta_2e^{-\lambda T}\right)}{\lambda\Delta}\right)\sum_{j=1}^{p}e^{\lambda t_j}\varphi_j^*(x(t_j)) - \left(\frac{\left(\alpha_2e^{-\lambda T}-\lambda\beta_2e^{-\lambda T}\right)}{\Delta}\right)\eta_1 \\
& + \left(\frac{\left(\alpha_1-\lambda\beta_1\right)}{\Delta}\right)\eta_2 \\
& - \sum_{j=k+1}^{p}\varphi_j(x(t_j)) - \left(\frac{1}{\lambda}\right)\sum_{j=k+1}^{p}\varphi_j^*(x(t_j)).
\end{aligned}
$$

$$
\begin{aligned}
e^{-\lambda t}A_k + B_k = & \, v_1(t)\int_0^T e^{-\lambda(T-s)}\mathbf{I}^{q-1}\rho(s)ds + v_2(t)\mathbf{I}^{q-1}\rho(T) + v_3(t)\sum_{j=1}^{p}\varphi_j(x(t_j)) \\
& + v_4(t)\sum_{j=1}^{p}\varphi_j^*(x(t_j) + \sum_{j=1}^{p}z_{1,j}\left(t\right)\varphi_j^*(x(t_j)) + \sum_{j=k+1}^{p}z_{2,j}\left(t\right)\varphi_j^*(x(t_j)) \\
& - \sum_{j=k+1}^{p}\varphi_j(x(t_j)) + z_3(t).
\end{aligned} \tag{13}
$$

Inserting (13) into (7), thus we obtain the desired formula (3).

The converse of the lemma follows by direct computation. This completes the proof. ■

3 Main results

This section deals with the existence and uniqueness of solutions for the problem (1). Before stating and proving the main results, we introduce the following hypotheses.

($\mathbf{H}_1$) the function $f : J \times R \to R$ is jointly continuous .

($\mathbf{H}_2$) there exists a constant $L_f > 0$ such that

$$|f(t,x) - f(t,y)| \le L_f |x-y|, \quad t \in J, \quad x, y \in R.$$

7

($\mathbf{H}_3$) There exist a positive constants $L_\varphi, L_{\varphi^*}, M_\varphi, M_{\varphi^*}$ such that

$$\left|\varphi_k(x)-\varphi_k(y)\right| \leq L_\varphi \left|x-y\right|, \left|\varphi_k^*(x)-\varphi_k^*(y)\right| \leq L_{\varphi^*} \left|x-y\right|; \left|\varphi_k(x)\right| \leq M_\varphi, \left|\varphi_k^*(x)\right| \leq M_{\varphi^*}.$$

From (G_1)-(G_3) it follows that

$$\begin{aligned} &\left|f\left(t,x\right)\right| \leq L_f \left|x\right| + M_f, \quad t\in J, \quad x\in R, \quad M_f := \sup\left\{\left|f\left(t,0\right)\right| : 0<t\leq T\right\}, \\ &\left|\varphi_k(x)\right| \leq L_\varphi \left|x\right| + M_\varphi, \quad \left|\varphi_k^*(x)\right| \leq L_{\varphi^*} \left|x\right| + M_{\varphi^*}. \end{aligned}$$

Theorem 6 *Suppose that (H_1), (H_2) and (H_3) hold. If*

$$\begin{aligned} L_{\mathfrak{T}} := &\left(\frac{T^{q-1}}{\lambda\Gamma\left(q\right)}\left(1-e^{-\lambda T}\right)\left(1+\left\|\nu_1\right\|\right) + \frac{T^{q-1}}{\Gamma\left(q\right)}\left\|\nu_2\right\|\right) L_f \\ &+ \left(1+\left\|\nu_3\right\|\right) pL_{_\varphi} + \left(\left\|v_4\right\| + \left\|z_{1j}\right\| + \left\|z_{2j}\right\|\right) pL_{\varphi^*} < 1, \end{aligned} \tag{14}$$

then the equation (1) has a unique solution on J.

Proof. In view of Lemma 5, we can transform problem (1) into a fixed point problem. Consider the operator $\mathfrak{T} : PC\left(J,R\right) \to PC\left(J,R\right)$ defined by

$$\begin{aligned} (\mathfrak{T}x)(t) := &\int_0^t e^{-\lambda(t-s)}\mathbf{I}^{q-1}f(s,x\left(s\right))ds + v_1(t)\int_0^T e^{-\lambda(T-s)}\mathbf{I}^{q-1}f(s,x\left(s\right))ds \\ &+ v_2(t)I^{q-1}f(T,x\left(T\right)) + v_3(t)\sum_{j=1}^p \varphi_j(x(t_j)) + v_4(t)\sum_{j=1}^p \varphi_j^*(x(t_j) \\ &+ \sum_{j=1}^p z_{1j}\left(t\right)\varphi_j^*(x(t_j)) + \sum_{j=k+1}^p z_{2j}\left(t\right)\varphi_j^*(x(t_j)) - \sum_{j=k+1}^p \varphi_j(x(t_j)) + z_3\left(t\right) \\ &, t\in J_k, \ k=0,1,...,p. \end{aligned} \tag{15}$$

It is obvious that $\mathfrak{T}$ is well defined due to (H_1) and sends $PC\left(J,R\right)$ into itself.

Step 1. $\mathfrak{T}$ maps $B_r = \left\{x\in PC\left(\left[0,T\right],R\right), \left\|x\right\| \leq r\right\}$ into itself for some $r>0$.

Let

$$\begin{aligned} r > &\left(1-L_{\mathfrak{T}}\right)^{-1}\left(\frac{T^{q-1}}{\lambda\Gamma\left(q\right)}\left(1-e^{-\lambda T}\right)\left(1+\left\|\nu_1\right\|\right) + \frac{T^{q-1}}{\Gamma\left(q\right)}\left\|\nu_2\right\|\right) E_f \\ &+ \left(1+\left\|\nu_3\right\|\right) p\left(L_\varphi r + M_\varphi\right) + \left(\left\|\nu_4\right\| + \left\|z_{1j}\right\| + \left\|z_{2j}\right\|\right) p\left(L_{\varphi^*} r + M_{\varphi^*}\right) + \left\|z_3\right\|. \end{aligned}$$

For $t\in J_k, k=0,1,...,p$; $x\in B_r$, we have

$$\begin{aligned} \left|(\mathfrak{T}x)(t)\right| \leq &\frac{1}{\Gamma\left(q-1\right)}\int_0^t e^{-\lambda(t-s)}\left(\int_0^s \left(s-\tau\right)^{q-2}\left|f(\tau,x(\tau))\right| d\tau\right) ds \\ &+ \frac{\left|v_1(t)\right|}{\Gamma\left(q-1\right)}\int_0^T e^{-\lambda(T-s)}\left(\int_0^s \left(s-\tau\right)^{q-2}\left|f(\tau,x(\tau))\right| d\tau\right) ds \\ &+ \frac{\left|v_2(t)\right|}{\Gamma\left(q-1\right)}\int_0^T (T-s)^{q-2}\left|f(s,x(s))\right| ds + \left|v_3(t)\right|\sum_{j=1}^p \left|\varphi_j(x(t_j))\right| \\ &+ \left|v_4(t)\right|\sum_{j=1}^p \left|\varphi_j^*(x(t_j)\right| + \sum_{j=1}^p \left|z_{1j}\left(t\right)\right|\left|\varphi_j^*(x(t_j))\right| \\ &+ \sum_{j=k+1}^p \left|z_{2j}\left(t\right)\right|\left|\varphi_j^*(x(t_j))\right| \sum_{j=k+1}^p \left|\varphi_j(x(t_j))\right| + \left|z_3\left(t\right)\right|, \end{aligned}$$

Thus

$$
\begin{aligned}
|(\mathfrak{T}x)(t)| &\leq \frac{t^{q-1}}{\lambda\Gamma(q)}\left(1-e^{-\lambda t}\right)(L_f r+M_f)+|v_1(t)|\frac{T^{q-1}}{\lambda\Gamma(q)}\left(1-e^{-\lambda T}\right)(L_f r+M_f)\\
&+|v_2(t)|\frac{T^{q-1}}{\Gamma(q)}(L_f r+M_f)+|v_3(t)|\,p\,(L_\varphi r+M_\varphi)+|v_4(t)|\,p\,(Lr+M_\varphi)\\
&+|z_{1j}(t)|\,p\,(L_\varphi r+M_\varphi)+|z_{2j}(t)|\,p\,(L_\varphi r+M_\varphi)+p\,(L_\varphi r+M_\varphi)+|z_3(t)|\\
&\leq \left(\frac{T^{q-1}}{\lambda\Gamma(q)}\left(1-e^{-\lambda T}\right)(1+\|\nu_1\|)+\frac{T^{q-1}}{\Gamma(q)}\|\nu_2\|\right)(L_f r+M_f)+(1+\|\nu_3\|)\,p\,(L_\varphi r+M_\varphi)\\
&+(\|\nu_4\|+\|z_{1j}\|+\|z_{2j}\|)\,p\,(L_{\varphi^*}r+M_{\varphi^*})+\|z_3\|
\end{aligned}
$$

We use the following estimation in what follows

$$
\begin{aligned}
&\left|\frac{1}{\Gamma(q-1)}\int_0^t e^{-\lambda(t-s)}\left(\int_0^s (s-\tau)^{q-2}\rho(\tau)\,d\tau\right)ds\right| \leq \frac{t^{q-1}}{\lambda\Gamma(q)}\left(1-e^{-\lambda t}\right)\|\rho\|_{PC} \\
&= \frac{T^{q-1}}{\lambda\Gamma(q)}\left(1-e^{-\lambda T}\right)\|\rho\|_{PC},\ \rho\in PC(J,R)
\end{aligned}
\tag{16}
$$

We obtain that

$$
\begin{aligned}
|(\mathfrak{T}x)(t)| &\leq \left(\frac{T^{q-1}}{\lambda\Gamma(q)}\left(1-e^{-\lambda T}\right)(1+\|\nu_1\|)+\frac{T^{q-1}}{\Gamma(q)}\|\nu_2\|\right)(L_f r+M_f)+(1+\|\nu_3\|)\,p\,(L_\varphi r+M_\varphi)\\
&+(\|\nu_4\|+\|z_{1j}\|+\|z_{2j}\|)\,p\,(L_{\varphi^*}r+M_{\varphi^*})+\|z_3\|<r.
\end{aligned}
$$

This implies that $\mathfrak{T}x\in B_r$. Thus $\mathfrak{T}B_r\subset B_r$.

Step 2. $\mathfrak{T}$ is a contraction operator on $PC(J,R)$.

Let $x,y\in B_r$. Then For each $t\in J$, we have

$$
\begin{aligned}
|(\mathfrak{T}x)(t)-(\mathfrak{T}y)(t)| := &\left|\int_0^t e^{-\lambda(t-s)}\mathbf{I}^{q-1}f(s,x(s))ds+v_1(t)\int_0^T e^{-\lambda(T-s)}\mathbf{I}^{q-1}f(s,x(s))ds\right.\\
&+v_2(t)I^{q-1}f(T,x(T))+v_3(t)\sum_{j=1}^p\varphi_j(x(t_j))+v_4(t)\sum_{j=1}^p\varphi_j^*(x(t_j)\\
&\left.+\sum_{j=1}^p z_{1j}(t)\,\varphi_j^*(x(t_j))+\sum_{j=k+1}^p z_{2j}(t)\,\varphi_j^*(x(t_j))-\sum_{j=k+1}^p\varphi_j(x(t_j))+z_3(t)\right|\\
&-\left|\int_0^t e^{-\lambda(t-s)}\mathbf{I}^{q-1}f(s,y(s))ds+v_1(t)\int_0^T e^{-\lambda(T-s)}\mathbf{I}^{q-1}f(s,y(s))ds\right.\\
&+v_2(t)I^{q-1}f(T,y(T))+v_3(t)\sum_{j=1}^p\varphi_j(y(t_j))+v_4(t)\sum_{j=1}^p\varphi_j^*(y(t_j)\\
&\left.+\sum_{j=1}^p z_{1j}(t)\,\varphi_j^*(y(t_j))+\sum_{j=k+1}^p z_{2j}(t)\,\varphi_j^*(y(t_j))+\sum_{j=k+1}^p\varphi_j(y(t_j))+z_3(t)\right|,
\end{aligned}
$$

9

$$
\begin{aligned}
|(\mathfrak{T}x)(t)-(\mathfrak{T}y)(t)| &:= \int_0^t e^{-\lambda(t-s)}\mathbf{I}^{q-1}\left|f(s,x(s))-f(s,y(s))\right|ds \\
&+|v_1(t)|\int_0^T e^{-\lambda(T-s)}\mathbf{I}^{q-1}\left|f(s,x(s))-f(s,y(s))\right|ds \\
&+|v_2(t)|\,I^{q-1}\left|f(T,x(T))-f(T,y(T))\right|+|v_3(t)|\sum_{j=1}^{p}|\varphi_j(x(t_j))-\varphi_j(y(t_j))| \\
&+v_4(t)\sum_{j=1}^{p}\left|\varphi_j^*(x(t_j)-\varphi_j^*(y(t_j))\right|+\sum_{j=1}^{p}|z_{1j}|\,(t)\left|\varphi_j^*(x(t_j))-\varphi_j^*(y(t_j))\right| \\
&+\sum_{j=k+1}^{p}|z_{2j}|\,(t)\left|\varphi_j^*(x(t_j))-\varphi_j^*(y(t_j))\right|+\sum_{j=k+1}^{p}|\varphi_j(x(t_j))|\,.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
|(\mathfrak{T}x)(t)-(\mathfrak{T}y)(t)| &\leq \left(\left(\frac{T^{q-1}}{\lambda\Gamma(q)}\left(1-e^{-\lambda T}\right)(1+\|\nu_1\|)+\frac{T^{q-1}}{\Gamma(q)}\|\nu_2\|\right)L_f\right. \\
&+(1+\|\nu_3\|)\,pL_\varphi+(\|\nu_4\|+\|z_{1j}\|+\|z_{2j}\|)\,pL_{\varphi^*})\,\|x-y\|_{PC} \\
&=L_{\mathfrak{T}}\,\|x-y\|_{PC}\,.
\end{aligned}
$$

Thus, $\mathfrak{T}$ is a contraction mapping on $PC(J,R)$ due to condition (14). By applying the well-known Banach's contraction mapping we see that the operator $\mathfrak{T}$ has a unique fixed point on $PC(J,R\ )$. Therefore, the problem (1) has a unique solution. This completes the proof. ■

The second result is based on a known result due to Krasnoselskii. We state the Krasnoselskii theorem which is needed to prove the existence of at least one solution of (1).

Theorem 7 . *Let M be a closed convex and nonempty subset of a Banach space X. Let $\mathfrak{T}_1$, $\mathfrak{T}_2$ be the operators such that:*

1. *$\mathfrak{T}_1x+\mathfrak{T}_2y\in M$ whenever $x,y\in M$;*
2. *$\mathfrak{T}_1$ is compact and continuous;*
3. *$\mathfrak{T}_2$ is a contraction mapping. Then there exists $z\in M$ such that $z=\mathfrak{T}_1z+\mathfrak{T}_2z$.*

Now, we replace (H_2) into the following condition:

$(\mathbf{H}_4)$ $|f(t,x)|\leq\mu(t)$ for $(t,x)\in J\times R$ where $\mu\in L^{\frac{1}{\sigma}}(J)\,,\sigma\in(0,q-1)\,.$

Theorem 8 *Suppose that (H_1),(H_3) and (H_4) hold. If*
$(1+\|\nu_3\|)\,pL_\varphi+(\|v_4\|+\|z_{1j}\|+\|z_{2j}\|)\,pL_{\varphi^*}<1.$
Then (1) has at least one solution on J.

Proof. Let $B_r=\{x\in PC(J,R),\|x\|_{PC}\leq r\}$. We choose

$$
\begin{aligned}
r \geq \frac{\|\mu\|_{L^{\frac{1}{\sigma}}}}{\Gamma(q)}&\left(\frac{T^{q-\sigma-1}\left(1-e^{-\lambda T}\right)}{\lambda\left(\frac{q-\sigma-1}{1-\sigma}\right)^{1-\sigma}}(1+\|v_1\|)+\frac{T^{q-\sigma-1}}{\left(\frac{q-\sigma-1}{1-\delta}\right)^{1-\sigma}}\|v_2\|\right) \\
&+(1+\|\nu_3\|)\,pL_\varphi+(\|v_4\|+\|z_{1j}\|+\|z_{2j}\|)\,pL_{\varphi^*}.
\end{aligned}
$$

The operators $\mathfrak{T}_1$ and $\mathfrak{T}_2$ on B_r are defined as:

$$(\mathfrak{T}_1 x)(t) = \int_0^t e^{-\lambda(t-s)} \mathbf{I}^{q-1} f(s, x(s)) ds + v_1(t) \int_0^T e^{-\lambda(T-s)} \mathbf{I}^{q-1} f(s, x(s)) ds \ + v_2(t) \mathbf{I}^{q-1} f(T, x(T)),$$

and

$$\begin{aligned}(\mathfrak{T}_2 x)(t) &:= v_3(t) \sum_{j=1}^{p} \varphi_j(x(t_j)) + v_4(t) \sum_{j=1}^{p} \varphi_j^*(x(t_j) + \sum_{j=1}^{p} z_{1j}(t) \varphi_j^*(x(t_j)) \\ &+ \sum_{j=k+1}^{p} z_{2j}(t) \varphi_j^*(x(t_j)) - \sum_{j=k+1}^{p} \varphi_j(x(t_j)), t \in J_k, k = 0, 1, ..., p.\end{aligned}$$

Step 1. $\mathfrak{T}_1 x + \mathfrak{T}_2 y \in B_r$ whenever $x, y \in B_r$.

For any $x, y \in B_r$ and $t \in J_k$, using the assumption (H$_4$) with the Holder inequality we get

$$\begin{aligned}&\left| \frac{1}{\Gamma(q-1)} \int_0^t e^{-\lambda(t-s)} \left(\int_0^s (s-\tau)^{q-2} |f(\tau, x(\tau))| d\tau \right) ds \right| \\ &\leq \left| \frac{1}{\Gamma(q-1)} \int_0^t e^{-\lambda(t-s)} \left(\int_0^s (s-\tau)^{\frac{q-2}{1-\sigma}} d\tau \right)^{1-\sigma} \left(\int_0^\tau |f(\tau, x(\tau))|^{\frac{1}{\sigma}} d\tau \right)^{\sigma} ds \right| \\ &\leq \frac{1}{\Gamma(q)} \frac{t^{q-\sigma-1}\left(1-e^{-\lambda t}\right)}{\lambda \left(\frac{q-\sigma-1}{1-\sigma}\right)^{1-\sigma}} \|\mu\|_{L^{\frac{1}{\sigma}}},\end{aligned}$$

$$\left| \int_0^T e^{-\lambda(T-s)} \left(\int_0^s \frac{(s-\tau)^{q-2}}{\Gamma(q-1)} f(\tau, x(\tau)) d\tau \right) ds \right| \leq \frac{1}{\Gamma(q)} \frac{T^{q-\sigma-1}\left(1-e^{-\lambda T}\right)}{\lambda \left(\frac{q-\sigma-1}{1-\sigma}\right)^{1-\sigma}} \|\mu\|_{L^{\frac{1}{\sigma}}}.$$

and

$$\left| \frac{v_2(t)}{\Gamma(q-1)} \int_0^T (T-s)^{q-2} f(s, x(s)) ds \right| \leq \frac{1}{\Gamma(q)} \frac{T^{q-\sigma-1}}{\left(\frac{q-\sigma-1}{1-\sigma}\right)^{1-\sigma}} \|\mu\|_{L^{\frac{1}{\sigma}}}.$$

Therefore,

$$\begin{aligned}\|\mathfrak{T}_1 x + \mathfrak{T}_2 y\|_{PC} &\leq \|\mu\|_{L^{\frac{1}{\sigma}}} \frac{1}{\Gamma(q)} \left(\frac{T^{q-\sigma-1}\left(1-e^{-\lambda T}\right)}{\lambda \left(\frac{q-\sigma-1}{1-\sigma}\right)^{1-\sigma}} (1 + \|\nu_1\|) + \frac{T^{q-\sigma-1}}{\left(\frac{q-\sigma-1}{1-\sigma}\right)^{1-\sigma}} \|\nu_2\| \right) \\ &+ ((1 + \|\nu_3\|) p M_\varphi + (\|\nu_4\| + \|z_{1j}\| + \|z_{2j}\|) p M_{\varphi^*} \leq r.\end{aligned}$$

Thus, $\|\mathfrak{T}_1 x + \mathfrak{T}_2 y\| \leq r$, so $\mathfrak{T}_1 x + \mathfrak{T}_2 y \in B_r$.

Step 2. $\mathfrak{T}_1$ is compact and continuous.

The continuity of f implies $\mathfrak{T}_1$ is continuous, also $\mathfrak{T}_1$ is uniformly bounded on B_r as

$$\|\mathfrak{T}_1 x\|_{PC} \leq \|\mu\|_{L^{\frac{1}{\sigma}}} \frac{1}{\Gamma(q)} \left(\frac{T^{q-\sigma-1}\left(1-e^{-\lambda T}\right)}{\lambda \left(\frac{q-\sigma-1}{1-\sigma}\right)^{1-\sigma}} (1 + \|\nu_1\|) + \frac{T^{q-\sigma-1}}{\left(\frac{q-\sigma-1}{1-\sigma}\right)^{1-\sigma}} \|\nu_2\| \right) \leq r.$$

For equicontinuity on $[0,t_1]$, let $x \in B_r$ and for any $s_1, s_2 \in [0,t_1]$, $s_1 < s_2$, we have

$$\begin{aligned}
|(\mathfrak{T}_1x)(s_2) - (\mathfrak{T}_1x)(s_1)| = \Bigg| & \int_0^{s_2} e^{-\lambda(s_2-s)} \left(\int_0^s \frac{(s-\tau)^{q-2}}{\Gamma(q-1)} f(\tau, x(\tau)) d\tau \right) ds \\
& + v_1(s_2) \int_0^T e^{-\lambda(T-s)} \left(\int_0^s \frac{(s-\tau)^{q-2}}{\Gamma(q-1)} f(\tau, x(\tau)) d\tau \right) ds \\
& + \frac{v_2(s_2)}{\Gamma(q-1)} \int_0^T (T-s)^{q-2} f(s, x(s)) ds \Bigg| \\
& - \Bigg| \int_0^{s1} e^{-\lambda(s_1-s)} \left(\int_0^s \frac{(s-\tau)^{q-1}}{\Gamma(q-1)} f(\tau, x(\tau)) d\tau \right) ds \\
& v_1(s_1) \int_0^T e^{-\lambda(T-s)} \left(\int_0^s \frac{(s-\tau)^{q-1}}{\Gamma(q-1)} f(\tau, x(\tau)) d\tau \right) ds \\
& + \frac{v_2(s_1)}{\Gamma(q-1)} \int_0^T (T-s)^{q-2} f(s, x(s)) ds \Bigg|,
\end{aligned}$$

$$\begin{aligned}
|(\mathfrak{T}_1x)(s_2) - (\mathfrak{T}_1x)(s_1)| \le & \left(e^{-\lambda(s_2)} - e^{-\lambda(s_1)} \right) \int_0^{s_1} e^{\lambda s} \left(\int_0^s \frac{(s-\tau)^{q-1}}{\Gamma(q-1)} d\tau \right) ds \\
& + \int_{s_1}^{s_2} e^{-\lambda(s_2-s)} \left(\int_0^s \frac{(s-\tau)^{q-1}}{\Gamma(q-1)} d\tau \right) ds \\
& + |v_1(s_2) - v_1(s_1)| \int_0^T e^{-\lambda(T-s)} \left(\int_0^s \frac{(s-\tau)^{q-1}}{\Gamma(q-1)} d\tau \right) ds \\
& + |v_2(s_2) - v_2(s_1)| \frac{v_2(s_1)}{\Gamma(q-1)} \int_0^T (T-s)^{q-2} ds.
\end{aligned}$$

It tends to zero as $s_1 \to s_2$. This implies that $\mathfrak{T}_1$ is equicontinuous on the interval $[0,t_1]$. In general, for the time interval $(t_k, t_{k+1}]$, we similarly obtain the same inequality, which yields that $\mathfrak{T}_1$ is equicontinuous on interval $(t_k, t_{k+1}]$. Together with the PC-type Arzela-Ascoli (Lemma 4) theorem, we can conclude that $\mathfrak{T}_1 : B_r \to B_r$ is continuous and compact.

Step 3. It is clearly that $\mathfrak{T}_2$ is contraction mapping.

Thus all the assumptions of the Krasnoselskii theorem are satisfied. In consequence, the the Krasnoselskii theorem is applied and hence the problem (1) has at least one solution on J. ■

Our second existence result is based on the nonlinear alternative of Leray-Schauder type. Assume that

($\mathbf{H}_5$) There exist $\vartheta_f \in PC(J,R)$ and $\Psi : R^+ \to R^+$ continuous and nondecreasing such that

$$|f(t,x)| \le \vartheta_f(t)\Psi(\|x\|), \text{for all } (t,x) \in J \times R,$$

($\mathbf{H}_6$) There exist an number $N > 0$ such that

$$\frac{N}{L_{\mathfrak{T}} \|\vartheta\| \Psi(N)} > 1.$$

Theorem 9 *Suppose that* (H_1)*,* (H_2)*,* (H_5)*,*(H_6) *are hold. Then our BVP in (1) has at least one solution on* J*.*

Proof. Consider the operator $\mathfrak{T} : PC(J,R) \to PC(J,R)$ defined by (15). It can be easily shown that $\mathfrak{T}$ is continuous and compact. maps bounded sets into bounded sets in $PC(J,R)$. Repeating the same process

in Step 2 of Theorem 8, we get

$$\begin{aligned}|(\mathfrak{T}x)(t)| \leq & \int_0^t e^{-\lambda(t-s)}\mathbf{I}^{q-1}\left|f(s,x(s))\right|ds+|v_1(t)|\int_0^T e^{-\lambda(T-s)}\mathbf{I}^{q-1}\left|f(s,x(s))\right|ds \\ & +|v_2(t)|\,\mathbf{I}^{q-1}\left|\rho(T)\right|+|v_3(t)|\sum_{j=1}^{p}|\varphi_j(x(t_j))|+|v_4(t)|\sum_{j=1}^{p}\left|\varphi_j^*(x(t_j)\right| \\ & +\sum_{j=1}^{p}|z_{1j}(t)|\left|\varphi_j^*(x(t_j))\right|+\sum_{j=k+1}^{p}|z_{2j}(t)|\left|\varphi_j^*(x(t_j))\right|+\sum_{j=k+1}^{p}|\varphi_j(x(t_j))|+|z_3(t)|,\end{aligned}$$

■

Theorem 10 ***Proof.***

$$\begin{aligned}&\leq \int_0^t e^{-\lambda(t-s)}\mathbf{I}^{q-1}\vartheta_f(s)\,\Psi(\|x\|)\,ds+|v_1(t)|\int_0^T e^{-\lambda(T-s)}\mathbf{I}^{q-1}\vartheta_f(s)\,\Psi(\|x\|)\,ds \\ &+|v_2(t)|\,\mathbf{I}^{q-1}\left|\rho(T)\right|\vartheta_f(s)\,\Psi(\|x\|)+|v_3(t)|\sum_{j=1}^{p}|\varphi_j(x(t_j))|+|v_4(t)|\sum_{j=1}^{p}\left|\varphi_j^*(x(t_j)\right| \\ &+\sum_{j=1}^{p}|z_{1j}(t)|\left|\varphi_j^*(x(t_j))\right|+\sum_{j=k+1}^{p}|z_{2j}(t)|\left|\varphi_j^*(x(t_j))\right|+\sum_{j=k+1}^{p}|\varphi_j(x(t_j))|+|z_3(t)|,\end{aligned}$$

$$\begin{aligned}|x(t)| \leq |(\mathfrak{T}x)(t)| \leq & \frac{1}{\Gamma(q)}\left(\frac{T^{q-\sigma-1}\left(1-e^{-\lambda T}\right)}{\lambda\left(\frac{q-\sigma-1}{1-\sigma}\right)^{1-\sigma}}(1+\|\nu_1\|)+\frac{T^{q-\sigma-1}}{\left(\frac{q-\sigma-1}{1-\sigma}\right)^{1-\sigma}}\|\nu_2\|\right)\|\vartheta\|\,\Psi(\|x\|) \\ & +(1+\|\nu_3\|)\,pM_{\varphi}+(\|\nu_4\|+\|z_{1j}\|+\|z_{2j}\|)\,pM_{\varphi^*}+\|z_3\|.\end{aligned}$$

Now, construct the set $\Lambda=\{x\in PC(J,R):\|x\|<N\}$ *.The operator* $\mathfrak{T}:\overline{\Lambda}\to PC(J,R)$ *is continuous and completely continuous. From the choice of* Λ*, there is no* $x\in\partial\Lambda$ *such that* $x=\lambda\mathfrak{T}x$*,* $0\leq\lambda\leq 1$*. As a consequence of the nonlinear alternative of Leray–Schauder type, we deduce that* $\mathfrak{T}$ *has a fixed point* $x\in\partial\Lambda$*, which implies that the problem (1) has at least one solution. This completes the proof.* ■

4 Example

In this section we give some examples to illustrate the usefulness of our main results.

Example 1. Consider the following ISFDE:

$$\begin{aligned}({}^cD^{\frac{3}{2}}+2\,{}^cD^{\frac{1}{2}})x(t) &= 0.01\left(t^2+\sin t+1+\tan^{-1}x(t)\right),\ t\in[0,1], \\ x(0)+x'(0) &= \eta_2, x(1)+x'(1)=\eta_2, \\ \Delta x(\frac{1}{4}) &= \Delta x(t_k)=0.01\frac{\|x\|}{1+\|x\|},\Delta x'(\frac{1}{4})=0.01\frac{\|x\|}{1+\|x\|},k=1,2,...,p.\end{aligned}\tag{17}$$

Here $t\in[0,1]$, let $\alpha_1=1,\alpha_2=1,\beta_1=1,\beta_2=1,\ \beta=\frac{3}{2},\lambda=2,T=1,\eta_1,\eta_2=0,\ L_{\varphi},L_{\varphi^*},=0.01$, $f(t,x))=L\left(t^2+\sin t+1+\tan^{-1}x\right)$.

A simple calculations show that

$$L_{\mathfrak{T}}:=\left(\frac{T^{\frac{3}{2}-1}}{2\Gamma(\frac{3}{2})}(1-e^{-2})\,(1+2.312)+\frac{1^{\frac{3}{2}-1}}{\Gamma(\frac{3}{2})}2.312\right)0.01+(1+1.312)\,0.01+(0.656+1.152+0.002)\,0.01<1,$$

where we used the inequality $0.88 < \Gamma(\frac{3}{2}) < 0.89$.

To apply Theorem 6 we need to show conditions $(\mathrm{H}_1)-(\mathrm{H}_3)$ are satisfied. Indeed, f is jointly continuous and

(H_1) $|f(t,x) - f(t,y)| = 0.01 \left|tan^{-1}x - tan^{-1}y\right| \leq 0.01|x-y|$.

(H_2) $L_{\mathfrak{T}} = 0.042 + 0.248 < 1$.

Therefore, by (6), ISFDE (17) has a unique solution on $[0,1]$.

References

[1] LakshmikanthamV, Leela S, Vasundhara Devi J: Theory of Fractional Dynamic Systems. Cambridge Scientific Publishers, Cambridge; 2009.

[2] Kilbas A. A, Srivastava H.M, Trujillo J., North-Holland Mathematics Studies 204. In Theory and Applications of Fractional Differential Equations. Elsevier, Amsterdam; 2006.

[3] Liang S, Zhang J: Existence of multiple positive solutions for m-point fractional boundary value problems on an infinite interval. Math. Comput. Model. 54, 1334-1346 (2011).

[4] Su X., Solutions to boundary value problem of fractional order on unbounded domains in a Banach space. Nonlinear Anal. 74, 2844-2852 (2011).

[5] Bai Z. B, Sun W., Existence and multiplicity of positive solutions for singular fractional boundary value problems. Comput. Math. Appl. 63, 1369-1381 (2012).

[6] Agarwal R.P, O'Regan D, Stanek S., Positive solutions for mixed problems of singular fractional differential equations. Math. Nachr. 285, 27-41 (2012).

[7] Podlubny I. Fractional Differential Equations. Academic Press, San Diego (1999).

[8] Kilbas A.A, Srivastava„ H.M, Trujillo„ J.J.,: Theory and Applications of Fractional Differential Equations. North-Holland Mathematics Studies, vol. 204. Elsevier, Amsterdam (2006).

[9] Bainov D, Simeonov P., Impulsive Differential Equations: Periodic Solutions and Applications. Pitman Monographs and Surveys in Pure and Applied Mathematics (1993).

[10] Zhang L, Wang G, Ahmad B, Agarwal R.P., Nonlinear fractional integro-differential equations on unbounded domains in a Banach space. J. Comput. Appl. Math. 249, 51-56 (2013).

[11] Yang, X-J: Fractional derivatives of constant and variable orders applied to anomalous relaxation models in heat-transfer problems. Therm. Sci. (2016).

[12] Yang X-J, Srivastava H.M, Machado J. A., A new fractional derivative without singular kernel: application to the modelling of the steady heat flow. Therm. Sci. 20, 753-756 (2016).

[13] Miller K.S, Ross B., An Introduction to the Fractional Calculus and Fractional Differential Equations. Wiley, New York (1993).

[14] Delbosco D, Rodino L., Existence and uniqueness for a nonlinear fractional differential equation, J. Math. Anal. Appl. 204 (1996), 609-625.

[15] TianY , Bai Z., :Existence results for the three-point impulsive boundary value problem involving fractional differential equations. 59, Issue 8, Pages 2601-2609, April (2010).

[16] Wang X., Impulsive boundary Value Problem for nonlinear differential Equations with Fractional order. Computers and Mathematics with Applications. Vol 62. September 2011. Pages 2383-2391.

[17] Mahmudov, N. I, Unul, S: On existence of BVP's for impulsive fractional differential equations. Advances in Difference Equations. 2017, 2017: Article ID 15.

[18] Mahmudov, N. I.; Mahmoud, H. Four-point impulsive multi-orders fractional boundary value problems, Journal of Computational Analysis and Applications, 2017, Volume: 22 Issue: 7 Pages: 1249-1260

[19] Ahmad B., Ntouyas S.K., Existence results for a coupled system of Caputo type sequential fractional differential equations with nonlocal integral boundary conditions, Appl. Math. Comput., 2015, 266, 615-622

[20] Ahmad B, Ntouyas S. K., On higher-order sequential fractional differential inclusions with nonlocal three-point boundary conditions. Abstr. Appl. Anal. 2014, Article ID 659405 (2014).

[21] Alsaedi A, Ahmad, B, Aqlan, M: Sequential fractional differential equations and unification of anti-periodic and multi-point boundary conditions. J. Nonlinear Sci. Appl. 10, 71–83, (2017).

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 27, NO.2, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC

The Differentiability and Gradient for Fuzzy Mappings Based on The Generalized Difference of Fuzzy Numbers *

Shexiang Hai†, Fangdi Kong

[a] School of Science, Lanzhou University of Technology, Lanzhou, 730050, P.R. China

Abstract In this paper, the concepts of differentiability and gradient for fuzzy mappings are presented and discussed using the characteristic theorem for generalized difference of n dimensional fuzzy numbers. The relationships of gradient, *support-function-wise* gradient and *level-wise* gradient are characterized.
Keywords: Fuzzy numbers, Fuzzy mappings, Differentiability, Gradient.

1. Introduction

Since the concept and operations of fuzzy set were introduced by Zadeh [1], many studies have focused on the theoretical aspects and applications of fuzzy sets. Soon after, Zadeh proposed the notion of fuzzy numbers in [2, 3, 4]. Since then, fuzzy numbers have been extensively investigated by many authors. Since then, fuzzy numbers have been extensively investigated by many authors. Fuzzy numbers are a powerful tool for modeling uncertainty and for processing vague or subjective information in mathematical models.

As part of the development of theories about fuzzy numbers and its applications, researchers began to study the differentiability and integrability of fuzzy mappings. Initially, the derivative for fuzzy mappings from an open subset of a normed space into the n dimension fuzzy number space E^n was developed by Puri and Ralescu [5], which generalized and extended the concept of Hukuhara differentiability for set-valued mappings. In 1987, Kaleva [6] discussed the G-derivative, and obtained a sufficient condition for the H-differentiability of the fuzzy mappings from $[a, b]$ into E^n as well as a necessary condition for the H-differentiability of fuzzy mapping from $[a, b]$ into E^1. In 2003, Wang and Wu [7] put forward the concepts of directional derivative, differential and sub-differential of fuzzy mappings from R^n into E^1 by using Hukuhara difference. However, the Hukuhara difference between two fuzzy numbers exists only under very restrictive conditions [6] and the H-difference of two fuzzy numbers does not always exist [8]. The g-difference proposed in [8, 9] overcomes these shortcomings of the above discussed concepts and the g-difference of two fuzzy numbers always exists. Based on the novel generalizations of the Hukuhara difference for fuzzy sets, Bede [10] introduced and studied new generalized differentiability concepts for fuzzy valued functions in 2013.

The purpose of the present paper is to use the fuzzy g-difference introduced in [10] to define and study differentiability and gradient for fuzzy mappings. First of all, we give the preliminary terminology used in the present paper. And then, in Section 3, the differentiability and gradient were presented and the relations among gradient, *support-function-wise* gradient and *level-wise* gradient for fuzzy mappings are examined.

2. Preliminaries

In this section, basic definitions and operations for fuzzy numbers are presented [11, 12, 13, 14].

Throughout this paper, $F(R^n)$ denote the set of all fuzzy subsets on n dimensional Euclidean space R^n. A fuzzy subset $\widetilde{u}$ (in short, a fuzzy set) on R^n is a function $\widetilde{u} : R^n \to [0, 1]$. For each fuzzy sets $\widetilde{u}$, we

*This work is supported by National Natural Science Fund of China (11761047).
†Corresponding author. Tel.: +86 931 2973590. E-mail address: haishexiang@lut.cn.

denote its r-level set as $[\widetilde{u}]^r = \{x \in R^n : \widetilde{u}(x) \geq r\}$ for any $r \in (0,1]$. The support of $\widetilde{u}$ is denoted by supp$\widetilde{u} = \{x \in R^n : \widetilde{u}(x) > 0\}$. The closure of supp$\widetilde{u}$ defines the 0-level of $\widetilde{u}$, i.e. $[\widetilde{u}]^0 = cl(\text{supp}\widetilde{u})$. Here $cl(M)$ denotes the closure of set M. Fuzzy set $\widetilde{u} \in F(R^n)$ is called a fuzzy number if
(1) $\widetilde{u}$ is a normal fuzzy set, i.e., there exists an $x_0 \in R^n$ such that $\widetilde{u}(x_0) = 1$,
(2) $\widetilde{u}$ is a convex fuzzy set, i.e., $\widetilde{u}(\lambda x + (1-\lambda)y) \geq \min\{\widetilde{u}(x), \widetilde{u}(y)\}$ for any $x, y \in R^n$ and $\lambda \in [0,1]$,
(3) $\widetilde{u}$ is upper semicontinuous ,
(4) $[\widetilde{u}]^0 = cl(\text{supp}\widetilde{u}) = cl(\bigcup_{r\in(0,1]}[\widetilde{u}]^r)$ is compact.

We will denote E^n the set of fuzzy numbers [11, 12, 13].

It is clear that any $u \in R^n$ can be regarded as a fuzzy number $\widetilde{u}$ defined by

$$\widetilde{u}(x) = \begin{cases} 1, & x = u, \\ 0, & \text{otherwise.} \end{cases}$$

In particular, the fuzzy number $\widetilde{0}$ is defined as $\widetilde{0}(x) = 1$ if $x = 0$, and $\widetilde{0}(x) = 0$ otherwise.

Theorem 2.1.[6, 13] If $\widetilde{u} \in E^n$, then
(1) $[\widetilde{u}]^r$ is a nonempty compact convex subset of R^n for any $r \in (0,1]$,
(2) $[\widetilde{u}]^{r_1} \subseteq [\widetilde{u}]^{r_2}$, whenever $0 \leq r_2 \leq r_1 \leq 1$,
(3) if $r_k > 0$ and r_k is a nondecreasing sequence converging to $r \in (0,1]$, then $\bigcap_{k=1}^{\infty}[\widetilde{u}]^{r_n} = [\widetilde{u}]^r$.
Conversely, if $\{[A]^r \subseteq R^n : r \in [0,1]\}$ satisfies the conditions (1)-(3), then there exists a unique $\widetilde{u} \in E^n$ such that $[\widetilde{u}]^r = [A]^r$ for each $r \in (0,1]$ and $[\widetilde{u}]^0 = cl(\bigcup_{r\in(0,1]}[\widetilde{u}]^r) \subseteq A^0$.

Let $\widetilde{u},\ \widetilde{v} \in E^n$ and $k \in R$. For any $x \in R^n$, the addition $\widetilde{u}+\widetilde{v}$ and scalar multiplication $k\widetilde{u}$ can be defined, respectively, as:

$$(\widetilde{u}+\widetilde{v})(x) = \sup_{s+t=x} \min\{\widetilde{u}(s), \widetilde{v}(t)\},$$

$$(k\widetilde{u})(x) = \widetilde{u}(\frac{x}{k}), k \neq 0,$$

$$(0\widetilde{u})(x) = \begin{cases} 0, & x \neq 0, \\ 1, & x = 0. \end{cases}$$

It is well known that for any $\widetilde{u}, \widetilde{v} \in E^n$ and $k \in R$, the addition $\widetilde{u}+\widetilde{v}$ and the scalar multiplication $k\widetilde{u}$ have the level sets

$$[\widetilde{u}+\widetilde{v}]^r = [\widetilde{u}]^r + [\widetilde{v}]^r = \{x + y : x \in [\widetilde{u}]^r, y \in [\widetilde{v}]^r\},$$

$$[k\widetilde{u}]^r = k[\widetilde{u}]^r = \{kx : x \in [\widetilde{u}]^r\},$$

for any $r \in [0,1]$.

The Hausdorff distance $D : E^n \times E^n \to [0, +\infty)$ on E^n is defined by

$$D(\widetilde{u}, \widetilde{v}) = \sup_{r\in[0,1]} d([\widetilde{u}]^r, [\widetilde{v}]^r),$$

where d is the Hausdorff metric given by

$$\begin{aligned} d([\widetilde{u}]^r, [\widetilde{v}]^r) &= \inf\{\varepsilon : [\widetilde{u}]^r \subset N([\widetilde{v}]^r, \varepsilon), [\widetilde{v}]^r \subset N([\widetilde{u}]^r, \varepsilon)\} \\ &= \max\{\sup_{a\in[\widetilde{u}]^r} \inf_{b\in[\widetilde{v}]^r} \|a-b\|, \sup_{b\in[\widetilde{v}]^r} \inf_{a\in[\widetilde{u}]^r} \|a-b\|\}. \end{aligned}$$

$N([\widetilde{u}]^r, \varepsilon) = \{x \in R^n : d(x, [\widetilde{u}]^r) = \inf_{y\in[\widetilde{u}]^r} d(x,y) \leq \varepsilon\}$ is the ε-neighborhood of $[\widetilde{u}]^r$. Then (E^n, D) is a complete metric space, and satisfies $D(\widetilde{u}+\widetilde{w}, \widetilde{v}+\widetilde{w}) = D(\widetilde{u}, \widetilde{v})$, $D(k\widetilde{u}, k\widetilde{v}) = |k|D(\widetilde{u}, \widetilde{v})$ for any $\widetilde{u}, \widetilde{v}, \widetilde{w} \in E^n$ and $k \in R$.

Let $S^{n-1} = \{x \in R^n : \|x\| = 1\}$ be the unit sphere of R^n and $\langle \cdot, \cdot \rangle$ be the inner product in R^n, i.e. $\langle x, y \rangle = \sum_{i=1}^n x_i y_i$, where $x = (x_1, x_2, \cdots, x_n) \in R^n$, $y = (y_1, y_2, \cdots, y_n) \in R^n$. Suppose $\widetilde{u} \in E^n$, $r \in [0,1]$ and $x \in S^{n-1}$, the support function of $\widetilde{u}$ is defined by

$$\widetilde{u}^*(r, x) = \sup_{a \in [\widetilde{u}]^r} \langle a, x \rangle.$$

Theorem 2.2.[14] Suppose $\widetilde{u} \in E^n$, $r \in [0,1]$, then

$$[\widetilde{u}]^r = \{y \in R^n : \langle y, x \rangle \leq \widetilde{u}^*(r, x), x \in S^{n-1}\}.$$

The theorem below will give some basic properties of the support function.

Theorem 2.3.[14, 15] Suppose $\widetilde{u} \in E^n$, then
(1) $\widetilde{u}^*(r, x + y) \leq \widetilde{u}^*(r, x) + \widetilde{u}^*(r, y)$,
(2) $\widetilde{u}^*(r, x) \leq \sup_{a \in [\widetilde{u}]^r} \| a \|$, i.e. $\widetilde{u}^*(r, x)$ is bounded on S^{n-1} for each fixed $r \in [0,1]$,
(3) $\widetilde{u}^*(r, x)$ is nonincreasing and left continuous in $r \in [0,1]$, right continuous at $r = 0$, for each fixed $x \in S^{n-1}$,
(4) $\widetilde{u}^*(r, x)$ is Lipschitz continuous in x, i.e.

$$|\widetilde{u}^*(r, x) - \widetilde{u}^*(r, y)| \leq (\sup_{a \in [\widetilde{u}]^r} \|a\|) \|x - y\|,$$

(5) if $\widetilde{u}, \widetilde{v} \in E^n$, $r \in [0,1]$, then

$$d([\widetilde{u}]^r, [\widetilde{v}]^r) = \sup_{x \in S^{n-1}} |\widetilde{u}^*(r, x) - \widetilde{v}^*(r, x)|,$$

(6) $(\widetilde{u} + \widetilde{v})^*(r, x) = \widetilde{u}^*(r, x) + \widetilde{v}^*(r, x)$,
(7) $(k\widetilde{u})^*(r, x) = k\widetilde{u}^*(r, x)$, for any $k \geq 0$,
(8) $-\widetilde{u}^*(r, -x) \leq \widetilde{u}^*(r, x)$,
(9) $(-\widetilde{u})^*(r, x) = \widetilde{u}^*(r, -x)$.

Definition 2.1. [10] The generalized difference (g-difference for short) of two fuzzy numbers $\widetilde{u}, \widetilde{v} \in E^n$ is given by its level sets as

$$[\widetilde{u} \ominus_g \widetilde{v}]^r = cl(\bigcup_{\beta \geq r} ([\widetilde{u}]^\beta \ominus_{gH} [\widetilde{v}]^\beta)), \quad r \in [0,1],$$

where the gH-difference $\ominus_{gH}$ is with interval operands $[\widetilde{u}]^\beta$ and $[\widetilde{v}]^\beta$.

Remark 2.1. A necessary condition for $\widetilde{u} \ominus_g \widetilde{v}$ to exist is that either $[\widetilde{u}]^r$ contains a translate of $[\widetilde{v}]^r$ or $[\widetilde{v}]^r$ contains a translate of $[\widetilde{u}]^r$ for any $r \in [0,1]$.

Theorem 2.4. [15] Let $\widetilde{u}, \widetilde{v} \in E^n$. If the g-difference $\widetilde{u} \ominus_g \widetilde{v}$ of $\widetilde{u}$ and $\widetilde{v}$ exists, then for any $r \in [0,1]$ and $x \in S^{n-1}$, we have

$$\begin{aligned} (\widetilde{u} \ominus_g \widetilde{v})^*(r, x) &= \begin{cases} \quad (1) & \sup_{\beta \geq r} (\widetilde{u}^*(\beta, x) - \widetilde{v}^*(\beta, x)), \\ \text{or } (2) & \sup_{\beta \geq r} ((-\widetilde{v})^*(\beta, x) - (-\widetilde{u})^*(\beta, x)), \end{cases} \\ &= \begin{cases} \quad (1) & \sup_{\beta \geq r} (\widetilde{u}^*(\beta, x) - \widetilde{v}^*(\beta, x)), \\ \text{or } (2) & \sup_{\beta \geq r} (\widetilde{v}^*(\beta, -x) - \widetilde{u}^*(\beta, -x)). \end{cases} \end{aligned}$$

Theorem 2.5.[15] Let $\widetilde{u}, \widetilde{v} \in E^n$. Then
(1) if the g-difference exists, it is unique,
(2) $\widetilde{u} \ominus_g \widetilde{u} = 0$,
(3) $(\widetilde{u} + \widetilde{v}) \ominus_g \widetilde{v} = \widetilde{u}$, $(\widetilde{u} + \widetilde{v}) \ominus_g \widetilde{u} = \widetilde{v}$,
(4) $\widetilde{u} \ominus_g \widetilde{v} = -(\widetilde{v} \ominus_g \widetilde{u})$.

3. The differentiability and gradient for fuzzy mappings

In [5], Puri and Ralescu defined the g-derivative of fuzzy mappings from an open subset of a normed space into n-dimension fuzzy number space E^n by using Hukuhara difference. In [7], Wang and Wu defined the directional g-derivative of fuzzy mappings from R^n into E^1. Based on the generalizations of the Hukuhara difference for fuzzy sets, Bede [10] introduced and studied new generalized differentiability concepts for fuzzy valued functions from R into E^1. The new generalized differentiability concept is a useful and applicable tool dealing with fuzzy differential equations and fuzzy optimization problems. In the following, using the characteristic theorem for generalized difference of n dimensional fuzzy numbers introduced in [15], we define and study differentiability and gradient for fuzzy mappings.

Definition 3.1. Let $\widetilde{F} : M \to E^n$, $t_0 = (t_1^0, t_2^0, \cdots, t_m^0) \in \mathrm{int}M$ and $t = (t_1, t_2, \cdots, t_m) \in \mathrm{int}M$. If g-difference $\widetilde{F}(t) \ominus_g \widetilde{F}(t_0)$ exists and there exist $\widetilde{u}_j \in E^n$ $(j = 1, 2, \cdots, m)$, such that

$$\lim_{t \to t_0} \frac{D(\widetilde{F}(t) \ominus_g \widetilde{F}(t_0), \sum_{j=1}^m \widetilde{u}_j(t_j - t_j^0))}{d(t, t_0)} = 0,$$

then we say that $\widetilde{F}$ is differentiable at t_0 and the fuzzy vector $(\widetilde{u}_1,\ \widetilde{u}_2, \cdots, \widetilde{u}_m)$ is the gradient of $\widetilde{F}$ at t_0, denoted by $\nabla\widetilde{F}(t_0)$, i.e., $\nabla\widetilde{F}(t_0) = (\widetilde{u}_1, \widetilde{u}_2, \cdots, \widetilde{u}_m)$.

Remark 3.1. Let $\widetilde{F} : M \to E^n$, $t_0 = (t_1^0, \cdots, t_j^0, \cdots, t_m^0) \in \mathrm{int}M$ and $h \in R$ with $t = (t_1^0, \cdots, t_j^0 + h, \cdots, t_m^0) \in \mathrm{int}M$. Then the gradient $\nabla\widetilde{F}(t_0)$ exists at t_0 if and only if $\widetilde{F}(t) \ominus_g \widetilde{F}(t_0)$ exists and there are $\widetilde{u}_j \in E^n$ $(j = 1, 2, \cdots, m)$, such that

$$\widetilde{u}_j = \lim_{h \to 0} \frac{\widetilde{F}(t_1^0, \cdots, t_j^0 + h, \cdots, t_m^0) \ominus_g \widetilde{F}(t_1^0, \cdots, t_j^0, \cdots, t_m^0)}{h}.$$

Here the limit is taken in the metric space (E^n, D).

Theorem 3.1. The gradient $\nabla\widetilde{F}(t)$ of fuzzy mapping $\widetilde{F} : M \to E^n$ is unique if it exists.

Proof. Suppose we have two gradients $(\widetilde{u}_1,\ \widetilde{u}_2, \cdots,\ \widetilde{u}_m)$ and $(\widetilde{v}_1,\ \widetilde{v}_2,\ \cdots,\ \widetilde{v}_m)$ for fuzzy mapping $\widetilde{F}$ at t_0. For any $\varepsilon > 0$, according to Remark 3.1, there exist two positive real numbers δ_1 and δ_2, when $|h| < \delta_1$, we have

$$D(\widetilde{F}(t_1^0, \cdots, t_j^0 + h, \cdots, t_m^0) \ominus_g \widetilde{F}(t_1^0, \cdots,\ t_j^0, \cdots,\ t_m^0), h\widetilde{u}_j) < \frac{|h|}{2}\varepsilon \quad (j = 1, 2, \cdots, m),$$

when $|h| < \delta_2$, we have

$$D(\widetilde{F}(t_1^0, \cdots, t_j^0 + h, \cdots, t_m^0) \ominus_g \widetilde{F}(t_1^0, \cdots,\ t_j^0, \cdots,\ t_m^0), h\widetilde{v}_j) < \frac{|h|}{2}\varepsilon \quad (j = 1, 2, \cdots, m).$$

Setting $|h| < \min(\delta_1, \delta_2)$, we obtain,

$$\begin{aligned}
& D(\widetilde{u}_j, \widetilde{v}_j) \\
= \ & \tfrac{1}{|h|} D(h\widetilde{u}_j, h\widetilde{v}_j) \\
\leq \ & \tfrac{1}{|h|} D(\widetilde{F}(t_1^0, \cdots, t_j^0 + h, \cdots, t_m^0) \ominus_g \widetilde{F}(t_1^0, \cdots,\ t_j^0, \cdots,\ t_m^0), h\widetilde{u}_j) \\
+ \ & \tfrac{1}{|h|} D(\widetilde{F}(t_1^0, \cdots, t_j^0 + h, \cdots, t_m^0) \ominus_g \widetilde{F}(t_1^0, \cdots,\ t_j^0, \cdots,\ t_m^0), h\widetilde{v}_j) \\
< \ & \varepsilon.
\end{aligned}$$

Then $\widetilde{u}_j = \widetilde{v}_j$ $\ (j = 1, 2, \cdots, m)$, which implies that the gradient $\nabla\widetilde{F}(t)$ of fuzzy mapping $\widetilde{F}$ at t_0 is unique.

Definition 3.2. Let $\widetilde{F} : M \to E^n$, $t_0 = (t_1^0, t_2^0, \cdots, t_m^0) \in \mathrm{int}M$ and $t = (t_1, t_2, \cdots, t_m) \in \mathrm{int}M$. If there exist $\widetilde{u}_j \in E^n$ $(j = 1, 2, \cdots, m)$, such that

$$\lim_{t \to t_0} \frac{|\widetilde{F}(t)^*(r, x) - \widetilde{F}(t_0)^*(r, x) - \sum_{j=1}^m \widetilde{u}_j^*(r, x)(t_j - t_j^0)|}{d(t, t_0)} = 0,$$

uniformly for any $r \in [0,1]$ and $x \in S^{n-1}$, then we say that $\widetilde{F}$ is *support-function-wise* differentiable (*s*-differentiable for short) at t_0 and the fuzzy vector $(\widetilde{u}_1,\ \widetilde{u}_2,\cdots,\widetilde{u}_m)$ is the *support-function-wise* gradient of $\widetilde{F}$ at t_0, denoted by $\nabla_s\widetilde{F}(t_0)$, i.e., $\nabla_s\widetilde{F}(t_0)=(\widetilde{u}_1,\widetilde{u}_2,\cdots,\widetilde{u}_m)$.

Remark 3.2. Let $\widetilde{F}: M\to E^n$, $t_0=(t_1^0,\cdots,t_j^0,\cdots,t_m^0)\in \mathrm{int}M$ and $h\in R$ with $t=(t_1^0,\cdots,t_j^0+h,\cdots,t_m^0)\in \mathrm{int}M$. Then the *support-function-wise*-gradient $\nabla_s\widetilde{F}(t_0)$ exists at t_0 if and only if there are $\widetilde{u}_j\in E^n$ $(j=1,2,\cdots,m)$, such that

$$\widetilde{u}_j^*(r,x)=\lim_{h\to 0}\frac{\widetilde{F}(t_1^0,\cdots,t_j^0+h,\cdots,t_m^0)^*(r,x)-\widetilde{F}(t_1^0,\cdots,\ t_j^0,\cdots,\ t_m^0)^*(r,x)}{h},$$

uniformly for any $r\in[0,1]$ any $x\in S^{n-1}$.

Theorem 3.2. The *support-function-wise* gradient $\nabla_s\widetilde{F}(t)$ of fuzzy mapping $\widetilde{F}$ is unique if it exists.

Theorem 3.3. If fuzzy mapping $\widetilde{F}: M\to E^n$ is *s*-differentiable at $t_0\in intM$, then $-\widetilde{F}$ is *s*-differentiable at t_0 and

$$\nabla_s(-\widetilde{F}(t_0))=-\nabla_s\widetilde{F}(t_0).$$

Proof. If $\widetilde{F}: M\to E^n$ is *s*-differentiable at t_0, then there exist $\widetilde{u}_j\in E^n$ $(j=1,2,\cdots,m)$, such that

$$\lim_{t\to t_0}\frac{|\widetilde{F}(t)^*(r,x)-\widetilde{F}(t_0)^*(r,x)-\sum_{j=1}^m\widetilde{u}_j^*(r,x)(t_j-t_j^0)|}{d(t,t_0)}=0,$$

uniformly for any $r\in[0,1]$ and $x\in S^{n-1}$, where $t=(t_1,\ t_2,\ \cdots,\ t_m)\in \mathrm{int}M$, then

$$\widetilde{F}(t)^*(r,x)-\widetilde{F}(t_0)^*(r,x)=\sum_{j=1}^m\widetilde{u}_j^*(r,x)(t_j-t_j^0)+\circ(d(t,t_0)),$$

uniformly for any $r\in[0,1]$ and $x\in S^{n-1}$. It follows from Theorem 2.3 that

$$\begin{aligned}
&(-\widetilde{F}(t))^*(r,x)-(-\widetilde{F}(t_0))^*(r,x)\\
=\ &\widetilde{F}(t)^*(r,-x)-\widetilde{F}(t_0)^*(r,-x)\\
=\ &\textstyle\sum_{j=1}^m\widetilde{u}_j^*(r,-x)(t_j-t_j^0)+\circ(d(t,t_0))\\
=\ &\textstyle\sum_{j=1}^m(-\widetilde{u}_j)^*(r,x)(t_j-t_j^0)+\circ(d(t,t_0)),
\end{aligned}$$

uniformly for any $r\in[0,1]$ and $x\in S^{n-1}$. Thus

$$\lim_{t\to t_0}\frac{|(-\widetilde{F}(t))^*(r,x)-(-\widetilde{F}(t_0))^*(r,x)-\sum_{j=1}^m(-\widetilde{u}_j)^*(r,x)(t_j-t_j^0)|}{d(t,t_0)}=0,$$

uniformly for any $r\in[0,1]$ and $x\in S^{n-1}$, which implies that $-\widetilde{F}$ is *s*-differentiable at t_0 and $\nabla_s(-\widetilde{F}(t_0))=-\nabla_s\widetilde{F}(t_0)$.

Theorem 3.4. Let $\widetilde{F}: M\to E^n$, $t_0=(t_1^0,\cdots,t_j^0,\cdots,t_m^0)\in \mathrm{int}M$ and $h\in R$ with $t=(t_1^0,\cdots,t_j^0+h,\cdots,t_m^0)\in \mathrm{int}M$. If the *support-function-wise* gradient $\nabla_s\widetilde{F}(t)$ exists at $t_0\in \mathrm{int}M$ and *g*-difference $\widetilde{F}(t_0+h)\ominus_g\widetilde{F}(t_0)$ exists, then the gradient $\nabla\widetilde{F}(t)$ of $\widetilde{F}$ exists at t_0 and we have

$$\widetilde{u}_j=\widetilde{v}_j\ \ (j=1,2,\cdots,m),$$

where $\nabla\widetilde{F}(t_0)=(\widetilde{u}_1,\ \widetilde{u}_2,\cdots,\ \widetilde{u}_m)$, $\nabla_s\widetilde{F}(t_0)=(\widetilde{v}_1,\ \widetilde{v}_2,\ \cdots,\ \widetilde{v}_m)$.

Proof. Let $\frac{\widetilde{F}(t)\ominus_g\widetilde{F}(t_0)}{h}=\frac{\widetilde{F}(t_1^0,\cdots,t_j^0+h,\cdots,t_m^0)\ominus_g\widetilde{F}(t_1^0,\cdots,\ t_j^0,\cdots,\ t_m^0)}{h}=(\widetilde{u}_j)_h\in E^n$. We can show that the class of sets

$$A_r=\{y\in R^n:\langle y,x\rangle\le\lim_{h\to 0}(\frac{\widetilde{F}(t)\ominus_g\widetilde{F}(t_0)}{h})^*(r,x),x\in S^{n-1}\}$$

satisfies the conditions of Theorem 2.1.

(1) It follows from Theorem 2.1 that

$$[(\widetilde{u}_j)_h]^r = \{y \in R^n : \langle y, x\rangle \leq (\frac{\widetilde{F}(t) \ominus_g \widetilde{F}(t_0)}{h})^*(r, x), x \in S^{n-1}\}$$

is a nonempty compact convex subset of R^n for any $r \in (0, 1]$, then

$$A_r = \{y \in R^n : \langle y, x\rangle \leq \lim_{h\to 0}(\frac{\widetilde{F}(t) \ominus_g \widetilde{F}(t_0)}{h})^*(r, x), x \in S^{n-1}\}$$

is also a nonempty compact convex subset of R^n for any $r \in (0, 1]$.

(2) When $0 \leq r_2 \leq r_1 \leq 1$, $[(\widetilde{u}_j)_h]^{r_1} \subseteq [(\widetilde{u}_j)_h]^{r_2}$, then

$$(\frac{\widetilde{F}(t) \ominus_g \widetilde{F}(t_0)}{h})^*(r_1, x) \leq (\frac{\widetilde{F}(t) \ominus_g \widetilde{F}(t_0)}{h})^*(r_2, x).$$

for any $x \in S^{n-1}$. Thus,

$$\lim_{h\to 0}(\frac{\widetilde{F}(t) \ominus_g \widetilde{F}(t_0)}{h})^*(r_1, x) \leq \lim_{h\to 0}(\frac{\widetilde{F}(t) \ominus_g \widetilde{F}(t_0)}{h})^*(r_2, x),$$

which implies that

$$\begin{aligned} A_{r_1} &= \{y \in R^n : \langle y, x\rangle \leq \lim_{h\to 0}(\tfrac{\widetilde{F}(t)\ominus_g\widetilde{F}(t_0)}{h})^*(r_1, x), x \in S^{n-1}\} \\ &\subseteq \{y \in R^n : \langle y, x\rangle \leq \lim_{h\to 0}(\tfrac{\widetilde{F}(t)\ominus_g\widetilde{F}(t_0)}{h})^*(r_2, x), x \in S^{n-1}\} \\ &= A_{r_2}. \end{aligned}$$

(3) For any r_k increasing to $r \in (0, 1]$, since $\bigcap_{k=1}^{\infty}[(\widetilde{u}_j)_h]^{r_k} = [(\widetilde{u}_j)_h]^r$, that

$$\lim_{k\to\infty}(\frac{\widetilde{F}(t) \ominus_g \widetilde{F}(t_0)}{h})^*(r_k, x) = (\frac{\widetilde{F}(t) \ominus_g \widetilde{F}(t_0)}{h})^*(r, x),$$

for any $x \in S^{n-1}$. Thus

$$\lim_{k\to\infty}\lim_{h\to 0}(\frac{\widetilde{F}(t) \ominus_g \widetilde{F}(t_0)}{h})^*(r_k, x) = \lim_{h\to 0}(\frac{\widetilde{F}(t) \ominus_g \widetilde{F}(t_0)}{h})^*(r, x),$$

which implies that

$$\bigcap_{k=1}^{\infty} A_{r_n} = A_r.$$

Then, there are $\widetilde{u}_j \in E^n$, such that $[\widetilde{u}_j]^r = A_r$ and $[\widetilde{u}_j]^0 = \overline{\bigcup_{r\in(0,1]}[\widetilde{u}]^r} \subseteq A_0$ $(j = 1, 2, \cdots, m)$ for any $r \in (0, 1]$.

When $h > 0$, it follows from Theorem 2.3 that,

$$(\frac{\widetilde{F}(t) \ominus_g \widetilde{F}(t_0)}{h})^*(r, x) = \frac{1}{h}(\widetilde{F}(t) \ominus_g \widetilde{F}(t_0))^*(r, x),$$

for any $r \in [0, 1]$ and $x \in S^{n-1}$. For any $r \in [0, 1]$ and $x \in S^{n-1}$, if taking

$$(\widetilde{F}(t) \ominus_g \widetilde{F}(t_0))^*(r, x) = \sup_{\beta\geq r}(\widetilde{F}(t)^*(\beta, x) - \widetilde{F}(t_0)^*(\beta, x)),$$

then

$$\begin{aligned} \widetilde{u}_j^*(r, x) &= \lim_{h\to 0}(\tfrac{\widetilde{F}(t)\ominus_g\widetilde{F}(t_0)}{h})^*(r, x) \\ &= \lim_{h\to 0}\sup_{\beta\geq r}\tfrac{\widetilde{F}(t)^*(\beta,x)-\widetilde{F}(t_0)^*(\beta,x)}{h} \\ &= \sup_{\beta\geq r}\widetilde{v}_j^*(\beta, x). \end{aligned}$$

According to Theorem 2.3, for any $\varepsilon > 0$, there is $\delta > 0$, when $h < \delta$, we have

$$\begin{aligned}
& D(\tfrac{\widetilde{F}(t)\ominus_g \widetilde{F}(t_0)}{h}, \widetilde{u}_j) \\
= \ & \sup_{r\in[0,1]} \sup_{x\in S^{n-1}} |(\tfrac{\widetilde{F}(t)\ominus_g \widetilde{F}(t_0)}{h})^*(r,x) - \widetilde{u}_j^*(r,x)| \\
= \ & \sup_{r\in[0,1]} \sup_{x\in S^{n-1}} |\sup_{\beta\geq r} \tfrac{\widetilde{F}(t)^*(\beta,x)-\widetilde{F}(t_0)^*(\beta,x)}{h} - \sup_{\beta\geq r} \widetilde{v}_j^*(\beta,x)| \\
< \ & \varepsilon.
\end{aligned}$$

Then, the gradient $\nabla\widetilde{F}(t_0) = (\widetilde{u}_1,\ \widetilde{u}_2, \cdots,\ \widetilde{u}_m)$ of $\widetilde{F}$ exists at t_0 and we have

$$\widetilde{u}_j^*(r,x) = \sup_{\beta\geq r} \widetilde{v}_j^*(\beta,x) = \widetilde{v}_j^*(r,x),$$

for any $r \in [0,1]$ and $x \in S^{n-1}$. On the other hand, for any $r \in [0,1]$ and $x \in S^{n-1}$, if taking

$$(\widetilde{F}(t) \ominus_g \widetilde{F}(t_0))^*(r,x) = \sup_{\beta\geq r}(\widetilde{F}(t_0)^*(\beta,-x) - \widetilde{F}(t)^*(\beta,-x)),$$

we have from Theorem 2.3 and Theorem 2.5 that

$$\begin{aligned}
\widetilde{u}_j^*(r,x) &= \lim_{h\to 0}(\tfrac{\widetilde{F}(t)\ominus_g \widetilde{F}(t_0)}{h})^*(r,x) \\
&= \lim_{h\to 0} \tfrac{1}{h}(\widetilde{F}(t) \ominus_g \widetilde{F}(t_0))^*(r,x) \\
&= \lim_{h\to 0} \tfrac{1}{h}[-(\widetilde{F}(t_0) \ominus_g \widetilde{F}(t))]^*(r,x) \\
&= \lim_{h\to 0} \tfrac{1}{h}(\widetilde{F}(t_0) \ominus_g \widetilde{F}(t))^*(r,-x) \\
&= \lim_{h\to 0} \sup_{\beta\geq r} \tfrac{\widetilde{F}(t)^*(\beta,x)-\widetilde{F}(t_0)^*(\beta,x)}{h} \\
&= \sup_{\beta\geq r} \lim_{h\to 0} \tfrac{\widetilde{F}(t)^*(\beta,x)-\widetilde{F}(t_0)^*(\beta,x)}{h} \\
&= \sup_{\beta\geq r} \widetilde{v}_j^*(\beta,x).
\end{aligned}$$

According to Theorem 2.3, for any $\varepsilon > 0$, there is $\delta > 0$, when $h < \delta$, we have

$$\begin{aligned}
& D(\tfrac{\widetilde{F}(t)\ominus_g \widetilde{F}(t_0)}{h}, \widetilde{u}_j) \\
= \ & \sup_{r\in[0,1]} \sup_{x\in S^{n-1}} |(\tfrac{\widetilde{F}(t)\ominus_g \widetilde{F}(t_0)}{h})^*(r,x) - \widetilde{u}_j^*(r,x)| \\
= \ & \sup_{r\in[0,1]} \sup_{x\in S^{n-1}} |\sup_{\beta\geq r} \tfrac{\widetilde{F}(t)^*(\beta,x)-\widetilde{F}(t_0)^*(\beta,x)}{h} - \sup_{\beta\geq r} \widetilde{v}_j^*(\beta,x)| \\
< \ & \varepsilon.
\end{aligned}$$

Then, the gradient $\nabla\widetilde{F}(t_0) = (\widetilde{u}_1,\ \widetilde{u}_2, \cdots,\ \widetilde{u}_m)$ of $\widetilde{F}$ exists at t_0 and we have

$$\widetilde{u}_j^*(r,x) = \sup_{\beta\geq r} \widetilde{v}_j^*(\beta,x) = \widetilde{v}_j^*(r,x),$$

for any $r \in [0,1]$ and $x \in S^{n-1}$. When $h < 0$, it follows from Theorem 2.3 and Theorem 2.5 that,

$$\begin{aligned}
(\tfrac{\widetilde{F}(t)\ominus_g \widetilde{F}(t_0)}{h})^*(r,x) &= -\tfrac{1}{h}(-(\widetilde{F}(t) \ominus_g \widetilde{F}(t_0)))^*(r,x) \\
&= -\tfrac{1}{h}(\widetilde{F}(t_0) \ominus_g \widetilde{F}(t))^*(r,x),
\end{aligned}$$

for any $r \in [0,1]$ and $x \in S^{n-1}$. For any $r \in [0,1]$ and $x \in S^{n-1}$, if taking

$$(\widetilde{F}(t) \ominus_g \widetilde{F}(t_0))^*(r,x) = \sup_{\beta\geq r}(\widetilde{F}(t)^*(\beta,x) - \widetilde{F}(t_0)^*(\beta,x)),$$

i.e.

$$(\widetilde{F}(t_0) \ominus_g \widetilde{F}(t))^*(r,x) = \sup_{\beta\geq r}(\widetilde{F}(t_0)^*(\beta,x) - \widetilde{F}(t)^*(\beta,x)),$$

then

$$\begin{aligned} \widetilde{u}_j^*(r,x) &= \lim_{h\to 0}(\tfrac{\widetilde{F}(t)\ominus_g \widetilde{F}(t_0)}{h})^*(r,x) \\ &= \lim_{h\to 0}\sup_{\beta\geq r}\tfrac{\widetilde{F}(t)^*(\beta,x)-\widetilde{F}(t_0)^*(\beta,x)}{h} \\ &= \sup_{\beta\geq r}\widetilde{v}_j^*(\beta,x). \end{aligned}$$

According to Theorem 2.3, for any $\varepsilon > 0$, there is $\delta > 0$, when $-h < \delta$, we have

$$\begin{aligned} & D(\tfrac{\widetilde{F}(t)\ominus_g \widetilde{F}(t_0)}{h}, \widetilde{u}_j) \\ = & \sup_{r\in[0,1]}\sup_{x\in S^{n-1}}|(\tfrac{\widetilde{F}(t)\ominus_g \widetilde{F}(t_0)}{h})^*(r,x) - \widetilde{u}_j^*(r,x)| \\ = & \sup_{r\in[0,1]}\sup_{x\in S^{n-1}}|\sup_{\beta\geq r}\tfrac{\widetilde{F}(t)^*(\beta,x)-\widetilde{F}(t_0)^*(\beta,x)}{h} - \sup_{\beta\geq r}\widetilde{v}_j^*(\beta,x)| \\ < & \ \varepsilon. \end{aligned}$$

Then, the gradient $\nabla\widetilde{F}(t_0) = (\widetilde{u}_1,\ \widetilde{u}_2, \cdots,\ \widetilde{u}_m)$ of $\widetilde{F}$ exists at t_0 and

$$\widetilde{u}_j^*(r,x) = \sup_{\beta\geq r}\widetilde{v}_j^*(\beta,x) = \widetilde{v}_j^*(r,x),$$

for any $r \in [0,1]$ and $x \in S^{n-1}$. On the other hand, for any $r \in [0,1]$ and $x \in S^{n-1}$, if taking

$$(\widetilde{F}(t) \ominus_g \widetilde{F}(t_0))^*(r,x) = \sup_{\beta\geq r}(\widetilde{F}(t_0)^*(\beta,-x) - \widetilde{F}(t)^*(\beta,-x)),$$

we have from Theorem 2.3 that

$$\begin{aligned} \widetilde{u}_j^*(r,x) &= \lim_{h\to 0}(\tfrac{\widetilde{F}(t)\ominus_g \widetilde{F}(t_0)}{h})^*(r,x) \\ &= \lim_{h\to 0}[-\tfrac{1}{h}(\widetilde{F}(t) \ominus_g \widetilde{F}(t_0))^*(r,x)] \\ &= \lim_{h\to 0}[-\tfrac{1}{h}\sup_{\beta\geq r}(\widetilde{F}(t_0)^*(\beta,-x) - \widetilde{F}(t)^*(\beta,-x))] \\ &= \lim_{h\to 0}\sup_{\beta\geq r}\tfrac{\widetilde{F}(t)^*(\beta,x)-\widetilde{F}(t_0)^*(\beta,x)}{h} \\ &= \sup_{\beta\geq r}\lim_{h\to 0}\tfrac{\widetilde{F}(t)^*(\beta,x)-\widetilde{F}(t_0)^*(\beta,x)}{h} \\ &= \sup_{\beta\geq r}\widetilde{v}_j^*(\beta,x). \end{aligned}$$

According to Theorem 2.3, for any $\varepsilon > 0$, there is $\delta > 0$, when $-h < \delta$, we have

$$\begin{aligned} & D(\tfrac{\widetilde{F}(t)\ominus_g \widetilde{F}(t_0)}{h}, \widetilde{u}_j) \\ = & \sup_{r\in[0,1]}\sup_{x\in S^{n-1}}|(\tfrac{\widetilde{F}(t)\ominus_g \widetilde{F}(t_0)}{h})^*(r,x) - \widetilde{u}_j^*(r,x)| \\ = & \sup_{r\in[0,1]}\sup_{x\in S^{n-1}}|\sup_{\beta\geq r}\tfrac{\widetilde{F}(t)^*(\beta,x)-\widetilde{F}(t_0)^*(\beta,x)}{h} - \sup_{\beta\geq r}\widetilde{v}_j^*(\beta,x)| \\ < & \ \varepsilon. \end{aligned}$$

Then, the gradient $\nabla\widetilde{F}(t_0) = (\widetilde{u}_1,\ \widetilde{u}_2, \cdots,\ \widetilde{u}_m)$ of $\widetilde{F}$ exists at t_0 and

$$\widetilde{u}_j^*(r,x) = \sup_{\beta\geq r}\widetilde{v}_j^*(\beta,x) = \widetilde{v}_j^*(r,x),$$

for any $r \in [0,1]$ and $x \in S^{n-1}$.

The converse result of Theorem 3.4 is not necessarily true, and hence the g-differentiability and the s-differentiability are not equivalent concepts.

Definition 3.3. Let $\widetilde{F}: M \to E^n$, $t_0 = (t_1^0, t_2^0, \cdots, t_m^0) \in \text{int}M$ and $t = (t_1, t_2, \cdots, t_m) \in \text{int}M$. If for any $r \in [0,1]$, $\widetilde{F}_r(t) \ominus_{gH} \widetilde{F}_r(t_0)$ ($\widetilde{F}_r(t) = [\widetilde{F}(t)]^r$) exist and there exist $\widetilde{u}_j \in E^n$ $(j = 1, 2, \cdots, m)$, such that

$$\lim_{t \to t_0} \frac{d(\widetilde{F}_r(t) \ominus_{gH} \widetilde{F}_r(t_0), \sum_{j=1}^m [\widetilde{u}_j]^r (t_j - t_j^0))}{d(t, t_0)} = 0,$$

uniformly for any $r \in [0,1]$, then we say that $\widetilde{F}$ is *level-wise* differentiable at t_0 and the fuzzy vector $(\widetilde{u}_1, \widetilde{u}_2, \cdots, \widetilde{u}_m)$ is the *level-wise* gradient of $\widetilde{F}$ at t_0, denoted by $\nabla_l \widetilde{F}(t_0)$, i.e., $\nabla_l \widetilde{F}(t_0) = (\widetilde{u}_1, \widetilde{u}_2, \cdots, \widetilde{u}_m)$.

Remark 3.3. Let $\widetilde{F}: M \to E^n$, $t_0 = (t_1^0, \cdots, t_j^0, \cdots, t_m^0) \in \text{int}M$ and $h \in R$ with $t = (t_1^0, \cdots, t_j^0 + h, \cdots, t_m^0) \in \text{int}M$. Then the *level-wise* gradient $\nabla_l \widetilde{F}(t)$ exists at t_0 if and only if for any $r \in [0,1]$, $\widetilde{F}_r(t) \ominus_{gH} \widetilde{F}_r(t_0)$ exist and there are $\widetilde{u}_j \in E^n$ $(j = 1, 2, \cdots, m)$, such that

$$[\widetilde{u}_j]^r = \lim_{h \to 0} \frac{\widetilde{F}_r(t_1^0, \cdots, t_j^0 + h, \cdots, t_m^0) \ominus_{gH} \widetilde{F}_r(t_1^0, \cdots, t_j^0, \cdots, t_m^0)}{h},$$

uniformly for any $r \in [0,1]$.

Here the limit is taken in the metric space $(\mathcal{K}_c^n, d)$.

Theorem 3.5. The *level-wise* gradient $\nabla_s \widetilde{F}(t)$ of fuzzy mapping $\widetilde{F}: M \to E^n$ is unique if it exists.

Theorem 3.6. Let $\widetilde{F}: M \to E^n$, $t_0 = (t_1^0, \cdots, t_j^0, \cdots, t_m^0) \in \text{int}M$ and $h \in R$ with $t = (t_1^0, \cdots, t_j^0 + h, \cdots, t_m^0) \in \text{int}M$. If the *level-wise* gradient $\nabla_l \widetilde{F}(t_0)$ exists at $t_0 \in \text{int}M$ and g-difference $\widetilde{F}(t) \ominus_g \widetilde{F}(t_0)$ exists, then the gradient $\nabla \widetilde{F}(t)$ of $\widetilde{F}$ exists at t_0 and we have

$$\widetilde{u}_j = \widetilde{v}_j \quad (j = 1, 2, \cdots, m),$$

where $\nabla \widetilde{F}(t_0) = (\widetilde{u}_1, \widetilde{u}_2, \cdots, \widetilde{u}_m)$, $\nabla_l \widetilde{F}(t_0) = (\widetilde{v}_1, \widetilde{v}_2, \cdots, \widetilde{v}_m)$.

Proof. According to Definition 2.1, for any $\varepsilon > 0$, there is $\delta > 0$, when $|h| < \delta$, we have

$$\begin{aligned}
& D\left(\tfrac{\widetilde{F}(t_1^0, \cdots, t_j^0 + h, \cdots, t_m^0) \ominus_g \widetilde{F}(t_1^0, \cdots, t_j^0, \cdots, t_m^0)}{h}, \widetilde{v}_j\right) \\
= \ & \sup_{r \in [0,1]} d\left(cl\left(\bigcup_{\beta \geq r} \tfrac{\widetilde{F}_\beta(t) \ominus_{gH} \widetilde{F}_\beta(t_0)}{h}\right), [\widetilde{v}_j]^r\right) \\
\leq \ & \sup_{r \in [0,1]} \sup_{\beta \geq r} d\left(\tfrac{\widetilde{F}_\beta(t) \ominus_{gH} \widetilde{F}_\beta(t_0)}{h}, [\widetilde{v}_j]^\beta\right) \\
< \ & \varepsilon.
\end{aligned}$$

Then, the gradient $\nabla \widetilde{F}(t_0) = (\widetilde{u}_1, \widetilde{u}_2, \cdots, \widetilde{u}_m)$ of $\widetilde{F}$ exists at t_0 and $\widetilde{u}_j = \widetilde{v}_j$ $(j = 1, 2, \cdots, m)$.

4. Conclusion

This article is to use the generalized difference of n dimensional fuzzy numbers introduced in Bede and Stefanini [10] to define the differentiability and gradient for fuzzy mappings. Additionally, we have examined the relationships between the concepts of gradient, *support-function-wise* gradient and *level-wise* gradient for fuzzy mappings. The results from our study can be applied directly to fuzzy differential equations. The next step for the continuation of the research direction proposed here is to investigate the sub-differential of n dimensional fuzzy mappings and applications in the convex fuzzy programming.

References

[1] L.A. Zadeh, Fuzzy sets, Information and Control. 8(1965)338-353.

[2] L.A. Zadeh, The concept of a linguistic variable and its application to approximate reasoning-I, Information Sciences. 8(1975)199-249.

[3] L.A. Zadeh, The concept of a linguistic variable and its application to approximate reasoning-II, Information Sciences. 8(1975)301-357.

[4] L.A. Zadeh, The concept of a linguistic variable and its application to approximate reasoning-III, Information Sciences. 9(1975)43-80.

[5] M.L. Puri, D.A. Ralescu, Differentials of Fuzzy Functions, Journal of Mathematical Analysis and Applications. 91(1983)552-558.

[6] O. Kaleva, Fuzzy differential equations, Fuzzy Sets and Systems. 24(1987)301-317.

[7] G.X. Wang, C.X. Wu, Directional derivatives and subdifferential of convex fuzzy mappings and application in convex fuzzy programming, Fuzzy Sets and Systems. 138(2003)559-591.

[8] L. Stefanini, A generalization of Hukuhara difference and division for interval and fuzzy arithmetic, Fuzzy Sets and Systems. 161(2010)1564-1584.

[9] L.T. Gomes, L.C. Barros, A note on the generalized difference and the generalized differentiability, Fuzzy Sets and Systems. 280(2015)142-145.

[10] B. Bede, L. Stefanini, Generalized differentiability of fuzzy-valued functions, Fuzzy Sets and Systems. 230(2013)119-141.

[11] P. Diamond, P. Kloeden, Characterization of compact subsets of fuzzy sets, Fuzzy Sets and Systems. 29(1989)341-348.

[12] M. Ma, On embedding problems of fuzzy number spaces: Part 5, Fuzzy Sets and Systems. 55(1993)313-318.

[13] C.V. Negoita, D.A. Ralescu, Application of Fuzzy Sets to Systems Analysis,Wiley Publishing, 1975.

[14] B.K. Zhang, On measurability of fuzzy-number-valued functions, Fuzzy Sets and Systems. 120(2001)505-509.

[15] S.X. Hai, Z.T. Gong, H.X. Li, Generalized differentiability for n dimensional fuzzy-number-valued functions and fuzzy optimization, Information Sciences. 374(2016)151–163.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 27, NO.2, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC

Global Attractivity and Periodic Nature of a Higher order Difference Equation

M. M. El-Dessoky[1,2], Abdul Khaliq[3], Asim Asiri[1] and Ansar Abbas[3]
[1]Mathematics Department, Faculty of Science, King Abdulaziz University,
P. O. Box 80203, Jeddah 21589, Saudi Arabia.
[2]Department of Mathematics, Faculty of Science,Mansoura University, Mansoura 35516, Egypt.
[3]Department of Mathematics, Riphah International University, Lahore,
Pakistan.
E-mails:dessokym@mans.edu.eg; khaliqsyed@gmail.com

ABSTRACT

Our aim in this paper is to study the global stability character and the periodic nature of the solutions of the difference equation

$$x_{n+1} = ax_{n-l} + \frac{b + cx_{n-k}}{dx_{n-s} + ex_{n-t}}, \quad n = 0, 1, ...,$$

where the initial conditions $x_{-r},\ x_{-r+1},\ x_{-r+2},\ ...,\ x_0$ are arbitrary positive real numbers, $r = \max\{l, k, s, t\}$ is nonnegative integer and $a,\ b,\ c,\ d,\ e$ are positive constants.Finally, some numerical examples are presented and graphed by Matlab.

Keywords: stability, periodic solutions, global attractor, difference equations.

Mathematics Subject Classification: 39A10; 40A05.

1. INTRODUCTION

Difference equations or discrete dynamical systems are diversed field which impact almost every branch of pure and applied mathematics. Every dynamical system $a_{n+1} = f(a_n)$ determines a difference equation and vice versa. Recently many researchers have studied the global attractivity, boundedness character and the periodic nature of nonlinear difference equations see for example [1-42]. One of the reasons for this is a prerequisite for some approaches, which can be used in inspecting equations arising in real life situations that can be model mathematically. The theory of difference equations and dynamical systems is developed during the last thirty years and there is no doubt that it will continue to play an important role in mathematical models describing real life situations and in many applied sciences, such as biology, physiology, ecology, engineering, economics, physics, probability theory, genetics, computers and resource allocation.

It is very interesting and attractive for the researcher to study the behavior and solution of nonlinear rational difference equations .Most of the real life phenomana has been solved by using these equations, examples include in [3,7,11,12] . Recently, many researchers have investigated the asymptotic behavior and periodic nature of rational difference equations for example in [36] R. Khallaf Allah investigated the asymptotic behavior and periodic nature of the following difference equation

$$x_{n+1} = \frac{x_{n-2}}{1 \pm x_n x_{n-1} x_{n-2}}.$$

G. Ladas et. al [8], investigated the asymptotic behavior and boundedness of the solution of the difference equation

$$x_n = \frac{(\alpha + \beta x_n + \gamma x_{n-1})}{(A + Bx_n + Cx_{n-1})}.$$

E. M. E. Zayed [33] studied thequalitative properties of the nonlinear difference equation

$$x_{n+1} = \frac{\alpha x_{n-\delta}}{\beta + \gamma x_{n-\tau}}.$$

Yalçınkaya [32] has studied the following difference equation

$$x_{n+1} = \alpha + \frac{x_{n-m}}{x_n^k}.$$

Taixiang Sun et al [39] considered the class of nonlinear delay difference equation

$$x_{n+1} = \frac{Af_1(x_n, ..., x_{n-k}) + Bf_2(x_n, ..., x_{n-k})f_3(x_n, ..., x_{n-k}) + C}{\alpha f_1(x_n, ..., x_{n-k})f_2(x_n, ..., x_{n-k}) + \beta f_3(x_n, ..., x_{n-k}) + \gamma}.$$

The goal of this paper is to determine the global stability character and the periodicity of the solutions of the difference equation

$$x_{n+1} = ax_{n-l} + \frac{b + cx_{n-k}}{dx_{n-s} + ex_{n-t}}, \quad n = 0,\ 1,\ ..., \tag{1}$$

where the initial conditions $x_{-r},\ x_{-r+1},\ x_{-r+2}, ...,\ x_0$ are arbitrary positive real numbers, $r = \max\{l,\ k,\ s,\ t\}$ is nonnegative integer and $a,\ b,\ c,\ d,\ e$ are positive constants.

" Here, we recall some basic definitions and some theorems that we need in the sequel.

Let I be some interval of real numbers and let

$$F : I^{r+1} \to I,$$

be a continuously differentiable function. Then for every set of initial conditions $x_{-r},\ x_{-r+1},\ ...,\ x_0 \in I$, the difference equation

$$x_{n+1} = F(x_n, x_{n-1}, ..., x_{n-r}), \quad n = 0, 1, ..., \tag{2}$$

has a unique solution $\{x_n\}_{n=-r}^{\infty}$.

A point $\overline{x} \in I$ is called an equilibrium point of Eq. (2) if

$$\overline{x} = f(\overline{x},\ \overline{x},\ ...,\ \overline{x}).$$

That is, $x_n = \overline{x}$ for $n \geq 0$, is a solution of Eq. (2), or equivalently $\overline{x}$ is a fixed point of f.

Definition 1.1. *(Periodicity) A sequence $\{x_n\}_{n=-k}^{\infty}$ is said to be periodic with period p if $x_{n+p} = x_n$ for all $n \geq -k$.*

Definition 1.2. *(Stability) (i) The equilibrium point $\overline{x}$ of Eq. (2) is locally stable if for every $\epsilon > 0$, there exists $\delta > 0$ such that for all $x_{-r},\ x_{-r+1},\ ..., x_{-1},\ x_0 \in I$ with*

$$|x_{-r} - \overline{x}| + |x_{-r+1} - \overline{x}| + ... + |x_0 - \overline{x}| < \delta,$$

we have

$$|x_n - \overline{x}| < \epsilon \quad \text{for all} \quad n \geq -r.$$

(ii) The equilibrium point $\overline{x}$ of Eq. (2) is locally asymptotically stable if $\overline{x}$ is locally stable solution of Eq.(2) and there exists $\gamma > 0$, such that for all $x_{-r},\ x_{-r+1},\ ..., x_{-1},\ x_0 \in I$ with

$$|x_{-r} - \overline{x}| + |x_{-r+1} - \overline{x}| + ... + |x_0 - \overline{x}| < \gamma,$$

we have

$$\lim_{n\to\infty} x_n = \overline{x}.$$

(iii) The equilibrium point $\overline{x}$ *of Eq. (2) is global attractor if for all* $x_{-r},\ x_{-r+1},\ ...,x_{-1},\ x_0 \in I$, *we have*

$$\lim_{n\to\infty} x_n = \overline{x}.$$

(iv) The equilibrium point $\overline{x}$ *of Eq. (2) is globally asymptotically stable if* $\overline{x}$ *is locally stable, and* $\overline{x}$ *is also a global attractor of Eq. (2).*
(v) The equilibrium point $\overline{x}$ *of Eq.(2) is unstable if* $\overline{x}$ *is not locally stable.*

The linearized equation of Eq. (2) about the equilibrium $\overline{x}$ is the linear difference equation

$$y_{n+1} = \sum_{i=0}^{r} \tfrac{\partial F(\overline{x},\overline{x},\ldots,\overline{x})}{\partial x_{n-i}} y_{n-i}. \tag{3}$$

Theorem A [26]: Assume that $p,\ q \in\ R$ and $r \in \{0,\ 1,\ 2,\ ...\}$. Then

$$|p| + |q| < 1,$$

is a sufficient condition for the asymptotic stability of the difference equation

$$x_{n+1} + px_n + qx_{n-r} = 0, \quad n = 0,\ 1,\ ....$$

Remark 1. *Theorem A can be easily extended to a general linear equations of the form*

$$x_{n+r} + p_1 x_{n+r-1} + ... + p_r x_n = 0, \ \ n = 0, 1, ..., \tag{4}$$

where $p_1, p_2, ..., p_r \in R$ *and* $r \in \{1, 2, ...\}$. *Then Eq.(4) is asymptotically stable provided that*

$$\sum_{i=1}^{r} |p_i| < 1.$$

Consider the following equation

$$x_{n+1} = g(x_n,\ x_{n-1},\ ...,\ x_{n-K}), \qquad n = 0,\ 1,\ 2,\ .... \tag{5}$$

The following theorem will be useful for the proof of our results in this paper.
Theorem B [27]: Let $[\alpha,\ \beta]$ be an interval of real numbers and assume that

$$g : [\alpha,\ \beta]^{k+1} \rightarrow [\alpha,\ \beta]$$

is a continuous function satisfying the following properties :

(a) $g(x_1,\ x_2,\ ...,\ x_{k+1})$ is non-increasing in one component (for example x_σ) for each x_r $(r \neq \sigma)$ in $[\alpha,\ \beta]$, and is non-increasing in the remaining components for each $x_\sigma \in [\alpha, \beta]$

(b) If $(m,\ M) \in [\alpha,\ \beta] \times [\alpha,\ \beta]$ is a solution of the system

$$M = g(m,\ m,\ ...,\ m,\ M,\ m,\ ...,\ m,\ m) \ \ and \ \ m = g(M,\ M,\ ...,\ M,\ m,\ M,\ ...,\ M,\ M),$$

then

$$m = M.$$

Then Eq. (5) has a unique equilibrium $\overline{x} \in [\alpha,\ \beta]$ and every solution of Eq. (5) converges to $\overline{x}$."

2. LOCAL STABILITY OF THE EQUILIBRIUM POINT OF EQ. (1)

In this section we study the local stability character of the solutions of Eq. (1). The equilibrium points of Eq. (1) are given by the relation

$$\overline{x} = a\overline{x} + \frac{b + c\overline{x}}{d\overline{x} + e\overline{x}}.$$

If $a \neq 1,\ d + e - ae - ad \neq 0$, then the positive equilibrium point of Eq. (1) is given by

$$\overline{x} = \frac{-c + \sqrt{4be + 4bd + c^2 - 4abe - 4abd}}{2(a-1)(d+e)}.$$

Let $f : (0,\ \infty)^4 \longrightarrow (0,\ \infty)$ be a function defined by

$$f(u_0,\ u_1,\ u_2,\ u_3) = au_0 + \frac{b + cu_1}{du_2 + eu_3}.$$

Then we see that at $\overline{x} = \frac{-c+\sqrt{4be+4bd+c^2-4abe-4abd}}{2(a-1)(d+e)}$

$$\begin{aligned}
\frac{\partial f(\overline{x},\ \overline{x},\ \overline{x},\ \overline{x})}{\partial u_0} &= a = -c_0,\ \ \frac{\partial f(\overline{x},\ \overline{x},\ \overline{x},\ \overline{x})}{\partial u_1} = \frac{2c(a(d+e)-1)}{\left(-c + \sqrt{(4be + 4bd + c^2 - 4abe - 4abd)}\right)(d+e)} = -c_1 \\
\frac{\partial f(\overline{x},\ \overline{x},\ \overline{x},\ \overline{x})}{\partial u_2} &= -\frac{(2(a(d+e)-1))d\left(c\sqrt{(4be + 4bd + c^2 - 4abe - 4abd)} + 2ba(d+e) - c^2 - 2b\right)}{(d+e)^2\left(-c + \sqrt{4be + 4bd + c^2 - 4abe - 4abd}\right)^2} = -c_2, \\
\frac{\partial f(\overline{x},\ \overline{x},\ \overline{x},\ \overline{x})}{\partial u_3} &= -\frac{(2(a(d+e)-1))e(c\sqrt{(4be + 4bd + c^2 - 4abe - 4abd)} + 2ba(d+e) - c^2 - 2b)}{(d+e)^2\left(-c + \sqrt{4be + 4bd + c^2 - 4abe - 4abd}\right)^2} = -c_3
\end{aligned}$$

Then the linearized equation of Eq.(1) about $\overline{x}$ is

$$y_{n+1} + c_0 y_{n-l} + c_1 y_{n-k} + c_2 y_{n-s} + c_3 y_{n-t} = 0.$$

3. EXISTENCE OF PERIODIC SOLUTIONS

In this section we study the existence of periodic solutions of Eq. (1).

Theorem 3.1. *Eq. (1) has a prime period two solutions if and only if*

$$c^2(a+1) - 4a^2 b(d+e) > 0, \qquad k,\ l,\ s,\ t - \text{even}. \tag{7}$$

Proof: First suppose that there exists a prime period two solution

$$...,\ p,\ q,\ p,\ q,\ ...,$$

of Eq. (1). We will prove that Condition (7) holds.

We see from Eq. (1) (when $k,\ l,\ s,\ t-$even) that

$$p = aq + \frac{b + cq}{dq + eq}, \qquad q = ap + \frac{b + cp}{dp + ep}$$

Then

$$p = aq + \frac{b + cq}{(d+e)q}, \qquad q = ap + \frac{b + cp}{(d+e)p}$$

$$(d+e)pq = a(d+e)q^2 + b + cq, \tag{8}$$

and

$$(d+e)pq = a(d+e)p^2 + b + cp. \tag{9}$$

Subtracting (9) from (8) gives

$$0 = a(d+e)(p^2 - q^2) + c(p-q).$$

Since $p \neq q$, it follows that

$$p + q = -\frac{c}{a(d+e)}. \tag{10}$$

Again, adding (8) and (9) yields

$$2(d+e)pq = a(d+e)(p+q)^2 - 2a(d+e)pq + 2b + c(p+q) \tag{11}$$

It follows by (10), (11) and the relation $p^2 + q^2 = (p+q)^2 - 2pq$ for all $p,\ q \in R$ that

$$pq = \frac{b}{(a+1)(d+e)}. \tag{12}$$

Now it is clear from Eq. (10) and Eq. (12) that p and q are the two positive distinct roots of the quadratic equation

$$t^2 + \left(\tfrac{c}{a(d+e)}\right) t + \left(\tfrac{b}{(a+1)(d+e)}\right) = 0, \tag{13}$$

$$a(d+e)(a+1)t^2 + c(a+1)t + ab = 0,$$

and so

$$(c(a+1))^2 - 4a^2 b(d+e)(a+1) > 0$$

thus

$$c^2(a+1) - 4a^2 b(d+e) > 0$$

Therefore Inequality (7) holds.

Second suppose that Inequality (7) is true. We will show that Eq. (1) has a prime period two solution. Assume that

$$p = \frac{-c(a+1) + \sqrt{\beta}}{2a(a+1)(d+e)} = \frac{-cA + \sqrt{\beta}}{2aAB},$$

and

$$q = \frac{-cA - \sqrt{\beta}}{2aAB}, \quad \text{where} \quad A = (a+1), \quad B = (d+e)$$

where $\beta = c^2(a+1)^2 - 4a^2 b(a+1)(d+e)$.

We see from Inequality (7) that

$$(c(a+1))^2 - 4a^2 b(d+e)(a+1) > 0$$

then after dividing by $(a+1)$we see that

$$\Rightarrow \qquad c^2 > 4a^2 b(d+e)$$

Therefore p and q are distinct real numbers.
Set

$$\begin{aligned} x_{-l} &= p,\ x_{-l+1} = q, \quad , x_{-k} = p,\ x_{-k+1} = q, \\ x_{-s} &= p,\ x_{-s+1} = q, \quad x_{-t} = p,\ x_{-t+1} = q \text{ and } \quad x_0 = p. \end{aligned}$$

We wish to show that

$$x_1 = x_{-1} = q \quad \text{and} \quad x_2 = x_0 = p.$$

It follows from Eq. (1) that

$$x_1 = ax_{-l} + \frac{b + cx_{-k}}{dx_{-s} + ex_{-t}} = ap + \frac{b + cp}{dp + ep} = ap + \frac{b + cp}{(d + e)p}$$

$$= ap + \frac{b + c(\frac{-cA+\sqrt{\beta}}{2aAB})}{(d + e)\left(\frac{-cA+\sqrt{\beta}}{2aAB}\right)}.$$

Multiplying the denominator and numerator of the right side by $2aAB$ gives

$$x_1 = ap + \frac{2abAB+c(-cA+\sqrt{\beta})}{(d+e)(-cA+\sqrt{\beta})},$$

Multiplying the denominator and numerator of the right side by $(-cA - \sqrt{\beta})$

and by Replacing $A = (a + 1)$, $B = (d + e)$ and $\beta = c^2(a + 1)^2 - 4a^2b(a + 1)(d + e)$in denominator and numerator of above equation gives

$$\begin{aligned} x_1 &= ap + \frac{2abAB(-cA-\sqrt{\beta})+c(c^2A^2-\beta)}{(d+e)(c^2A^2-\beta)}, \\ &= ap + \frac{2ab(a+1)(d+e)(-cA-\sqrt{\beta})+c(c^2(a+1)^2-c^2(a+1)^2+4a^2b(a+1)(d+e))}{(d+e)(c^2(a+1)^2-c^2(a+1)^2+4a^2b(a+1)(d+e))}, \end{aligned}$$

$$= ap + \frac{2ab(a+1)(d+e)(-cA-\sqrt{\beta})+4a^2bc(a+1)(d+e)}{4a^2b(a+1)(d+e)^2},$$

Dividing numerator and denominator by $(2ab(a + 1)(d + e))$ we get

$$= ap + \frac{-cA-\sqrt{\beta}+2ac}{2a(d+e)}$$

$$= \frac{2a^2(d+e)p-cA-\sqrt{\beta}+2ac}{2a(d+e)}$$

Now inserting the value of p we get

$$\begin{aligned} x_1 &= \frac{1}{2a(d + e)} \left(\frac{-ca(a+1)+a\sqrt{\beta}-c(a+1)^2-(a+1)\sqrt{\beta}+2ac(a+1)}{(a+1)}\right) \\ &= \frac{1}{2a(a + 1)(d + e)} \left(-\sqrt{\beta} - c(a + 1)^2 + ca(a + 1)\right) \\ &= \frac{-\sqrt{\beta} - c(a + 1)}{2a(a + 1)(d + e)} \end{aligned}$$

But $(a + 1) = A$ and $(d + e) = B$ we get

$$x_1 = \frac{-cA-\sqrt{\beta}}{2aAB} = q$$

Similarly as before one can easily show that

$$x_2 = p.$$

Then it follows by induction that

$$x_{2n} = p \quad \text{and} \quad x_{2n+1} = q \quad \text{for all} \quad n \geq -1.$$

Thus Eq. (1) has the positive prime period two solution

$$..., \ p, \ q, \ p, \ q, \ ...,$$

where p and q are the distinct roots of the quadratic equation (13) and the proof is completed.

The following Theorems can be proved similarly.

Theorem 3.2. *Eq. (1) has a prime period two solutions if and only if*

$$c^2 + 4b(d+e)(1-a) > 0 \qquad (l, \ k, \ s, \ t - odd).$$

Theorem 3.3. *Eq. (1) has a prime period two solutions if and only if*

$$c^2(d-e)(1+a) - 4(b(ad+e)^2 - ec^2) > 0 \qquad (l, \ k, \ s - even \text{ and } t - odd).$$

Theorem 3.4. *Eq. (1) has a prime period two solutions if and only if*

$$c^2(e-d)(1+a) - 4(b(ae+d)^2 - c^2 d) > 0 \qquad (l, \ k, \ t - even \text{ and } s - odd).$$

Theorem 3.5. *Eq. (1) has a prime period two solutions if and only if*

$$c^2(1+a) - 4a(ab(d+e) + c^2) > 0 \qquad (l, \ s, \ t - even \text{ and } k - odd).$$

Theorem 3.6. *Eq. (1) has a prime period two solutions if and only if*

$$c^2(e-d) - 4bd^2(1-a) > 0 \qquad (l, \ k, \ s - odd \text{ and } t - even).$$

Theorem 3.7. *Eq. (1) has a prime period two solutions if and only if*

$$c^2(d-e) - 4be^2(1-a) > 0 \qquad (l, \ k, \ t - odd \text{ and } s - even).$$

Theorem 3.8. *Eq. (1) has a prime period two solutions if and only if*

$$c^2 - 4(b(d+e)(a-1) + c^2) > 0 \qquad (l, \ s, \ t - odd \text{ and } k - even).$$

Theorem 3.9. *Eq. (1) has a prime period two solutions if and only if*

$$c^2(1+a) + 4b(d+e) > 0 \qquad (k, \ s, \ t - odd \text{ and } l - even).$$

Theorem 3.10. *Eq. (1) has a prime period two solutions if and only if*

$$c^2(1+a) - 4(c^2 - b(d+e)) > 0 \qquad (l, \ k - even \text{ and } s, \ t - odd).$$

Theorem 3.11. *Eq. (1) has a prime period two solutions if and only if*

$$c^2(1+a)(d-e) - 4(b(ad+e)^2 + ac^2 d) > 0 \qquad (l, \ s - even \text{ and } k, \ t - odd).$$

THEOREM 3.12. *Eq. (1) has a prime period two solutions if and only if*

$$c^2(e-d)-4e(be(a-1)+c^2)>0 \qquad (s,\ k-\textit{even and }\ l,\ t-\textit{odd}).$$

THEOREM 3.13. *Eq. (1) has a prime period two solutions if and only if*

$$c^2(d-e)-4d(bd(a-1)+c^2)>0 \qquad (l,\ s-\textit{odd and }\ k,\ t-\textit{even}).$$

THEOREM 3.14. *Eq. (1) has a prime period two solutions if and only if*

$$c^2(a+1)(e-d)-4(b(ae+d)^2+ac^2e)>0 \qquad (s,\ k-\textit{odd and }\ l,\ t-\textit{even}).$$

THEOREM 3.15. *Eq. (1) has no prime period two solutions if one of the following statements holds*

$$(i)\ c\neq 0 \qquad (k,\ s,\ t-\textit{even and }\ l-\textit{odd}),$$

$$(ii)\ c\neq 0 \qquad (s,\ t-\textit{even and }\ l,\ k-\textit{odd}).$$

4. GLOBAL ATTRACTIVITY OF THE EQUILIBRIUM POINT OF EQ. (1)

In this section we investigate the global attractivity character of solutions of Eq. (1).

THEOREM 4.1. *The equilibrium point $\overline{x}$ of Eq. (1) is global attractor.*

Proof: Let $p,\ q$ are a real numbers and assume that $f:[p,\ q]^4\longrightarrow[p,\ q]$ be a function defined by

$$f(u_0,\ u_1,\ u_2,\ u_3)=au_0+\frac{b+cu_1}{du_2+eu_3}.$$

We can easily see that the function $f(u_0,\ u_1,\ u_2,\ u_3)$ increasing in $u_0,\ u_1$ and decreasing in $u_2,\ u_3$.

Suppose that $(m,\ M)$ is a solution of the system

$$m=f(m,\ m,\ M,\ M) \quad\text{and}\quad M=f(M,\ M,\ m,\ m).$$

Then from Eq. (1), we see that

$$m=am+\frac{b+cm}{(d+e)M},\quad M=aM+\frac{b+cM}{(d+e)m},$$

That is

$$(1-a)m=\frac{b+cm}{(d+e)M},\quad (1-a)M=\frac{b+cM}{(d+e)m},$$

or,

$$b+cm=b+cM$$

Thus $m=M$. It follows by the Theorem B that $\overline{x}$ is a global attractor of Eq. (1) and then the proof is complete.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 27, NO.2, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC

5. NUMERICAL EXAMPLES

For confirming the results of this paper, we consider numerical examples which represent different types of solutions to Eq. (1).

Example 1. We assume $l = 5,\ k = 4,\ s = 3,\ t = 5,\ \ x_{-5} = 6,\ x_{-4} = 9,\ x_{-3} = 8,\ x_{-2} = 9,\ x_{-1} = 12,\ x_{-1} = 4,\ a = 0.1,\ b = 0.2,\ c = 0.9,\ d = 0.7\ \ e = 0.8$**.** [See Fig. 1]

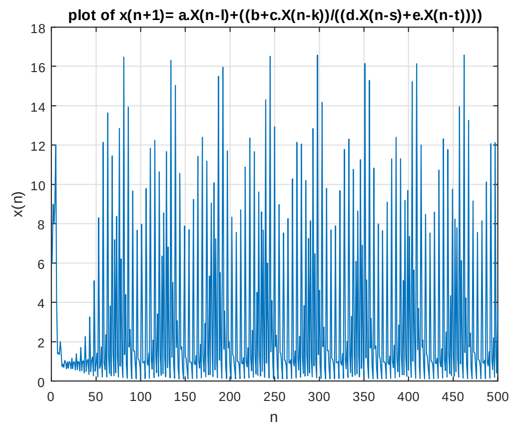


Figure 1.

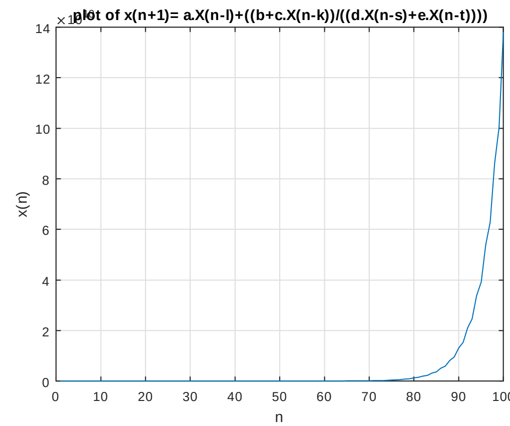


Figure 2.

Example 2. See Fig. 2, since $l = 1,\ k = 2,\ s = 1,\ t = 3,\ \ x_{-3} = 1.2,\ x_{-2} = 0.7,\ x_{-1} = 8.5,\ x_0 = 5,\ a = 1.6,\ b = 0.2,\ c = 0.9,\ d = 0.09,\ e = 0.01.$

Example 3. See Fig. 3, since $l = 1,\ k = 2, s = 1,\ t = 1\ x_{-3} = 12,\ x_{-2} = 7,\ x_{-1} = 8,\ x_0 = 3,\ a = 0.1,\ b = 0.2,\ c = 0.5,\ d = 0.6,\ e = 0.2.$

Example 4. Fig. 4. shows the solutions when $a = 0.1,\ b = 0.2,\ c = 0.5,\ d = 0.6, e = 0.9,\ l = 4,\ k = 2,\ s = 4,\ t = 2,\ \ x_{-4} = p,\ x_{-3} = q,\ x_{-2} = p,\ x_{-1} = q,\ x_0 = p.$

Since $\left(p, q = \frac{-c(a+1)\pm\sqrt{c^2(a+1)^2-4a^2b(a+1)(d+e)}}{2a(a+1)(d+e)}\right)$

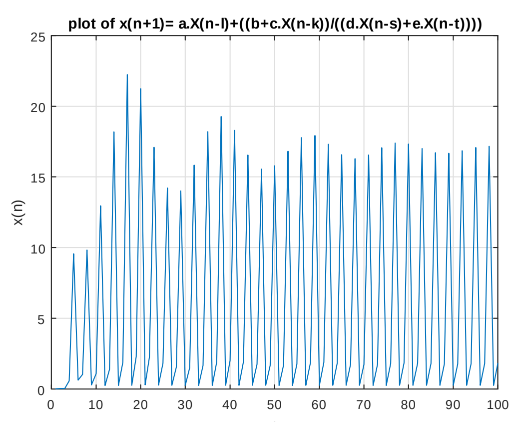


Figure 3.

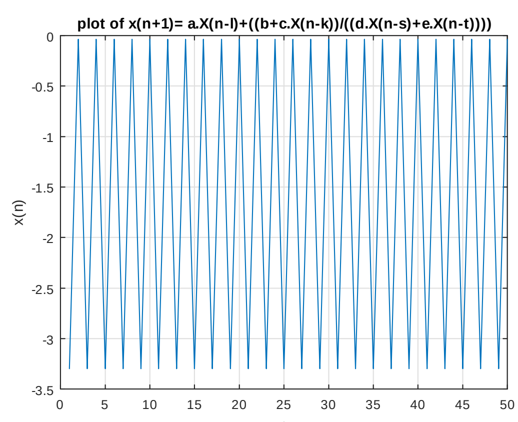


Figure 4.

Acknowledgements

This article was funded by the Deanship of Scientific Research (DSR), King Abdulaziz University, Jeddah. The authors, therefore, acknowledge with thanks DSR technical and financial support.

REFERENCES

1. R. Abo-Zeid, Global Attractivity of a Higher-Order Difference Equation, Discrete Dyn. Nat. Soc., Vol., 2012 (2012), Article ID 930410, 11 pages.
2. M. Aloqeili, Global stability of a rational symmetric difference equation, Appl. Math. Comp., 215, (2009), 950-953.
3. M. Aprahamian, D. Souroujon, and S. Tersian, Decreasing and fast solutions for a second-order difference equation related to Fisher-Kolmogorov's equation, J. Math. Anal. Appl., 363, (2010), 97-110.
4. H. Chen and H. Wang, Global attractivity of the difference equation $x_{n+1} = \frac{x_n+\alpha x_{n-1}}{\beta+x_n}$, Appl. Math. Comp., 181, (2006), 1431–1438.
5. C. Cinar, On the positive solutions of the difference equation $x_{n+1} = \frac{ax_{n-1}}{1+bx_nx_{n-1}}$, Appl. Math. Comp., 156, (2004), 587-590.
6. S. E. Das and M.Bayram, On a System of Rational Difference Equations, World Applied Sciences Journal, 10(11), (2010), 1306-1312.
7. Q. Din, and E. M. Elsayed, Stability analysis of a discrete ecological model, Computational Ecology and Software, 4 (2) (2014), 89–103.
8. G. Ladas, E. Camouzis and H. D. Voulov, On the dynamic of $x_n = \frac{(\alpha+\beta x_n+\gamma x_{n-1})}{(A+Bx_n+Cx_{n-1})}$ J. Difference Equations and Appl., 9, (2003), 731-738.
9. E. M. E. Zayed and M. A. El-Moneam, On the qualitative study of the nonlinear difference equation $x_{n+1} = \dfrac{\alpha x_{n-\delta}}{\beta+\gamma x_{n-\tau}}$ Fasciculi Mathematici, 50, (2013), 137-147.
10. E. M. Elabbasy, H. El-Metwally and M. Elsayed, On the Difference Equation $x_{n+1} = \frac{a_0x_n+a_1x_{n-1}+\ldots+a_kx_{n-k}}{b_0x_n+b_1x_{n-1}+\ldots+b_kx_{n-k}}$, Mathematica Bohemica, 133(2), (2008), 133-147.
11. Miron B. Bekker, Martin J. Bohner, Hristo D. Voulov, Asymptotic behavior of solutions of a rational system of dierence equations, J. Nonlinear Sci. Appl., 7, (2014), 379-382.
12. H. El-Metwally and M. M. El-Afifi, On the behavior of some extension forms of some population models, Chaos, Solitons and Fractals, 36, (2008), 104–114.
13. H. El-Metwally and E. M. Elsayed, Form of solutions and periodicity for systems of difference equations, Journal of Computational Analysis and Applications, 15(5), (2013), 852-857.
14. E. A. Grove and G. Ladas, Periodicities in Nonlinear Difference Equations, Chapman & Hall / CRC Press, 2005.
15. A. S. Kurbanli, C. Cinar and I. Yalçinkaya, On the behavior of positive solutions of the system of rational difference equations, Math. Comput. Mod., 53, (2011),1261-1267.
16. E. M. Elsayed and H. El-Metwally, Global Behavior and Periodicity of Some Difference Equations, Journal of Computational Analysis and Applications, 19 (2), (2015), 298-309.
17. E. M. Elsayed, Dynamics of a Recursive Sequence of Higher Order, Communications on Applied Nonlinear Analysis, 16 (2), (2009), 37–50.
18. I. Yalcinkaya, On the global asymptotic stability of a second-order system of difference equations, Discrete Dyn. Nat. Soc., Vol. 2008, (2008), Article ID 860152,12 pages.
19. E. M. Elsayed, Qualitative behavior of difference equation of order two, Mathematical and Computer Modelling, 50, (2009), 1130–1141.
20. R. Karatas, C. Cinar and D. Simsek, On positive solutions of the difference equation $x_{n+1} = \dfrac{x_{n-5}}{1+x_{n-2}x_{n-5}}$, Int. J. Contemp. Math. Sci., 1(10), (2006), 495-500.

21. V. L. Kocic and G. Ladas, Global Behavior of Nonlinear Difference Equations of Higher Order with Applications, Kluwer Academic Publishers, Dordrecht, 1993.
22. M. R. S. Kulenovic and G. Ladas, Dynamics of Second Order Rational Difference Equations with Open Problems and Conjectures, Chapman & Hall / CRC Press, 2001.
23. A. S. Kurbanli, On the Behavior of Solutions of the System of Rational Difference Equations, World Applied Sciences Journal, 10 (11). (2010), 1344-1350.
24. R. Memarbashi, Sufficient conditions for the exponential stability of nonautonomous difference equations, Appl. Math. Lett., 21, (2008), 232–235.
25. A. Neyrameh, H. Neyrameh, M. Ebrahimi and A. Roozi, Analytic solution diffusivity equation in rational form, World Applied Sciences Journal, 10 (7), (2010), 764-768.
26. M. Saleh and M. Aloqeili, On the difference equation $y_{n+1} = A + \frac{y_n}{y_{n-k}}$ with $A < 0$, Appl. Math. Comp., 176, (2006), 359–363.
27. M. Saleh and M. Aloqeili, On the difference equation $x_{n+1} = A + \frac{x_n}{x_{n-k}}$, Appl. Math. Comp., 171, (2005), 862-869.
28. T. Sun and H. Xi, On convergence of the solutions of the difference equation $x_{n+1} = 1 + \frac{x_{n-1}}{x_n}$, J. Math. Anal. Appl., 325, (2007), 1491–1494.
29. C. Wang, S. Wang, L. LI, Q. Shi, Asymptotic behavior of equilibrium point for a class of nonlinear difference equation, Adv. Differ. Equ., Vol. 2009, (2009), Article ID 214309, 8 pages.
30. C. Wang, S. Wang, Z. Wang, H. Gong, R. Wang, Asymptotic stability for a class of nonlinear difference equation, Discrete Dyn. Nat. Soc., Vol. 2010, (2010), Article ID 791610, 10 pages.
31. I. Yalçınkaya, On the global asymptotic stability of a second-order system of difference equations, Discrete Dyn. Nat. Soc., Vol. 2008, (2008), Article ID 860152, 12 pages.
32. I. Yalçınkaya, On the difference equation $x_{n+1} = \alpha + \frac{x_{n-m}}{x_n^k}$, Discrete Dyn. Nat. Soc., Vol. 2008, (2008) Article ID 805460, 8 pages, doi: 10.1155/2008/ 805460.
33. E. M. E. Zayed and M. A. El-Moneam, On the rational recursive sequence $x_{n+1} = \frac{\alpha+\beta x_n+\gamma x_{n-1}}{A+Bx_n+Cx_{n-1}}$, Communications on Applied Nonlinear Analysis, 12 (4), (2005), 15–28.
34. E. M. E. Zayed and M. A. El-Moneam, On the rational recursive sequence $x_{n+1} = \frac{\alpha x_n+\beta x_{n-1}+\gamma x_{n-2}+\delta x_{n-3}}{Ax_n+Bx_{n-1}+Cx_{n-2}+Dx_{n-3}}$; Comm. Appl. Nonlinear Analysis, 12, (2005), 15-28.
35. M. Garic-Demirovic,M. Nurkanovic, Dynamics of an anti-competitive two dimensional rational system of difference equations, Sarajevo J. Math., 7(19), (2011),39-56.
36. R. Khalllaf Allah, Asynmptotic behavior and periodic nature of two difference equations, Ukranian Mathematical Journal 61(6), (2009), 988-993.
37. M. M. El-Dessoky, Dynamics and Behavior of the Higher Order Rational Difference equation, J. Comput. Anal. Appl., Vol., 21(4), (2016), 743-760.
38. M. M. El-Dessoky and Aatef Hobiny, On the Difference equation $x_{n+1} = \alpha x_n+\beta x_{n-l}+\gamma x_{n-k}+\frac{ax_{n-l}+bx_{n-k}}{cx_{n-l}+dx_{n-k}}$, J. Comput. Anal. Appl., Vol., 24(4), (2018), 644-655.
39. Taixiang Sun, H. Xi, Hu. Wu, C. Han, Stability of solutions for a family of nonlinear delay difference equations, Dyn. Cont. Discrete. Impu. Syst., 15, (2008), 345-351.
40. M. M. El-Dessoky, On the Difference equation $x_{n+1} = ax_{n-l} + bx_{n-k} + \frac{cx_{n-s}}{dx_{n-s}-e}$, Math. Meth. Appl. Sci., 40(3), (2017), 535–545.
41. E. M. Elsayed, M. M. El-Dessoky and Asim Asiri, Dynamics and Behavior of a Second Order Rational Difference equation, J. Comput. Anal. Appl., Vol., 16(4), (2014), 794-807.
42. E. M. Elsayed, M. M. El-Dessoky and Ebraheem O. Alzahrani, The Form of The Solution and Dynamics of a Rational Recursive Sequence, J. Comput. Anal. Appl., Vol., 17(1), (2014), 172-186.

Asymptotic Representations for Fourier Approximation of Functions on the Unit Square *

Zhihua Zhang
College of Global Change and Earth System Science, Beijing Normal University, Beijing, China, 100875
E-mail: zhangzh@bnu.edu.cn

Abstract. In this paper, for any smooth function on $[0,1]^2$, we give an asymptotic representation of hyperbolic cross approximations of its Fourier series whose principal part is determined by the values of the function at vertexes of $[0,1]^2$ and present a novel approach to estimates of the upper bounds of approximation errors. At the same time, we also give an asymptotic formula of partial sum approximations whose principal part is determined by not only partial derivatives at vertexes of $[0,1]^2$, but also mean values on each side. Comparing asymptotic representations of these two kinds of approximation, we find that although in general the hyperbolic cross approximation is better than the partial sum approximation, the partial sum approximation possibly work better under some cases, and we also give the corresponding necessary and sufficient condition to characterize these cases.

1. Introduction

For a function f on $[0,1]^2$, regardless of how smooth it is, by the Riemann-Lebesgue lemma, we only know that its Fourier coefficients $c_{mn}(f) = o(1)$. In this paper, we first obtain a precise asymptotic formula of the Fourier coefficients (see Theorem 2.2) by using our novel decomposition formula of f:

$$f(x,y) = \begin{cases} q(x,y) + \tau(x,y) & (x,y) \in [0,1]^2, \\ q(x,y) & (x,y) \in \partial([0,1]^2), \end{cases}$$

where $q(x,y)$ is a combination of the boundary function and four simple polynomial factors $x, 1-x, y$, and $1-y$. After that, we will discuss further two kinds of Fourier approximations of functions on the unit square.

The sparse approximation has received much attention in recent years [1,6,7,8]. As an approximation tool, hyperbolic cross truncations of Fourier series has obvious advantages over partial sums of Fourier

*Zhihua Zhang is a full professor at Beijing Normal University, China. He has published more than 50 first-authored papers in applied mathematics, signal processing and climate change. His research is supported by National Key Science Program No.2013CB956604; Fundamental Research Funds for the Central Universities (Key Program) No.105565GK; Beijing Young Talent fund and Scientific Research Foundation for the Returned Overseas Chinese Scholars, State Education Ministry. Zhihua Zhang is an associate editor of “EURASIP Journal on Advances in Signal Processing” (Springer, SCI-indexed), an editorial board member in applied mathematics of “SpringerPlus” (Springer, SCI-indexed) and an editorial board member of “Journal of Applied Mathematics” (Hindawi).

series since the hyperbolic cross truncations [8]:

$$s_N^{(h)}(f;x,y) = \sum_{|m|=0}^{N} c_{m0}(f)\, e^{2\pi imx} + \sum_{|n|=1}^{N} c_{0n}(f)\, e^{2\pi iny} + \sum_{1\le|mn|\le N} c_{mn}(f)\, e^{2\pi i(mx+ny)} \tag{1.1}$$

can make full use of the decay of Fourier coefficients to reconstruct the target function f.

Throughout this paper, we always assume that $f \in C^{(3,3)}([0,1]^2)$ which means that $\frac{\partial f^{i+j}}{\partial x^i \partial y^j}(0 \le i,j \le 3)$ are continuous on $[0,1]^2$. We will show that, for the hyperbolic cross truncations of its Fourier series, the following asymptotic representation holds (see Theorem 3.1):

$$\| f - s_N^{(h)}(f) \|_2^2 = \frac{1}{4\pi^4}(f(0,0)+f(1,1)-f(0,1)-f(1,0))^2 \frac{\log^2 N_d}{N_d} + O\left(\frac{\log N_d}{N_d}\right), \tag{1.2}$$

where N_d is the number of Fourier coefficients in $s_N^{(h)}(f)$ and $\| F \|_2^2 = \int_0^1\int_0^1 |F(x,y)|^2 \mathrm{d}x\mathrm{d}y$.

For the partial sum approximation of the Fourier series of f on $[0,1]^2$, we will give another asymptotic representation. The corresponding principal part will become more complicated. It depends on not only values of function f and its partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ at vertexes of $[0,1]^2$, but also the mean values of f on each side of the boundary $\partial([0,1]^2)$ (in detail, see Theorem 4.1).

Comparing asymptotic representations of two kinds of Fourier approximations, we find that for hyperbolic cross approximation, the approximation order is $\frac{\log^2 N_d}{N_d}$, while for the partial sum approximation, in general the approximation order is $\frac{1}{\sqrt{N_d}}$, and under some cases the approximation order is $\frac{1}{N_d}$. Moreover, we further give a corresponding necessary and sufficient condition for these cases (see Corollary 4.2).

2. Asymptotic representation of Fourier coefficients

Let $f \in C^{(3,3)}([0,1]^2)$. Expand f into Fourier series: $f(x,y) = \sum_{m,n} c_{mn}\, e^{2\pi i(mx+ny)}$, where

$$c_{mn}(f) = \int_0^1\int_0^1 f(x,y)\, e^{-2\pi i(mx+ny)} \mathrm{d}x\mathrm{d}y$$

and $\sum_{m,n}$ means $\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}$. We extend f from $[0,1]^2$ to $\mathbb{R}^2$. Then f is a function on the whole plane $\mathbb{R}^2$ with period 1 and f is discontinuous at the integral points $\{m,n\}_{m,n\in\mathbb{Z}}$. By the Riemann-Lebesgue lemma, we only know that $c_{mn}(f) = o(1)$ as $m \to 0$ or $n \to \infty$, where "o" means high-order infinitesimal. To obtain the precise asymptotic formula of Fourier coefficients, we construct a combination $q(x,y)$ of the boundary functions $f(x,0), f(x,1), f(0,y), f(1,y)$ and factors $x, (1-x), y, (1-y)$ such that the difference $f(x,y) - q(x,y)$ vanishes on the boundary $\partial([0,1]^2)$.

Now we define three functions as follows.

$$\begin{aligned} q_1(x,y) &= (f(x,0) - f(0,0)(1-x) - f(1,0)x)(1-y) + (f(x,1) - f(0,1)(1-x) - f(1,1)x)y, \\ q_2(x,y) &= (f(0,y) - f(0,0)(1-y) - f(0,1)y)(1-x) + (f(1,y) - f(1,0)(1-y) - f(1,1)y)x, \\ q_3(x,y) &= f(0,0)(1-x)(1-y) + f(0,1)(1-x)y + f(1,0)x(1-y) + f(1,1)xy. \end{aligned} \tag{2.1}$$

Then $q(x,y) = q_1(x,y) + q_2(x,y) + q_3(x,y)$ is the desired function, i.e., we have the following theorem.

2

Theorem 2.1. Let f be defined on $[0,1]^2$ and $q(x,y)$ be stated as above. Then $\tau(x,y) = f(x,y) - q(x,y)$ vanished on the boundary $\partial([0,1]^2)$.

From this, we deduce that if $f \in C^{(3,3)}([0,1]^2)$, then $\tau(x,y) \in C^{(3,3)}([0,1]^2)$ and satisfies that for $i = 1,2,3$,

$$\begin{aligned} &\frac{\partial^i \tau}{\partial x^i}(x,0) = \frac{\partial^i \tau}{\partial x^i}(x,1) = 0 \qquad (0 \le x \le 1), \\ &\frac{\partial^i \tau}{\partial y^i}(0,y) = \frac{\partial^i \tau}{\partial y^i}(1,y) = 0 \qquad (0 \le y \le 1). \end{aligned} \tag{2.2}$$

Now we further explain the relationship between $q(x,y)$ and $f(x,y)$. By (2.1), it follows that

$$\begin{aligned} \frac{\partial q}{\partial x}(x,y) &= \frac{\partial f}{\partial x}(x,0)(1-y) + \frac{\partial f}{\partial x}(x,1)y - f(0,y) + f(1,y) \\ &\quad + (f(0,0) - f(1,0))(1-y) + (f(0,1) - f(1,1))y, \\ \frac{\partial q}{\partial y}(x,y) &= \frac{\partial f}{\partial y}(0,y)(1-x) + \frac{\partial f}{\partial y}(1,y)x - f(x,0) + f(x,1) \\ &\quad + (f(0,0) - f(0,1))(1-x) + (f(1,0) - f(1,1))x, \\ \frac{\partial^2 q}{\partial x \partial y}(x,y) &= -\frac{\partial f}{\partial x}(x,0) + \frac{\partial f}{\partial x}(x,1) - \frac{\partial f}{\partial y}(0,y) + \frac{\partial f}{\partial y}(1,y) \\ &\quad - f(0,0) + f(1,0) + f(0,1) - f(1,1), \\ \frac{\partial^3 q}{\partial x^2 \partial y}(x,y) &= -\frac{\partial^2 f}{\partial x^2}(x,0) + \frac{\partial^2 f}{\partial x^2}(x,1), \\ \frac{\partial^3 q}{\partial x \partial y^2}(x,y) &= -\frac{\partial^2 f}{\partial y^2}(0,y) + \frac{\partial^2 f}{\partial y^2}(1,y), \\ \frac{\partial^4 q}{\partial x^2 \partial y^2}(x,y) &= 0. \end{aligned}$$

From this, we get

$$\begin{aligned} &\frac{\partial^2 q}{\partial x \partial y}(1,1) - \frac{\partial^2 q}{\partial x \partial y}(1,0) - \frac{\partial^2 q}{\partial x \partial y}(0,1) + \frac{\partial^2 q}{\partial x \partial y}(0,0) = 0, \\ &\frac{\partial q}{\partial x}(1,y) - \frac{\partial q}{\partial x}(0,y) = \left(\frac{\partial f}{\partial x}(1,0) - \frac{\partial f}{\partial x}(0,0)\right)(1-y) + \left(\frac{\partial f}{\partial x}(1,1) - \frac{\partial f}{\partial x}(0,1)\right)y, \\ &\frac{\partial q}{\partial y}(x,1) - \frac{\partial q}{\partial y}(x,0) = \left(\frac{\partial f}{\partial y}(0,1) - \frac{\partial f}{\partial y}(0,0)\right)(1-x) + \left(\frac{\partial f}{\partial y}(1,1) - \frac{\partial f}{\partial y}(1,0)\right)x. \end{aligned} \tag{2.3}$$

Since $c_{mn}(f) = c_{mn}(q) + c_{mn}(\tau)$ and $c_{mn}(q) = c_{mn}(q_1) + c_{mn}(q_2) + c_{mn}(q_3)$, by (2.1),

$$c_{mn}(q_1) = c_m(R(x,0))c_n(1-y) + c_m(R(x,1))c_n(y), \tag{2.4}$$

where

$$R(x,\nu) = f(x,\nu) - f(0,\nu)(1-x) - f(1,\nu)x \qquad (\nu = 0,1). \tag{2.5}$$

Since $R(0,\nu) = R(1,\nu)$,

$$\begin{aligned} c_m(R(x,\nu)) &= \int_0^1 R(x,\nu)\, e^{-2\pi i m x} \mathrm{d}x = \frac{1}{2\pi i m} \int_0^1 \frac{\partial R}{\partial x}(x,\nu)\, e^{-2\pi i m x} \mathrm{d}x \\ &= \frac{1}{4\pi^2 m^2} \left(\frac{\partial R}{\partial x}(1,\nu) - \frac{\partial R}{\partial x}(0,\nu) - \int_0^1 \frac{\partial^2 R}{\partial x^2}(x,\nu)\, e^{-2\pi i m x} \mathrm{d}x \right) \qquad (m \ne 0), \\ c_0(R(x,\nu)) &= \int_0^1 f(x,\nu) \mathrm{d}x - \frac{1}{2}(f(0,\nu) + f(1,\nu)). \end{aligned}$$

3

Noticing that $\frac{\partial R}{\partial x}(x,\nu) = \frac{\partial f}{\partial x}(x,\nu) + f(0,0) - f(1,\nu)$, we get

$$\frac{\partial R}{\partial x}(1,\nu) - \frac{\partial R}{\partial x}(0,\nu) = \frac{\partial f}{\partial x}(1,\nu) - \frac{\partial f}{\partial x}(0,\nu),$$

$$\frac{\partial^2 R}{\partial x^2}(x,\nu) = \frac{\partial^2 f}{\partial x^2}(x,\nu) \qquad (\nu = 0, 1).$$

Since mth Fourier coefficients of $(1-x)$ and x are $\frac{1}{2\pi i m}$ and $-\frac{1}{2\pi i m}$ $(m \neq 0)$, respectively, we get by (2.5)

$$c_m(R(x,\nu)) = \frac{1}{4\pi^2 m^2}\left(\frac{\partial f}{\partial x}(1,\nu) - \frac{\partial f}{\partial x}(0,\nu) - \int_0^1 \frac{\partial^2 f}{\partial x^2}(x,\nu)\, e^{-2\pi i m x}\mathrm{d}x\right)$$

while

$$\int_0^1 \frac{\partial^2 f}{\partial x^2}(x,\nu)\, e^{-2\pi i m x}\mathrm{d}x = -\frac{1}{2\pi i m}\left(\frac{\partial^2 f}{\partial x^2}(1,\nu) - \frac{\partial^2 f}{\partial x^2}(0,\nu) - \int_0^1 \frac{\partial^3 f}{\partial x^3}(x,\nu)\, e^{2\pi i m x}\mathrm{d}x\right).$$

So

$$c_m(R(x,\nu)) = \frac{1}{4\pi^2 m^2}\left(\frac{\partial f}{\partial x}(1,\nu) - \frac{\partial f}{\partial x}(0,\nu)\right) + O\left(\frac{1}{m^3}\right) \qquad (m \neq 0),$$

$$c_0(R(x,\nu)) = \int_0^1 f(x,\nu)\mathrm{d}x - \frac{1}{2}(f(0,0) + f(1,\nu)).$$

From this and (2.4), it follows that

$$c_{mn}(q_1) = -\frac{i}{8\pi^3 m^2 n}\left(\frac{\partial f}{\partial x}(1,0) + \frac{\partial f}{\partial x}(0,1) - \frac{\partial f}{\partial x}(0,0) - \frac{\partial f}{\partial x}(1,1)\right) + O\left(\frac{1}{m^3 n}\right) \qquad (m \neq 0, n \neq 0),$$

$$c_{m0}(q_1) = \frac{1}{8\pi^2 m^2}\left(\frac{\partial f}{\partial x}(1,0) - \frac{\partial f}{\partial x}(0,1) - \frac{\partial f}{\partial x}(0,0) + \frac{\partial f}{\partial x}(1,1)\right) + O\left(\frac{1}{m^3}\right) \qquad (m \neq 0),$$

$$c_{0n}(q_1) = -\frac{i}{2\pi n}\left(\int_0^1 (f(x,0) - f(x,1))\mathrm{d}x - \frac{1}{2}(f(0,1) + f(1,0) - f(0,1) - f(1,1))\right) \qquad (n \neq 0).$$

Similarly, we have

$$c_{mn}(q_2) = -\frac{i}{8\pi^3 m n^2}\left(\frac{\partial f}{\partial y}(1,0) + \frac{\partial f}{\partial y}(0,1) - \frac{\partial f}{\partial y}(0,0) - \frac{\partial f}{\partial y}(1,1)\right) + O\left(\frac{1}{m n^3}\right) \qquad (m \neq 0, n \neq 0),$$

$$c_{0n}(q_2) = \frac{1}{8\pi^2 n^2}\left(\frac{\partial f}{\partial y}(0,1) - \frac{\partial f}{\partial y}(1,0) - \frac{\partial f}{\partial y}(0,0) + \frac{\partial f}{\partial y}(1,1)\right) + O\left(\frac{1}{n^3}\right) \qquad (n \neq 0),$$

$$c_{m0}(q_2) = -\frac{i}{2\pi m}\left(\int_0^1 (f(0,y) - f(1,y))\mathrm{d}y - \frac{1}{2}(f(0,0) + f(0,1) - f(1,0) - f(1,1))\right) \qquad (m \neq 0).$$

and

$$c_{mn}(q_3) = \frac{1}{4\pi^2 m n}(f(1,0) + f(0,1) - f(0,0) - f(1,1)) \qquad (m \neq 0,\ n \neq 0),$$

$$c_{m0}(q_3) = -\frac{i}{4\pi m}(f(0,0) - f(0,1) - f(1,0) + f(1,1)) \qquad (m \neq 0),$$

$$c_{0n}(q_3) = -\frac{i}{4\pi n}(f(0,0) - f(0,1) + f(1,0) - f(1,1)) \qquad (n \neq 0).$$

From this, we get an asymptotic representation of $c_{mn}(q)$ by $q(x,y) = q_1(x,y) + q_2(x,y) + q_3(x,y)$. Finally, we write out the asymptotic representation of $c_{mn}(\tau)$.

Using the integration by parts, it follows by Theorem 2.1, (2,2) and (2.4) that

(i) For $m \neq 0, n \neq 0$,

$$c_{mn}(\tau) = \frac{1}{16\pi^4 m^2 n^2}\left(\frac{\partial^2 f}{\partial x \partial y}(1,1) - \frac{\partial^2 f}{\partial x \partial y}(1,0) - \frac{\partial^2 f}{\partial x \partial y}(0,1) + \frac{\partial^2 f}{\partial x \partial y}(0,0)\right) + O\left(\frac{1}{m^2 n^2}\right)\left(\frac{1}{m} + \frac{1}{n}\right);$$

4

(ii) For $m \neq 0$,

$$c_{m0}(\tau)=\frac{1}{4\pi^2m^2}\left(\int_0^1\left(\frac{\partial f}{\partial x}(1,y)-\frac{\partial f}{\partial x}(0,y)\right)\mathrm{d}y+\frac{1}{2}\left(\frac{\partial f}{\partial x}(0,0)-\frac{\partial f}{\partial x}(1,0)+\frac{\partial f}{\partial x}(0,1)-\frac{\partial f}{\partial x}(1,1)\right)\right)+O\left(\frac{1}{m^3}\right)$$

(iii) For $n \neq 0$,

$$c_{0n}(\tau)=\frac{1}{4\pi^2n^2}\left(\int_0^1\left(\frac{\partial f}{\partial y}(x,1)-\frac{\partial f}{\partial y}(x,0)\right)\mathrm{d}x+\frac{1}{2}\left(\frac{\partial f}{\partial y}(0,0)-\frac{\partial f}{\partial y}(0,1)+\frac{\partial f}{\partial y}(1,0)-\frac{\partial f}{\partial y}(1,1)\right)\right)+O\left(\frac{1}{n^3}\right).$$

From this and $c_{mn}(f) = c_{mn}(q) + c_{mn}(\tau)$, we get the following asymptotic representation of Fourier coefficients of $f(x, y)$.

Theorem 2.2. Let $f \in C^{(3,3)}([0, 1]^2)$. Then Fourier coefficients of $f(x, y)$ satisfy

(i) for $m \neq 0$, $n \neq 0$,

$$c_{mn}(f) = \frac{1}{4\pi^2mn}(-\alpha + i\frac{\beta}{2\pi m} + i\frac{\gamma}{2\pi n} + \frac{\delta}{4\pi^2mn}) + O\left(\frac{1}{m^2n}\right)\left(\frac{1}{m} + \frac{1}{n}\right),$$

where

$$\alpha = f(0,0) - f(0,1) - f(1,0) + f(1,1),$$

$$\beta = \tfrac{\partial f}{\partial x}(0,0) - \tfrac{\partial f}{\partial x}(0,1) - \tfrac{\partial f}{\partial x}(1,0) + \tfrac{\partial f}{\partial x}(1,1),$$

$$\gamma = \tfrac{\partial f}{\partial y}(0,0) - \tfrac{\partial f}{\partial y}(0,1) - \tfrac{\partial f}{\partial y}(1,0) + \tfrac{\partial f}{\partial y}(1,1),$$

$$\delta = \tfrac{\partial^2 f}{\partial x\partial y}(0,0) - \tfrac{\partial^2 f}{\partial x\partial y}(0,1) - \tfrac{\partial^2 f}{\partial x\partial y}(1,0) + \tfrac{\partial^2 f}{\partial x\partial y}(1,1);$$

(ii) for $m \neq 0$,

$$c_{m0}(f) = i\frac{a}{2\pi m} + \frac{b}{4\pi^2m^2} + O\left(\frac{1}{m^3}\right),$$

where

$$a = f(0,1) - f(1,0) - \textstyle\int_0^1 (f(0,y) - f(1,y))\mathrm{d}y,$$

$$b = \textstyle\int_0^1 \left(\frac{\partial f}{\partial x}(1,y) - \frac{\partial f}{\partial x}(0,y)\right)\mathrm{d}y;$$

(iii) for $n \neq 0$,

$$c_{0n}(f) = i\frac{c}{2\pi n} + \frac{d}{4\pi^2n^2} + O\left(\frac{1}{n^3}\right),$$

where

$$c = f(1,0) - f(0,1) - \textstyle\int_0^1 (f(x,0) - f(x,1))\mathrm{d}x,$$

$$d = \textstyle\int_0^1 \left(\frac{\partial f}{\partial y}(x,1) - \frac{\partial f}{\partial y}(x,0)\right)\mathrm{d}x + O\left(\frac{1}{n^3}\right).$$

Now we compute $|c_{mn}(f)|^2$. Since f is a real-valued function, it is clear that $\alpha, \beta, \gamma, \delta$ and a, b, c, d in Theorem 2.2 are all real numbers. So we get the following corollary.

Corollary 2.3. Let $f \in C^{(3,3)}([0, 1]^2)$. Then

$$|c_{mn}(f)|^2 = \tfrac{1}{16\pi^4m^2n^2}\left(\alpha^2 + \tfrac{\beta\gamma-\alpha\delta}{2\pi^2mn} + \tfrac{\beta^2}{4\pi^2m^2} + \tfrac{\gamma^2}{4\pi^2n^2}\right) + O\left(\tfrac{1}{m^3n^3}\right)\left(\tfrac{1}{m} + \tfrac{1}{n}\right),$$

$$|c_{m0}(f)|^2 = \tfrac{a^2}{4\pi^2m^2} + O\left(\tfrac{1}{m^4}\right),$$

$$|c_{0n}(f)|^2 = \tfrac{c^2}{4\pi^2n^2} + O\left(\tfrac{1}{n^4}\right),$$

where $\alpha,\beta,\gamma,\delta$ and a,b,c,d are stated as above.

3. Asymptotic representation of hyperbolic cross approximation

Let $f\in C^{(3,3)}([0,1]^2)$. We expand it into a Fourier series. Consider the hyperbolic cross truncations of its Fourier series:

$$s_N^{(h)}(f;x,y)=\sum_{|m|=0}^{N} c_{m0}(f)\,e^{2\pi imx}+\sum_{|n|=1}^{N} c_{0n}(f)\,e^{2\pi iny}$$

$$+\sum_{|n|=1}^{N}\sum_{|m|\le \frac{N}{|n|}} c_{mn}(f)\,e^{2\pi i(mx+ny)},$$

where $c_{mn}(f)=\int_0^1\int_0^1 f(x,y)\,e^{-2\pi i(mx+ny)}\mathrm{d}x\mathrm{d}y$. So

$$f(x,y)-s_N^{(h)}(f;x,y)=\sum_{|m|\ge N+1} c_{m0}(f)\,e^{2\pi imx}+\sum_{|n|=N+1}^{\infty} c_{0n}(f)\,e^{2\pi iny}$$

$$+\sum_{|n|\ge N+1}\sum_{|m|=1}^{\infty} c_{mn}(f)\,e^{2\pi i(mx+ny)}+\sum_{|n|=1}^{N}\sum_{|m|>\frac{N}{|n|}} c_{mn}(f)\,e^{2\pi i(mx+ny)}.$$

Using the Parseval identity [4,5,9] of bivariate Fourier series,

$$\begin{aligned}\| f-s_N^{(h)}\|_2^2 &= \sum_{|n|\ge N+1}(|c_{0n}(f)|^2+|c_{n0}(f)|^2)\\ &\quad+\sum_{|n|\ge N+1}\sum_{|m|=1}^{\infty}|c_{mn}(f)|^2+\sum_{|n|=1}^{N}\sum_{|m|>\frac{N}{|n|}}|c_{mn}(f)|^2 \qquad (3.1)\\ &=: P_N+Q_N+R_N.\end{aligned}$$

By Corollary 2.3,

$$|c_{mn}(f)|^2=\frac{\alpha^2}{16\pi^4m^2n^2}+O\left(\frac{1}{m^3}\right)\frac{1}{n^2}+O\left(\frac{1}{n^3}\right)\frac{1}{m^2}.$$

We first compute R_N:

$$\begin{aligned}R_N &= \sum_{|n|=1}^{N}\sum_{|m|>\frac{N}{|n|}}|c_{mn}(f)|^2\\ &= \tfrac{\alpha^2}{16\pi^4}\sum_{|n|=1}^{N}\tfrac{1}{n^2}\sum_{|m|>\frac{N}{|n|}}\tfrac{1}{m^2}+O(1)\sum_{|n|=1}^{N}\tfrac{1}{n^4}\sum_{|m|>\frac{N}{|n|}}\tfrac{1}{m^3}+O(1)\sum_{|n|=1}^{N}\tfrac{1}{n^3}\sum_{|m|>\frac{N}{|n|}}\tfrac{1}{m^4} \qquad (3.2)\\ &=: R_N^{(1)}+R_N^{(2)}+R_N^{(3)}.\end{aligned}$$

Note that

$$R_N^{(1)}=\tfrac{\alpha^2}{16\pi^4}\sum_{|n|=1}^{N}\tfrac{1}{n^2}\sum_{|m|\ge\frac{N}{|n|}}\tfrac{1}{m^2},$$

$$\sum_{|m|\ge\frac{N}{|n|}}\tfrac{1}{m^2}=2\int_{\frac{N}{|n|}}^{\infty}\tfrac{1}{t^2}\mathrm{d}t+O\left(\tfrac{n^2}{N^2}\right)=\tfrac{2|n|}{N}+O\left(\tfrac{n^2}{N^2}\right).$$

6

This implies that

$$R_N^{(1)} = \frac{\alpha^2}{4\pi^4}\sum_{n=1}^{N}\frac{1}{nN} + O\left(\frac{1}{N}\right) = \frac{\alpha^2 \log N}{4\pi^4 N} + O\left(\frac{1}{N}\right).$$

Similarly, $R_N^{(2)} = O\left(\frac{1}{N}\right)$ and $R_N^{(3)} = O\left(\frac{1}{N}\right)$. So

$$R_N = \frac{\alpha^2 \log N}{4\pi^4 N} + O\left(\frac{1}{N}\right).$$

By $|c_{mn}|^2 = O\left(\frac{1}{m^2n^2}\right)$, it follows that

$$Q_N = O(1)\left(\sum_{|n|\geq N+1}\frac{1}{n^2}\right)\left(\sum_{|m|=1}\frac{1}{m^2}\right) + O\left(\frac{1}{N^2}\right) = O\left(\frac{1}{N}\right).$$

From $|c_{0n}(f)|^2 = O\left(\frac{1}{n^2}\right)$ and $|c_{m0}(f)|^2 = O\left(\frac{1}{m^2}\right)$, it is easy to deduce that

$$P_N = \sum_{|n|\geq N+1}|c_{0n}(f)|^2 + \sum_{|m|\geq N+1}|c_{m0}(f)| = O\left(\frac{1}{N}\right).$$

Therefore, by (3,1),

$$\| f - s_N^{(h)}(f) \|_2^2 = \frac{\alpha^2 \log N}{4\pi^4 N} + O\left(\frac{1}{N}\right).$$

The number N_d of Fourier coefficients in the hyperbolic cross truncation $s_N^{(h)}(f)$ is equal to

$$N_d = 2N + 1 + \sum_{n_1=1}^{N}\left[\frac{N}{|n_1|}\right] = 2N\log N + O(N).$$

Theorem 3.1. Let $f \in C^{(3,3)}([0,1]^2)$. Then the asymptotic representation of the hyperbolic cross approximation of Fourier series of f is

$$\| f - s_N^{(h)}(f) \|_2^2 = \frac{\alpha^2 \log^2 N_d}{4\pi^4 N_d}\left(1 + O\left(\frac{1}{\log N_d}\right)\right), \tag{3.3}$$

where N_d is the number of Fourier coefficients in hyperbolic cross truncation $s_N^{(h)}(f)$ and $\alpha = f(0,0) - f(0,1) - f(1,0) + f(1,1)$.

Corollary 3.2. Let $f \in C^{(2,2)}([0,1]^2)$. Then

(i) $\| f - s_N^{(h)}(f) \|_2^2 = O\left(\frac{\log N_d}{N_d}\right)$ if and only if $f(0,0) + f(1,1) = f(0,1) + f(1,0)$.

(ii) when $F(x,y) = f(x,y) + (f(0,1) + f(1,0) - f(0,0) - f(1,1))xy$,

$$\| F - s_N^{(h)}(F) \|_2^2 = O\left(\frac{\log N_d}{N_d}\right).$$

Now we show an approach to estimates of the bound of the term " O" in Theorem 3.1 using the Sobolev norm. For $\frac{\partial^6 f}{\partial x^3 \partial y^3} \in C([0,1]^2)$, its Sobolev norm is defined as

$$M(f) = \max_{\substack{x,y\in\partial([0,1]^2)\\ i,j=0,1,2,3}}\left|\frac{\partial^{i+j} f}{\partial x^i \partial y^j}\right|.$$

By Theorem 2.1 and (2.2), and (2.4), we get

$$c_{mn}(\tau) = \int_0^1 \int_0^1 \tau(x,y)\, e^{-2\pi i(mx+ny)}\mathrm{d}x\mathrm{d}y = \frac{\delta}{16\pi^4 m^2 n^2} + J_{mn},$$

where

$$\delta = \frac{\partial^2 f}{\partial x \partial y}(0,0) - \frac{\partial^2 f}{\partial x \partial y}(0,1) - \frac{\partial^2 f}{\partial x \partial y}(1,0) + \frac{\partial^2 f}{\partial x \partial y}(1,1)$$

and

$$\begin{aligned} J_{mn} = {} & \tfrac{1}{32\pi^5 m^2 n^3}\left(\tfrac{\partial^3 f}{\partial x \partial y^2}(1,1) - \tfrac{\partial^3 f}{\partial x \partial y^2}(0,1) - \tfrac{\partial^3 f}{\partial x \partial y^2}(1,0) + \tfrac{\partial^3 f}{\partial x \partial y^2}(0,0)\right) \\ & - \tfrac{1}{32\pi^5 m^2 n^3}\int_0^1 \left(\tfrac{\partial^4 f}{\partial x \partial y^3}(1,y) - \tfrac{\partial^4 f}{\partial x \partial y^2}(0,y)\right) e^{-2\pi i n y}\mathrm{d}y \\ & + \tfrac{1}{32 i\pi^5 m^3 n^2}\left(\tfrac{\partial^3 f}{\partial x^2 \partial y}(1,0) - \tfrac{\partial^3 f}{\partial x^2 \partial y}(0,0)\right) \\ & - \tfrac{1}{32 i\pi^5 m^3 n^2}\int_0^1 \left(\tfrac{\partial^4 f}{\partial x^2 \partial y^2}(1,y) - \tfrac{\partial^4 f}{\partial x^2 \partial y^2}(0,y)\right) e^{-2\pi i n y}\mathrm{d}y \\ & + \tfrac{1}{32 i\pi^5 m^3 n^2}\int_0^1 \left(\tfrac{\partial^4 f}{\partial x^3 \partial y}(x,1) - \tfrac{\partial^4 f}{\partial x^3 \partial y}(x,0)\right) e^{-2\pi i m x}\mathrm{d}x \\ & - \tfrac{1}{32 i\pi^5 m^3 n^2}\int_0^1\int_0^1 \tfrac{\partial^5 f}{\partial x^3 \partial y^2}(x,y)\, e^{-2\pi i(mx+ny)}\mathrm{d}x\mathrm{d}y \end{aligned}$$

So

$$|J_{mn}| \le \frac{6M(f)}{32\pi^5 m^2 n^3} + \frac{7M(f)}{32\pi^5 m^3 n^2} \le \frac{13M(f)}{32\pi^5 m^2 n^2}\left(\frac{1}{m} + \frac{1}{n}\right).$$

For c_{m0} and c_{0n}, we have

$$c_{m0}(\tau) = \tfrac{1}{(2\pi m)^2}\left(\int_0^1 \left(\tfrac{\partial f}{\partial x}(1,y) - \tfrac{\partial f}{\partial x}(0,y)\right)\mathrm{d}y + \tfrac{1}{2}\beta\right) + T_m^{(1)},$$

$$c_{0n}(\tau) = \tfrac{1}{(2\pi n)^2}\left(\int_0^1 \left(\tfrac{\partial f}{\partial y}(x,1) - \tfrac{\partial f}{\partial y}(x,0)\right)\mathrm{d}x + \tfrac{1}{2}\gamma\right) + T_n^{(2)},$$

where

$$\begin{aligned} T_m^{(1)} = {} & \tfrac{1}{(2\pi m)^3 i}\int_0^1 \left(\tfrac{\partial^2 f}{\partial x^2}(1,y) - \tfrac{\partial^2 f}{\partial x^2}(0,y)\right)\mathrm{d}y \\ & - \tfrac{1}{2(2\pi m)^3 i}\left(\tfrac{\partial^2 f}{\partial x^2}(1,0) - \tfrac{\partial^2 f}{\partial x^2}(0,0) - \tfrac{\partial^2 f}{\partial x^2}(1,1) + \tfrac{\partial^2 f}{\partial x^2}(0,1)\right) \\ & - \tfrac{1}{(2\pi m)^3 i}\int_0^1\int_0^1 \left(\tfrac{\partial^3 f}{\partial x^3}(x,y) - \tfrac{1}{2}\tfrac{\partial^3 f}{\partial x^3}(x,0) - \tfrac{1}{2}\tfrac{\partial^3 f}{\partial x^3}(x,1)\right)\mathrm{d}x\mathrm{d}y. \\ T_n^{(2)} = {} & \tfrac{1}{(2\pi n)^3 i}\int_0^1 \left(\tfrac{\partial^2 f}{\partial y^2}(x,1) - \tfrac{\partial^2 f}{\partial y^2}(x,0)\right)\mathrm{d}x \\ & - \tfrac{1}{2(2\pi n)^3 i}\left(\tfrac{\partial^2 f}{\partial y^2}(0,1) - \tfrac{\partial^2 f}{\partial y^2}(0,0) - \tfrac{\partial^2 f}{\partial y^2}(1,1) + \tfrac{\partial^2 f}{\partial y^2}(1,0)\right) \\ & - \tfrac{1}{(2\pi n)^3 i}\int_0^1\int_0^1 \left(\tfrac{\partial^3 f}{\partial y^3}(x,y) - \tfrac{1}{2}\tfrac{\partial^3 f}{\partial y^3}(0,y) - \tfrac{1}{2}\tfrac{\partial^3 f}{\partial y^3}(1,y)\right)\mathrm{d}x\mathrm{d}y. \end{aligned}$$

So

$$|T_m^{(1)}| \le \tfrac{6M(f)}{(2\pi m)^3},$$

$$|T_n^{(2)}| \le \tfrac{6M(f)}{(2\pi n)^3}.$$

Now we estimate $c_{mn}(q)$. Note that

$$c_m(R(x,\nu)) = \frac{1}{4\pi^2 m^2}\left(\frac{\partial f}{\partial x}(1,\nu) - \frac{\partial f}{\partial x}(0,\nu)\right) + L_m^{(\nu)} \qquad (\nu = 0,1),$$

where

$$L_m^{(\nu)} = \frac{1}{8\pi^3 m^3 i}\left(\frac{\partial^2 f}{\partial x^2}(1,\nu) - \frac{\partial^2 f}{\partial x^2}(0,\nu) - \int_0^1 \frac{\partial^3 f}{\partial x^3}(x,\nu)\, e^{-2\pi i m x}\mathrm{d}x\right) \qquad (\nu = 0,1).$$

Then $|L_m^{(\nu)}| \le \frac{5M(f)}{8\pi^3 m^3}$. This implies that

$$c_{mn}(q_1) = -\tfrac{\beta}{8\pi^3 m^2 n i} + H_{mn}^{(1)},$$

$$c_{mn}(q_2) = -\tfrac{\gamma}{8\pi^3 m n^2 i} + H_m^{(2)},$$

where

$$|H_{,mn}^{(1)}| \le \tfrac{5M(f)}{8\pi^4 m^3 n},$$

$$|H_{mn}^{(2)}| \le \tfrac{5M(f)}{8\pi^4 m n^3}.$$

From this and $c_{mn}(q_3) = \frac{\alpha}{4\pi^2 mn}$, we get

$$c_{mn}(q) = \frac{1}{4\pi^2 mn}\left(\alpha - \frac{\beta}{2\pi m i} - \frac{\gamma}{2\pi n i}\right) + H_{mn},$$

where $|H_{mn}| \le \frac{5M(f)}{8\pi^4 mn}\left(\frac{1}{m^2} + \frac{1}{n^2}\right)$.

Similarly, we may estimate $c_{m0}(q)$ and $c_{0n}(q)$. Using $c_{mn}(f) = c_{mn}(q) + c_{mn}(\tau)$ and the above estimates, we easily obtain the estimates of upper bounds of $|c_{mn}(f)|^2$. Again, using the method of argument in Theorem 3.1, we finally can give the bound of the term " O " in (3.3).

4. Asymptotic representation of square errors of partial sums

Let $f \in C^{(3,3)}([0,1]^2)$. Consider the partial sums of its Fourier series:

$$s_N(f;x,y) = \sum_{|m|\le N}\sum_{|n|\le N} c_{mn}(f)\, e^{2\pi i(mx+ny)}.$$

Then the square errors are equal to

$$\begin{aligned}\| f - s_N(f) \|_2^2 &= \sum_{|n|\ge N+1} |c_{0n}(f)|^2 + \sum_{|m|\ge N+1} |c_{m0}(f)|^2 \\ &\quad + \sum_{|n|\ge N+1}\sum_{|m|=1}^{\infty} |c_{mn}(f)|^2 + \sum_{|n|=1}^{N}\sum_{|m|\ge N+1} |c_{mn}(f)|^2 \qquad (4.1) \\ &=: K_N + L_N + I_N + J_N.\end{aligned}$$

By Corollary 2,3 (ii) and (iii),

$$K_N = \tfrac{c^2}{2\pi^2 N} + O\left(\tfrac{1}{N^3}\right),$$

$$L_N = \tfrac{a^2}{2\pi^2 N} + O\left(\tfrac{1}{N^3}\right).$$

9

By Corollary 2.3 (i),

$$|c_{mn}(f)|^2 = \frac{1}{16\pi^4 m^2 n^2}\left(\alpha^2 + \frac{\beta^2}{4\pi^2 m^2} + \frac{\gamma^2}{4\pi^2 n^2}\right) + O\left(\frac{1}{m^3 n^3}\right),$$

and so

$$I_N = \tfrac{\alpha^2}{48\pi^2}\left(\sum_{|n|>N}\tfrac{1}{n^2}\right) + \tfrac{\beta^2}{64\pi^6}\left(\sum_{|n|>N}\tfrac{1}{n^2}\right)\zeta(4) + \tfrac{\gamma^2}{192\pi^4}\left(\sum_{|n|>N}\tfrac{1}{n^4}\right) + O\left(\tfrac{1}{N^2}\right)$$

$$= \tfrac{1}{8\pi^2}\left(\tfrac{\alpha^2}{3} + \tfrac{\beta^2}{2\pi^4}\zeta(4)\right)\tfrac{1}{N} + O\left(\tfrac{1}{N^2}\right),$$

$$J_N = \tfrac{\alpha^2}{16\pi^4}\left(\sum_{|n|=1}^{N}\tfrac{1}{n^2}\right)\left(\sum_{|m|>N}\tfrac{1}{m^2}\right) + \tfrac{\beta^2}{64}\left(\sum_{|n|=1}^{N}\tfrac{1}{n^2}\right)\left(\sum_{|m|>N}\tfrac{1}{m^4}\right)$$

$$+\tfrac{\gamma^2}{64\pi^6}\left(\sum_{|n|=1}^{N}\tfrac{1}{n^4}\right)\left(\sum_{|m|>N}\tfrac{1}{m^2}\right) + O\left(\tfrac{1}{N^2}\right),$$

where $\zeta(s) = \sum\limits_{n=1}^{\infty}\frac{1}{n^s}$ is the Riemann-Zeta function. Note that

$$\sum_{|n|=1}^{N}\tfrac{1}{n^2} = \sum_{|n|=1}^{\infty}\tfrac{1}{n^2} - \sum_{|n|>N}\tfrac{1}{n^2} = \tfrac{\pi^2}{3} + O\left(\tfrac{1}{N}\right),$$

$$\sum_{|n|=1}^{N}\tfrac{1}{n^4} = \sum_{|n|=1}^{\infty}\tfrac{1}{n^4} - \sum_{|n|>N}\tfrac{1}{n^4} = \zeta(4) + O\left(\tfrac{1}{N^3}\right).$$

Then

$$J_N = \frac{1}{8\pi^2}\left(\frac{\alpha^2}{3} + \frac{\gamma^2}{\pi^4}\zeta(4)\right)\frac{1}{N} + O\left(\frac{1}{N^2}\right).$$

Finally, by (4.1), we get the following theorem.

Theorem 4.1. Let $f \in C^{(3,3)}([0,1]^2)$. Then the partial sums $s_N(f)$ of its Fourier series satisfy

$$\| f - s_N(f) \|_2^2 = \left(\frac{a^2+c^2}{2\pi^2} + \frac{\alpha^2}{24\pi^2} + \frac{\beta^2+\gamma^2}{8\pi^6}\zeta(4)\right)\frac{1}{N} + O\left(\frac{1}{N^2}\right),$$

where $a, c, \alpha, \beta, \gamma$ are stated in Theorem 2.2 and $\zeta(4)$ is the Riemann-Zeta function.

Note that the number N_d of Fourier coefficients in the sum $s_N(f)$ is $(2N+1)^2$. From Theorem 4.1, it follows that

$$\| f - s_N(f) \|_2^2 \sim \frac{1}{\sqrt{N_d}}.$$

Again, by Theorem 4.1, we get the following corollary.

Corollary 4.2. Let $f \in C^{(3,3)}([0,1]^2)$. Then the partial sums $s_N(f)$ of its Fourier series satisfy

$$\| f - s_N(f) \|_2^2 = O\left(\frac{1}{N^2}\right)$$

10

if and only if $a = c = \alpha = \beta = \gamma = 0$, i.e.,

$$\begin{aligned} &f(1,0) - f(0,1) = \textstyle\int_0^1 (f(x,0) - f(x,1))\mathrm{d}x, \\ &f(0,1) - f(1,0) = \textstyle\int_0^1 (f(0,y) - f(1,y))\mathrm{d}y, \\ &f(0,1) + f(1,0) = f(0,0) + f(1,1), \\ &\tfrac{\partial f}{\partial x}(0,1) + \tfrac{\partial f}{\partial x}(1,0) = \tfrac{\partial f}{\partial x}(0,0) + \tfrac{\partial f}{\partial x}(1,1), \\ &\tfrac{\partial f}{\partial y}(0,1) + \tfrac{\partial f}{\partial y}(1,0) = \tfrac{\partial f}{\partial y}(0,0) + \tfrac{\partial f}{\partial y}(1,1). \end{aligned} \tag{4.2}$$

Since the number of Fourier coefficients is $2N+1$ in $s_N(f)$, it is clear that when (4.2) holds,

$$\| f - s_N(f) \|_2^2 = O\left(\frac{1}{N_d}\right).$$

Comparing it with Theorem 3.1, we see that in this case the partial sum approximation is better than the hyperbolic cross approximation.

References

[1] V. Barthelmann, E. Novak, and K. Ritter, High dimensional polynomial interpolation on sparse grids, Advances in Computation Mathematics, 12(4) (2000), 273-288.

[2] B. Boashash, Time-frequency signal analysis and processing, Second edition, Academic press, 2016.

[3] W. Cheney and W. Light, A course in approximation theory, Thomson Learning, 2000.

[4] A. DoVore and G. G. Lorentz, Constructive approximation, Vol. 303 of Grundlehren, Springer, Hcidelberg, 1993.

[5] G.G. Lorentz, M.von Golitschck, and Ju. Makovoz, Constructive approximation, Advanced Problems, Springer, Borlin, 1996.

[6] M. Griebil and J. Hamaekers, Sparse grids for the Schrödinger equation, ESAIM Math. Model, Numer. Anal. 41 (2007).

[7] J. Shen and H. Yu, Efficient spectral sparse grid methods and applications to high-dimensional elliptic problems, SIAM Journal on Scientific Computing, 32(6) (2010), 3228-3250.

[8] J. Shen and L. Wang, Sparse spectral approximation of high-dimensional problems based on the hyperbolic cross, SIAM, J. Num. Anal., 48 (2010).

[9] E. M. Stein and G. Weiss, Introduction to Fourier analysis on Euclidean spaces, Princeton Univ. Press, 1971.

[10] P. Stoica and R. Moses, Spectral analysis of signals, Prentice Hall, 2005.

[11] A. F. Timan, Theory at approximation of Functions of a real variable, Pergamon, 1963.

[12] Z. Zhang and John C. Moore, Mathematical and physical fundamentals of climate change, Elsevier, 2015.

[13] Z. Zhang, Approximation of bivariate functions via smooth extensions, The Scientific World Journal, vol. 2014, Article ID 102062, 2014. doi:10.1155/2014/102062. 119-136.

[14] Z. Zhang, Environmental Data Analysis, DeGruyter, December 2016.

[15] Z. Zhang, P. Jorgensen, Modulated Haar wavelet analysis of climatic background noise, Acta Appl Math, 140, 71-93, 2015

Khatri-Rao Products and Selection Operators

Arnon Ploymukda and Pattrawut Chansangiam*
Department of Mathematics, Faculty of Science,
King Mongkut's Institute of Technology Ladkrabang,
Bangkok 10520, Thailand.

Abstract

We develop further theory for Khatri-Rao products of Hilbert space operators in connections with selection operators. We provide two constructions related to selection operators. Then we establish certain identities and inequalities involving Khatri-Rao and Tracy-Singh products. As consequences, we obtain some characterizations for the mixed product property concerning the Khatri-Rao product of operators.

Keywords: tensor product, Khatri-Rao product, Tracy-Singh product, operator matrix
Mathematics Subject Classifications 2010: 47A80, 15A69, 47A05.

1 Introduction

This paper concerns operator extensions of certain matrix products, namely, the Kronecker (tensor) product, the Tracy-Singh product, and the Khatri-Rao product. Fundamental theory for these matrix products are collected, for instance, in [1, 2, 4, 5, 10, 11, 12] and references therein. Denote by $M_{m,n}(\mathbb{C})$ the algebra of m-by-n complex matrices. Recall that the Kronecker product of $A = [a_{ij}] \in M_{m,n}(\mathbb{C})$ and $B \in M_{p,q}(\mathbb{C})$ is given by

$$A \,\hat{\otimes}\, B \;=\; [a_{ij}B]_{ij} \,.$$

Consider partitioned matrices A and B such that the (i,j)th block of A is A_{ij} and the (k,l)th block of B is B_{kl}. The Tracy-Singh product [9] of A and B is defined by

$$A \,\hat{\boxtimes}\, B \;=\; \big[[A_{ij}\hat{\otimes}B_{kl}]_{kl}\big]_{ij} \,. \tag{1}$$

The Khatri-Rao product [3] is defined for two partitioned matrices $A = [A_{ij}]$ and $B = [B_{ij}]$ as follows

$$A \hat{\boxdot} B \;=\; \big[A_{ij}\hat{\otimes}B_{ij}\big]_{ij} \,. \tag{2}$$

*Corresponding author. Email: pattrawut.ch@kmitl.ac.th

1

The tensor product of Hilbert space operators can be viewed as an extension of the Kronecker product of complex matrices. Recall that the tensor product of $A \in \mathbb{B}(\mathcal{H}, \mathcal{H}')$ and $B \in \mathbb{B}(\mathcal{K}, \mathcal{K}')$ is the unique bounded linear operator from $\mathcal{H} \otimes \mathcal{K}$ into $\mathcal{H}' \otimes \mathcal{K}'$ such that $(A \otimes B)(x \otimes y) = Ax \otimes By$ for all $x \in \mathcal{H}$ and $y \in \mathcal{K}$. Recently, the Tracy-Singh product and the Khatri-Rao product for matrices were generalized to those for operators acting on the direct sum of Hilbert spaces, see [6, 7, 8]. Fundamental algebraic and order properties of operator Khatri-Rao products are investigated in [8]. That paper also provides a construction of a unital positive linear map taking the Tracy-Singh product of two operators to their Khatri-Rao product. Such a linear map appears in the form $X \mapsto Z^*AZ$ where Z is an isometry, called a selection operator. See details in Section 2.

The present paper contains further development on operator Khatri-Rao products in relations with Tracy-Singh products and selection operators. First, we provide two constructions related to selection operators (see Section 3). Consequently, we establish some operator identities and inequalities involving Khatri-Rao and Tracy-Singh products (see Section 4). Finally, we obtain some characterizations for the mixed product property concerning the Khatri-Rao product of operators (see Section 5).

2 Tracy-Singh products and Khatri-Rao products for operators

Throughout this paper, let $\mathcal{H}$, $\mathcal{H}'$, $\mathcal{K}$ and $\mathcal{K}'$ be complex separable Hilbert spaces. When $\mathcal{X}$ and $\mathcal{Y}$ are Hilbert spaces, let us denote by $\mathbb{B}(\mathcal{X}, \mathcal{Y})$ the space of all bounded linear operators from $\mathcal{X}$ into $\mathcal{Y}$ and abbreviate $\mathbb{B}(\mathcal{X}, \mathcal{X})$ to $\mathbb{B}(\mathcal{X})$. Capital letters always denote a Hilbert space operator. In particular, I and O stand for the identity and the zero operator, respectively.

In order to define Tracy-Singh products of operators, we fix the following decompositions

$$\mathcal{H} = \bigoplus_{j=1}^{n} \mathcal{H}_j, \quad \mathcal{H}' = \bigoplus_{i=1}^{m} \mathcal{H}'_i, \quad \mathcal{K} = \bigoplus_{j=1}^{q} \mathcal{K}_j, \quad \mathcal{K}' = \bigoplus_{i=1}^{p} \mathcal{K}'_i. \tag{3}$$

where all of $\mathcal{H}_j, \mathcal{H}'_i, \mathcal{K}_l, \mathcal{K}'_k$ are Hilbert spaces. For each j and l, let $M_j : \mathcal{H}_j \to \mathcal{H}$ and $N_l : \mathcal{K}_l \to \mathcal{K}$ be the canonical injections. For each i and k, let $P_i : \mathcal{H}' \to \mathcal{H}'_i$ and $Q_k : \mathcal{K}' \to \mathcal{K}'_k$ be the canonical projections. Given $A \in \mathbb{B}(\mathcal{H}, \mathcal{H}')$, put $A_{ij} = P_i A M_j \in \mathbb{B}(\mathcal{H}_j, \mathcal{H}'_i)$ for each i, j. Thus we can write A in the operator-matrix form $A = [A_{ij}]_{i,j=1}^{m,n}$. Similarly, given $B \in \mathbb{B}(\mathcal{K}, \mathcal{K}')$, let $B_{kl} = Q_k B N_l \in \mathbb{B}(\mathcal{K}_l, \mathcal{K}'_k)$ for each $k = 1, \dots, p$ and $l = 1, \dots, q$. We can identify B with the operator matrix $B = [B_{kl}]_{k,l=1}^{p,q}$.

Definition 1. *The Tracy-Singh product of A and B is defined to be the bounded linear operator from $\bigoplus_{j,l=1}^{n,q} \mathcal{H}_j \otimes \mathcal{K}_l$ to $\bigoplus_{i,k=1}^{m,p} \mathcal{H}'_i \otimes \mathcal{K}'_k$ represented by*

$$A \boxtimes B = \left[[A_{ij} \otimes B_{kl}]_{kl} \right]_{ij}. \tag{4}$$

If both factor A and B consist of only one block, then $A \boxtimes B = A \otimes B$.

Lemma 2 ([6])**.** *The following properties of the Tracy-Singh product for operators hold (provided that each term is well-defined):*

1. *Compatibility with adjoints:* $(A \boxtimes B)^* = A^* \boxtimes B^*$.
2. *Mixed-product property:* $(A \boxtimes B)(C \boxtimes D) = AC \boxtimes BD$.
3. *Monotonicity: if* $A \geqslant B \geqslant 0$ *and* $C \geqslant D \geqslant 0$*, then* $A \boxtimes B \geqslant C \boxtimes D \geqslant 0$.

From now on, we fix the decomposition (3), and assume $n = q$ and $m = p$.

Definition 3. *The Khatri-Rao product of* $A = [A_{ij}]_{i,j=1}^{m,n}$ *and* $B = [B_{ij}]_{i,j=1}^{m,n}$ *is defined to be a bounded linear operator from* $\bigoplus_{j=1}^{n} \mathcal{H}_j \otimes \mathcal{K}_j$ *to* $\bigoplus_{i=1}^{m} \mathcal{H}'_i \otimes \mathcal{K}'_i$ *represented by the operator matrix*

$$A \boxdot B = [A_{ij} \otimes B_{ij}]_{i,j=1}^{m,n}. \tag{5}$$

Lemma 4 ([8])**.** *For* $A \in \mathbb{B}(\mathcal{H}, \mathcal{H}')$ *and* $B \in \mathbb{B}(\mathcal{K}, \mathcal{K}')$*, we have* $(A \boxdot B)^* = A^* \boxdot B^*$.

Fix an ordered tuple $(\mathcal{H}, \mathcal{H}', \mathcal{K}, \mathcal{K}')$ of Hilbert spaces. Define the ordered pair (Z_1, Z_2) of selection operators associated with $(\mathcal{H}, \mathcal{H}', \mathcal{K}, \mathcal{K}')$ by [8]:

$$Z_1 = \begin{bmatrix} E_1 \\ \vdots \\ E_m \end{bmatrix} \quad \text{and} \quad Z_2 = \begin{bmatrix} F_1 \\ \vdots \\ F_n \end{bmatrix}. \tag{6}$$

Here, for each $r = 1, ..., m$

$$E_r = \left[E_{gh}^{(r)} \right]_{g,h=1}^{m,m} : \bigoplus_{k=1}^{m} \mathcal{H}'_k \otimes \mathcal{K}'_k \to \bigoplus_{l=1}^{m} \mathcal{H}'_r \otimes \mathcal{K}'_l$$

with $E_{gh}^{(r)}$ is an identity operator if $g = h = r$ and the others are zero operators. For each $s = 1, ..., n$, the operator F_s is defined by

$$F_s = \left[F_{gh}^{(s)} \right]_{g,h=1}^{n,n} : \bigoplus_{i=1}^{n} \mathcal{H}_i \otimes \mathcal{K}_i \to \bigoplus_{j=1}^{n} \mathcal{H}_s \otimes \mathcal{K}_j$$

with $F_{gh}^{(s)}$ is an identity operator if $g = h = s$ and the others are zero operators. From the construction, the operator Z_i is an isometry and $Z_i Z_i^* \leqslant I$ for $i = 1, 2$. When $\mathcal{H} = \mathcal{H}'$ and $\mathcal{K} = \mathcal{K}'$, we have $Z_1 = Z_2$.

Lemma 5 ([8])**.** *Let* (Z_1, Z_2) *be the ordered pair of selection operators associated with the ordered tuple* $(\mathcal{H}, \mathcal{H}', \mathcal{K}, \mathcal{K}')$*. For any operator matrices* $A \in \mathbb{B}(\mathcal{H}, \mathcal{H}')$ *and* $B \in \mathbb{B}(\mathcal{K}, \mathcal{K}')$*, we have*

$$A \boxdot B = Z_1^* (A \boxtimes B) Z_2. \tag{7}$$

For the case $\mathcal{H} = \mathcal{H}'$ *and* $\mathcal{K} = \mathcal{K}'$*, we have* $Z_1 = Z_2 := Z$ *and hence for any* $A \in \mathbb{B}(\mathcal{H})$ *and* $B \in \mathbb{B}(\mathcal{K})$*,*

$$A \boxdot B = Z^* (A \boxtimes B) Z. \tag{8}$$

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 27, NO.2, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC

3 Two constructions related to selection operators

In this section, we construct certain operators related to selection operators.

Theorem 6. *Let (Z_1, Z_2) be the ordered pair of selection operators associated with an ordered tuple $(\mathcal{H}, \mathcal{H}', \mathcal{K}, \mathcal{K}')$. Then there exist operators*

$$V : \bigoplus_{i=1}^{m-1} \bigoplus_{j=1}^{m} \mathcal{H}'_i \otimes \mathcal{K}'_j \to \bigoplus_{i=1}^{m} \bigoplus_{j=1}^{m} \mathcal{H}'_i \otimes \mathcal{K}'_j,$$
$$W : \bigoplus_{i=1}^{n-1} \bigoplus_{j=1}^{n} \mathcal{H}_i \otimes \mathcal{K}_j \to \bigoplus_{i=1}^{n} \bigoplus_{j=1}^{n} \mathcal{H}_i \otimes \mathcal{K}_j$$

such that $Z_1^* V = 0$, $Z_2^* W = 0$, $Z_1 Z_1^* + VV^* = I$ and $Z_2 Z_2^* + WW^* = I$. If, in addition, $\mathcal{H} = \mathcal{H}'$ and $\mathcal{K} = \mathcal{K}'$, we have $V = W$.

Proof. Let

$$V = \begin{bmatrix} V_1 \\ \vdots \\ V_m \end{bmatrix} \tag{9}$$

where

$$V^{(r)} = \left[V_{kl}^{(r)} \right]_{k,l=1}^{m,m^2-1} : \bigoplus_{i=1}^{m} \bigoplus_{\substack{j=1 \\ i+j<m^2}}^{m} \mathcal{H}'_i \otimes \mathcal{K}'_i \to \bigoplus_{i=1}^{m} \mathcal{H}'_r \otimes \mathcal{K}'_i$$

for $r = 1, ..., m$, with $V_{kl}^{(r)}$ is an identity operator if $k \neq r$ and $l = m(r-1) + k$ and the others are zero operators. Note that

$$\begin{aligned} & E_1^* V_1 \\ = & \begin{bmatrix} I & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & & 0 & 0 & \cdots & 0 \\ 0 & I & \cdots & 0 & 0 & 0 & \cdots & 0 & & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \cdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & I & 0 & 0 & \cdots & 0 & & 0 & 0 & \cdots & 0 \end{bmatrix} \\ = & \ 0. \end{aligned}$$

For each r, we have

$$V_r V_r^* = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & I & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & I \end{bmatrix}.$$

Then we obtain

$$Z_1^* V = \begin{bmatrix} E_1^* & E_2^* & \cdots & E_m^* \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ \vdots \\ V_m \end{bmatrix} = E_1^* V_1 + E_2^* V_2 + \ldots + E_m^* V_m = 0,$$

$$\begin{aligned} & Z_1 Z_1^* + VV^* \\ &= \begin{bmatrix} E_1E_1^* & E_1E_2^* & \cdots & E_1E_m^* \\ E_2E_1^* & E_2E_2^* & \cdots & E_2E_m^* \\ \vdots & \vdots & \ddots & \vdots \\ E_mE_1^* & E_mE_2^* & \cdots & E_mE_m^* \end{bmatrix} + \begin{bmatrix} V_1V_1^* & V_1V_2^* & \cdots & V_1V_m^* \\ V_2V_1^* & V_2V_2^* & \cdots & V_2V_m^* \\ \vdots & \vdots & \ddots & \vdots \\ V_mEV_1^* & V_mV_2^* & \cdots & V_mV_m^* \end{bmatrix} \\ &= \begin{bmatrix} I & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & I & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & I \end{bmatrix} = \begin{bmatrix} I & 0 & \cdots & 0 \\ 0 & I & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & I \end{bmatrix}. \end{aligned}$$

Now, let

$$W = \begin{bmatrix} W_1 \\ \vdots \\ W_m \end{bmatrix} \tag{10}$$

where

$$W^{(s)} = \left[W_{kl}^{(s)} \right]_{k,l=1}^{n,n^2-1} : \bigoplus_{i=1}^{n} \bigoplus_{\substack{j=1 \\ i+j<n^2}}^{n} \mathcal{H}_i \otimes \mathcal{K}_i \to \bigoplus_{i=1}^{n} \mathcal{H}_s \otimes \mathcal{K}_i$$

for $s = 1, ..., n$, with $W_{kl}^{(s)}$ is an identity operator if $k \neq s$ and $l = n(s-1) + k$ and others are zero operators. A direct computation shows that $Z_2^* W = 0$ and $Z_2 Z_2^* + WW^* = I$. When $\mathcal{H} = \mathcal{H}'$ and $\mathcal{K} = \mathcal{K}'$, we have $V_i = W_i$ for all $i = 1, \ldots, m$, i.e. $V = W$. □

Theorem 7. *Fix the decomposition* (3) *with* $n = q$ *and* $m = p$. *Suppose further that* $\mathcal{H}_i = \mathcal{X}$, $\mathcal{K}_i = \mathcal{Y}$, $\mathcal{H}'_j = \mathcal{X}'$ *and* $\mathcal{K}'_j = \mathcal{Y}'$ *for all* $i = 1, \ldots, n$ *and* $j = 1, \ldots, m$. *Let* (Z_1, Z_2) *be the ordered pair of associated selection operators. Then there exist operators*

$$\begin{aligned} Q_1 &: \bigoplus_{i=1}^{m-1} \bigoplus_{j=1}^{m} \mathcal{X}' \otimes \mathcal{Y}' \to \bigoplus_{i=1}^{m} \bigoplus_{j=1}^{m} \mathcal{X}' \otimes \mathcal{Y}', \\ Q_2 &: \bigoplus_{i=1}^{n-1} \bigoplus_{j=1}^{n} \mathcal{X} \otimes \mathcal{Y} \to \bigoplus_{i=1}^{n} \bigoplus_{j=1}^{n} \mathcal{X} \otimes \mathcal{Y} \end{aligned}$$

such that $Z_i^* Q_i = 0, Q_i^* Q_i = I$ *and* $Z_i Z_i^* + Q_i Q_i^* = I$ *for* $i = 1, 2$. *If, in addition,* $\mathcal{H} = \mathcal{H}'$ *and* $\mathcal{K} = \mathcal{K}'$, *we have* $Q_1 = Q_2$.

Proof. Consider

$$Q_1 = \begin{bmatrix} E_2 & E_3 & \cdots & E_m \\ E_3 & E_4 & \cdots & E_1 \\ \vdots & \vdots & \ddots & \vdots \\ E_1 & E_2 & \cdots & E_{m-1} \end{bmatrix}, \quad Q_2 = \begin{bmatrix} F_2 & F_3 & \cdots & F_n \\ F_3 & F_4 & \cdots & F_1 \\ \vdots & \vdots & \ddots & \vdots \\ F_1 & F_2 & \cdots & F_{n-1} \end{bmatrix}. \tag{11}$$

Then calculations reveal that

$$Z_1^* Q_1 = \begin{bmatrix} E_1^* & E_2^* & \cdots & E_m^* \end{bmatrix} \begin{bmatrix} E_2 & E_3 & \cdots & E_m \\ E_3 & E_4 & \cdots & E_1 \\ \vdots & \vdots & \ddots & \vdots \\ E_1 & E_2 & \cdots & E_{m-1} \end{bmatrix} = \begin{bmatrix} 0 & 0 & \cdots & 0 \end{bmatrix},$$

$$\begin{aligned} Q_1^* Q_1 &= \begin{bmatrix} E_2^* & E_3^* & \cdots & E_1^* \\ E_3^* & E_4^* & \cdots & E_2^* \\ \vdots & \vdots & \ddots & \vdots \\ E_m^* & E_1^* & \cdots & E_{m-1}^* \end{bmatrix} \begin{bmatrix} E_2 & E_3 & \cdots & E_m \\ E_3 & E_4 & \cdots & E_1 \\ \vdots & \vdots & \ddots & \vdots \\ E_1 & E_2 & \cdots & E_{m-1} \end{bmatrix} \\ &= \begin{bmatrix} \sum E_i^* E_i & 0 & \cdots & 0 \\ 0 & \sum E_i^* E_i & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sum E_i^* E_i \end{bmatrix} = I, \end{aligned}$$

$$\begin{aligned} &Q_1^* Q_1 + Z_1 Z_1^* \\ &= \begin{bmatrix} E_2^* & E_3^* & \cdots & E_1^* \\ E_3^* & E_4^* & \cdots & E_2^* \\ \vdots & \vdots & \ddots & \vdots \\ E_m^* & E_1^* & \cdots & E_{m-1}^* \end{bmatrix} \begin{bmatrix} E_2 & E_3 & \cdots & E_m \\ E_3 & E_4 & \cdots & E_1 \\ \vdots & \vdots & \ddots & \vdots \\ E_1 & E_2 & \cdots & E_{m-1} \end{bmatrix} + \begin{bmatrix} E_1 \\ E_2 \\ \vdots \\ E_m \end{bmatrix} \begin{bmatrix} E_1^* & E_2^* & \cdots & E_m^* \end{bmatrix} \\ &= \begin{bmatrix} \sum_{i \neq 1} E_i E_j^* & 0 & \cdots & 0 \\ 0 & \sum_{i \neq 2} E_i E_j^* & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sum_{i \neq m} E_i E_j^* \end{bmatrix} + \begin{bmatrix} E_1 E_1^* & 0 & \cdots & 0 \\ 0 & E_2 E_2^* & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & E_m E_m^* \end{bmatrix} \\ &= \begin{bmatrix} \sum E_i E_i^* & 0 & \cdots & 0 \\ 0 & \sum E_i E_i^* & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sum E_i E_i^* \end{bmatrix} = I. \end{aligned}$$

Similarly, we have $Z_2^* Q_2 = 0$, $Q_2^* Q_2 = I$ and $Z_2 Z_2^* + Q_2 Q_2^* = I$. When $\mathcal{H} = \mathcal{H}'$ and $\mathcal{K} = \mathcal{K}'$, we have $E_i = F_i$ for all $i = 1, \ldots, m$, i.e. $Q_1 = Q_2$. $\square$

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 27, NO.2, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC

4 Operator identities and inequalities concerning Khatri-Rao products, Tracy-Singh products, and selection operators

In this section, we apply the construction in Section 3 to establish certain operator identities and inequalities concerning Khatri-Rao products, Tracy-Singh products, and selection operators.

Theorem 8. *Let* (Z_1, Z_2) *be the ordered pair of selection operators associated with an ordered tuple* $(\mathcal{H}, \mathcal{H}', \mathcal{K}, \mathcal{K}')$*. Let* V *and* W *be operator matrices defined by* (9) *and* (10)*. For any operator matrices* $A \in \mathbb{B}(\mathcal{H}, \mathcal{H}')$ *and* $B \in \mathbb{B}(\mathcal{K}, \mathcal{K}')$*, we have*

$$\begin{aligned} AA^* \boxdot BB^* &= (A \boxdot B)(A^* \boxdot B^*) + Z_1^*(A \boxtimes B)WW^*(A^* \boxtimes B^*)Z_1, && (12)\\ A^*A \boxdot B^*B &= (A^* \boxdot B^*)(A \boxdot B) + Z_2^*(A^* \boxtimes B^*)VV^*(A \boxtimes B)Z_2. && (13) \end{aligned}$$

Proof. Since $AA^* \in \mathbb{B}(\mathcal{H}')$ and $BB^* \in \mathbb{B}(\mathcal{K}')$, the ordered pair of selection operators associated with $(\mathcal{H}', \mathcal{H}', \mathcal{K}', \mathcal{K}')$ is given by (Z_1, Z_1). By using Lemmas 2 and 5, and Theorem 6, we get

$$\begin{aligned} AA^* \boxdot BB^* &= Z_1^* \left(AA^* \boxtimes BB^*\right) Z_1 \\ &= Z_1^*(A \boxtimes B)(A \boxtimes B)^* Z_1 \\ &= Z_1^*(A \boxtimes B)\left(Z_2Z_2^* + WW^*\right)(A \boxtimes B)^* Z_1 \\ &= Z_1^*(A \boxtimes B)Z_2Z_2^*(A \boxtimes B)^* Z_1 + Z_1^*(A \boxtimes B)WW^*(A \boxtimes B)^* Z_1 \\ &= (A \boxdot B)(A \boxdot B)^* + Z_1^*(A \boxtimes B)WW^*(A \boxtimes B)^* Z_1. \end{aligned}$$

Now, for inequality (13), note that $A^*A \in \mathbb{B}(\mathcal{H})$ and $B^*B \in \mathbb{B}(\mathcal{K})$. In this case, the pair of associated selection operators is (Z_2, Z_2). It follows that

$$\begin{aligned} A^*A \boxdot B^*B &= Z_2^* \left(A^*A \boxtimes B^*B\right) Z_2 \\ &= Z_2^*(A \boxtimes B)^*(A \boxtimes B)Z_2 \\ &= Z_2^*(A \boxtimes B)^*\left(Z_1Z_1^* + VV^*\right)(A \boxtimes B)Z_2 \\ &= Z_2^*(A \boxtimes B)^*Z_1Z_1^*(A \boxtimes B)Z_2 + Z_2^*(A \boxtimes B)^*VV^*(A \boxtimes B)Z_2 \\ &= (A^* \boxdot B^*)(A \boxdot B) + Z_2^*(A^* \boxtimes B^*)VV^*(A \boxtimes B)Z_2. \end{aligned}$$

□

Corollary 9. *Let* $A \in \mathbb{B}(\mathcal{H}, \mathcal{H}')$ *and* $B \in \mathbb{B}(\mathcal{K}, \mathcal{K}')$ *be operator matrices. Then*

$$AA^* \boxdot BB^* \geqslant (A \boxdot B)(A^* \boxdot B^*). \tag{14}$$

Proof. It follows immediately from Theorem 8. □

Theorem 10. *Assume the hypothesis of Theorem 7. For any* $A \in \mathbb{B}(\mathcal{H}, \mathcal{H}')$ *and* $B \in \mathbb{B}(\mathcal{K}, \mathcal{K}')$*, we have*

$$\begin{aligned} AA^* \boxdot BB^* &= (A \boxdot B)(A^* \boxdot B^*) + Z_1^*(A \boxtimes B)Q_2Q_2^*(A^* \boxtimes B^*)Z_1, && (15)\\ A^*A \boxdot B^*B &= (A^* \boxdot B^*)(A \boxdot B) + Z_2^*(A^* \boxtimes B^*)Q_1Q_1^*(A \boxtimes B)Z_2, && (16) \end{aligned}$$

where Q_1 *and* Q_2 *are operator matrices in* (11)*.*

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 27, NO.2, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC

Proof. The proof is similar to that of Theorem 8. Instead of Theorem 6, we apply Theorem 7. □

5 Characterizations of the mixed product property for Khatri-Rao products

In general, the mixed product property

$$(A \boxdot B)(C \boxdot D) \;=\; AC \boxdot BD$$

does not hold for compatible operator matrices A, B, C, D. It is interesting to find necessary and sufficient conditions for which this property holds. Indeed, we have the following assertions.

Theorem 11. *Assume the notations in Theorem 8. For any operator matrices $A \in \mathbb{B}(\mathcal{H}, \mathcal{H}')$ and $B \in \mathbb{B}(\mathcal{K}, \mathcal{K}')$, the following statements are equivalent:*

(i) $AC \boxdot BD \;=\; (A \boxdot B)(C \boxdot D)$ *for all* $C \in \mathbb{B}(\mathcal{H}', \mathcal{H})$ *and* $D \in \mathbb{B}(\mathcal{K}', \mathcal{K})$,

(ii) $AA^* \boxdot BB^* \;=\; (A \boxdot B)(A^* \boxdot B^*)$,

(iii) $Z_1^*(A \boxtimes B)W \;=\; 0$.

Proof. It is clear that (i)⇒(ii). To prove (ii)⇒(iii), suppose (ii). By Theorem 8, (ii) holds only if

$$[Z_1^*(A \boxtimes B)W]\,[W^*(A^* \boxtimes B^*)Z_1] = 0,$$

i.e., $Z_1^*(A \boxtimes B)W = 0$.
(iii)⇒(i): Assume the condition (iii) holds. Note that by Theorem 6 we have

$$Z_1^*(A \boxtimes B)(I - Z_2 Z_2^*) \;=\; Z_1^*(A \boxtimes B)WW^* \;=\; 0,$$

and hence $Z_1^*(A \boxtimes B) = Z_1^*(A \boxtimes B)Z_2 Z_2^*$. For any $C \in \mathbb{B}(\mathcal{H}', \mathcal{H})$ and $D \in \mathbb{B}(\mathcal{K}', \mathcal{K})$, we have by Lemmas 2 and 5 that

$$\begin{aligned} AC \boxdot BD &= Z_1^*(AC \boxtimes BD)Z_1 \\ &= Z_1^*(A \boxtimes B)(C \boxtimes D)Z_1 \\ &= Z_1^*(A \boxtimes B)Z_2 Z_2^*(C \boxtimes D)Z_1 \\ &= (A \boxdot B(C \boxdot D). \end{aligned}$$

Thus we arrive at (i). □

Theorem 12. *Assume the notations in Theorem 8. For any operator matrices $A \in \mathbb{B}(\mathcal{H}, \mathcal{H}')$ and $B \in \mathbb{B}(\mathcal{K}, \mathcal{K}')$, the following statements are equivalent:*

(i) $CA \boxdot DB \;=\; (C \boxdot D)(A \boxdot B)$ *for all* $C \in \mathbb{B}(\mathcal{H}', \mathcal{H})$ *and* $D \in \mathbb{B}(\mathcal{K}', \mathcal{K})$,

(ii) $A^*A \boxdot B^*B \;=\; (A^* \boxdot B^*)(A \boxdot B)$,

(iii) $V^*(A \boxtimes B)Z_2 \;=\; 0.$

Proof. Clearly,(i)⇒(ii). The assertion (ii)⇒(iii) follows from Theorem 8. Now, suppose that (iii) holds. Then $VV^*(A \boxtimes B)Z_2 = 0$. Using Theorem 6, we get

$$(I - Z_1 Z_1^*)(A \boxtimes B)Z_1 \;=\; VV^*(A \boxtimes B)Z_1 \;=\; 0$$

which implies $(A \boxtimes B)Z_1 = Z_1 Z_1^*(A \boxtimes B)Z_1$. For any $C \in \mathbb{B}(\mathcal{H}', \mathcal{H})$ and $D \in \mathbb{B}(\mathcal{K}', \mathcal{K})$, we have by Lemmas 2 and 5 that

$$\begin{aligned} CA \boxdot DB \;&=\; Z_2^*(CA \boxtimes DB)Z_2 \\ &=\; Z_2^*(C \boxtimes D)(A \boxtimes B)Z_2 \\ &=\; Z_2^*(C \boxtimes D)Z_1 Z_1^*(A \boxtimes B)Z_2 \\ &=\; (C \boxdot D)(A \boxdot B). \end{aligned}$$

□

Theorem 13. *Assume the hypothesis of Theorem 7. For any operator matrices $A \in \mathbb{B}(\mathcal{H}, \mathcal{H}')$ and $B \in \mathbb{B}(\mathcal{K}, \mathcal{K}')$, the following conditions are equivalent:*

(i) $AC \boxdot BD \;=\; (A \boxdot B)(C \boxdot D)$ *for all* $C \in \mathbb{B}(\mathcal{H}', \mathcal{H})$ *and* $D \in \mathbb{B}(\mathcal{K}', \mathcal{K})$*,*

(ii) $AA^* \boxdot BB^* \;=\; (A \boxdot B)(A^* \boxdot B^*)$*,*

(iii) $Z_1^*(A \boxtimes B)Q_2 \;=\; 0.$

Proof. The proof is similar to that of Theorem 11. □

Theorem 14. *Assume the hypothesis of Theorem 7. For any operator matrices $A \in \mathbb{B}(\mathcal{H}, \mathcal{H}')$ and $B \in \mathbb{B}(\mathcal{K}, \mathcal{K}')$, the following statements are equivalent:*

(i) $CA \boxdot DB \;=\; (C \boxdot D)(A \boxdot B)$ *for all* $C \in \mathbb{B}(\mathcal{H}', \mathcal{H})$ *and* $D \in \mathbb{B}(\mathcal{K}', \mathcal{K})$*,*

(ii) $A^*A \boxdot B^*B \;=\; (A^* \boxdot B^*)(A \boxdot B)$*,*

(iii) $Q_1^*(A \boxtimes B)Z_2 \;=\; 0.$

Proof. The proof is similar to that of Theorem 12. □

Acknowledgement. This research was supported King Mongkut's Institute of Technology Ladkrabang Research Fund.

References

[1] Z. A. Al Zhour, A. Kilicman, Extension and generalization inequalities involving the Khatri-Rao product of several positive matrices, *J. Inequal. Appl.*, 2006, 21 pages (2006).

[2] Z. A. Al Zhour, A. Kilicman, Matrix equalities and inequalities involving Khatri-Rao and Tracy-Singh sums, *J. Inequal. Appl.*, 7, 11–17 (2006).

[3] C. G. Khatri, C. R. Rao, Solutions to some functional equations and their applications to characterization of probability distributions, *Sankhyā*, 26, 167–180 (1968).

[4] S. Liu, Matrix results on the Khatri-Rao and Tracy-Singh products, *Linear Algebra Appl.*, 289, 267–277 (1999).

[5] S. Liu, G. Trenkler, Hadamard, Khatri-Rao, Kronecker and other matrix products, *International Journal of Information and Systems Sciences*, 4, 160–177 (2008).

[6] A. Ploymukda, P. Chansangaim, W. Lewkeeratiyutkul, Algebraic and order properties of Tracy-Singh products for operator matrices, *J. Comput. Anal. Appl.,* 24, 656–664 (2018).

[7] A. Ploymukda, P. Chansangaim, W. Lewkeeratiyutkul, Analytic properties of Tracy-Singh products for operator matrices, *J. Comput. Anal. Appl.,* 24, 665–674 (2018).

[8] A. Ploymukda, P. Chansangaim, Khatri-Rao products for operator matrices acting on the direct sum of Hilbert spaces, *Journal of Mathematics*, 2016, Article ID 8301709, 7 pages, http://dx.doi.org/10.1155/2016/8301709, (2016).

[9] D. S. Tracy, R. P. Singh, A new matrix product and its applications in partitioned matrix differentiation, *Stat. Neerl.*, 26, 143–157 (1972).

[10] C. F. Van Loan, The ubiquitous Kronecker product, *Int. J. Comput. Appl. Math.*, 123, 85–100 (2000).

[11] G. Visick, A quantitative version of the observation that the Hadamard product is a principal submatrix of the Kronecker product, *Linear Algebra Appl.*, 304, 45–68 (2000).

[12] H. Zhang, F. Ding, On the Kronecker products and their applications, *J. Appl. Math.*, 2013, 8 pages (2013).

Some new coupled fixed point theorems in partially ordered complete Menger probabilistic G-metric spaces

Gang Wang, Chuanxi Zhu*, Zhaoqi Wu

Department of Mathematics, Nanchang University, Nanchang, 330031, P. R. China

gangwang0904@126.com (G. Wang), chuanxizhu@126.com (C. X. Zhu)

Abstract. In this paper, we study the mapping satisfying mixed g-monotone property in partially ordered complete Menger probabilistic G-metric spaces. By weakening the notion of Ψ, we prove some new coupled coincidence point theorems and coupled common fixed point theorems. Finally, we provide an example to illustrate our results.

Keywords: partially ordered; coupled fixed point; mixed g-monotone mapping; Menger PGM-space

1 Introduction

The notions of mixed monotone mappings and coupled fixed point were first introduced by Bhaskar and Lakshmikantham [1], which was extended to the partially ordered metric spaces. Since then, some results have been presented about the existence and uniqueness of coupled fixed points (see [2]-[8]). In 2009, Lakshmikantham and *Ćirić* [7] introduced the concept of a mixed g-monotone mapping, which generalized and extended the notion of mixed monotone mappings and the coupled fixed point in [1]. In 2010, Jachymski [9] established a fixed point theorem for φ-contractions and gave a characterization of a function φ, satisfying probabilistic φ-contraction. On the other hand, Choudhury and Das [2] gave a fixed point theorem by using an altering distance function. In addition, by taking advantage of the notion of the notion of φ-contractive mapping in Menger PM-spaces, some fixed point theorems were brought by Ktbi and Gopal [6]. And Jin [10] put forward a new fixed point theorems for φ-contraction in KM fuzzy metric spaces. For other results in the direction, we refer to [11]-[14].

†Corresponding author: Chuanxi Zhu. Email: chuanxizhu@126.com.

†Supported by the National Natural Science Foundation of China (11361042,11461045,11071108) and the Provincial Natural Science Foundation of Jiangxi, China (20132BAB201001, 2010GZS0147) and the Innovation Program of the Graduate student of Nanchang University(colonel-level project)

In this paper, we generalize the results of other scholars ([8],[14]) by weakening the notion of Ψ in [4]. We study compatibility of the mappings g and T, where T is a mixed g-monotone mapping. We also establish some new coupled coincidences point theorems and coupled common fixed point theorems in partially ordered Menger probabilistic G-metric spaces. Finally, an example is given to illustrate our main results.

2 preliminaries

At this stage, we recall some well-known definitions and results in the theory of partially ordered set and PGM-space.

Let $\mathbb{R}$ be the set of all real numbers, $\mathbb{R}^+$ be the set of all nonnegative real numbers, $\mathbb{Z}^+$ be the set of all positive integers.

A mapping $F : \mathbb{R} \to \mathbb{R}^+$ is called a distribution function if it is nondecreasing and left continuous with $\sup\limits_{t\in\mathbb{R}} F(t) = 1$ and $\inf\limits_{t\in\mathbb{R}} F(t) = 1$. We will denote $\mathcal{D}$ by the set of all distributions function.

Let H denote the specific distribution function defined by

$$H(t) = \begin{cases} 0, \text{ if } t \leq 0, \\ 1, \text{ if } t > 0. \end{cases}$$

Definition 2.1 ([9]). A function $\triangle : [0,1] \times [0,1] \to [0,1]$ is called a triangular norm (for short, $t - norm$) if the following conditions are satisfied for any $a, b, c, d \in [0,1]$:

$(\triangle - 1)$ $\triangle(a,1) = a$;

$(\triangle - 2)$ $\triangle(a,b) = \triangle(b,a)$;

$(\triangle - 3)$ $\triangle(a,b) \geq \triangle(c,d)$, for $a \geq c, b \geq d$;

$(\triangle - 4)$ $\triangle(\triangle(a,b),c) = \triangle(a,\triangle(b,c))$.

Definition 2.2 ([2]). Let Φ denote the class of all functions $\phi : R^+ \to R^+$ satisfies the following conditions:

(i) $\phi(t) = 0$ if and only if $t = 0$;

(ii) $\phi(t)$ is strictly increasing and $\phi(t) \to \infty$ as $t \to \infty$;

(iii) ϕ is left continuous in $(0, +\infty)$;

(iv) ϕ is continuous at 0.

Definition 2.3 ([8]). Let Ψ denote the class of all functions $\psi : R^+ \to R^+$ satisfies the following conditions:

(1)ψ is non-decreasing;
(2) $\psi(t+s) \leq \psi(t)+\psi(s)$ for all $t, s \in [0,1)$.

Remark 2.1 ([8]). Ψ also satisfies that Ψ is continuous and $\Psi(t)=0$ if and only if $t=0$. It is easy to see that the notion of Ψ is stronger than Definition 2.3 in [8]. And it is obvious that the following condition holds:
(3) $\psi(p+q+t+s) \leq \psi(p)+\psi(q)+\psi(t)+\psi(s)$ for all $p, q, t, s \in [0,1)$.

Definition 2.4 ([18]). A Menger probabilistic G-metric space (briefly, a PGM-space) is a triple $(X, G^*, \triangle)$, where X is a nonempty set, $\triangle$ is a continuous t-norm, and G^* is a mapping from $X \times X \times X$ into $\mathscr{D}^+$ ($G^*_{x,y,z}$ denotes the value of G^* at the point (x,y,z)) satisfying the following conditions:
(PGM-1) $G^*_{x,y,z}(t)=1$ for $x, y, z \in X$ and $t>0$ if and only if $x=y=z$;
(PGM-2) $G^*_{x,x,y}(t) \geq G^*_{x,y,z}(t)$ for $x, y, z \in X$ with $z \neq y$ and $t>0$;
(PGM-3) $G^*_{x,y,z}(t)=G^*_{x,z,y}(t)=G^*_{y,x,z}(t)=\cdots$ (symmetry in all three variables);
(PGM-4) $G^*_{x,y,z}(t+s) \geq \triangle(G^*_{x,a,a}(t), G^*_{a,y,z}(s))$ for $x, y, z, a \in X$ and $s, t>0$.

Definition 2.5 ([1]). Let $(X, G^*, \triangle)$ be a PGM-space, and $\{x_n\}$ is a sequence in X. (1) $\{x_n\}$ is said to be convergent to $x \in X$ (write $x_n \to x$), if for any $\varepsilon>0$ and $0<\delta<1$, there exists a positive integer $M_{\varepsilon,\lambda}$ such that $x_n \in N_{x_0}(\varepsilon, \lambda)$ whenever $n>M_{\varepsilon,\lambda}$;
(2) $\{x_n\}$ is said to be *Cauchy* sequence, if for any $\varepsilon>0$ and $0<\delta<1$, there exists a positive integer $M_{\varepsilon,\lambda}$ such that $G^*_{x_n,x_m,x_l}>1-\delta$ whenever $n, m, l>M_{\varepsilon,\lambda}$;
(3) $(X, G^*, \triangle)$ is said to be complete, if every *Cauchy* sequence in X converges to a point in X.

Definition 2.6 ([7]). Let X be a non-empty set and $F: X \times X \to X$ and $g: X \to X$. We say F and g are commutative if

$$g(F(x,y))=F(g(x), g(y)) \quad \text{for all } x, y \in X.$$

Definition 2.7 ([7]). Let $(X, \leq)$ be a partially ordered set and $F: X \times X \to X$ is said to possess the mixed monotone property if F is monotone non-decreasing in its first argument and is monotone non-increasing in its second argument, that is, for any $x, y \in X$,

$$x_1, x_2 \in X, \quad x_1 \leq x_2 \Rightarrow F(x_1, y) \leq F(x_2, y)$$

and

$$y_1, y_2 \in X, \quad y_1 \leq y_2 \Rightarrow F(x, y_2) \leq F(x, y_1)$$

Definition 2.8 ([11]). Let $(X, \leq)$ be a partially ordered set and $F : X \times X \to X$ is said to have the mixed $g-$monotone property if F is monotone $g-$non-decreasing in its first argument and is monotone $g-$non-decreasing in its second argument, that is, for any $x, y \in X$.

$$x_1, x_2 \in X,\ \ g(x_1) \leq g(x_2) \Rightarrow F(x_1, y) \leq F(x_2, y)$$

and

$$y_1, y_2 \in X,\ \ g(y_1) \leq g(y_2) \Rightarrow F(x, y_2) \leq F(x, y_1).$$

3 Coupled coincidence point results in partially ordered complete Menger probabilistic G-metric spaces

In this section, We begin with the following definition which is useful to prove some new coupled coincidence point theorems and coupled fixed point theorems in partially ordered complete Menger probabilistic G-metric spaces.

Definition 3.1 Let $(X, G^*, \triangle)$ be a Menger PGM-space with $\triangle$ (a continuous $t-norm$), $T : X^4 \to X$ and $g : X \to X$ be two mappings satisfying the following condition:

$$\begin{aligned}\psi(\frac{1}{G^*_{T(x,y,z,w),T(u,v,p,q),T(a,b,c,d)}(\phi(\lambda t))} - 1) \leq \frac{1}{4}\psi(&\frac{1}{G^*_{g(x),g(u),g(a)}(\phi(t))} - 1 + \frac{1}{G^*_{g(y),g(v),g(b)}(\phi(t))} - 1 \\ &+ \frac{1}{G^*_{g(z),g(p),g(c)}(\phi(t))} - 1 + \frac{1}{G^*_{g(w),g(q),g(d)}(\phi(t))} - 1).\end{aligned} \tag{3.1}$$

for all $t > 0$, and $x, y, z, w, u, v, p, q, a, b, c, d \in X, g(x) \leq g(u) \leq g(a), g(y) \geq g(v) \geq g(b), g(z) \leq g(p) \leq g(c)$ and $g(w) \geq g(q) \geq g(d)$, where $\lambda \in (0, 1), \psi \in \Psi$ and $\phi \in \Phi$. Then mappings T and g are said to satisfy ψ-contractive condition.

Theorem 3.1 Let$(X, \leq)$ be a partially ordered set and $(X, G^*, \triangle)$ be a complete PGM-space with a continuous $t - norm$. suppose that $T : X^4 \to X$ and $g : X \to X$ are the mappings with mixed $g-monotone$ property and satisfy ψ-contractive condition, such that $G^*_{g(x),g(u),g(a)} > 0, G^*_{g(y),g(v),g(b)} > 0, G^*_{g(z),g(p),g(c)} > 0, G^*_{g(w),g(q),g(d)} > 0$. Suppose $T(X^4) \subseteq g(X)$, g is continuous and commutes with T. Assuming that either

(a) T is continuous, or

(b) X has the following properties:

(I) If a non-decreasing sequence $x_n \to x, z_n \to z$, then $x_n \leq x, z_n \leq z$ for all n;

(II) If a non-increasing sequence $y_n \to y, w_n \to w$, then $y_n \leq y, w_n \leq w$ for all n.

If there exist $x_0, y_0, z_0, w_0 \in X$, such that $g(x_0) \leq T(x_0, y_0, z_0, w_0), g(z_0) \leq T(z_0, w_0, x_0, y_0)$, $g(y_0) \geq T(y_0, z_0, w_0, x_0)$ and $g(w_0) \geq T(w_0, x_0, y_0, z_0)$, then there exist $x, y, z, w \in X$, such that

$$g(x) = T(x, y, z, w), g(y) = T(y, z, w, x), g(z) = T(z, w, x, y), g(w) = T(w, x, y, z),$$

that is, T and g have a coupled coincidence point.

Proof Let $x_0, y_0, z_0, w_0 \in X$, such that $g(x_0) \leq T(x_0, y_0, z_0, w_0), g(z_0) \leq T(z_0, w_0, x_0, y_0)$ and $g(y_0) \geq T(y_0, z_0, w_0, x_0), g(w_0) \geq T(w_0, x_0, y_0, z_0)$, since $T(X^4) \subseteq g(X)$, we can choose $x_1, y_1, z_1, w_1 \in X$, such that

$$g(x_1) = T(x_0, y_0, z_0, w_0), g(y_1) = T(y_0, z_0, w_0, x_0), \tag{3.2}$$

$$g(z_1) = T(z_0, w_0, x_0, y_0), g(w_1) = T(w_0, x_0, y_0, z_0). \tag{3.3}$$

Continuing this process we can construct sequences $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ and $\{w_n\}$ in X, such that

$$g(x_{n+1}) = T(x_n, y_n, z_n, w_n), g(y_{n+1}) = T(y_n, z_n, w_n, x_n) \quad \text{for all } n \geq 0,$$
$$g(z_{n+1}) = T(z_n, w_n, x_n, y_n), g(w_{n+1}) = T(w_n, x_n, y_n, z_n) \quad \text{for all } n \geq 0,$$

we shall show that

$$g(x_n) \leq g(x_{n+1}),\ g(y_n) \geq g(y_{n+1}),\ g(z_n) \leq g(z_{n+1}),\ g(w_n) \geq g(w_{n+1}). \tag{3.4}$$

We shall use the mathematical induction to show that (3.4) holds.

Let $n = 0$, since

$$g(x_0) \leq T(x_0, y_0, z_0, w_0), g(y_0) \geq T(y_0, z_0, w_0, x_0),$$
$$g(z_0) \leq T(z_0, w_0, x_0, y_0), g(w_0) \geq T(w_0, x_0, y_0, z_0),$$

by (3.2) and (3.3), we have

$$g(x_0) \leq g(x_1),\quad g(y_0) \geq g(y_1),\quad g(z_0) \leq g(z_1),\quad g(w_0) \geq g(w_1).$$

Thus (3.4) holds for $n = 0$.

Now we suppose that (3.4) holds for some $n = i, i \in \mathbb{Z}^+$, we get

$$g(x_i) \leq g(x_{i+1}),\ g(y_i) \geq g(y_{i+1}), g(z_i) \leq g(z_{i+1}),\ g(w_i) \geq g(w_{i+1}).$$

Let $n = i+1$, owing to the property of mixed g-monotone, we have

$$g(x_{i+2}) = T(x_{i+1}, y_{i+1}, z_{i+1}, w_{i+1}) \geq T(x_i, y_{i+1}, z_i, w_{i+1}) \geq T(x_i, y_i, z_i, w_i) = g(x_{i+1}),$$
$$g(y_{i+2}) = T(y_{i+1}, z_{i+1}, w_{i+1}, x_{i+1}) \leq T(y_i, z_{i+1}, w_i, x_{i+1}) \leq T(y_i, z_n, w_i, x_i) = g(y_{i+1}).$$

Similarly, we obtain

$$g(z_{i+2}) \geq g(z_{i+1}), \quad g(w_{i+2}) \leq g(w_{i+1}).$$

By the mathematical induction, we conclude that (3.4) holds for all $n > 0$. Therefore

$$g(x_0) \leq g(x_1) \leq g(x_2) \leq ... \leq g(x_n) \leq g(x_{n+1}) \leq \cdots;$$
$$g(y_0) \geq g(y_1) \geq g(y_2) \geq ... \geq g(y_n) \geq g(y_{n+1}) \leq \cdots;$$
$$g(z_0) \leq g(z_1) \leq g(z_2) \leq ... \leq g(z_n) \leq g(z_{n+1}) \leq \cdots;$$
$$g(w_0) \geq g(w_1) \geq g(w_2) \geq ... \geq g(w_n) \geq g(w_{n+1}) \leq \cdots.$$

In view of the fact, we have

$$\sup_{t\in\mathbb{R}} G^*_{g(x_2),g(x_1),g(x_0)}(t) = 1, \ \sup_{t\in\mathbb{R}} G^*_{g(y_2),g(y_1),g(y_0)}(t) = 1,$$
$$\sup_{t\in\mathbb{R}} G^*_{g(z_2),g(z_1),g(z_0)}(t) = 1, \ \sup_{t\in\mathbb{R}} G^*_{g(w_2),g(w_1),g(w_0)}(t) = 1,$$

and by (ii) of Definition 2.2, we can find some $t > 0$, such that

$$G^*_{g(x_2),g(x_1),g(x_0)}(\phi(t)) > 0, \ G^*_{g(y_2),g(y_1),g(y_0)}(\phi(t)) > 0,$$
$$G^*_{g(z_2),g(z_1),g(z_0)}(\phi(t)) > 0, \ G^*_{g(w_2),g(w_1),g(w_0)}(\phi(t)) > 0,$$

for

$$g(x_0) \leq g(x_1) \leq g(x_2), g(y_0) \geq g(y_1) \geq g(y_2),$$
$$g(z_0) \leq g(z_1) \leq g(z_2, g(w_0) \geq g(w_1) \geq g(w_2),$$

which implies that

$$G^*_{g(x_2),g(x_1),g(x_0)}(\phi(\frac{t}{\lambda})) > 0, G^*_{g(y_2),g(y_1),g(y_0)}(\phi(\frac{t}{\lambda})) > 0,$$
$$G^*_{g(z_2),g(z_1),g(z_0)}(\phi(\frac{t}{\lambda})) > 0, G^*_{g(w_2),g(w_1),g(w_0)}(\phi(\frac{t}{\lambda})) > 0.$$

Then by (3.1), we get

$$\begin{aligned}\psi(\frac{1}{G^*_{g(x_3),g(x_2),g(x_1)}(\phi(t))} - 1) &= \psi(\frac{1}{G^*_{T(x_2,y_2,z_2,w_2),T(x_1,y_1,z_1,w_1),T(x_0,y_0,z_0,w_0)}(\phi(t))} - 1) \\ &\leq \frac{1}{4}\psi(G^*_{g(x_2),g(x_1),g(x_0)}(\phi(\frac{t}{\lambda})) - 1 + G^*_{g(y_2),g(y_1),g(y_0)}(\phi(\frac{t}{\lambda})) - 1 \\ &\quad + G^*_{g(z_2),g(z_1),g(z_0)}(\phi(\frac{t}{\lambda})) - 1 + G^*_{(w_2),g(w_1),g(w_0)}(\phi(\frac{t}{\lambda})) - 1).\end{aligned} \tag{3.5}$$

Similarly,

$$\begin{aligned}\psi(\frac{1}{G^*_{g(y_3),g(y_2),g(y_1)}(\phi(t))}-1)\leq \frac{1}{4}\psi(G^*_{g(y_2),g(y_1),g(y_0)}(\phi(\frac{t}{\lambda}))-1+G^*_{g(z_2),g(z_1),g(z_0)}(\phi(\frac{t}{\lambda}))-1\\ +G^*_{g(w_2),g(w_1),g(w_0)}(\phi(\frac{t}{\lambda}))-1+G^*_{(x_2),g(x_1),g(x_0)}(\phi(\frac{t}{\lambda}))-1),\end{aligned}\tag{3.6}$$

$$\begin{aligned}\psi(\frac{1}{G^*_{g(z_3),g(z_2),g(z_1)}(\phi(t))}-1)\leq \frac{1}{4}\psi(G^*_{g(z_2),g(z_1),g(z_0)}(\phi(\frac{t}{\lambda}))-1+G^*_{g(w_2),g(w_1),g(w_0)}(\phi(\frac{t}{\lambda}))-1\\ +G^*_{g(x_2),g(x_1),g(x_0)}(\phi(\frac{t}{\lambda}))-1+G^*_{(y_2),g(y_1),g(y_0)}(\phi(\frac{t}{\lambda}))-1),\end{aligned}\tag{3.7}$$

$$\begin{aligned}\psi(\frac{1}{G^*_{g(w_3),g(w_2),g(w_1)}(\phi(t))}-1)\leq \frac{1}{4}\psi(G^*_{g(w_2),g(w_1),g(w_0)}(\phi(\frac{t}{\lambda}))-1+G^*_{g(x_2),g(x_1),g(x_0)}(\phi(\frac{t}{\lambda}))-1\\ +G^*_{g(y_2),g(y_1),g(y_0)}(\phi(\frac{t}{\lambda}))-1+G^*_{(z_2),g(z_1),g(z_0)}(\phi(\frac{t}{\lambda}))-1).\end{aligned}\tag{3.8}$$

From (3.5)-(3.8), we have

$$\begin{aligned}&\psi(\frac{1}{G^*_{g(x_3),g(x_2),g(x_1)}(\phi(t))}-1)+\psi(\frac{1}{G^*_{g(y_3),g(y_2),g(y_1)}(\phi(t))}-1)+\psi(\frac{1}{G^*_{g(z_3),g(z_2),g(z_1)}(\phi(t))}-1)\\ &\quad+\psi(\frac{1}{G^*_{g(w_3),g(w_2),g(w_1)}(\phi(t))}-1)\\ &\leq\psi(\frac{1}{G^*_{g(x_2),g(x_1),g(x_0)}(\phi(\frac{t}{\lambda}))}-1+\frac{1}{G^*_{g(y_2),g(y_1),g(y_0)}(\phi(\frac{t}{\lambda}))}-1+\frac{1}{G^*_{g(z_2),g(z_1),g(z_0)}(\phi(\frac{t}{\lambda}))}-1\\ &\quad+\frac{1}{G^*_{g(w_2),g(w_1),g(w_0)}(\phi(\frac{t}{\lambda}))}-1).\end{aligned}$$

By (3) of Remark 2.1, we have

$$\begin{aligned}&\psi(\frac{1}{G^*_{g(x_3),g(x_2),g(x_1)}(\phi(t))}-1+\frac{1}{G^*_{g(y_3),g(y_2),g(y_1)}(\phi(t))}-1+\frac{1}{G^*_{g(z_3),g(z_2),g(z_1)}(\phi(t))}-1\\ &\quad+\frac{1}{G^*_{g(w_3),g(w_2),g(w_1)}(\phi(t))}-1)\\ &\leq\psi(\frac{1}{G^*_{g(x_3),g(x_2),g(x_1)}(\phi(t))}-1)+\psi(\frac{1}{G^*_{g(y_3),g(y_2),g(y_1)}(\phi(t))}-1)+\psi(\frac{1}{G^*_{g(z_3),g(z_2),g(z_1)}(\phi(t))}-1)\\ &\quad+\psi(\frac{1}{G^*_{g(w_3),g(w_2),g(w_1)}(\phi(t))}-1),\end{aligned}$$

which implies that

$$\begin{aligned}&\psi(\frac{1}{G^*_{g(x_3),g(x_2),g(x_1)}(\phi(t))}-1+\frac{1}{G^*_{g(y_3),g(y_2),g(y_1)}(\phi((t)))}-1+\frac{1}{G^*_{g(z_3),g(z_2),g(z_1)}(\phi(t))}-1\\ &\quad+\frac{1}{G^*_{g(w_3),g(w_2),g(w_1)}(\phi(t))}-1)\\ &\leq\psi(\frac{1}{G^*_{g(x_2),g(x_1),g(x_0)}(\phi(\frac{t}{\lambda}))}-1+\frac{1}{G^*_{g(y_2),g(y_1),g(y_0)}(\phi(\frac{t}{\lambda}))}-1+\frac{1}{G^*_{g(z_2),g(z_1),g(z_0)}(\phi(\frac{t}{\lambda}))}-1\\ &\quad+\frac{1}{G^*_{g(w_2),g(w_1),g(w_0)}(\phi(\frac{t}{\lambda}))}-1).\end{aligned}$$

Using the fact that ψ is non-decreasing, we get

$$\begin{aligned}&\frac{1}{G^*_{g(x_3),g(x_2),g(x_1)}(\phi(t))}-1+\frac{1}{G^*_{g(y_3),g(y_2),g(y_1)}(\phi(t))}-1+\frac{1}{G^*_{g(z_3),g(z_2),g(z_1)}(\phi(t))}-1\\&\quad+\frac{1}{G^*_{g(w_3),g(w_2),g(w_1)}(\phi(t))}-1\\&\leq\frac{1}{G^*_{g(x_2),g(x_1),g(x_0)}(\phi(\frac{t}{\lambda}))}-1+\frac{1}{G^*_{g(y_2),g(y_1),g(y_0)}(\phi(\frac{t}{\lambda}))}-1+\frac{1}{G^*_{g(z_2),g(z_1),g(z_0)}(\phi(\frac{t}{\lambda}))}-1\\&\quad+\frac{1}{G^*_{g(w_2),g(w_1),g(w_0)}(\phi(\frac{t}{\lambda}))}-1.\end{aligned}$$

From the above inequalities we deduce that

$$\begin{aligned}&G^*_{g(x_3),g(x_2),g(x_1)}(\phi(t))>0,G^*_{g(y_3),g(y_2),g(y_1)}(\phi(t))>0,\\&G^*_{g(z_3),g(z_2),g(z_1)}(\phi(t))>0,G^*_{g(w_3),g(w_2),g(w_1)}(\phi(t))>0,\end{aligned}$$

and

$$\begin{aligned}&G^*_{g(x_3),g(x_2),g(x_1)}(\phi(\frac{t}{\lambda}))>0,G^*_{g(y_3),g(y_2),g(y_1)}(\phi(\frac{t}{\lambda}))>0,\\&G^*_{g(z_3),g(z_2),g(z_1)}(\phi(\frac{t}{\lambda}))>0,G^*_{g(w_3),g(w_2),g(w_1)}(\phi(\frac{t}{\lambda}))>0.\end{aligned}$$

Again, by using (3.1), we have

$$\begin{aligned}&\frac{1}{G^*_{g(x_4),g(x_3),g(x_2)}(\phi(t))}-1+\frac{1}{G^*_{g(y_4),g(y_3),g(y_2)}(\phi(t))}-1+\frac{1}{G^*_{g(z_4),g(z_3),g(z_2)}(\phi(t))}-1\\&\quad+\frac{1}{G^*_{g(w_4),g(w_3),g(w_2)}(\phi(t))}-1\\&\leq\frac{1}{G^*_{g(x_3),g(x_2),g(x_1)}(\phi(\frac{t}{\lambda}))}-1+\frac{1}{G^*_{g(y_3),g(y_2),g(y_1)}(\phi(\frac{t}{\lambda}))}-1+\frac{1}{G^*_{g(z_3),g(z_2),g(z_1)}(\phi(\frac{t}{\lambda}))}-1\\&\quad+\frac{1}{G^*_{g(w_3),g(w_2),g(w_1)}(\phi(\frac{t}{\lambda}))}-1\\&\leq\frac{1}{G^*_{g(x_2),g(x_1),g(x_0)}(\phi(\frac{t}{\lambda^2}))}-1+\frac{1}{G^*_{g(y_2),g(y_1),g(y_0)}(\phi(\frac{t}{\lambda^2}))}-1+\frac{1}{G^*_{g(z_2),g(z_1),g(z_0)}(\phi(\frac{t}{\lambda^2}))}-1\\&\quad+\frac{1}{G^*_{g(w_2),g(w_1),g(w_0)}(\phi(\frac{t}{\lambda^2}))}-1.\end{aligned}$$

Repeating the above procedure successively, we obtain

$$\begin{aligned}&\frac{1}{G^*_{g(x_{n+2}),g(x_{n+1}),g(x_n)}(\phi(t))}-1+\frac{1}{G^*_{g(y_{n+2}),g(y_{n+1}),g(y_n)}(\phi(t))}-1+\frac{1}{G^*_{g(z_{n+2}),g(z_{n+1}),g(z_n)}(\phi(t))}-1\\&\quad+\frac{1}{G^*_{g(w_{n+2}),g(w_{n+1}),g(w_n)}(\phi(t))}-1\\&\leq\frac{1}{G^*_{g(x_2),g(x_1),g(x_0)}(\phi(\frac{t}{\lambda^n}))}-1+\frac{1}{G^*_{g(y_2),g(y_1),g(y_0)}(\phi(\frac{t}{\lambda^n}))}-1+\frac{1}{G^*_{g(z_2),g(z_1),g(z_0)}(\phi(\frac{t}{\lambda^n}))}-1\\&\quad+\frac{1}{G^*_{g(w_2),g(w_1),g(w_0)}(\phi(\frac{t}{\lambda^n}))}-1.\end{aligned}$$

If we replace x_0 with x_k in the previous inequalities, then for all $n > k$, we get

$$\begin{aligned}
&\frac{1}{G^*_{g(x_{n+2}),g(x_{n+1}),g(x_n)}(\phi(\lambda^k t))} - 1 + \frac{1}{G^*_{g(y_{n+2}),g(y_{n+1}),g(y_n)}(\phi(\lambda^k t))} - 1 \\
&\quad + \frac{1}{G^*_{g(z_{n+2}),g(z_{n+1}),g(z_n)}(\phi(\lambda^k t))} - 1 + \frac{1}{G^*_{g(w_{n+2}),g(w_{n+1}),g(w_n)}(\phi(\lambda^k t))} - 1 \\
&\leq \frac{1}{G^*_{g(x_{k+2}),g(x_{k+1}),g(x_k)}(\phi(\frac{\lambda^k t}{\lambda^{n-k}}))} - 1 + \frac{1}{G^*_{g(y_{k+2}),g(y_{k+1}),g(y_k)}(\phi(\frac{\lambda^k t}{\lambda^{n-k}}))} - 1 \\
&\quad + \frac{1}{G^*_{g(z_{k+2}),g(z_{k+1}),g(z_k)}(\phi(\frac{\lambda^k t}{\lambda^{n-k}}))} - 1 + \frac{1}{G^*_{g(w_{k+2}),g(w_{k+1}),g(w_k)}(\phi(\frac{\lambda^k t}{\lambda^{n-k}}))} - 1.
\end{aligned}$$

Since $\phi(\frac{\lambda^k t}{\lambda^{n-k}}) \to \infty$ as $n \to \infty$ for all $0 < k < n$, we have

$$\begin{aligned}
&\lim_{n\to\infty} G^*_{g(x_{k+2}),g(x_{k+1}),g(x_k)}(\phi(\frac{\lambda^k t}{\lambda^{n-k}})) = 1, \quad \lim_{n\to\infty} G^*_{g(y_{k+2}),g(y_{k+1}),g(y_k)}(\phi(\frac{\lambda^k t}{\lambda^{n-k}})) = 1, \\
&\lim_{n\to\infty} G^*_{g(z_{k+2}),g(z_{k+1}),g(z_k)}(\phi(\frac{\lambda^k t}{\lambda^{n-k}})) = 1, \quad \lim_{n\to\infty} G^*_{g(w_{k+2}),g(w_{k+1}),g(w_k)}(\phi(\frac{\lambda^k t}{\lambda^{n-k}})) = 1.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\lim_{n\to\infty} (\frac{1}{G^*_{g(x_{n+2}),g(x_{n+1}),g(x_n)}(\phi(\lambda^k t))} - 1) \\
&\leq \lim_{n\to\infty} (\frac{1}{G^*_{g(x_{n+2}),g(x_{n+1}),g(x_n)}(\phi(\lambda^k t))} - 1 + \frac{1}{G^*_{g(y_{n+2}),g(y_{n+1}),g(y_n)}(\phi(\lambda^k t))} - 1 \\
&\quad + \frac{1}{G^*_{g(z_{n+2}),g(z_{n+1}),g(z_n)}(\phi(\lambda^k t))} - 1 + \frac{1}{G^*_{g(w_{n+2}),g(w_{n+1}),g(w_n)}(\phi(\lambda^k t))} - 1) \leq 0, \\
&\lim_{n\to\infty} (\frac{1}{G^*_{g(y_{n+2}),g(y_{n+1}),g(y_n)}(\phi(\lambda^k t))} - 1) \\
&\leq \lim_{n\to\infty} (\frac{1}{G^*_{g(y_{n+2}),g(y_{n+1}),g(y_n)}(\phi(\lambda^k t))} - 1 + \frac{1}{G^*_{g(z_{n+2}),g(z_{n+1}),g(z_n)}(\phi(\lambda^k t))} - 1 \\
&\quad + \frac{1}{G^*_{g(w_{n+2}),g(w_{n+1}),g(w_n)}(\phi(\lambda^k t))} - 1 + \frac{1}{G^*_{g(x_{n+2}),g(x_{n+1}),g(x_n)}(\phi(\lambda^k t))} - 1) \leq 0,
\end{aligned}$$

similarly

$$\lim_{n\to\infty} (\frac{1}{G^*_{g(z_{n+2}),g(z_{n+1}),g(z_n)}(\phi(\lambda^k t))} - 1) \leq 0, \quad \lim_{n\to\infty} (\frac{1}{G^*_{g(w_{n+2}),g(w_{n+1}),g(w_n)}(\phi(\lambda^k t))} - 1) \leq 0,$$

which implies that

$$\lim_{n\to\infty} (G^*_{g(x_{n+2}),g(x_{n+1}),g(x_n)}(\phi(\lambda^k t)) = 1, \lim_{n\to\infty} (G^*_{g(y_{n+2}),g(y_{n+1}),g(y_n)}(\phi(\lambda^k t)) = 1, \tag{3.9}$$

$$\lim_{n\to\infty} (G^*_{g(z_{n+2}),g(z_{n+1}),g(z_n)}(\phi(\lambda^k t)) = 1, \lim_{n\to\infty} (G^*_{g(w_{n+2}),g(w_{n+1}),g(w_n)}(\phi(\lambda^k t)) = 1. \tag{3.10}$$

Now, let $\epsilon > 0$ be given, by (i) and (iv) of Definition 2.2, we can find $k \in \mathbb{Z}^+$ such that $\phi(\lambda^k t) < \epsilon$, it follows from (3.9) and (3.10) that

$$\begin{aligned}
\lim_{n\to\infty} (G^*_{g(x_{n+2}),g(x_{n+1}),g(x_n)}(\epsilon)) &\geq \lim_{n\to\infty} (G^*_{g(x_{n+2}),g(x_{n+1}),g(x_n)}(\phi(\lambda^k t)) = 1, \\
\lim_{n\to\infty} (G^*_{g(y_{n+2}),g(y_{n+1}),g(y_n)}(\epsilon)) &\geq \lim_{n\to\infty} (G^*_{g(y_{n+2}),g(y_{n+1}),g(y_n)}(\phi(\lambda^k t)) = 1,
\end{aligned}$$

similarly,

$$\lim_{n\to\infty}(G^*_{g(w_{n+2}),g(w_{n+1}),g(w_n)}(\epsilon)) \geq 1, \quad \lim_{n\to\infty}(G^*_{g(z_{n+2}),g(z_{n+1}),g(z_n)}(\epsilon)) \geq 1.$$

By using Menger triangle inequality, we obtain

$$G^*_{g(x_{n+p}),g(x_{n+1}),g(x_n)}(\epsilon) \geq \triangle(G^*_{g(x_{n+p}),g(x_{n+p-1}),g(x_{n+p-1})}(\frac{\epsilon}{p}), \triangle(G^*_{g(x_{n+p-1}),g(x_{n+p-2}),g(x_{n+p-2})}(\frac{\epsilon}{p}) \\ \cdots, G^*_{g(x_{n+2}),g(x_{n+1}),g(x_n)}(\frac{\epsilon}{p})).$$

Thus, letting $n \to \infty$ and making use of (3.9) and (3.10), for any integer, we get

$$\lim_{n\to\infty} G^*_{g(x_{n+p}),g(x_{n+1}),g(x_n)}(\epsilon) = 1 \quad \text{for every } \epsilon > 0.$$

Hence $g(x_n)$ is a Cauchy sequence. Similarly, we can prove that $g(y_n), g(z_n), g(w_n)$ are also Cauchy sequences. Since $(X, G^*, \triangle)$ is complete, there exist $x, y, z, w \in X$ such that

$$\lim_{n\to\infty} g(x_n) = x, \quad \lim_{n\to\infty} g(y_n) = y, \quad \lim_{n\to\infty} g(z_n) = z, \quad \lim_{n\to\infty} g(w_n) = w. \tag{3.11}$$

From (3.11) and the continuity of g, we have

$$\lim_{n\to\infty} g(g(x_n)) = g(x), \quad \lim_{n\to\infty} g(g(y_n)) = g(y), \quad \lim_{n\to\infty} g(g(z_n)) = g(z), \quad \lim_{n\to\infty} g(g(w_n)) = g(w).$$

From (3.2), (3.3) and the commutativity of T and g, we have

$$g(g(x_{n+1})) = g(T(x_n, y_n, z_n, w_n)) = T(g(x_n), g(y_n), g(z_n), g(w_n)), \tag{3.12}$$

$$g(g(y_{n+1})) = g(T(y_n, z_n, w_n, x_n)) = T(g(y_n), g(z_n), g(w_n), g(x_n)), \tag{3.13}$$

$$g(g(z_{n+1})) = g(T(z_n, w_n, x_n, y_n)) = T(g(z_n), g(w_n), g(x_n), g(y_n)), \tag{3.14}$$

$$g(g(w_{n+1})) = g(T(w_n, x_n, y_n, z_n)) = T(g(w_n), g(x_n), g(y_n), g(z_n)). \tag{3.15}$$

Now,we show that

$$g(x) = T(x, y, z, w), \ g(y) = T(y, z, w, x), \ g(z) = T(z, w, x, y), \ g(w) = T(w, x, y, z).$$

Suppose that the assumption (a) holds. Taking the limit of (3.11) as $n \to \infty$, by (3.12) $\sim$ (3.15) and the continuity of T, we get

$$g(x) = \lim_{n\to\infty} g(g(x_{n+1})) = \lim_{n\to\infty} T(g(x_n, y_n, z_n, w_n)) = T(\lim_{n\to\infty} g(x_n), \lim_{n\to\infty} g(y_n), \lim_{n\to\infty} g(z_n), \lim_{n\to\infty} g(w_n)) \\ = T(x, y, z, w),$$

10

$$g(y) = \lim_{n\to\infty} g(g(y_{n+1})) = \lim_{n\to\infty} T(g(y_n, z_n, w_n, x_n)) = T(\lim_{n\to\infty} g(y_n), \lim_{n\to\infty} g(z_n), \lim_{n\to\infty} g(w_n), \lim_{n\to\infty} g(x_n))$$
$$= T(y, z, w, x).$$

Similarly,

$$g(z) = T(z, w, x, y),\ g(w) = T(w, x, y, z).$$

Thus we prove that

$$g(x) = T(x, y, z, w),\ g(y) = T(y, z, w, x),\ g(z) = T(z, w, x, y),\ g(w) = T(w, x, y, z).$$

Suppose now that (b) holds, since

$$G^*_{g(x),T(x,y,z,w),T(x,y,z,w)}(\epsilon) \geq \triangle(G^*_{g(x),g(g(x_{n+1})),g(g(x_{n+1}))}(\frac{\epsilon}{2}), G^*_{g(g(x_{n+1})),T(x,y,z,w),T(x,y,z,w)}(\frac{\epsilon}{2})). \quad (3.16)$$

and using (i) of Definition 2.2, we find some $s > 0$ such that $\phi(s) < \frac{\epsilon}{2}$, since

$$\lim_{n\to\infty} g(g(x_n)) = g(x), \lim_{n\to\infty} g(g(y_n)) = g(y), \lim_{n\to\infty} g(g(z_n)) = g(z), \lim_{n\to\infty} g(g(w_n)) = g(w).$$

then there exists $n_0 \in \mathbb{Z}^+$, such that

$$G^*_{g(g(x_n)),g(x),g(x)}(\phi(s)) > 0, \quad G^*_{g(g(y_n)),g(y),g(y)}(\phi(s)) > 0,$$
$$G^*_{g(g(z_n)),g(z),g(z)}(\phi(s)) > 0, \quad G^*_{g(g(w_n)),g(w),g(w)}(\phi(s)) > 0.$$

for all $n > n_0$. Since $\{g(x_n)\}, \{g(z_n)\}$ is non-decreasing and as $\{g(y_n)\}, \{g(w_n)\}$ is non-increasing and

$$g(x_n) \to x,\ g(y_n) \to y,\ g(z_n) \to z,\ g(w_n) \to w.$$

By (3.1) and (3.12)-(3,15), we get

$$\psi(\frac{1}{G^*_{g(g(x_{n+1})),T(x,y,z,w),T(x,y,z,w)}(\phi(s))} - 1) = \psi(\frac{1}{G^*_{T(g(x_n),g(y_n),g(z_n),g(w_n)),T(x,y,z,w),T(x,y,z,w)}(\phi(s))} - 1)$$
$$\leq \frac{1}{4}\psi(\frac{1}{G^*_{g(g(x_n)),g(x),g(x)}(\phi(\frac{s}{\lambda}))} - 1 + \frac{1}{G^*_{g(g(y_n)),g(y),g(y)}(\phi(\frac{s}{\lambda}))} - 1 + \frac{1}{G^*_{g(g(z_n)),g(z),g(z)}(\phi(\frac{s}{\lambda}))} - 1$$
$$+ \frac{1}{G^*_{g(g(w_n)),g(w),g(w)}(\phi(\frac{s}{\lambda}))} - 1).$$

By the same way, we obtain

$$\psi(\frac{1}{G^*_{g(g(y_{n+1})),T(y,z,w,x),T(y,z,w,x)}(\phi(s))} - 1)$$
$$\leq \frac{1}{4}\psi(\frac{1}{G^*_{g(g(x_n)),g(x),g(x)}\phi(\frac{s}{\lambda})} - 1 + \frac{1}{G^*_{g(g(y_n)),g(y),g(y)}\phi(\frac{s}{\lambda})} - 1 + \frac{1}{G^*_{g(g(z_n)),g(z),g(z)}\phi(\frac{s}{\lambda})} - 1$$
$$+ \frac{1}{G^*_{g(g(w_n)),g(w),g(w)}\phi(\frac{s}{\lambda})} - 1),$$

$$\begin{aligned}
&\psi(\frac{1}{G^*_{g(g(z_{n+1})),T(z,w,x,y),T(z,w,x,y)}(\phi(s))}-1)\\
&\le \frac{1}{4}\psi(\frac{1}{G^*_{g(g(x_n)),g(x),g(x)}\phi(\frac{s}{\lambda})}-1+\frac{1}{G^*_{g(g(y_n)),g(y),g(y)}\phi(\frac{s}{\lambda})}-1+\frac{1}{G^*_{g(g(z_n)),g(z),g(z)}\phi(\frac{s}{\lambda})}-1\\
&\quad+\frac{1}{G^*_{g(g(w_n)),g(w),g(w)}\phi(\frac{s}{\lambda})}-1),\\
&\psi(\frac{1}{G^*_{g(g(w_{n+1})),T(w,x,y,z),T(w,x,y,z)}(\phi(s))}-1)\\
&\le \frac{1}{4}\psi(\frac{1}{G^*_{g(g(x_n)),g(x),g(x)}\phi(\frac{s}{\lambda})}-1+\frac{1}{G^*_{g(g(y_n)),g(y),g(y)}\phi(\frac{s}{\lambda})}-1+\frac{1}{G^*_{g(g(z_n)),g(z),g(z)}\phi(\frac{s}{\lambda})}-1\\
&\quad+\frac{1}{G^*_{g(g(w_n)),g(w),g(w)}\phi(\frac{s}{\lambda})}-1).
\end{aligned}$$

By the above inequalities and (3) of Remark 2.1, we have

$$\begin{aligned}
&\frac{1}{G^*_{g(g(x_{n+1})),T(x,y,z,w),T(x,y,z,w)}(\phi(\frac{\epsilon}{2}))}-1\le\frac{1}{G^*_{g(g(x_{n+1})),T(x,y,z,w),T(x,y,z,w)}(\phi(s))}-1\\
&\le\frac{1}{G^*_{g(g(x_{n+1})),T(x,y,z,w),T(x,y,z,w)}(\phi(s))}-1+\frac{1}{G^*_{g(g(y_{n+1})),T(y,z,w,x),T(y,z,w,x)}(\phi(s))}-1\\
&\quad+\frac{1}{G^*_{g(g(z_{n+1})),T(z,w,x,y),T(z,w,x,y)}(\phi(s))}-1+\frac{1}{G^*_{g(g(w_{n+1})),T(w,x,y,z),T(w,x,y,z)}(\phi(s))}-1\\
&\le\frac{1}{G^*_{g(g(x_n)),g(x),g(x)}(\phi(\frac{s}{\lambda}))}-1+\frac{1}{G^*_{g(g(y_n)),g(y),g(y)}(\phi(\frac{s}{\lambda}))}-1+\frac{1}{G^*_{g(g(z_n)),g(z),g(z)}(\phi(\frac{s}{\lambda}))}-1\\
&\quad+\frac{1}{G^*_{g(g(w_n)),g(w),g(w)}(\phi(\frac{s}{\lambda}))}-1.
\end{aligned}\tag{3.17}$$

Letting $n\to\infty$ in above inequalities (3.17), we obtain

$$\lim_{n\to\infty}G^*_{g(g(x_{n+1})),T(x,y,z,w),T(x,y,z,w)}(\frac{\epsilon}{2})=1.\tag{3.18}$$

From (3.16) and (3.18), we get $G^*_{g(x),T(x,y,z,w),T(x,y,z,w)}(\epsilon)=1$ for every $\epsilon>0$, which implies that $g(x)=T(x,y,z,w)$. Similarly, we show that $g(y)=T(y,z,w,x), g(z)=T(z,w,x,y), g(w)=T(w,x,y,z)$. Thus we prove that g and T have a coupled coincidence point.

Corollary 3.1 Let $(X,\le)$ be a partially ordered set and $(X,G^*,\triangle)$ be a complete PGM-space with a continuous $t-norm$. Assume that $T:X^4\to X$ has the mixed monotone property, and satisfying the following:

$$\frac{1}{G^*_{T(x,y,z,w),T(u,v,p,q),T(a,b,c,d)}(\frac{t}{2})}-1\le\frac{1}{4}(\frac{1}{G^*_{x,u,a}(t)}-1+\frac{1}{G^*_{y,v,b}(t)}-1+\frac{1}{G^*_{z,p,c}(t)}-1+\frac{1}{G^*_{w,q,d}(t)}-1)$$

for $t>0$, $G^*_{x,u,a}(t)>0, G^*_{y,v,b}(t)>0, G^*_{z,p,c}(t)>0, G^*_{w,q,d}(t)>0$ and $x,y,z,w,u,v,p,q,a,b,c,d\in X$ satisfying $x\le u\le a, z\le p\le c$, $y\ge v\ge b$ and $w\ge q\ge d$. Suppose that either

(a) T is continuous, or

(b) X has the following properties:

(I) if non-decreasing sequences $\{x_n\} \to x, \{z_n\} \to z$, then $x_n \leq x, z_n \leq z$ for all n,

(II) if non-increasing sequences $\{y_n\} \to y, \{w_n\} \to w$, then $y_n \leq y, w_n \leq w$ for all n.

If there exist $x_0, y_0, z_0, w_0 \in X$, such that $x_0 \leq T(x_0, y_0, z_0, w_0), z_0 \leq T(z_0, w_0, x_0, y_0)$ and $y_0 \geq T(y_0, z_0, w_0, x_0), w_0 \geq T(w_0, x_0, y_0, z_0)$, then their exist $x, y, z, w \in X$, such that

$$x = T(x, y, z, w), y = T(y, z, w, x), z = T(z, w, x, y), w = T(w, x, y, z),$$

that is, T has a coupled coincidence point.

Proof Taking $g = I_X$(the identity mapping on X), $\lambda = \frac{1}{2}$ and $\phi(t) = \varphi(t) = t$ for all $t \geq 0$ in Theorem 3.1, we can easily obtain the above corollary.

4 Coupled common fixed point results in partially ordered complete Menger probabilistic G-metric spaces

In the section, we prove the existence and uniqueness theorem of a coupled fixed point in partially ordered complete Menger probabilistic G-metric spaces.

Theorem 3.2 In addition to the hypotheses of Theorem 3.1, suppose that for every $(x, y, z, w), (x^*, y^*, z^*, w^*) \in X^4$ there exists a $(u, v, p, q) \in X^4$, such that $(T(u, v, p, q), T(v, p, q, u), T(p, q, u, v), T(q, u, v, p))$ are comparable to $(T(x, y, z, w), T(y, z, w, x), T(z, w, x, y), T(w, x, y, z))$ and $(T(x^*, y^*, z^*, w^*), T(y^*, z^*, w^*, x^*), T(z^*, w^*, x^*, y^*), T(w^*, x^*, y^*, z^*))$. Then T and g have a unique coupled common fixed point, that is, there exists a unique $(x, y, z, w) \in X^4$, such that

$$x = g(x) = T(x, y, z, w), y = g(y) = T(y, z, w, x), z = g(z) = T(z, w, x, y), w = g(w) = T(w, x, y, z).$$

Proof From Theorem 3.1, the set of coupled coincidences is non-empty, we shall first show that if (x, y, z, w) and (x^*, y^*, z^*, w^*) are coupled coincidence points, that is, if

$$g(x) = T(x, y, z, w),\ g(y) = T(y, z, w, x),\ g(z) = T(z, w, x, y),\ g(w) = T(w, x, y, z)$$

and

$$g(x^*) = T(x^*, y^*, z^*, w^*), g(y^*) = T(y^*, z^*, w^*, x^*),$$
$$g(z^*) = T(z^*, w^*, x^*, y^*), g(w^*) = T(w^*, x^*, y^*, z^*),$$

then

$$g(x) = g(x^*), \quad g(y) = g(y^*), \quad g(z) = g(z^*), \quad g(w) = g(w^*). \tag{4.1}$$

By assumption, there exists a $(u, v, p, q) \in X^4$, such that $(T(u, v, p, q), T(v, p, q, u), T(p, q, u, v), T(q, u, v, p))$ is comparable to $(T(x, y, z, w), T(y, z, w, x), T(z, w, x, y), T(w, x, y, z))$ and $(T(x^*, y^*, z^*, w^*), T(y^*, z^*, w^*, x^*), T(z^*, w^*, x^*, y^*), T(w^*, x^*, y^*, z^*))$. Putting $u_0 = u, v_0 = v, p_0 = p, q_0 = q$ and $u_1, v_1, p_1, q_1 \in X$, such that $g(u_1) = T(u_0, v_0, p_0, q_0), g(v_1) = T(v_0, p_0, q_0, u_0), g(p_1) = T(p_0, q_0, u_0, v_0), g(q_1) = T(q_0, u_0, v_0, p_0)$. The proof of Theorems is similar to Theorem 3.1. We inductively define sequences $\{g(u_n)\}, \{g(v_n)\}, \{g(p_n)\}, \{g(q_n)\}$, such that

$$\begin{aligned} g(u_{n+1}) &= T(u_n, v_n, p_n, q_n),\ g(v_{n+1}) = T(v_n, p_n, q_n, u_n), \\ g(p_{n+1}) &= T(p_n, q_n, u_n, v_n),\ g(q_{n+1}) = T(q_n, u_n, v_n, p_n). \end{aligned}$$

Similarly, setting $x_0 = x, y_0 = y, z_0 = z, w_0 = w$, and $x_0^* = x^*, y_0^* = y^*, z_0^* = z^*, w_0^* = w^*$. We also define sequences $\{g(x_n)\}, \{g(y_n)\}, \{g(z_n)\}, \{g(w_n)\}$ and $\{g(x_n^*)\}, \{g(y_n^*)\}, \{g(z_n^*)\}, \{g(w_n^*)\}$, then it is easy to show that

$$g(x_n) = T(x, y, z, w),\quad g(y_n) = T(y, z, w, x),\quad g(z_n) = T(z, w, x, y),\quad g(w_n) = T(w, x, y, z)$$

and

$$\begin{aligned} g(x_n^*) &= T(x^*, y^*, z^*, w^*),\ g(y_n^*) = T(y^*, z^*, w^*, x^*), \\ g(z_n^*) &= T(z^*, w^*, x^*, y^*),\ g(w_n^*) = T(w^*, x^*, y^*, z^*). \end{aligned}$$

Since $(T(x, y, z, w), T(y, z, w, x), T(z, w, x, y), T(w, x, y, z)) = (g(x_1), g(y_1), g(z_1), g(w_1)) = (g(x), g(y), g(z), g(w))$ and $(T(u, v, p, q), T(v, p, q, u), T(p, q, u, v), T(q, u, v, p)) = (g(u_1), g(v_1), g(p_1), g(q_1))$ are comparable, then we have $g(x) \le g(u_1), g(z) \le g(p_1)$, $g(y) \ge g(v_1)$ and $g(w) \ge g(q_1)$. It is easy to show that $(g(x), g(y), g(w), g(z))$ and $(g(u_n), g(v_n), g(p_n), g(q_n))$ are comparable, that is, $g(x) \le g(x_n), g(z) \le g(z_n)$, $g(y) \ge g(y_n)$ and $g(w) \ge g(w_n)$, for all $n \ge 1$. Following the proof of Theorem 3.1, we can find some $t > 0$ such that

$$\begin{aligned} &G^*_{g(x),g(u_n,g(u_n)}(\phi(\frac{t}{\lambda})) > 0,\ G^*_{g(y),g(v_n,g(v_n)}(\phi(\frac{t}{\lambda})) > 0 \ \text{ for all } n \ge 0, \\ &G^*_{g(z),g(p_n,g(p_n)}(\phi(\frac{t}{\lambda})) > 0,\ G^*_{g(z),g(q_n,g(q_n)}(\phi(\frac{t}{\lambda})) > 0 \ \text{ for all } n \ge 0. \end{aligned}$$

Thus from (3.1)

$$\begin{aligned} &\psi(\frac{1}{G^*_{g(x),g(u_{n+1}),g(u_{n+1})}(\phi(t))} - 1) = \psi(\frac{1}{G^*_{T(x,y,z,w),T(u_n,v_n,p_n,q_n),T(u_n,v_n,p_n,q_n)}(\phi(t))} - 1) \\ &\le \frac{1}{4}\psi(\frac{1}{G^*_{x,u_n,u_n}(\phi(\frac{t}{\lambda}))} - 1 + \frac{1}{G^*_{y,v_n,v_n}(\phi(\frac{t}{\lambda}))} - 1 + \frac{1}{G^*_{z,p_n,p_n}(\phi(\frac{t}{\lambda}))} - 1 + \frac{1}{G^*_{w,q_n,q_n}(\phi(\frac{t}{\lambda}))} - 1). \end{aligned}$$

By Remark 2.4, we get

$$
\begin{aligned}
&\frac{1}{G^*_{g(x),g(u_{n+1}),g(u_{n+1})}(\phi(t))}-1+\frac{1}{G^*_{g(y),g(v_{n+1}),g(v_{n+1})}(\phi(t))}-1+\frac{1}{G^*_{g(z),g(p_{n+1}),g(p_{n+1})}(\phi(t))}-1\\
&\quad+\frac{1}{G^*_{g(w),g(q_{n+1}),g(q_{n+1})}(\phi(t))}-1\\
&\le\frac{1}{G^*_{g(x),g(u_n),g(u_n)}(\phi(\frac{t}{\lambda}))}-1+\frac{1}{G^*_{g(y),g(v_n),g(v_n)}(\phi(\frac{t}{\lambda}))}-1+\frac{1}{G^*_{g(z),g(p_n),g(p_n)}(\phi(\frac{t}{\lambda}))}-1\\
&\quad+\frac{1}{G^*_{g(w),g(q_n),g(q_n)}(\phi(\frac{t}{\lambda}))}-1 \qquad (4.2)\\
&\vdots\\
&\le\frac{1}{G^*_{g(x),g(u_0),g(u_0)}(\phi(\frac{t}{\lambda^n}))}-1+\frac{1}{G^*_{g(y),g(v_0),g(v_0)}(\phi(\frac{t}{\lambda^n}))}-1+\frac{1}{G^*_{g(z),g(p_0),g(p_0)}(\phi(\frac{t}{\lambda^n}))}-1\\
&\quad+\frac{1}{G^*_{g(w),g(q_0),g(q_0)}(\phi(\frac{t}{\lambda^n}))}-1.
\end{aligned}
$$

We replace u_k with u_0 in (4.2), we get

$$
\begin{aligned}
&\frac{1}{G^*_{g(x),g(u_{n+1}),g(u_{n+1})}\phi(\lambda^k t)}-1+\frac{1}{G^*_{g(y),g(v_{n+1}),g(v_{n+1})}\phi(\lambda^k t)}-1+\frac{1}{G^*_{g(z),g(p_{n+1}),g(p_{n+1})}\phi(\lambda^k t)}-1\\
&\quad+\frac{1}{G^*_{g(w),g(q_{n+1}),g(q_{n+1})}\phi(\lambda^k t)}-1\\
&\le\frac{1}{G^*_{g(x),g(u_k),g(u_k)}(\phi(\frac{\lambda^k t}{\lambda^{n-k}}))}-1+\frac{1}{G^*_{g(y),g(v_k),g(v_k)}(\phi(\frac{\lambda^k t}{\lambda^{n-k}}))}-1+\frac{1}{G^*_{g(z),g(p_k),g(p_k)}(\phi(\frac{\lambda^k t}{\lambda^{n-k}}))}-1\\
&\quad+\frac{1}{G^*_{g(w),g(q_k),g(q_k)}(\phi(\frac{\lambda^k t}{\lambda^{n-k}}))}-1,
\end{aligned}
$$

for all $n > k$. Letting $n \to \infty$, we obtain

$$
\begin{aligned}
&\lim_{n\to\infty} G^*_{g(x),g(u_{n+1},g(u_{n+1})(\phi(\lambda^k t))}=1,\ \lim_{n\to\infty} G^*_{g(y),g(v_{n+1},g(v_{n+1})(\phi(\lambda^k t))}=1.\\
&\lim_{n\to\infty} G^*_{g(z),g(p_{n+1},g(p_{n+1})(\phi(\lambda^k t))}=1,\ \lim_{n\to\infty} G^*_{g(w),g(q_{n+1},g(q_{n+1})(\phi(\lambda^k t))}=1.
\end{aligned}
$$

Let $\epsilon > 0$ be given. By (i) and (iv) of Definition 2.2, there exists $k \in \mathbb{Z}^+$, such that $\phi(\lambda^k t) < \frac{\epsilon}{2}$. Then we have

$$
\lim_{n\to\infty} G^*_{g(x),g(u_{n+1}),g(u_{n+1})}(\frac{\epsilon}{2}) \ge \lim_{n\to\infty} G^*_{g(x),g(u_{n+1}),g(u_{n+1})}(\phi(\lambda^k t)) = 1, \tag{4.3}
$$

$$
\lim_{n\to\infty} G^*_{g(y),g(v_{n+1}),g(v_{n+1})}(\frac{\epsilon}{2}) \ge \lim_{n\to\infty} G^*_{g(y),g(v_{n+1}),g(v_{n+1})}(\phi(\lambda^k t)) = 1. \tag{4.4}
$$

Similarly, we prove that

$$
\lim_{n\to\infty} G^*_{g(x^*),g(u_{n+1}),,g(u_{n+1})}(\frac{\epsilon}{2}) = 1,\ \lim_{n\to\infty} G^*_{g(y^*),g(v_{n+1}),,g(v_{n+1})}(\frac{\epsilon}{2}) = 1. \tag{4.5}
$$

$$
\lim_{n\to\infty} G^*_{g(z^*),g(p_{n+1}),,g(p_{n+1})}(\frac{\epsilon}{2}) = 1,\ \lim_{n\to\infty} G^*_{g(w^*),g(q_{n+1}),,g(q_{n+1})}(\frac{\epsilon}{2}) = 1. \tag{4.6}
$$

By using Menger triangle inequality, and (4.3)-(4.6), we get

$$G^*_{g(x),g(u_{n+1}),g(x^*)}(\epsilon) \geq \triangle(G^*_{g(x),g(u_{n+1}),g(u_{n+1})}(\frac{\epsilon}{2}), G^*_{g(u_{n+1}),g(u_{n+1}),g(x^*)}(\frac{\epsilon}{2})) \to 1 \quad \text{as } n \to \infty,$$
$$G^*_{g(y),g(v_{n+1}),g(y^*)}(\epsilon) \geq \triangle(G^*_{g(y),g(v_{n+1}),g(v_{n+1})}(\frac{\epsilon}{2}), G^*_{g(v_{n+1}),g(v_{n+1}),g(y^*)}(\frac{\epsilon}{2})) \to 1 \quad \text{as } n \to \infty,$$
$$G^*_{g(z),g(p_{n+1}),g(z^*)}(\epsilon) \geq \triangle(G^*_{g(z),g(p_{n+1}),g(p_{n+1})}(\frac{\epsilon}{2}), G^*_{g(p_{n+1}),g(p_{n+1}),g(z^*)}(\frac{\epsilon}{2})) \to 1 \quad \text{as } n \to \infty,$$
$$G^*_{g(w),g(q_{n+1}),g(w^*)}(\epsilon) \geq \triangle(G^*_{g(w),g(q_{n+1}),g(q_{n+1})}(\frac{\epsilon}{2}), G^*_{g(q_{n+1}),g(q_{n+1}),g(w^*)}(\frac{\epsilon}{2})) \to 1 \quad \text{as } n \to \infty.$$

Hence $g(x) = g(x^*), g(y) = g(y^*), g(z) = g(z^*), g(w) = g(w^*)$, thus (4.1) holds. Since $g(x) = T(x, y, z, w), g(y) = T(y, z, w, x), g(z) = T(z, w, x, y), g(w) = T(w, x, y, z)$, by commutativity of T and g, we have

$$g(g(x)) = g(T(x, y, z, w)) = T(g(x), g(y), g(z), g(w)), \tag{4.7}$$
$$g(g(y)) = g(T(y, z, w, x)) = T(g(y), g(z), g(w), g(x)), \tag{4.8}$$
$$g(g(z)) = g(T(z, w, x, y)) = T(g(z), g(w), g(x), g(y)), \tag{4.9}$$
$$g(g(w)) = g(T(w, x, y, z)) = T(g(w), g(x), g(y), g(z)). \tag{4.10}$$

Denote $g(x) = \alpha, g(y) = \beta, g(z) = \gamma, g(w) = \sigma$. From (4.7)-(4.10), we obtain

$$g(\alpha) = T(\alpha, \beta, \gamma, \sigma), g(\beta) = T(\beta, \gamma, \sigma, \alpha), g(\gamma) = T(\gamma, \sigma, \alpha, \beta), g(\sigma) = T(\sigma, \alpha, \beta, \gamma), \tag{4.11}$$

thus $(\alpha, \beta, \gamma, \sigma)$ is a coupled coincidence point. Owing to (4.1) with $x^* = \alpha, y^* = \beta, z^* = \gamma$, and $w^* = \sigma$, it follows

$$g(\alpha) = g(x),\ g(\beta) = g(y),\ g(\gamma) = g(z),\ g(\sigma) = g(w),$$

that is

$$g(\alpha) = \alpha, g(\beta) = \beta, g(\gamma) = \gamma, g(\sigma) = \sigma. \tag{4.12}$$

From (4.11) and (4.12), we have

$$\alpha = g(\alpha) = T(\alpha, \beta, \gamma, \sigma), \beta = g(\beta) = T(\beta, \gamma, \sigma, \alpha), \gamma = g(\gamma) = T(\gamma, \sigma, \alpha, \beta), \sigma = g(\sigma) = T(\sigma, \alpha, \beta, \gamma).$$

Therefore, $(\alpha, \beta, \gamma, \sigma)$ is a coupled common fixed point of T and g. Suppose that $(\alpha^*, \beta^*, \gamma^*, \sigma^*)$ is another coupled common fixed point. By (4.1), we have

$$\alpha^* = g(\alpha^*) = g(x) = x, \beta^* = g(\beta^*) = g(y) = y, \gamma^* = g(\gamma^*) = g(z) = z, \sigma^* = g(\sigma^*) = g(w) = w,$$

which implies that T and g has a unique coupled common fixed point.

This completes the proof.

5 An example

In this section, an example are presented to verify the effectiveness and applicability of Theorem 3.1.

Example 5.1 Let $X = [0, 1]$ be given. Define $G(x, y, z) = |x - y| + |y - z| + |z - x|$. A mapping $T : X^4 \to X$ define by $T(x_1, x_2, x_3, x_4) = \frac{x_1+x_2+x_3+x_4}{16}$. And $g : X \to X$ define by $g(x) = \frac{x}{2}$. Define

$$G^*_{x,y,z}(t) = \begin{cases} \frac{t}{t+G(x,y,z)}, & \text{if } t > 0, \\ 0, & \text{if } t < 0. \end{cases}$$

for $x_1, x_2, x_3, x_4, x, y, z \in X$, where $T(X^4) \subset g(X)$. Then $(X, G^*, \triangle_m)$ is a complete Menger PGM-space with a continuous t-norm $\triangle_m$. Let $\lambda = \frac{1}{2}$, $\varphi(t) = \frac{9t}{10}$ and $\phi(t) = \frac{t}{2}$ be given for all $t > 0$. Then we have

$$\begin{aligned}
&\psi(\frac{1}{G^*_{T(x,y,z,w),T(u,v,p,q),T(a,b,c,d)}(\phi(\lambda t))} - 1) = \psi(\frac{1}{\phi(\lambda t)}(G(T(x, y, z, w), T(u, v, p, q), T(a, b, c, d)))) \\
&= \frac{9}{40t}(|x + y + z + w - u - v - p - q| + |u + v + p + q - a - b - c - d| \\
&\quad + |a + b + c + d - x - y - z - w|),
\end{aligned} \tag{5.1}$$

$$\begin{aligned}
&\frac{1}{4}\psi(\frac{1}{G^*_{g(x),g(u),g(a)}\phi(t)} - 1 + \frac{1}{G^*_{g(y),g(v),g(b)}\phi(t)} - 1 + \frac{1}{G^*_{g(z),g(p),g(c)}\phi(t)} - 1 + \frac{1}{G^*_{g(w),g(q),g(d)}\phi(t)} - 1) \\
&= \frac{9}{40t}(|x - u| + |u - a| + |a - x| + |y - v| + |v - b| + |b - y| + |z - p| + |p - c| + |c - z| \\
&\quad + |w - q| + |q - d| + |d - w|).
\end{aligned} \tag{5.2}$$

By (5.1) and (5.2), we obtain

$$\begin{aligned}
\psi(\frac{1}{G^*_{T(x,y,z,w),T(u,v,p,q),T(a,b,c,d)}(\phi(\lambda t))} - 1) \leq &\frac{1}{4}\psi(\frac{1}{G^*_{g(x),g(u),g(a)}\phi(t)} - 1 + \frac{1}{G^*_{g(y),g(v),g(b)}\phi(t)} - 1 \\
&+ \frac{1}{G^*_{g(z),g(p),g(c)}\phi(t)} - 1 + \frac{1}{G^*_{g(w),g(q),g(d)}\phi(t)} - 1),
\end{aligned}$$

which implies that T and g satisfy ψ-contractive condition. Thus, all the conditions of Theorem 3.1 are satisfied. And (0,0,0,0) is the coupled coincidence point of T and g.

Acknowledgement

The authors would like to thank the editor and the referees for their constructive comments and suggestions. The research was supported by the Natural Science Foundation of China (11361042,11071108, 11461045,71363043), the Natural Science Foundation of Jiangxi Province of China (2010GZS0147,2011

4BAB 201003, 20142BAB211016, 20132BAB201001), and partly supported by the NSF of Education Department of Jiangxi Province of China (GJJ150008).

References

[1] T.G. Bhaskar and V. Lakshmikantham, Fixed point theory in partially ordered metric spaces and applications, *Nonlinear Anal.* 65, 1379-1393 (2006).

[2] B.S. Choudhury and K. Das, A new contraction principle in Menger spaces, *Acta Math. Sin.* 24, 1379-1386 (2008).

[3] E. Karapinar, Coupled fixed point theorems for nonlinear contractions in cone metric spaces, *Comput. Math. Appl.* 59, 3656-3668 (2010).

[4] S.S. Chang, Y.J. Cho and S.M. Kang, Nonlinear Operator Theory in Probabilistic Metric Space, *Nova Science, Huntington, NY, USA.* (2001).

[5] B.S Choudhury and A. Kundu, A coupled coincidence point result in partially ordered metric spaces for compatible mappings, *Nonlinear Anal.* 73, 2524-2531 (2010).

[6] M.A. Kutbi, D. Gopal, C. Vetro and W. Sintunavarat, Further generalization of fixed point theorems in Menger PM-spaces, *Fixed Point Theory Appl.* 2015, 32 (2015).

[7] V. Lakshmikantham and L.*Ćirić*, Coupled fxed point theorems for nonlinear contractions in partially ordered metric spaces, *Nonlinear Anal.* 70, 4341-4349 (2009).

[8] V. Luong and N. Thuan, Coupled fixed points in partially ordered metric spaces and application, *Nonlinear Anal.* 74, 983-992 (2011).

[9] J. Jachymski, On probabilistic φ-contractions on Menger spaces, *Nonlinear Anal.* 73, 2199-2203 (2010).

[10] J.M. Jin, C.X. Zhu and Z.Q. Wu, New fixed point theorem for phi-contractions in KM-fuzzy metric spaces, *J. of Nonlinear Sci. Appl.* 9, 6204-6209 (2016).

[11] J.Z. Xiao, X.H. Zhu and Y.F. Cao, Common coupled fixed point results for probabilistic ϕ-contractions in Menger spaces, *Nonlinear Anal.* 74, 4589-4600 (2011).

[12] B.S. Choudhury and P. Maity, coupled fixed point results in generalized metric spaces, *Math. Comput. Modelling.* 54, 73-79 (2011).

[13] J.H. Cheng and X.J. Huang, coupled fixed point theorems for compatible mappings in partially ordered G-metric spaces, *J.of Nonlinear Sci. Appl.* 8, 130-141 (2015).

[14] Q.Tu, C. X. Zhu and Z. Q. Wu, Some new coupled fixed point theorems in partially ordered complete probabilistic metric spaces, *J. of Nonlinear Sci. Appl.* 9 1116-1128 (2016).

[15] N. Wairojjana, T. Došenović, D. Rakić, D. Gopal and P. Kuman, An altering distance function in fuzzy metric fixed point theorems, *Fixed Point Theory Appl.* 2015, 69 (2015).

[16] C.X. Zhu, Several nonlinear operator problems in the Menger PN space, *Nonlinear Anal.* 65, 1281-1284 (2006).

[17] C.X. Zhu, Research on some problems for nonlinear operators, *Nonlinear Anal.* 71, 4568-4571 (2009).

[18] C.L. Zhou, S.H. Wang, L. Ćirić and S.M. Alsulami, Generalized probabilistic metric spaces and fixed point theorems, *Fixed point Theory Appl.* 2014, 91 (2014).

FOURIER SERIES OF SUMS OF PRODUCTS OF HIGHER-ORDER EULER FUNCTIONS

TAEKYUN KIM[1], DAE SAN KIM[2], GWAN-WOO JANG[3], AND JONGKYUM KWON[4,*]

Abstract. In this paper, we consider three types of functions given by sums of products of higher-order Euler functions and derive their Fourier series expansions. Moreover, we express each of them in terms of Bernoulli functions.

1. Introduction

Let r be a nonnegative integer. The Euler polynomials $E_m^{(r)}(x)$ of order r are defined by the generating function (see [2, 9–12, 17, 19])

$$\left(\frac{2}{e^t+1}\right)^r e^{xt} = \sum_{m=0}^{\infty} E_m^{(r)}(x)\frac{t^m}{m!}. \tag{1.1}$$

When $x = 0$, $E_m^{(r)} = E_m^{(r)}(0)$ are called the Euler numbers of order r. For $r = 1$, $E_m(x) = E_m^{(1)}(x)$, and $E_m = E_m^{(1)}$ are called Euler polynomials and numbers, respectively.

From (1.1), it is immediate to see that

$$\frac{d}{dx}E_m^{(r)}(x) = mE_{m-1}^{(r)}(x),\ m \geq 1, E_m^{(r)}(x+1)+E_m^{(r)}(x) = 2E_m^{(r-1)}(x),\ m \geq 0. \tag{1.2}$$

These in turn imply that

$$E_m^{(r)}(1) = 2E_m^{(r-1)} - E_m^{(r)},\ (m \geq 0), \tag{1.3}$$

and

$$\int_0^1 E_m^{(r)}(x)dx = \frac{2}{m+1}\left(E_{m+1}^{(r-1)} - E_{m+1}^{(r)}\right),\ (m \geq 0). \tag{1.4}$$

For any real number x, the fractional part of x is denoted by

$$< x >= x - [x] \in [0, 1). \tag{1.5}$$

We will need the following facts about the Fourier series expansion of the Bernoulli function $B_m(< x >)$:

2010 *Mathematics Subject Classification.* 11B68, 42A16.
Key words and phrases. Fourier series, sums of products of higher-order Euler functions.
* corresponding author.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 27, NO.2, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC

(a) for $m \geq 2$,

$$B_m(<x>) = -m! \sum_{n=-\infty, n\neq 0}^{\infty} \frac{e^{2\pi inx}}{(2\pi in)^m}, \tag{1.6}$$

(b) for $m = 1$,

$$-\sum_{n=-\infty, n\neq 0}^{\infty} \frac{e^{2\pi inx}}{2\pi in} = \begin{cases} B_1(<x>), & \text{for } x \notin \mathbb{Z}, \\ 0, & \text{for } x \in \mathbb{Z}. \end{cases} \tag{1.7}$$

In the present paper, we will study the following three types of sums of products of higher-order Euler functions and find Fourier series expansions for them. Furthermore, we will express them in terms of Bernoulli functions. In the following, we let r, s be positive integers.

(1) $\alpha_m(<x>) = \sum_{k=0}^{m} E_k^{(r)}(<x>)E_{m-k}^{(s)}(<x>)$, $(m \geq 1)$;
(2) $\beta_m(<x>) = \sum_{k=0}^{m} \frac{1}{k!(m-k)!} E_k^{(r)}(<x>)E_{m-k}^{(s)}(<x>)$, $(m \geq 1)$;
(3) $\gamma_m(<x>) = \sum_{k=1}^{m-1} \frac{1}{k(m-k)} E_k^{(r)}(<x>)E_{m-k}^{(s)}(<x>)$, $(m \geq 2)$.

For elementary facts about Fourier analysis, the reader may refer to any book (for example, see [1, 20]).

As to $\gamma_m(<x>)$, we note that the polynomial identity (1.8) follows immediately from the Fourier series expansion of $\gamma_m(<x>)$ in Theorems 4.1 and 4.2:

$$\begin{aligned} &\sum_{k=1}^{m-1} \frac{1}{k(m-k)} E_k^{(r)}(x) E_{m-k}^{(s)}(x) \\ &= \frac{1}{m} \sum_{k=0}^{m} \binom{m}{k} \Big\{ \Lambda_{m-k+1} + \frac{2(H_{m-1} - H_{m-k})}{m-k+1} \\ &\times (E_{m-k+1}^{(r-1)} + E_{m-k+1}^{(s-1)} - E_{m-k+1}^{(r)} - E_{m-k+1}^{(s)}) \Big\} B_k(x), \end{aligned} \tag{1.8}$$

where, for each integer $l \geq 2$,

$$\Lambda_l = \sum_{k=1}^{l-1} \frac{2}{k(l-k)} \big(2E_k^{(r-1)} E_{l-k}^{(s-1)} - E_k^{(r)} E_{l-k}^{(s-1)} - E_k^{(r-1)} E_{l-k}^{(s)} \big), \tag{1.9}$$

and $H_m = \sum_{j=1}^{m} \frac{1}{j}$ are the harmonic numbers.

The obvious polynomial identities can be derived also for $\alpha_m(<x>)$ and $\beta_m(<x>)$ from Theorems 2.1 and 2.2, and Theorems 3.1 and 3.2, respectively. It is noteworthy that from the Fourier series expansion of the function

$$\sum_{k=1}^{m-1} \frac{1}{k(m-k)} B_k(\langle x \rangle) B_{m-k}(\langle x \rangle) \tag{1.10}$$

we can derive the famous Faber-Pandharipande-Zagier identity (see [4, 7, 8]) and the Miki's identity (see [3, 5, 7, 8, 18]). Hence our problem here is a natural extension of the previous works which lead to a simple proof for the important Faber-Pandharipande-Zagier and Miki's identities (see [15]). Some related recent works can be found in [6, 13–16].

2. The function $\alpha_m(< x >)$

Let $\alpha_m(x) = \sum_{k=0}^{m} E_k^{(r)}(x)E_{m-k}^{(s)}(x)$, $(m \geq 1)$. Then we will consider the function

$$\alpha_m(< x >) = \sum_{k=0}^{m} E_k^{(r)}(< x >)E_{m-k}^{(s)}(< x >),\ (m \geq 1), \tag{2.1}$$

defined on $\mathbb{R}$, which is periodic with period 1.

The Fourier series of $\alpha_m(< x >)$ is

$$\sum_{n=-\infty}^{\infty} A_n^{(m)} e^{2\pi i n x}, \tag{2.2}$$

where

$$A_n^{(m)} = \int_0^1 \alpha_m(< x >)e^{-2\pi i n x}dx = \int_0^1 \alpha_m(x)e^{-2\pi i n x}dx. \tag{2.3}$$

To proceed further, we need to observe the following.

$$\begin{aligned}
\alpha'_m(x) &= \sum_{k=0}^{m} \Big(kE_{k-1}^{(r)}(x)E_{m-k}^{(s)}(x) + (m-k)E_k^{(r)}(x)E_{m-k-1}^{(s)}(x)\Big) \\
&= \sum_{k=1}^{m} kE_{k-1}^{(r)}(x)E_{m-k}^{(s)}(x) + \sum_{k=0}^{m-1}(m-k)E_k^{(r)}(x)E_{m-k-1}^{(s)}(x) \\
&= \sum_{k=0}^{m-1}(k+1)E_k^{(r)}(x)E_{m-1-k}^{(s)}(x) + \sum_{k=0}^{m-1}(m-k)E_k^{(r)}(x)E_{m-1-k}^{(s)}(x) \\
&= (m+1)\sum_{k=0}^{m-1} E_k^{(r)}(x)E_{m-1-k}^{(s)}(x) \\
&= (m+1)\alpha_{m-1}(x).
\end{aligned} \tag{2.4}$$

From this, we have

$$\left(\frac{\alpha_{m+1}(x)}{m+2}\right)' = \alpha_m(x), \tag{2.5}$$

and

$$\int_0^1 \alpha_m(x)dx = \frac{1}{m+2}\left(\alpha_{m+1}(1) - \alpha_{m+1}(0)\right). \tag{2.6}$$

For $m \geq 1$, we put

$$\begin{aligned}
\Delta_m &= \alpha_m(1) - \alpha_m(0) \\
&= \sum_{k=0}^{m} \left(E_k^{(r)}(1)E_{m-k}^{(s)}(1) - E_k^{(r)}E_{m-k}^{(s)}\right) \\
&= \sum_{k=0}^{m} \left((2E_k^{(r-1)} - E_k^{(r)})(2E_{m-k}^{(s-1)} - E_{m-k}^{(s)}) - E_k^{(r)}E_{m-k}^{(s)}\right) \\
&= 2\sum_{k=0}^{m} \left(2E_k^{(r-1)}E_{m-k}^{(s-1)} - E_k^{(r)}E_{m-k}^{(s-1)} - E_k^{(r-1)}E_{m-k}^{(s)}\right).
\end{aligned} \tag{2.7}$$

We now see that

$$\alpha_m(0) = \alpha_m(1) \Longleftrightarrow \Delta_m = 0, \tag{2.8}$$

and

$$\int_0^1 \alpha_m(x)dx = \frac{1}{m+2}\Delta_{m+1}. \tag{2.9}$$

Next, we want to determine the Fourier coefficients $A_n^{(m)}$.

Case 1 : $n \neq 0$.

$$\begin{aligned} A_n^{(m)} &= \int_0^1 \alpha_m(x)e^{-2\pi inx}dx \\ &= -\frac{1}{2\pi in}[\alpha_m(x)e^{-2\pi inx}]_0^1 + \frac{1}{2\pi in}\int_0^1 \alpha'_m(x)e^{-2\pi inx}dx \\ &= -\frac{1}{2\pi in}\left(\alpha_m(1) - \alpha_m(0)\right) + \frac{m+1}{2\pi in}\int_0^1 \alpha_{m-1}(x)e^{-2\pi inx}dx \\ &= \frac{m+1}{2\pi in}A_n^{(m-1)} - \frac{1}{2\pi in}\Delta_m, \end{aligned} \tag{2.10}$$

from which by induction on m, we can easily derive that

$$A_n^{(m)} = -\frac{1}{m+2}\sum_{j=1}^{m}\frac{(m+2)_j}{(2\pi in)^j}\Delta_{m-j+1}. \tag{2.11}$$

Case 2 : $n = 0$.

$$A_0^{(m)} = \int_0^1 \alpha_m(x)dx = \frac{1}{m+2}\Delta_{m+1}. \tag{2.12}$$

$\alpha_m(< x >), (m \geq 1)$ is piecewise C^∞. In addition, $\alpha_m(< x >)$ is continuous for those positive integers m with $\Delta_m = 0$, and discontinuous with jump discontinuities at integers for those positive integers with $\Delta_m \neq 0$.

Assume first that $\Delta_m = 0$, for a positive integer m. Then $\alpha_m(0) = \alpha_m(1)$. Hence $\alpha_m(< x >)$ is piecewise C^∞, and continuous. Thus the Fourier series of $\alpha_m(< x >)$ converges uniformly to $\alpha_m(< x >)$, and

$$\begin{aligned} \alpha_m(< x >) &= \frac{1}{m+2}\Delta_{m+1} + \sum_{n=-\infty, n\neq 0}^{\infty}\left(-\frac{1}{m+2}\sum_{j=1}^{m}\frac{(m+2)_j}{(2\pi in)^j}\Delta_{m-j+1}\right)e^{2\pi inx} \\ &= \frac{1}{m+2}\Delta_{m+1} + \frac{1}{m+2}\sum_{j=1}^{m}\binom{m+2}{j}\Delta_{m-j+1}\left(-j!\sum_{n=-\infty,n\neq 0}^{\infty}\frac{e^{2\pi inx}}{(2\pi in)^j}\right) \\ &= \frac{1}{m+2}\Delta_{m+1} + \frac{1}{m+2}\sum_{j=2}^{m}\binom{m+2}{j}\Delta_{m-j+1}B_j(< x >) \\ &+ \Delta_m \times \begin{cases} B_1(< x >), & \text{for } x \notin \mathbb{Z}, \\ 0, & \text{for } x \in \mathbb{Z}. \end{cases} \end{aligned} \tag{2.13}$$

We are now going to state our first result.

Theorem 2.1. *For each positive integer l, we let*

$$\Delta_l = 2\sum_{k=0}^{l}\left(2E_k^{(r-1)}E_{l-k}^{(s-1)} - E_k^{(r)}E_{l-k}^{(s-1)} - E_k^{(r-1)}E_{l-k}^{(s)}\right).$$

Assume that $\Delta_m = 0$, for a positive integer m. Then we have the following.
(a) $\sum_{k=0}^{m} E_k^{(r)}(<x>)E_{m-k}^{(s)}(<x>)$ has the Fourier series expansion

$$\begin{aligned}&\sum_{k=0}^{m} E_k^{(r)}(<x>)E_{m-k}^{(s)}(<x>)\\ =&\frac{1}{m+2}\Delta_{m+1} + \sum_{n=-\infty,n\neq 0}^{\infty}\left(-\frac{1}{m+2}\sum_{j=1}^{m}\frac{(m+2)_j}{(2\pi in)^j}\Delta_{m-j+1}\right)e^{2\pi inx},\end{aligned} \tag{2.14}$$

for all $x \in \mathbb{R}$, where the convergence is uniform.

$$(b)\sum_{k=0}^{m} E_k^{(r)}(<x>)E_{m-k}^{(s)}(<x>) = \frac{1}{m+2}\sum_{j=0,j\neq 1}^{m}\binom{m+2}{j}\Delta_{m-j+1}B_j(<x>), \tag{2.15}$$

for all x in $\mathbb{R}$.

Assume next that $\Delta_m \neq 0$, for a positive integer m. Then $\alpha_m(0) \neq \alpha_m(1)$. Hence $\alpha_m(<x>)$ is piecewise C^∞, and discontinuous with jump discontinuities at integers.
Then the Fourier series of $\alpha_m(<x>)$ converges pointwise to $\alpha_m(<x>)$, for $x \notin \mathbb{Z}$,
and converges to

$$\frac{1}{2}\left(\alpha_m(0)+\alpha_m(1)\right) = \alpha_m(0) + \frac{1}{2}\Delta_m, \tag{2.16}$$

for $x \in \mathbb{Z}$.

Now, we are going to state our second result.

Theorem 2.2. *For each positive integer l, we let*

$$\Delta_l = 2\sum_{k=0}^{l}\left(2E_k^{(r-1)}E_{l-k}^{(s-1)} - E_k^{(r)}E_{l-k}^{(s-1)} - E_k^{(r-1)}E_{l-k}^{(s)}\right).$$

Assume that $\Delta_m \neq 0$, for a positive integer m. Then we have the following.

$$\begin{aligned}&(a)\frac{1}{m+2}\Delta_{m+1} + \sum_{n=-\infty,n\neq 0}^{\infty}\left(-\frac{1}{m+2}\sum_{j=1}^{m}\frac{(m+2)_j}{(2\pi in)^j}\Delta_{m-j+1}\right)e^{2\pi inx}\\ &=\begin{cases}\sum_{k=0}^{m} E_k^{(r)}(<x>)E_{m-k}^{(s)}(<x>), & \text{for } x \notin \mathbb{Z},\\ \sum_{k=0}^{m} E_k^{(r)}E_{m-k}^{(s)} + \frac{1}{2}\Delta_m, & \text{for } x \in \mathbb{Z}.\end{cases}\end{aligned} \tag{2.17}$$

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 27, NO.2, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC

$$(b)\frac{1}{m+2}\sum_{j=0}^{m}\binom{m+2}{j}\Delta_{m-j+1}B_j(<x>)=\sum_{k=0}^{m}E_k^{(r)}(<x>)E_{m-k}^{(s)}(<x>),\ for\ x\notin\mathbb{Z}; \tag{2.18}$$

$$\frac{1}{m+2}\sum_{j=0,j\neq 1}^{m}\binom{m+2}{j}\Delta_{m-j+1}B_j(<x>)=\sum_{k=0}^{m}E_k^{(r)}E_{m-k}^{(s)}+\frac{1}{2}\Delta_m,\ for\ x\in\mathbb{Z}. \tag{2.19}$$

3. The function $\beta_m(<x>)$

Let $\beta_m(x)=\sum_{k=0}^{m}\frac{1}{k!(m-k)!}E_k^{(r)}(x)E_{m-k}^{(s)}(x),\ (m\geq 1)$. Then we will consider the function

$$\beta_m(<x>)=\sum_{k=0}^{m}\frac{1}{k!(m-k)!}E_k^{(r)}(<x>)E_{m-k}^{(s)}(<x>),\ (m\geq 1),$$

defined on $\mathbb{R}$, which is periodic with period 1.

The Fourier series of $\beta_m(<x>)$ is

$$\sum_{n=-\infty}^{\infty}B_n^{(m)}e^{2\pi inx}, \tag{3.1}$$

where

$$B_n^{(m)}=\int_0^1\beta_m(<x>)e^{-2\pi inx}dx=\int_0^1\beta_m(x)e^{-2\pi inx}dx. \tag{3.2}$$

Before continuing further, we need to note the following.

$$\begin{aligned}\beta_m'(x)&=\sum_{k=0}^{m}\left\{\frac{k}{k!(m-k)!}E_{k-1}^{(r)}(x)E_{m-k}^{(s)}(x)+\frac{(m-k)}{k!(m-k)!}E_k^{(r)}(x)E_{m-k-1}^{(s)}(x)\right\}\\&=\sum_{k=1}^{m}\frac{1}{(k-1)!(m-k)!}E_{k-1}^{(r)}(x)E_{m-k}^{(s)}(x)\\&+\sum_{k=0}^{m-1}\frac{1}{k!(m-k-1)!}E_k^{(r)}(x)E_{m-k-1}^{(s)}(x)\\&=\sum_{k=0}^{m-1}\frac{1}{k!(m-1-k)!}E_k^{(r)}(x)E_{m-1-k}^{(s)}(x)\\&+\sum_{k=0}^{m-1}\frac{1}{k!(m-1-k)!}E_k^{(r)}(x)E_{m-1-k}^{(s)}(x)\\&=2\beta_{m-1}(x).\end{aligned} \tag{3.3}$$

From this, we have

$$\left(\frac{\beta_{m+1}(x)}{2}\right)'=\beta_m(x), \tag{3.4}$$

and

$$\int_0^1 \beta_m(x)dx = \frac{1}{2}\Big(\beta_{m+1}(1) - \beta_{m+1}(0)\Big). \tag{3.5}$$

For $m \geq 1$, we set

$$\begin{aligned}\Omega_m &= \beta_m(1) - \beta_m(0)\\ &= \sum_{k=0}^{m} \frac{1}{k!(m-k)!}\big(E_k^{(r)}(1)E_{m-k}^{(s)}(1) - E_k^{(r)}E_{m-k}^{(s)}\big)\\ &= \sum_{k=0}^{m} \frac{1}{k!(m-k)!}\Big((2E_k^{(r-1)} - E_k^{(r)})(2E_{m-k}^{(s-1)} - E_{m-k}^{(s)}) - E_k^{(r)}E_{m-k}^{(s)}\Big)\\ &= \sum_{k=0}^{m} \frac{2}{k!(m-k)!}\big(2E_k^{(r-1)}E_{m-k}^{(s-1)} - E_k^{(r)}E_{m-k}^{(s-1)} - E_k^{(r-1)}E_{m-k}^{(s)}\big).\end{aligned} \tag{3.6}$$

Now, it is immediate to see that

$$\beta_m(0) = \beta_m(1) \Longleftrightarrow \Omega_m = 0, \tag{3.7}$$

and

$$\int_0^1 \beta_m(x)dx = \frac{1}{2}\Omega_{m+1}. \tag{3.8}$$

We now would like to determine the Fourier coefficients $B_n^{(m)}$.

Case 1:$n \neq 0$

$$\begin{aligned}B_n^{(m)} &= \int_0^1 \beta_m(x)e^{-2\pi i n x}dx\\ &= -\frac{1}{2\pi i n}\Big[\beta_m(x)e^{-2\pi i n x}\Big]_0^1 + \frac{1}{2\pi i n}\int_0^1 \beta_m'(x)e^{-2\pi i n x}dx\\ &= -\frac{1}{2\pi i n}\Big(\beta_m(1) - \beta_m(0)\Big) + \frac{2}{2\pi i n}\int_0^1 \beta_{m-1}(x)e^{-2\pi i n x}dx\\ &= \frac{2}{2\pi i n}B_n^{(m-1)} - \frac{1}{2\pi i n}\Omega_m,\end{aligned} \tag{3.9}$$

from which by induction on m gives

$$B_n^{(m)} = -\sum_{j=1}^{m} \frac{2^{j-1}}{(2\pi i n)^j}\Omega_{m-j+1}. \tag{3.10}$$

Case 2: $n = 0$

$$B_0^{(m)} = \int_0^1 \beta_m(x)dx = \frac{1}{2}\Omega_{m+1}. \tag{3.11}$$

$\beta_m(< x >)$, $(m \geq 1)$ is piecewise C^∞. Further, $\beta_m(< x >)$ is continuous for those positive integers m with $\Omega_m = 0$, and discontinuous with jump discontinuities at integers for those positive integers m with $\Omega_m \neq 0$.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 27, NO.2, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC

Assume first that $\Omega_m = 0$, for a positive integer m. Then $\beta_m(0) = \beta_m(1)$. Hence $\beta_m(< x >)$ is piecewise C^∞, and continuous. Thus the Fourier series of $\beta_m(< x >)$ converges uniformly to $\beta_m(< x >)$, and

$$\begin{aligned}
&\beta_m(< x >)\\
&= \frac{1}{2}\Omega_{m+1} + \sum_{n=-\infty, n\neq 0}^{\infty} \Big(- \sum_{j=1}^{m} \frac{2^{j-1}}{(2\pi i n)^j} \Omega_{m-j+1} \Big) e^{2\pi i n x}\\
&= \frac{1}{2}\Omega_{m+1} + \sum_{j=1}^{m} \frac{2^{j-1}}{j!} \Omega_{m-j+1} \Big(-j! \sum_{n=-\infty, n\neq 0}^{\infty} \frac{e^{2\pi n x}}{(2\pi i n)^j} \Big)\\
&= \frac{1}{2}\Omega_{m+1} + \sum_{j=2}^{m} \frac{2^{j-1}}{j!} \Omega_{m-j+1} B_j(< x >)\\
&+ \Omega_m \times \begin{cases} B_1(< x >), & \text{for } x \notin \mathbb{Z}, \\ 0, & \text{for } x \in \mathbb{Z}. \end{cases}
\end{aligned} \tag{3.12}$$

Now, we are going to state our first result.

Theorem 3.1. *For each positive integer l, we let*

$$\Omega_l = \sum_{k=0}^{l} \frac{2}{k!(l-k)!} \big(2E_k^{(r-1)} E_{l-k}^{(s-1)} - E_k^{(r)} E_{l-k}^{(s-1)} - E_k^{(r-1)} E_{l-k}^{(s)} \big). \tag{3.13}$$

Assume that $\Omega_m = 0$, for a positive integer m. Then we have the following.

(a) $\sum_{k=0}^{m} \frac{1}{k!(m-k)!} E_k^{(r)}(< x >) E_{m-k}^{(s)}(< x >)$ *has the Fourier series expansion*

$$\begin{aligned}
&\sum_{k=0}^{m} \frac{1}{k!(m-k)!} E_k^{(r)}(< x >) E_{m-k}^{(s)}(< x >)\\
&= \frac{1}{2}\Omega_{m+1} + \sum_{n=-\infty, n\neq 0}^{\infty} \Big(- \sum_{j=1}^{m} \frac{2^{j-1}}{(2\pi i n)^j} \Omega_{m-j+1} \Big) e^{2\pi i n x},
\end{aligned} \tag{3.14}$$

for all $x \in \mathbb{R}$, where the convergence is uniform.

$$\begin{aligned}
(b) &\sum_{k=0}^{m} \frac{1}{k!(m-k)!} E_k^{(r)}(< x >) E_{m-k}^{(s)}(< x >)\\
&= \sum_{j=0, j\neq 1}^{m} \frac{2^{j-1}}{j!} \Omega_{m-j+1} B_j(< x >),
\end{aligned} \tag{3.15}$$

for all $x \in \mathbb{R}$.

Assume next that $\Omega_m \neq 0$, for a positive integer m. Then, $\beta_m(0) \neq \beta_m(1)$. Hence $\beta_m(< x >)$ is piecewise C^∞, and discontinuous with jump discontinuities at integers. Then the Fourier series of $\beta_m(< x >)$ converges pointwise to $\beta_m(< x >)$, for $x \notin \mathbb{Z}$, and converges to

$$\frac{1}{2}(\beta_m(0) + \beta_m(1)) = \beta_m(0) + \frac{1}{2}\Omega_m, \tag{3.16}$$

for $x \in \mathbb{Z}$.

Next, we are going to state our second result.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 27, NO.2, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC

Theorem 3.2. *For each positive integer l, we let*

$$\Omega_l = \sum_{k=0}^{l} \frac{2}{k!(l-k)!}\left(2E_k^{(r-1)}E_{l-k}^{(s-1)} - E_k^{(r)}E_{l-k}^{(s-1)} - E_k^{(r-1)}E_{l-k}^{(s)}\right). \tag{3.17}$$

Assume that $\Omega_m \neq 0$, for a positive integer m. Then we have the following.

$$\begin{aligned}(a)\ &\frac{1}{2}\Omega_{m+1} + \sum_{n=-\infty, n\neq 0}^{\infty}\left(-\sum_{j=1}^{m}\frac{2^{j-1}}{(2\pi i n)^j}\Omega_{m-j+1}\right)e^{2\pi i n x}\\ &= \begin{cases}\sum_{k=0}^{m}\frac{1}{k!(m-k)!}E_k^{(r)}(<x>)E_{m-k}^{(s)}(<x>), & \text{for } x \notin \mathbb{Z},\\ \sum_{k=0}^{m}\frac{1}{k!(m-k)!}E_k^{(r)}E_{m-k}^{(s)} + \frac{1}{2}\Omega_m, & \text{for } x \in \mathbb{Z}.\end{cases}\end{aligned} \tag{3.18}$$

$$\begin{aligned}(b)\ &\sum_{j=0}^{m}\frac{2^{j-1}}{j!}\Omega_{m-j+1}B_j(<x>)\\ &= \sum_{k=0}^{m}\frac{1}{k!(m-k)!}E_k^{(r)}(<x>)E_{m-k}^{(s)}(<x>), \quad \text{for } x \notin \mathbb{Z};\\ &\sum_{j=0, j\neq 1}^{m}\frac{2^{j-1}}{j!}\Omega_{m-j+1}B_j(<x>)\\ &= \sum_{k=0}^{m}\frac{1}{k!(m-k)!}E_k^{(r)}E_{m-k}^{(s)} + \frac{1}{2}\Omega_m, \quad \text{for } x \in \mathbb{Z}.\end{aligned} \tag{3.19}$$

4. The function $\gamma_m(<x>)$

Let $\gamma_m(x) = \sum_{k=1}^{m-1}\frac{1}{k(m-k)}E_k^{(r)}(x)E_{m-k}^{(s)}(x)$, $(m \geq 2)$. Then we will consider the function

$$\gamma_m(<x>) = \sum_{k=1}^{m-1}\frac{1}{k(m-k)}E_k^{(r)}(<x>)E_{m-k}^{(s)}(<x>),$$

defined on $\mathbb{R}$, which is periodic with period 1.

The Fourier series of $\gamma_m(<x>)$ is

$$\sum_{n=-\infty}^{\infty} C_n^{(m)} e^{2\pi i n x}, \tag{4.1}$$

where

$$C_n^{(m)} = \int_0^1 \gamma_m(<x>)e^{-2\pi i n x}dx = \int_0^1 \gamma_m(x)e^{-2\pi i n x}dx. \tag{4.2}$$

To proceed further, we need to observe the following.

$$\begin{aligned}\gamma_m'(x) &= \sum_{k=1}^{m-1} \frac{1}{m-k} E_{k-1}^{(r)}(x)E_{m-k}^{(s)}(x) + \sum_{k=1}^{m-1} \frac{1}{k} E_k^{(r)}(x)E_{m-k-1}^{(s)}(x) \\ &= \sum_{k=0}^{m-2} \frac{1}{m-1-k} E_k^{(r)}(x)E_{m-1-k}^{(s)}(x) + \sum_{k=1}^{m-1} \frac{1}{k} E_k^{(r)}(x)E_{m-1-k}^{(s)}(x) \\ &= (m-1)\sum_{k=1}^{m-2} \frac{1}{k(m-1-k)} E_k^{(r)}(x)E_{m-1-k}^{(s)}(x) + \frac{1}{m-1}E_{m-1}^{(s)}(x) + \frac{1}{m-1}E_{m-1}^{(r)}(x) \\ &= (m-1)\gamma_{m-1}(x) + \frac{1}{m-1}E_{m-1}^{(s)}(x) + \frac{1}{m-1}E_{m-1}^{(r)}(x).\end{aligned} \tag{4.3}$$

From this, we easily see that

$$\left(\frac{1}{m}\big(\gamma_{m+1}(x) - \frac{1}{m(m+1)}E_{m+1}^{(r)}(x) - \frac{1}{m(m+1)}E_{m+1}^{(s)}(x)\big)\right)' = \gamma_m(x), \tag{4.4}$$

and

$$\begin{aligned}&\int_0^1 \gamma_m(x)dx \\ &= \frac{1}{m}\Big[\gamma_{m+1}(x) - \frac{1}{m(m+1)}E_{m+1}^{(r)}(x) - \frac{1}{m(m+1)}E_{m+1}^{(s)}(x)\Big]_0^1 \\ &= \frac{1}{m}\Big(\gamma_{m+1}(1) - \gamma_{m+1}(0) - \frac{1}{m(m+1)}(E_{m+1}^{(r)}(1) - E_{m+1}^{(r)}(0)) \\ &- \frac{1}{m(m+1)}(E_{m+1}^{(s)}(1) - E_{m+1}^{(s)}(0))\Big) \\ &= \frac{1}{m}\Big(\gamma_{m+1}(1) - \gamma_{m+1}(0) - \frac{2}{m(m+1)}(E_{m+1}^{(r-1)} - E_{m+1}^{(r)}) \\ &- \frac{2}{m(m+1)}(E_{m+1}^{(s-1)} - E_{m+1}^{(s)})\Big).\end{aligned} \tag{4.5}$$

Let $\Lambda_1 = 0$, and for $m \geq 2$, we let

$$\begin{aligned}\Lambda_m &= \gamma_m(1) - \gamma_m(0) \\ &= \sum_{k=1}^{m-1} \frac{1}{k(m-k)}\Big(E_k^{(r)}(1)E_{m-k}^{(s)}(1) - E_k^{(r)}E_{m-k}^{(s)}\Big) \\ &= \sum_{k=1}^{m-1} \frac{1}{k(m-k)}\Big((2E_k^{(r-1)} - E_k^{(r)})(2E_{m-k}^{(s-1)} - E_{m-k}^{(s)}) - E_k^{(r)}E_{m-k}^{(s)}\Big) \\ &= \sum_{k=1}^{m-1} \frac{2}{k(m-k)}\big(2E_k^{(r-1)}E_{m-k}^{(s-1)} - E_k^{(r)}E_{m-k}^{(s-1)} - E_k^{(r-1)}E_{m-k}^{(s)}\big).\end{aligned} \tag{4.6}$$

Then we have

$$\gamma_m(0) = \gamma_m(1) \iff \Lambda_m = 0, \ (m \geq 2), \tag{4.7}$$

and

$$\int_0^1 \gamma_m(x)dx = \frac{1}{m}\left(\Lambda_{m+1} - \frac{2}{m(m+1)}\Big(E_{m+1}^{(r-1)} - E_{m+1}^{(r)}\Big) - \frac{2}{m(m+1)}\Big(E_{m+1}^{(s-1)} - E_{m+1}^{(s)}\Big)\right). \tag{4.8}$$

We now want to determine the Fourier coefficients $C_n^{(m)}$.
Case 1: $n \neq 0$

$$\begin{aligned}
C_n^{(m)} &= \int_0^1 \gamma_m(x)e^{-2\pi inx}dx \\
&= -\frac{1}{2\pi in}\Big[\gamma_m(x)e^{-2\pi inx}\Big]_0^1 + \frac{1}{2\pi in}\int_0^1 \gamma_m'(x)e^{-2\pi inx}dx \\
&= -\frac{1}{2\pi in}\Big(\gamma_m(1) - \gamma_m(0)\Big) \\
&+ \frac{1}{2\pi in}\int_0^1 \{(m-1)\gamma_{m-1}(x) + \frac{1}{m-1}E_{m-1}^{(r)}(x) + \frac{1}{m-1}E_{m-1}^{(s)}(x)\}e^{-2\pi inx}dx \\
&= \frac{m-1}{2\pi in}C_n^{(m-1)} - \frac{1}{2\pi in}\Lambda_m + \frac{1}{2\pi in(m-1)}\int_0^1 E_{m-1}^{(r)}(x)e^{-2\pi inx}dx \\
&+ \frac{1}{2\pi in(m-1)}\int_0^1 E_{m-1}^{(s)}(x)e^{-2\pi inx}dx \\
&= \frac{m-1}{2\pi in}C_n^{(m-1)} - \frac{1}{2\pi in}\Lambda_m - \frac{1}{2\pi in(m-1)}\Big(\Phi_m^{(r)} + \Phi_m^{(s)}\Big),
\end{aligned} \tag{4.9}$$

where

$$\begin{aligned}
\Phi_m^{(r)} &= \sum_{k=1}^{m-1}\frac{2(m-1)_{k-1}}{(2\pi in)^k}\Big(E_{m-k}^{(r-1)} - E_{m-k}^{(r)}\Big), \\
&\int_0^1 E_l^{(r)}(x)e^{-2\pi inx}dx \\
&= \begin{cases} -\sum_{k=1}^{l}\frac{2(l)_{k-1}}{(2\pi in)^k}\Big(E_{l-k+1}^{(r-1)} - E_{l-k+1}^{(r)}\Big), & \text{for } n \neq 0, \\ \frac{2}{l+1}\Big(E_{l+1}^{(r-1)} - E_{l+1}^{(r)}\Big), & \text{for } n = 0. \end{cases}
\end{aligned} \tag{4.10}$$

Thus we have shown that

$$C_n^{(m)} = \frac{m-1}{2\pi in}C_n^{(m-1)} - \frac{1}{2\pi in}\Lambda_m - \frac{1}{2\pi in(m-1)}\Big(\Phi_m^{(r)} + \Phi_m^{(s)}\Big). \tag{4.11}$$

An easy induction on m now gives

$$C_n^{(m)} = -\sum_{j=1}^{m-1}\frac{(m-1)_{j-1}}{(2\pi in)^j}\Lambda_{m-j+1} - \sum_{j=1}^{m-1}\frac{(m-1)_{j-1}}{(2\pi in)^j(m-j)}(\Phi_{m-j+1}^{(r)} + \Phi_{m-j+1}^{(s)}). \tag{4.12}$$

To find a more explicit expression for $C_n^{(m)}$, we need to observe the following.

$$\begin{aligned}
&\sum_{j=1}^{m-1} \frac{(m-1)_{j-1}}{(2\pi in)^j (m-j)} \Phi_{m-j+1}^{(r)} \\
=&\sum_{j=1}^{m-1} \frac{(m-1)_{j-1}}{(2\pi in)^j (m-j)} \sum_{k=1}^{m-j} \frac{2(m-j)_{k-1}}{(2\pi in)^k} (E_{m-j-k+1}^{(r-1)} - E_{m-j-k+1}^{(r)}) \\
=&2\sum_{j=1}^{m-1} \frac{1}{m-j} \sum_{k=1}^{m-j} \frac{(m-1)_{j+k-2}}{(2\pi in)^{j+k}} (E_{m-j-k+1}^{(r-1)} - E_{m-j-k+1}^{(r)}) \\
=&2\sum_{j=1}^{m-1} \frac{1}{m-j} \sum_{k=j+1}^{m} \frac{(m-1)_{k-2}}{(2\pi in)^{k}} (E_{m-k+1}^{(r-1)} - E_{m-k+1}^{(r)}) \\
=&2\sum_{k=2}^{m} \frac{(m-1)_{k-2}}{(2\pi in)^{k}} (E_{m-k+1}^{(r-1)} - E_{m-k+1}^{(r)}) \sum_{j=1}^{k-1} \frac{1}{m-j} \\
=&2\sum_{k=1}^{m} \frac{(m-1)_{k-2}}{(2\pi in)^{k}} (E_{m-k+1}^{(r-1)} - E_{m-k+1}^{(r)}) (H_{m-1} - H_{m-k}) \\
=&\frac{2}{m}\sum_{k=1}^{m} \frac{(m)_{k}}{(2\pi in)^{k}} \frac{E_{m-k+1}^{(r-1)} - E_{m-k+1}^{(r)}}{m-k+1} (H_{m-1} - H_{m-k}) .
\end{aligned} \tag{4.13}$$

Recalling that $\Lambda_1 = 0$, we get the following expression of $C_n^{(m)}$: for $n \neq 0$,

$$\begin{aligned}
C_n^{(m)} =& -\frac{1}{m} \sum_{k=1}^{m} \frac{(m)_k}{(2\pi in)^k} \Big(\Lambda_{m-k+1} + \frac{2(H_{m-1} - H_{m-k})}{m-k+1} \\
&\times (E_{m-k+1}^{(r-1)} + E_{m-k+1}^{(s-1)} - E_{m-k+1}^{(r)} - E_{m-k+1}^{(s)}) \Big).
\end{aligned} \tag{4.14}$$

Case 2: $n = 0$

$$C_0^{(m)} = \int_0^1 \gamma_m(x)dx = \frac{1}{m} \left(\Lambda_{m+1} - \frac{2}{m(m+1)} (E_{m+1}^{(r-1)} + E_{m+1}^{(s-1)} - E_{m+1}^{(r)} - E_{m+1}^{(s)}) \right). \tag{4.15}$$

$\gamma_m(< x >)$, $(m \geq 2)$ is piecewise C^∞. Furthermore, $\gamma_m(< x >)$ is continuous for those integers $m \geq 2$ with $\Lambda_m = 0$, and discontinuous with jump discontinuities at integers for those integer $m \geq 2$ with $\Lambda_m \neq 0$.

Assume first that $\Lambda_m = 0$, for an integer $m \geq 2$. Then $\gamma_m(0) = \gamma_m(1)$. Hence $\gamma_m(< x >)$ is piecewise C^∞, and continuous. Thus the Fourier series of $\gamma_m(< x >)$

converges uniformly to $\gamma_m(<x>)$, and

$$\begin{aligned}
&\gamma_m(<x>)\\
&=\frac{1}{m}\left(\Lambda_{m+1}-\frac{2}{m(m+1)}(E_{m+1}^{(r-1)}+E_{m+1}^{(s-1)}-E_{m+1}^{(r)}-E_{m+1}^{(s)})\right)\\
&-\frac{1}{m}\sum_{n=-\infty,n\neq 0}^{\infty}\Big\{\sum_{k=1}^{m}\frac{(m)_k}{(2\pi in)^k}\Big(\Lambda_{m-k+1}+\frac{2(H_{m-1}-H_{m-k})}{m-k+1}\\
&\times(E_{m-k+1}^{(r-1)}+E_{m-k+1}^{(s-1)}-E_{m-k+1}^{(r)}-E_{m-k+1}^{(s)})\Big)\Big\}e^{2\pi inx}\\
&=\frac{1}{m}\left(\Lambda_{m+1}-\frac{2}{m(m+1)}(E_{m+1}^{(r-1)}+E_{m+1}^{(s-1)}-E_{m+1}^{(r)}-E_{m+1}^{(s)})\right)\\
&+\frac{1}{m}\sum_{k=1}^{m}\binom{m}{k}\Big\{\Lambda_{m-k+1}+\frac{2(H_{m-1}-H_{m-k})}{m-k+1}\\
&\times(E_{m-k+1}^{(r-1)}+E_{m-k+1}^{(s-1)}-E_{m-k+1}^{(r)}-E_{m-k+1}^{(s)})\Big\}\Big(-k!\sum_{n=-\infty,n\neq 0}^{\infty}\frac{e^{2\pi inx}}{(2\pi in)^k}\Big)\\
&=\frac{1}{m}\left(\Lambda_{m+1}-\frac{2}{m(m+1)}(E_{m+1}^{(r-1)}+E_{m+1}^{(s-1)}-E_{m+1}^{(r)}-E_{m+1}^{(s)})\right)\\
&+\frac{1}{m}\sum_{k=2}^{m}\binom{m}{k}\Big\{\Lambda_{m-k+1}+\frac{2(H_{m-1}-H_{m-k})}{m-k+1}\\
&\times(E_{m-k+1}^{(r-1)}+E_{m-k+1}^{(s-1)}-E_{m-k+1}^{(r)}-E_{m-k+1}^{(s)})\Big\}B_k(<x>)\\
&+\Lambda_m\times\begin{cases}B_1(<x>), & \text{for } x\notin\mathbb{Z},\\ 0, & \text{for } x\in\mathbb{Z}\end{cases}\\
&=\frac{1}{m}\sum_{k=0,k\neq 1}^{m}\binom{m}{k}\Big\{\Lambda_{m-k+1}+\frac{2(H_{m-1}-H_{m-k})}{m-k+1}\\
&\times(E_{m-k+1}^{(r-1)}+E_{m-k+1}^{(s-1)}-E_{m-k+1}^{(r)}-E_{m-k+1}^{(s)})\Big\}B_k(<x>)\\
&+\Lambda_m\times\begin{cases}B_1(<x>), & \text{for } x\notin\mathbb{Z},\\ 0, & \text{for } x\in\mathbb{Z}.\end{cases}
\end{aligned}\tag{4.16}$$

Now, we can state our first result.

Theorem 4.1. *For each integer $l\geq 2$, we let*

$$\Lambda_l=\sum_{k=1}^{l-1}\frac{2}{k(l-k)}\left(2E_k^{(r-1)}E_{l-k}^{(s-1)}-E_k^{(r)}E_{l-k}^{(s-1)}-E_k^{(r-1)}E_{l-k}^{(s)}\right),\tag{4.17}$$

with $\Lambda_1=0$. Assume that $\Lambda_m=0$, for an integer $m\geq 2$. Then we have the following.

(a) $\sum_{k=1}^{m-1}\frac{1}{k(m-k)}E_k^{(r)}(<x>)E_{m-k}^{(s)}(<x>)$ has the Fourier series expansion

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 27, NO.2, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC

$$\begin{aligned}
&\sum_{k=1}^{m-1} \frac{1}{k(m-k)} E_k^{(r)}(<x>) E_{m-k}^{(s)}(<x>) \\
&= \frac{1}{m}\left(\Lambda_{m+1} - \frac{2}{m(m+1)}(E_{m+1}^{(r-1)} + E_{m+1}^{(s-1)} - E_{m+1}^{(r)} - E_{m+1}^{(s)})\right) \\
&- \frac{1}{m} \sum_{n=-\infty, n\neq 0}^{\infty} \Big\{ \sum_{k=1}^{m} \frac{(m)_k}{(2\pi i n)^k} \big(\Lambda_{m-k+1} + \frac{2(H_{m-1} - H_{m-k})}{m-k+1} \\
&\times (E_{m-k+1}^{(r-1)} + E_{m-k+1}^{(s-1)} - E_{m-k+1}^{(r)} - E_{m-k+1}^{(s)})\big)\Big\} e^{2\pi i n x},
\end{aligned} \tag{4.18}$$

for all $x \in \mathbb{R}$, where the convergence is uniform.

$$\begin{aligned}
(b)\ &\sum_{k=1}^{m-1} \frac{1}{k(m-k)} E_k^{(r)}(<x>) E_{m-k}^{(s)}(<x>) \\
&= \frac{1}{m} \sum_{k=0, k\neq 1}^{m} \binom{m}{k} \Big\{ \Lambda_{m-k+1} + \frac{2(H_{m-1} - H_{m-k})}{m-k+1} \\
&\times (E_{m-k+1}^{(r-1)} + E_{m-k+1}^{(s-1)} - E_{m-k+1}^{(r)} - E_{m-k+1}^{(s)})\Big\} B_k(<x>)
\end{aligned} \tag{4.19}$$

for all $x \in \mathbb{R}$.

Assume next that $\Lambda_m \neq 0$, for an integers $m \geq 2$. Then $\gamma_m(0) \neq \gamma_m(1)$. Hence $\gamma_m(<x>)$ is piecewise C^∞, and discontinuous with jump discontinuities at integers. Thus the Fourier series of $\gamma_m(<x>)$ converges pointwise to $\gamma_m(<x>)$, for $x \notin \mathbb{Z}$, and converges to

$$\frac{1}{2}(\gamma_m(0) + \gamma_m(1)) = \gamma_m(0) + \frac{1}{2}\Lambda_m, \tag{4.20}$$

for $x \in \mathbb{Z}$.

We can now state our second result.

Theorem 4.2. *For each integer $l \geq 2$, let*

$$\Lambda_l = \sum_{k=1}^{l-1} \frac{2}{k(l-k)} (2E_k^{(r-1)} E_{l-k}^{(s-1)} - E_k^{(r)} E_{l-k}^{(s-1)} - E_k^{(r-1)} E_{l-k}^{(s)}), \tag{4.21}$$

with $\Lambda_1 = 0$. Assume that $\Lambda_m \neq 0$, for an integer $m \geq 2$.
Then we have the following.

$$\begin{aligned}
(a)\ &\frac{1}{m}\left(\Lambda_{m+1} - \frac{2}{m(m+1)}(E_{m+1}^{(r-1)} + E_{m+1}^{(s-1)} - E_{m+1}^{(r)} - E_{m+1}^{(s)})\right) \\
&- \frac{1}{m} \sum_{n=-\infty, n\neq 0}^{\infty} \Big\{ \sum_{k=1}^{m} \frac{(m)_k}{(2\pi i n)^k} \big(\Lambda_{m-k+1} + \frac{2(H_{m-1} - H_{m-k})}{m-k+1} \\
&\times (E_{m-k+1}^{(r-1)} + E_{m-k+1}^{(s-1)} - E_{m-k+1}^{(r)} - E_{m-k+1}^{(s)})\big)\Big\} e^{2\pi i n x} \\
&= \begin{cases} \sum_{k=1}^{m-1} \frac{1}{k(m-k)} E_k^{(r)}(<x>) E_{m-k}^{(s)}(<x>), & \text{for } x \notin \mathbb{Z}, \\ \sum_{k=1}^{m-1} \frac{1}{k(m-k)} E_k^{(r)} E_{m-k}^{(s)} + \frac{1}{2}\Lambda_m, & \text{for } x \in \mathbb{Z}. \end{cases}
\end{aligned} \tag{4.22}$$

$$
\begin{aligned}
(b)\ &\frac{1}{m}\sum_{k=0}^{m}\binom{m}{k}\Big\{\Lambda_{m-k+1}+\frac{2(H_{m-1}-H_{m-k})}{m-k+1}\\
&\times(E_{m-k+1}^{(r-1)}+E_{m-k+1}^{(s-1)}-E_{m-k+1}^{(r)}-E_{m-k+1}^{(s)})\Big\}B_k(<x>)\\
&=\sum_{k=1}^{m-1}\frac{1}{k(m-k)}E_k^{(r)}(<x>)E_{m-k}^{(s)}(<x>),\ for\ x\notin\mathbb{Z};\\
&\frac{1}{m}\sum_{k=0,k\neq 1}^{m}\binom{m}{k}\Big\{\Lambda_{m-k+1}+\frac{2(H_{m-1}-H_{m-k})}{m-k+1}\\
&\times(E_{m-k+1}^{(r-1)}+E_{m-k+1}^{(s-1)}-E_{m-k+1}^{(r)}-E_{m-k+1}^{(s)})\Big\}B_k(<x>)\\
&=\sum_{k=1}^{m-1}\frac{1}{k(m-k)}E_k^{(r)}E_{m-k}^{(s)}+\frac{1}{2}\Lambda_m,\ for\ x\in\mathbb{Z}.
\end{aligned}
\tag{4.23}
$$

References

[1] M. Abramowitz, IA. Stegun, *Handbook of Mathematical Functions*, Dover, New York, **1970**.

[2] A. Bayad, T. Kim, *Identities involving values of Bernstein, q-Bernoulli, and q-Euler polynomials*, Russ. J. Math. Phys., **18**(2011), no. 2, 133-143.

[3] G. V. Dunne, C. Schubert, *Bernoulli number identities from quantum field theory and topological string theory,* Commun. Number Theory Phys., **7(2)**(2013), 225-249.

[4] C. Faber, R. Pandharipande, *Hodge integrals and Gromov-Witten theory,* Invent. Math. **139(1)**(2000), 173-199.

[5] I. M. Gessel, *On Miki's identities for Bernoulli numbers,* J. Number Theory, **110(1)**(2005), 75-82.

[6] G.-W. Jang, T. Kim, D.S. Kim, T. Mansour, *Fourier series of functions related to Bernoulli polynomials,* Adv. Stud. Contemp. Math., **27**(2017), no.1, 49-62.

[7] D.S. Kim, T. Kim, *Some identities of higher order Euler polynomials arising from Euler basis,* Integral Transforms Spec. Funct., **24(9)** (2013), 734-738.

[8] D.S. Kim, T. Kim, *Euler basis, identities, and their applications,* Int. J. Math. Math. Sci. 2012, Art. ID 343981.

[9] D.S. Kim, T. Kim, Y.H. Lee, *Some arithemetic properties of Bernoulli and Euler nembers,* Adv. Stud. Contemp. Math., **22**(2010), no.4, 467-480.

[10] T. Kim, *Euler numbers and polynomials associated with zeta functions*, Abstr. Appl. Anal., 2008, Art. ID 581582, 11pp.

[11] T. Kim, *Some identities for the Bernoulli, the Euler and Genocchi numbers and polynomials*, Adv. Stud. Contemp. Math., **20**(2015), no.1, 23-28.

[12] T. Kim, *On the Multiple q-Genocchi and Euler Numbers*, Russ. J. Math. Phys., **15**(2008), 481-486.

[13] T. Kim, D.S. Kim, D.Dolgy, and J.-W. Park, *Fourier series of sums of products of poly-Bernoulli functions and their applications,* J. Nonlinear Sci. Appl., **10**(2017), no.4, 2384-2401.

[14] T. Kim, D.S. Kim, D.Dolgy, and J.-W. Park, *Fourier series of sums of products of ordered Bell and poly-Bernoulli functions,* J. Inequal. Appl., **2017** Article ID 13660, 17pages,(2017).

[15] T. Kim, D.S. Kim, G.-W. Jang, and J. Kwon, *Fourier series of sums of products of Genocchi functions and their applications,* J. Nonlinear Sci. Appl., **10**(2017), no.4, 1683-1694.

[16] T. Kim, D.S. Kim, S.-H. Rim, and D.Dolgy, *Fourier series of higher-order Bernoulli functions and their applications,* J. Inequal. Appl., **2017** Article ID 71452, 8pages,(2017).

[17] H. Liu, and W. Wang, *Some identities on the Bernoulli, Euler and Genocchi poloynomials via power sums and alternate power sums,* Disc. Math., **309**(2009), 3346-3363.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 27, NO.2, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC

[18] K. Shiratani, S. Yokoyama, *An application of p-adic convolutions,* Mem. Fac. Sci. Kyushu Univ. Ser. A **36(1)**(1982), 7383.

[19] H. M. Srivastava, *Some generalizations and basic extensions of the Bernoulli, Euler and Genocchi polynomials,* Appl. Math. and Inf. Sci., **5**(2011), no. 3, 390-414.

[20] D. G. Zill, M. R. Cullen, *Advanced Engineering Mathematics,* Jones and Bartlett Publishers 2006.

[1] Department of Mathematics, College of Science, Tianjin Polytechnic University, Tianjin 300160, China, Department of Mathematics, Kwangwoon University, Seoul, 139-701, Republic of Korea

E-mail address: tkkim@kw.ac.kr

[2] Department of Mathematics, Sogang University, Seoul, 121-742, Republic of Korea

E-mail address: dskim@sogang.ac.kr

[3] Department of Mathematics, Kwangwoon University, Seoul, 139-701, Republic of Korea

E-mail address: jgw5687@naver.com

[4,*] Department of Mathematics Education and ERI, Gyeongsang National University, Jinju, Gyeongsangnamdo, 52828, Republic of Korea

E-mail address: mathkjk26@gnu.ac.kr

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 27, NO.2, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC

Some symmetric identities for (p, q)-Euler zeta function

Cheon Seoung Ryoo

Department of Mathematics, Hannam University, Daejeon 306-791, Korea

Abstract : In this paper we obtain several symmetric identities of the (p, q)-Euler zeta function. We also give some new interesting properties, explicit formulas, a connection with (p, q)-Euler numbers and polynomials.

Key words : Euler numbers and polynomials, q-Euler numbers and polynomials, (p, q)-Euler numbers and polynomials, (p, q)-analogue of Euler zeta function.

2000 Mathematics Subject Classification : 11B68, 11S40, 11S80.

1. Introduction

Many mathematicians have studied in the area of the Bernoulli numbers and polynomials, Euler numbers and polynomials, Genocchi numbers and polynomials, tangent numbers and polynomials(see [1-10]). The Euler numbers and the Euler polynomials have been extensively worked in many different contexts in such branches of mathematics as, for instance, complex analytic number theory, elementary number theory, differential topology, q-adic analytic number theory and quantum physics. In this paper, we obtain symmetric properties of the (p, q)-Euler zeta function. As applications of these properties, we study some interesting identities for the (p, q)-Euler polynomials and numbers.

Throughout this paper, we always make use of the following notations: $\mathbb{N}$ denotes the set of natural numbers, $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$ denotes the set of nonnegative integers, $\mathbb{Z}_0^- = \{0, -1, -2, -2, \ldots\}$ denotes the set of nonpositive integers, $\mathbb{Z}$ denotes the set of integers, $\mathbb{R}$ denotes the set of real numbers, and $\mathbb{C}$ denotes the set of complex numbers. We remember that the classical Euler numbers E_n and Euler polynomials $E_n(x)$ are defined by the following generating functions

$$\frac{2}{e^t + 1} = \sum_{n=0}^{\infty} E_n \frac{t^n}{n!}, \quad (|t| < \pi). \tag{1.1}$$

and

$$\left(\frac{2}{e^t + 1}\right) e^{xt} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}, \quad (|t| < \pi). \tag{1.2}$$

respectively.

Some interesting properties of the (p, q)-Euler numbers and polynomials were first investigated by Ryoo[6]. The (p, q)-number is defined by

$$[n]_{p,q} = \frac{p^n - q^n}{p - q}.$$

It is clear that (p, q)-number contains symmetric property, and this number is q-number when $p = 1$. In particular, we can see $\lim_{q \to 1} [n]_{p,q} = n$ with $p = 1$.

By using (p, q)-number, we introduced the (p, q)-Euler polynomials and numbers, which generalized the previously known numbers and polynomials, including the Carlitz's type q-Euler numbers

and polynomials. We begin by recalling here the Carlitz's type (p,q)-Euler numbers and polynomials(see [2]).

Definition 1. For $0 < q < p \leq 1$, the Carlitz's type (p,q)-Euler numbers $E_{n,p,q}$ and polynomials $E_{n,p,q}(x)$ are defined by means of the generating functions

$$F_{p,q}(t) = \sum_{n=0}^{\infty} E_{n,p,q}\frac{t^n}{n!} = [2]_q \sum_{m=0}^{\infty}(-1)^m q^m e^{[m]_{p,q}t}. \tag{1.1}$$

and

$$F_{p,q}(t,x) = \sum_{n=0}^{\infty} E_{n,p,q}(x)\frac{t^n}{n!} = [2]_q \sum_{m=0}^{\infty}(-1)^m q^m e^{[m+x]_{p,q}t}, \tag{1.2}$$

respectively.

The following elementary properties of Carlitz's type (p,q)-Euler numbers $E_{n,p,q}$ and polynomials $E_{n,p,q}(x)$ are readily derived from (1.1) and (1.2). We, therefore, choose to omit the details involved. More studies and results in this subject we may see reference [6].

Theorem 2. For $n \in \mathbb{Z}_+$, we have

$$E^{(h)}_{n,p,q}(x) = [2]_q \left(\frac{1}{p-q}\right)^n \sum_{l=0}^{n}\binom{n}{l}(-1)^l q^{xl} p^{(n-l)x}\frac{1}{1+q^{l+1}p^{n-l+h}}.$$

Theorem 3 (Distribution relation). For any positive integer $m(=\text{odd})$, we have

$$E_{n,p,q}(x) = \frac{[2]_q}{[2]_{q^m}}[m]_{p,q}^n \sum_{a=0}^{m-1}(-1)^a q^a E_{n,p^m,q^m}\left(\frac{a+x}{m}\right), \quad n \in \mathbb{N}_0.$$

Next, we introduce Carlitz's type (h,p,q)-Euler polynomials $E^{(h)}_{n,p,q}(x)$. The Carlitz's type (h,p,q)-Euler polynomials $E^{(h)}_{n,p,q}(x)$ are defined by

$$E^{(h)}_{n,p,q}(x) = [2]_q \sum_{m=0}^{\infty}(-1)^m q^m p^{hm}[m+x]_{p,q}^n.$$

By (p,q)-number, we have the following theorem.

Theorem 4. For $n \in \mathbb{Z}_+$, we have

$$E_{n,p,q}(x) = \sum_{l=0}^{n}\binom{n}{l}[x]_{p,q}^{n-l}q^{xl}E^{(n-l)}_{l,p,q}.$$

By using Carlitz's type (p,q)-Euler numbers and polynomials, (p,q)-Euler zeta function and Hurwitz (p,q)-Euler zeta functions are defined. These functions interpolate the Carlitz's type (p,q)-Euler numbers $E_{n,p,q}$, and polynomials $E_{n,p,q}(x)$, respectively. From (1.1), we note that

$$\begin{aligned}\left.\frac{d^k}{dt^k}F_{p,q}(t)\right|_{t=0} &= [2]_q \sum_{m=0}^{\infty}(-1)^n q^m [m]_{p,q}^k \\ &= E_{k,p,q}, (k \in \mathbb{N}).\end{aligned}$$

By using the above equation, we are now ready to define (p,q)-Euler zeta function.

Definition 5. Let $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 0$.

$$\zeta_{p,q}(s) = [2]_q \sum_{n=1}^{\infty} \frac{(-1)^n q^n}{[n]_{p,q}^s}. \tag{1.3}$$

Note that $\zeta_{p,q}(s)$ is a meromorphic function on $\mathbb{C}$. Note that, if $p = 1, q \to 1$, then $\zeta_{p,q}(s) = \zeta_E(s)$ which is the Euler zeta function(see [3, 4]). Relation between $\zeta_{p,q}(s)$ and $E_{k,p,q}$ is given by the following theorem.

Theorem 6. For $k \in \mathbb{N}$, we have

$$\zeta_{p,q}(-k) = E_{k,p,q}.$$

Observe that $\zeta_{p,q}(s)$ function interpolates $E_{k,p,q}$ numbers at non-negative integers. By using (1.2), we note that

$$\left.\frac{d^k}{dt^k} F_{p,q}(t,x)\right|_{t=0} = [2]_q \sum_{m=0}^{\infty} (-1)^m q^m [m+x]_{p,q}^k \tag{1.4}$$

and

$$\left.\left(\frac{d}{dt}\right)^k \left(\sum_{n=0}^{\infty} E_{n,p,q}(x) \frac{t^n}{n!}\right)\right|_{t=0} = E_{k,p,q}(x), \text{ for } k \in \mathbb{N}. \tag{1.5}$$

By (1.4) and (1.5), we are now ready to define the Hurwitz (p,q)-Euler zeta function.

Definition 7. Let $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 0$ and $x \notin \mathbb{Z}_0^-$.

$$\zeta_{p,q}(s,x) = [2]_q \sum_{n=0}^{\infty} \frac{(-1)^n q^n}{[n+x]_{p,q}^s}. \tag{1.6}$$

Note that $\zeta_{p,q}(s,x)$ is a meromorphic function on $\mathbb{C}$.

Obverse that, if $p = 1$ and $q \to 1$, then $\zeta_{p,q}(s,x) = \zeta_E(s,x)$ which is the Hurwitz Euler zeta function(see [3, 4]). Relation between $\zeta_{p,q}(s,x)$ and $E_{k,p,q}(x)$ is given by the following theorem.

Theorem 8. For $k \in \mathbb{N}$, we have

$$\zeta_{p,q}(-k,x) = E_{k,p,q}(x).$$

Observe that $\zeta_{p,q}(-k,x)$ function interpolates $E_{k,p,q}(x)$ numbers at non-negative integers.

2. Symmetric properties about (p,q)-analogue of Euler zeta functions

In this section, we are going to obtain the main results of (p,q)-Euler zeta function. We also establish some interesting symmetric identities for (p,q)-Euler polynomials by using (p,q)-Euler zeta function.

Observe that $[xy]_{p,q} = [x]_{p^y,q^y}[y]_{p,q}$ for any $x, y \in \mathbb{C}$.
By substitute $w_1x + \frac{w_1 i}{w_2}$ for x in Definition 7, replace p by p^{w_2} and replace q by q^{w_2}, respectively, we derive

$$\begin{aligned}
&\zeta_{p^{w_2},q^{w_2}}\left(s, w_1x + \frac{w_1 i}{w_2}\right) \\
&= [2]_{q^{w_2}} \sum_{n=0}^{\infty} \frac{(-1)^n q^{w_2 n}}{[w_1x + \frac{w_1 i}{w_2} + n]_{p^{w_2},q^{w_2}}^s} \\
&= [2]_{q^{w_2}} [w_2]_{p,q}^s \sum_{n=0}^{\infty} \frac{(-1)^n q^{w_2 n}}{[w_1w_2x + w_1 i + w_2 n]_{p,q}^s}.
\end{aligned}$$

Since for any non-negative integer m and odd positive integer w_1, there exist unique non-negative integer r such that $m = w_1 r + j$ with $0 \leq j \leq w_1 - 1$. Hence, this can be written as

$$\begin{aligned}
&\zeta_{p^{w_2},q^{w_2}}\left(s, w_1 x + \frac{w_1 i}{w_2}\right) \\
&= [2]_{q^{w_2}}[w_2]_{p,q}^s \sum_{\substack{w_1 r + j = 0 \\ 0 \leq j \leq w_1 - 1}}^{\infty} \frac{(-1)^{w_1 r + j} q^{w_2(w_1 r + j)}}{[w_2(w_1 r + j) + w_1 w_2 x + w_1 i]_{p,q}^s} \\
&= [2]_{q^{w_2}}[w_2]_{p,q}^s \sum_{j=0}^{w_1 - 1} \sum_{r=0}^{\infty} \frac{(-1)^{w_1 r + j} q^{w_2(w_1 r + j)}}{[w_1 w_2(r + x) + w_1 i + w_2 j]_{p,q}^s}.
\end{aligned}$$

It follows from the above equation that

$$\begin{aligned}
&[2]_{q^{w_1}}[w_1]_{p,q}^s \sum_{i=0}^{w_2-1} (-1)^i q^{w_1 i} \zeta_{p^{w_2},q^{w_2}}\left(s, w_1 x + \frac{w_1 i}{w_2}\right) \\
&= [2]_{q^{w_1}}[2]_{q^{w_2}}[w_1]_{p,q}^s [w_2]_{p,q}^s \sum_{i=0}^{w_2-1} \sum_{j=0}^{w_1-1} \sum_{r=0}^{\infty} \frac{(-1)^{r+i+j} q^{(w_1 w_2 r + w_1 i + w_2 j)}}{[w_1 w_2(r + x) + w_1 i + w_2 j]_q^s}.
\end{aligned} \tag{2.1}$$

From the similar method, we can have that

$$\begin{aligned}
\zeta_{p^{w_1},q^{w_1}}\left(s, w_2 x + \frac{w_2 j}{w_1}\right) &= [2]_{q^{w_1}} \sum_{n=0}^{\infty} \frac{(-1)^n q^{w_1 n}}{[w_2 x + \frac{w_2 j}{w_1} + n]_{p^{w_1},q^{w_1}}^s} \\
&= [2]_{q^{w_1}}[w_1]_{p,q}^s \sum_{n=0}^{\infty} \frac{(-1)^n q^{w_1 n}}{[w_1 w_2 x + w_2 j + w_1 n]_{p,q}^s}.
\end{aligned}$$

After some calculations in the above, we have

$$\begin{aligned}
&[2]_{q^{w_2}}[w_2]_{p,q}^s \sum_{j=0}^{w_1-1} (-1)^j q^{w_2 j} \zeta_{p^{w_1},q^{w_1}}^{(h)}\left(s, w_2 x + \frac{w_2 j}{w_1}\right) \\
&= [2]_{q^{w_1}}[2]_{q^{w_2}}[w_1]_{p,q}^s [w_2]_{p,q}^s \sum_{i=0}^{w_2-1} \sum_{j=0}^{w_1-1} \sum_{r=0}^{\infty} \frac{(-1)^{r+i+j} q^{(w_1 w_2 r + w_1 i + w_2 j)}}{[w_1 w_2(r + x) + w_1 i + w_2 j]_{p,q}^s}.
\end{aligned} \tag{2.2}$$

Thus, we have the following theorem from (2.1) and (2.2).

Theorem 9. Let $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 0$ and w_1, w_2 : odd positive integers. Then one has

$$\begin{aligned}
&[2]_{q^{w_1}}[w_1]_{p,q}^s \sum_{i=0}^{w_2-1} (-1)^i q^{w_1 i} \zeta_{p^{w_2},q^{w_2}}\left(s, w_1 x + \frac{w_1 i}{w_2}\right) \\
&= [2]_{q^{w_2}}[w_2]_{p,q}^s \sum_{j=0}^{w_1-1} (-1)^j q^{w_2 j} \zeta_{p^{w_1},q^{w_1}}\left(s, w_2 x + \frac{w_2 j}{w_1}\right).
\end{aligned}$$

In Theorem 9, we get the following formulas for the (p, q)-tangent zeta function.

Corollary 10. Let $w_2 = 1$ in Theorem 9. Then we get

$$\zeta_{p,q}(s, x) = [w_1]_{p,q}^{-s} \sum_{j=0}^{w_1-1} (-1)^j q^j \zeta_{p^{w_1},q^{w_1}}\left(s, \frac{x + j}{w_1}\right).$$

Corollary 11. Let $w_1 = 2, w_2 = 1$ in Theorem 9. Then we have

$$\zeta_{p^2,q^2}\left(s, \frac{x}{2}\right) - q\zeta_{p^2,q^2}\left(s, \frac{x+1}{2}\right) = [2]_{q^2}[2]_q^{-1}[2]_{p,q}^s \zeta_{p,q}(s, x).$$

For $n \in \mathbb{N}$, we have

$$\zeta_{p,q}(-n,x)=E_{n,p,q}(x), (\text{see Theorem 8}).$$

By substituting $E_{n,p,q}(x)$ for $\zeta_{p,q}(s,x)$ in Theorem 9, we can derive that

$$\begin{aligned}&[2]_{q^{w_1}}[w_1]_{p,q}^{-n}\sum_{i=0}^{w_2-1}(-1)^i q^{w_1 i}\zeta_{p^{w_2},q^{w_2}}\left(-n,w_1x+\frac{w_1 i}{w_2}\right)\\&=[2]_{q^{w_1}}[w_1]_{p,q}^{-n}\sum_{i=0}^{w_2-1}(-1)^i q^{w_1 i}E_{n,p^{w_2},q^{w_2}}\left(w_1x+\frac{w_1 i}{w_2}\right),\end{aligned}$$

and

$$\begin{aligned}&[2]_{q^{w_2}}[w_2]_{p,q}^{-n}\sum_{j=0}^{w_1-1}(-1)^j q^{w_2 j}\zeta_{p^{w_1},q^{w_1}}\left(-n,w_2x+\frac{w_2 j}{w_1}\right)\\&=[2]_{q^{w_2}}[w_2]_{p,q}^{-n}\sum_{j=0}^{w_1-1}(-1)^j q^{w_2 j}E_{n,p^{w_1},q^{w_1}}\left(w_2x+\frac{w_2 j}{w_1}\right).\end{aligned}$$

Thus, we obtain the following theorem from Theorem 9.

Theorem 12. Let w_1, w_2 be any odd positive integer. Then for non-negative integers n, one has

$$\begin{aligned}&[2]_{q^{w_1}}[w_2]_{p,q}^{n}\sum_{i=0}^{w_2-1}(-1)^i q^{w_1 i}E_{n,p^{w_2},q^{w_2}}\left(w_1x+\frac{w_1 i}{w_2}\right)\\&=[2]_{q^{w_2}}[w_1]_{p,q}^{n}\sum_{j=0}^{w_1-1}(-1)^j q^{w_2 j}E_{n,p^{w_1},q^{w_1}}\left(w_2x+\frac{w_2 j}{w_1}\right).\end{aligned}$$

Considering $w_1=1$ in the Theorem 12, we obtain as below equation(see Theorem 3).

$$E_{n,p,q}(x)=\frac{[2]_q}{[2]_{q^{w_2}}}[w_2]_{p,q}^{n}\sum_{j=1}^{w_2-1}(-1)^j q^j E_{n,p^{w_2},q^{w_2}}\left(\frac{x+j}{w_2}\right).$$

We obtain another result by applying the addition theorem for the Carlitz's type (h,p,q)-tangent polynomials $E_{n,p,q}^{(h)}(x)$. From the Theorem 12, we have

$$\begin{aligned}&[2]_{q^{w_1}}[w_2]_{p,q}^{n}\sum_{i=0}^{w_2-1}(-1)^i q^{w_1 i}E_{n,p^{w_2},q^{w_2}}\left(w_1x+\frac{w_1 i}{w_2}\right)\\&=[2]_{q^{w_1}}[w_2]_{p,q}^{n}\sum_{i=0}^{w_2-1}(-1)^i q^{w_1 i}\sum_{l=0}^{n}\binom{n}{l}q^{w_1(n-l)i}p^{w_1w_2xl}E_{n-l,p^{w_2},q^{w_2}}^{(l)}(w_1x)\left(\frac{[w_1]_{p,q}}{[w_2]_{p,q}}\right)^l[i]_{p^{w_1},q^{w_1}}^l\\&=[2]_{q^{w_1}}[w_2]_{p,q}^{n}\sum_{l=0}^{n}\binom{n}{l}\left(\frac{[w_1]_{p,q}}{[w_2]_{p,q}}\right)^l p^{w_1w_2xl}E_{n-l,p^{w_2},q^{w_2}}^{(l)}(w_1x)\sum_{i=0}^{w_2-1}(-1)^i q^{w_1 i}q^{(n-l)w_1 i}[i]_{p^{w_1},q^{w_1}}^l.\end{aligned}$$

Therefore, we obtain that

$$\begin{aligned}&[2]_{q^{w_1}}[w_2]_{p,q}^{n}\sum_{i=0}^{w_2-1}(-1)^i q^{w_1 i}E_{n,p^{w_2},q^{w_2}}\left(w_1x+\frac{w_1 i}{w_2}\right)\\&=[2]_{q^{w_1}}\sum_{l=0}^{n}\binom{n}{l}[w_1]_{p,q}^{l}[w_2]_{p,q}^{n-l}p^{w_1w_2xl}E_{n-l,p^{w_2},q^{w_2}}^{(l)}(w_1x)\mathcal{E}_{n,l,p^{w_1},q^{w_1}}(w_2),\end{aligned}\tag{2.3}$$

and

$$\begin{aligned}&[2]_{q^{w_2}}[w_1]^n_{p,q}\sum_{j=0}^{w_1-1}(-1)^j q^{w_2 j}E_{n,p^{w_1},q^{w_1}}\left(w_2x+\frac{w_2 j}{w_1}\right)\\&=[2]_{q^{w_2}}\sum_{l=0}^{n}\binom{n}{l}[w_2]^l_{p,q}[w_1]^{n-l}_{p,q}p^{w_1w_2xl}E^{(l)}_{n-l,p^{w_1},q^{w_1}}(w_2x)\mathcal{E}_{n,l,p^{w_2},q^{w_2}}(w_1).\end{aligned}\tag{2.4}$$

where $\mathcal{E}_{n,l,p,q}(k)=\sum_{i=0}^{k-1}(-1)^i q^{(1+n-l)i}[i]^l_{p,q}$ is called as the sums of powers.

Hence, from (2.3) and (2.4), we have the following theorem.

Theorem 13. Let w_1, w_2 be any odd positive integer. Then we have

$$\begin{aligned}&[2]_{q^{w_2}}\sum_{l=0}^{n}\binom{n}{l}[w_2]^l_{p,q}[w_1]^{n-l}_{p,q}p^{w_1w_2xl}E^{(l)}_{n-l,p^{w_1},q^{w_1}}(w_2x)\mathcal{E}_{n,l,p^{w_2},q^{w_2}}(w_1)\\&=[2]_{q^{w_1}}\sum_{l=0}^{n}\binom{n}{l}[w_1]^l_{p,q}[w_2]^{n-l}_{p,q}p^{w_1w_2xl}E^{(l)}_{n-l,p^{w_2},q^{w_2}}(w_1x)\mathcal{E}_{n,l,p^{w_1},q^{w_1}}(w_2).\end{aligned}$$

Acknowledgement: This work was supported by the National Research Foundation of Korea(NRF) grant funded by the Korea government(MEST) (No. 2017R1A2B4006092).

REFERENCES

1. R. P. Agarwal, J. Y. Kang, C. S. Ryoo, Some properties of (p,q)-tangent polynomials, J. Computational Analysis and Applications, **24** (2018), 1439-1454.

2. N. S. Jung, C. S. Ryoo, A research on a new approach to Euler polynomials and Bernstein polynomials with variable $[x]_q$, J. Appl. Math. & Informatics, **35** (2017), 205-215.

3. A. M. Robert, A Course in p-adic Analysis, Graduate Text in Mathematics, Vol. 198, Springer, 2000.

4. H. Ozden, Y. Simsek, A new extension of q-Euler numbers and polynomials related to their interpolation functions, Appl. Math. Letters, **21** (2008), 934-938.

5. C. S. Ryoo, A numerical investigation on the zeros of the tangent polynomials, J. Appl. Math. & Informatics, **32** (2014), 315-322.

6. C. S. Ryoo, On the (p,q)-analogue of Euler zeta function, J. Appl. Math. & Informatics, **35** (2017), 303-311.

7. C. S. Ryoo, On degenerate q-tangent polynomials of higher order, J. Appl. Math. & Informatics **35** (2017), 113-120.

8. C. S. Ryoo, R .P. Agarwal, Some identities involving q-poly-tangent numbers and polynomials and distribution of their zeros, Advances in Difference Equations **2017:213** (2017), 1-14.

9. H. Shin, J. Zeng, The q-tangent and q-secant numbers via continued fractions, European J. Combin., **31** (2010), 1689-1705.

10. P. T. Young, Degenerate Bernoulli polynomials, generalized factorial sums, and their applications, Journal of Number Theorey, **128** (2008), 738-758.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 27, NO.2, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC

ADDITIVE (ρ_1, ρ_2)-FUNCTIONAL INEQUALITIES IN COMPLEX BANACH SPACES

CHOONKIL PARK, DONG YUN SHIN*, AND GEORGE A. ANASTASSIOU

ABSTRACT. In this paper, we introduce and solve the following additive (ρ_1, ρ_2)-functional inequalities

$$\begin{aligned}\|f(x+y+z)-f(x)-f(y)-f(z)\| &\geq \|\rho_1(f(x+y-z)-f(x)-f(y)+f(z))\| \\ &+ \|\rho_2(f(x-y+z)-f(x)+f(y)-f(z))\|, \qquad (0.1)\end{aligned}$$

where ρ_1 and ρ_2 are fixed complex numbers with $|\rho_1| \cdot |\rho_2| > 1$, and

$$\begin{aligned}\|f(x+y-z)-f(x)-f(y)+f(z)\| &\geq \|\rho_1(f(x+y+z)-f(x)-f(y)-f(z))\| \\ &+ \|\rho_2(f(x-y+z)-f(x)+f(y)-f(z))\| \qquad (0.2)\end{aligned}$$

where ρ_1 and ρ_2 are fixed complex numbers with $|\rho_1| > 1$.

Using the fixed point method and the direct method, we prove the Hyers-Ulam stability of the additive (ρ_1, ρ_2)-functional inequalities (0.1) and (0.2) in complex Banach spaces.

1. Introduction and preliminaries

The stability problem of functional equations originated from a question of Ulam [29] concerning the stability of group homomorphisms.

The functional equation $f(x + y) = f(x) + f(y)$ is called the *Cauchy equation*. In particular, every solution of the Cauchy equation is said to be an *additive mapping*. Hyers [13] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [2] for additive mappings and by Rassias [23] for linear mappings by considering an unbounded Cauchy difference. A generalization of the Rassias theorem was obtained by Găvruta [12] by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassias' approach. The stability of quadratic functional equation was proved by Skof [28] for mappings $f : E_1 \to E_2$, where E_1 is a normed space and E_2 is a Banach space. Cholewa [8] noticed that the theorem of Skof is still true if the relevant domain E_1 is replaced by an Abelian group.

Park [18, 19] defined additive ρ-functional inequalities and proved the Hyers-Ulam stability of the additive ρ-functional inequalities in Banach spaces and non-Archimedean Banach spaces. The stability problems of various functional equations have been extensively investigated by a number of authors (see [1, 3, 7, 10, 11, 15, 17, 20, 21, 24, 25, 26, 27, 30, 31, 32]).

We recall a fundamental result in fixed point theory.

Theorem 1.1. [4, 9] *Let (X, d) be a complete generalized metric space and let $J : X \to X$ be a strictly contractive mapping with Lipschitz constant $\alpha < 1$. Then for each given element $x \in X$, either*

$$d(J^n x, J^{n+1}x) = \infty$$

for all nonnegative integers n or there exists a positive integer n_0 such that

(1) $d(J^n x, J^{n+1}x) < \infty, \qquad \forall n \geq n_0$;

(2) *the sequence $\{J^n x\}$ converges to a fixed point y^* of J;*

2010 *Mathematics Subject Classification.* Primary 39B62, 47H10, 39B52.

Key words and phrases. Hyers-Ulam stability; additive (ρ_1, ρ_2)-functional inequality; fixed point method; direct method; Banach space.

*Corresponding author (Dong Yun Shin).

(3) y^* *is the unique fixed point of* J *in the set* $Y = \{y \in X \mid d(J^{n_0}x, y) < \infty\}$;
(4) $d(y, y^*) \leq \frac{1}{1-\alpha} d(y, Jy)$ *for all* $y \in Y$.

In 1996, G. Isac and Th.M. Rassias [14] were the first to provide applications of stability theory of functional equations for the proof of new fixed point theorems with applications. By using fixed point methods, the stability problems of several functional equations have been extensively investigated by a number of authors (see [5, 6, 22]).

In Section 2, we solve the additive (ρ_1, ρ_2)-functional inequality (0.1) and prove the Hyers-Ulam stability of the additive (ρ_1, ρ_2)-functional inequality (0.1) in Banach spaces by using the fixed point method.

In Section 3, we prove the Hyers-Ulam stability of the additive (ρ_1, ρ_2)-functional inequality (0.1) in Banach spaces by using the direct method.

In Section 4, we solve the additive (ρ_1, ρ_2)-functional inequality (0.1) and prove the Hyers-Ulam stability of the additive (ρ_1, ρ_2)-functional inequality (0.1) in Banach spaces by using the fixed point method.

In Section 5, we prove the Hyers-Ulam stability of the additive (ρ_1, ρ_2)-functional inequality (0.1) in Banach spaces by using the direct method.

Throughout this paper, let X be a real or complex normed space with norm $\|\cdot\|$ and Y a complex Banach space with norm $\|\cdot\|$. Assume that ρ_1 and ρ_2 are fixed complex numbers with $|\rho_1| \cdot |\rho_2| > 1$.

2. Additive (ρ_1, ρ_2)-functional inequality (0.1): a fixed point method

In this section, we solve and investigate the additive (ρ_1, ρ_2)-functional inequality (0.1) in complex Banach spaces.

Lemma 2.1. *If a mapping* $f : X \to Y$ *satisfies* $f(0) = 0$ *and*

$$\begin{aligned} \|f(x+y+z) - f(x) - f(y) - f(z)\| &\geq \|\rho_1(f(x+y-z) - f(x) - f(y) + f(z))\| \\ &\quad + \|\rho_2(f(x-y+z) - f(x) + f(y) - f(z))\| \end{aligned} \tag{2.1}$$

for all $x, y, z \in X$, *then* $f : X \to Y$ *is additive.*

Proof. Assume that $f : X \to Y$ satisfies (2.1).

Since $|\rho_1| \cdot |\rho_2| > 1$, $|\rho_1| > 1$ or $|\rho_2| > 1$.

(i) Assume that $|\rho_1| > 1$. Letting $z = 0$ in (4.1), we get

$$(1 - |\rho_1|)\|f(x+y) - f(x) - f(y)\| \geq |\rho_2|\|f(x-y) - f(x) + f(y)\|$$

for all $x, y \in X$. So $f(x+y) = f(x) + f(y)$ for all $x, y \in X$, since $|\rho_1| > 1$. So f is additive.

(ii) Assume that $|\rho_2| > 1$. Letting $y = 0$ in (4.1), we get

$$(1 - |\rho_2|)\|f(x+z) - f(x) - f(z)\| \geq |\rho_1|\|f(x-z) - f(x) + f(z)\|$$

for all $x, z \in X$. So $f(x+z) = f(x) + f(z)$ for all $x, z \in X$, since $|\rho_2| > 1$. So f is additive. □

Using the fixed point method, we prove the Hyers-Ulam stability of the additive (ρ_1, ρ_2)-functional inequality (2.1) in complex Banach spaces.

Since $|\rho_1| \cdot |\rho_2| > 1$, $|\rho_1| > 1$ or $|\rho_2| > 1$. One can exchange y and z and from now on, one can assume that $|\rho_1| > 1$.

Theorem 2.2. *Let* $\varphi : X^3 \to [0, \infty)$ *be a function such that there exists an* $L < 1$ *with*

$$\varphi\left(\frac{x}{2}, \frac{y}{2}, \frac{z}{2}\right) \leq \frac{L}{2}\varphi(x, y, z) \tag{2.2}$$

for all $x, y, z \in X$. *Let* $f : X \to Y$ *be a mapping satisfying* $f(0) = 0$ *and*

$$\begin{aligned} &\|\rho_1(f(x+y-z) - f(x) - f(y) + f(z))\| + \|\rho_2(f(x-y+z) - f(x) + f(y) - f(z))\| \\ &\leq \|f(x+y+z) - f(x) - f(y) - f(z)\| + \varphi(x, y, z) \end{aligned} \tag{2.3}$$

for all $x, y, z \in X$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \leq \frac{L}{2(1-L)(|\rho_1|-1)}\varphi(x,x,0)$$

for all $x \in X$.

Proof. Letting $z = 0$ and $y = x$ in (2.3), we get

$$\|f(2x) - 2f(x)\| \leq \frac{1}{|\rho_1|-1}\varphi(x,x,0) \tag{2.4}$$

for all $x \in X$.

Consider the set

$$S := \{h : X \to Y, \quad h(0) = 0\}$$

and introduce the generalized metric on S:

$$d(g,h) = \inf\left\{\mu \in \mathbb{R}_+ : \|g(x) - h(x)\| \leq \mu\varphi(x,x,0),\ \forall x \in X\right\},$$

where, as usual, $\inf \phi = +\infty$. It is easy to show that (S,d) is complete (see [16]).

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := 2g\left(\frac{x}{2}\right)$$

for all $x \in X$.

Let $g, h \in S$ be given such that $d(g,h) = \varepsilon$. Then

$$\|g(x) - h(x)\| \leq \varepsilon\varphi(x,x,0)$$

for all $x \in X$. Hence

$$\begin{aligned}\|Jg(x) - Jh(x)\| &= \left\|2g\left(\frac{x}{2}\right) - 2h\left(\frac{x}{2}\right)\right\| \leq 2\varepsilon\varphi\left(\frac{x}{2},\frac{x}{2},0\right)\\ &\leq 2\varepsilon\frac{L}{2}\varphi(x,x,0) = L\varepsilon\varphi(x,x,0)\end{aligned}$$

for all $x \in X$. So $d(g,h) = \varepsilon$ implies that $d(Jg,Jh) \leq L\varepsilon$. This means that

$$d(Jg,Jh) \leq Ld(g,h)$$

for all $g, h \in S$.

It follows from (2.4) that

$$\left\|f(x) - 2f\left(\frac{x}{2}\right)\right\| \leq \frac{1}{|\rho_1|-1}\varphi\left(\frac{x}{2},\frac{x}{2},0\right) \leq \frac{L}{2(|\rho_1|-1)}\varphi(x,x,0)$$

for all $x \in X$ So $d(f,Jf) \leq \frac{L}{2(|\rho_1|-1)}$.

By Theorem 1.1, there exists a mapping $A : X \to Y$ satisfying the following:

(1) A is a fixed point of J, i.e.,

$$A(x) = 2A\left(\frac{x}{2}\right) \tag{2.5}$$

for all $x \in X$. The mapping A is a unique fixed point of J in the set

$$M = \{g \in S : d(f,g) < \infty\}.$$

This implies that A is a unique mapping satisfying (2.5) such that there exists a $\mu \in (0,\infty)$ satisfying

$$\|f(x) - A(x)\| \leq \mu\varphi(x,x,0)$$

for all $x \in X$;

(2) $d(J^l f, A) \to 0$ as $l \to \infty$. This implies the equality

$$\lim_{l\to\infty} 2^n f\left(\frac{x}{2^n}\right) = A(x)$$

for all $x \in X$;

(3) $d(f, A) \le \frac{1}{1-L} d(f, Jf)$, which implies

$$\|f(x) - A(x)\| \le \frac{L}{2(1-L)(|\rho_1|-1)} \varphi(x, x, 0)$$

for all $x \in X$.

It follows from (2.2) and (2.3) that

$$\begin{aligned}
&\|A(x+y+z) - A(x) - A(y) - A(z)\| \\
&= \lim_{n\to\infty} 2^n \left\| f\left(\frac{x+y+z}{2^n}\right) - f\left(\frac{x}{2^n}\right) - f\left(\frac{y}{2^n}\right) - f\left(\frac{z}{2^n}\right)\right\| + \lim_{n\to\infty} 2^n \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}, \frac{z}{2^n}\right) \\
&\ge \lim_{n\to\infty} 2^n |\rho_1| \left\| f\left(\frac{x+y-z}{2^n}\right) - f\left(\frac{x}{2^n}\right) - f\left(\frac{y}{2^n}\right) + f\left(\frac{z}{2^n}\right)\right\| \\
&+ \lim_{n\to\infty} 2^n |\rho_2| \left\| f\left(\frac{x-y+z}{2^n}\right) - f\left(\frac{x}{2^n}\right) + f\left(\frac{y}{2^n}\right) - f\left(\frac{z}{2^n}\right)\right\| \\
&= \|\rho_1(A(x+y-z) - A(x) - A(y) + A(z))\| \\
&\quad + \|\rho_2(A(x-y+z) - A(x) + A(y) - A(z))\|
\end{aligned}$$

for all $x, y, z \in X$. So

$$\begin{aligned}
\|A(x+y+z) - A(x) - A(y) - A(z)\| &\ge \|\rho_1(A(x+y-z) - A(x) - A(y) + A(z))\| \\
&+ \|\rho_2(A(x-y+z) - A(x) + A(y) - A(z))\|
\end{aligned}$$

for all $x, y, z \in X$. By Lemma 2.1, the mapping $A : X \to Y$ is additive. □

Corollary 2.3. *Let $r > 1$ and θ be nonnegative real numbers, and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and*

$$\begin{aligned}
&\|\rho_1(f(x+y-z) - f(x) - f(y) + f(z))\| + \|\rho_2(f(x-y+z) - f(x) + f(y) - f(z))\| \\
&\le \|f(x+y+z) - f(x) - f(y) - f(z)\| + \theta(\|x\|^r + \|y\|^r + \|z\|^r) \qquad (2.6)
\end{aligned}$$

for all $x, y, z \in X$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \le \frac{2\theta}{(2^r-2)(|\rho_1|-1)} \|x\|^r$$

for all $x \in X$.

Proof. The proof follows from Theorem 2.2 by taking $\varphi(x, y, z) = \theta(\|x\|^r + \|y\|^r + \|z\|^r)$ for all $x, y, z \in X$. Choosing $L = 2^{1-r}$, we obtain the desired result. □

Theorem 2.4. *Let $\varphi : X^3 \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x, y, z) \le 2L\varphi\left(\frac{x}{2}, \frac{y}{2}, \frac{z}{2}\right) \qquad (2.7)$$

for all $x, y, z \in X$. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and (2.3). Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \le \frac{1}{2(1-L)(|\rho_1|-1)} \varphi(x, x, 0)$$

for all $x \in X$.

Proof. Let (S, d) be the generalized metric space defined in the proof of Theorem 2.2.

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := \frac{1}{2} g(2x)$$

for all $x \in X$.

It follows from (2.4) that

$$\left\| f(x) - \frac{1}{2} f(2x) \right\| \le \frac{1}{2(|\rho_1| - 1)} \varphi(x, x, 0)$$

for all $x \in X$.

The rest of the proof is similar to the proof of Theorem 2.2. $\square$

Corollary 2.5. *Let $r < 1$ and θ be positive real numbers, and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and (2.6). Then there exists a unique additive mapping $A : X \to Y$ such that*

$$\|f(x) - A(x)\| \le \frac{2\theta}{(2 - 2^r)(|\rho_1| - 1)} \|x\|^r$$

for all $x \in X$.

Proof. The proof follows from Theorem 2.4 by taking $\varphi(x, y, z) = \theta(\|x\|^r + \|y\|^r + \|z\|^r)$ for all $x, y, z \in X$. Choosing $L = 2^{r-1}$, we obtain the desired result. $\square$

Remark 2.6. If ρ_1 and ρ_2 are real numbers such that $|\rho_1| \cdot |\rho_2| > 1$ and Y is a real Banach space, then all the assertions in this section remain valid.

3. Additive (ρ_1, ρ_2)-functional inequality (0.1): a direct method

In this section, we prove the Hyers-Ulam stability of the additive (ρ_1, ρ_2)-functional inequality (2.1) in complex Banach spaces by using the direct method.

Theorem 3.1. *Let $\varphi : X^3 \to [0, \infty)$ be a function such that*

$$\Psi(x, y, z) := \sum_{j=1}^{\infty} 2^j \varphi\left(\frac{x}{2^j}, \frac{y}{2^j}, \frac{z}{2^j}\right) < \infty \tag{3.1}$$

for all $x, y, z \in X$. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and (2.3). Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \le \frac{1}{2(|\rho_1| - 1)} \Psi(x, x, 0) \tag{3.2}$$

for all $x \in X$.

Proof. Letting $z = y$ and $x = 0$ in (2.3), we get

$$\|f(2x) - 2f(x)\| \le \frac{1}{|\rho_1| - 1} \varphi(x, x, 0) \tag{3.3}$$

and so

$$\left\| f(x) - 2f\left(\frac{x}{2}\right) \right\| \le \frac{1}{|\rho_1| - 1} \varphi\left(\frac{x}{2}, \frac{x}{2}, 0\right)$$

for all $x \in X$. Thus

$$\begin{aligned} \left\| 2^l f\left(\frac{x}{2^l}\right) - 2^m f\left(\frac{x}{2^m}\right) \right\| &\le \sum_{j=l}^{m-1} \left\| 2^j f\left(\frac{x}{2^j}\right) - 2^{j+1} f\left(\frac{x}{2^{j+1}}\right) \right\| \\ &\le \sum_{j=l}^{m-1} \frac{2^j}{|\rho_1| - 1} \varphi\left(\frac{x}{2^{j+1}}, \frac{x}{2^{j+1}}, 0\right) \end{aligned} \tag{3.4}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (3.4) that the sequence $\{2^k f(\frac{x}{2^k})\}$ is Cauchy for all $x \in X$. Since Y is a Banach space, the sequence $\{2^k f(\frac{x}{2^k})\}$ converges. So one can define the mapping $A : X \to Y$ by

$$A(x) := \lim_{k\to\infty} 2^k f\left(\frac{x}{2^k}\right)$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (3.4), we get (3.2).

It follows from (2.3) and (3.1) that

$$\begin{aligned}
&\|A(x+y+z) - A(x) - A(y) - A(z)\| \\
&= \lim_{n\to\infty} 2^n \left\| f\left(\frac{x+y+z}{2^n}\right) - f\left(\frac{x}{2^n}\right) - f\left(\frac{y}{2^n}\right) - f\left(\frac{z}{2^n}\right)\right\| + \lim_{n\to\infty} 2^n \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}, \frac{z}{2^n}\right) \\
&\geq \lim_{n\to\infty} 2^n |\rho_1| \left\| f\left(\frac{x+y-z}{2^n}\right) - f\left(\frac{x}{2^n}\right) - f\left(\frac{y}{2^n}\right) + f\left(\frac{z}{2^n}\right)\right\| \\
&+ \lim_{n\to\infty} 2^n |\rho_2| \left\| f\left(\frac{x-y+z}{2^n}\right) - f\left(\frac{x}{2^n}\right) + f\left(\frac{y}{2^n}\right) - f\left(\frac{z}{2^n}\right)\right\| \\
&= \|\rho_1(A(x+y-z) - A(x) - A(y) + A(z))\| \\
&\quad + \|\rho_2(A(x-y+z) - A(x) + A(y) - A(z))\|
\end{aligned}$$

for all $x, y, z \in X$. So

$$\begin{aligned}
\|A(x+y+z) - A(x) - A(y) - A(z)\| \geq \|\rho_1(A(x+y-z) - A(x) - A(y) + A(z))\| \\
+ \|\rho_2(A(x-y+z) - A(x) + A(y) - A(z))\|
\end{aligned}$$

for all $x, y, z \in X$. By Lemma 2.1, the mapping $A : X \to Y$ is additive.

Now, let $T : X \to Y$ be another additive mapping satisfying (3.2). Then we have

$$\begin{aligned}
\|A(x) - T(x)\| &= \left\| 2^q A\left(\frac{x}{2^q}\right) - 2^q T\left(\frac{x}{2^q}\right)\right\| \\
&\leq \left\| 2^q A\left(\frac{x}{2^q}\right) - 2^q f\left(\frac{x}{2^q}\right)\right\| + \left\| 2^q T\left(\frac{x}{2^q}\right) - 2^q f\left(\frac{x}{2^q}\right)\right\| \\
&\leq \frac{2^q}{|\rho_1| - 1} \Psi\left(\frac{x}{2^q}, \frac{x}{2^q}, 0\right),
\end{aligned}$$

which tends to zero as $q \to \infty$ for all $x \in X$. So we can conclude that $A(x) = T(x)$ for all $x \in X$. This proves the uniqueness of A. □

Corollary 3.2. *Let $r > 1$ and θ be nonnegative real numbers, and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and (2.6). Then there exists a unique additive mapping $A : X \to Y$ such that*

$$\|f(x) - A(x)\| \leq \frac{2\theta}{(2^r - 2)(|\rho_1| - 1)} \|x\|^r$$

for all $x \in X$.

Theorem 3.3. *Let $\varphi : X^3 \to [0, \infty)$ be a function and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$, (2.3) and*

$$\Psi(x, y, z) := \sum_{j=0}^{\infty} \frac{1}{2^j} \varphi(2^j x, 2^j y, 2^j z) < \infty \tag{3.5}$$

for all $x, y, z \in X$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \leq \frac{1}{2(|\rho_1| - 1)} \Psi(x, x, 0)$$

for all $x \in X$.

Proof. It follows from (3.3) that

$$\left\| f(x) - \frac{1}{2} f(2x) \right\| \leq \frac{1}{2(|\rho_1| - 1)} \varphi(x, x)$$

for all $x \in X$. Hence

$$\begin{aligned} \left\| \frac{1}{2^l} f(2^l x) - \frac{1}{2^m} f(2^m x) \right\| &\leq \sum_{j=l}^{m-1} \left\| \frac{1}{2^j} f\left(2^j x\right) - \frac{1}{2^{j+1}} f\left(2^{j+1} x\right) \right\| \\ &\leq \sum_{j=l}^{m-1} \frac{1}{2^{j+1}(|\rho_1| - 1)} \varphi(2^j x, 2^j x, 0) \end{aligned} \tag{3.6}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (3.6) that the sequence $\{\frac{1}{2^n} f(2^n x)\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{\frac{1}{2^n} f(2^n x)\}$ converges. So one can define the mapping $A : X \to Y$ by

$$A(x) := \lim_{n \to \infty} \frac{1}{2^n} f(2^n x)$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (3.6), we get (3.6).

The rest of the proof is similar to the proof of Theorem 3.1. □

Corollary 3.4. *Let $r < 1$ and θ be positive real numbers, and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and (2.6). Then there exists a unique additive mapping $A : X \to Y$ such that*

$$\|f(x) - A(x)\| \leq \frac{2\theta}{(2 - 2^r)(|\rho_1| - 1)} \|x\|^r$$

for all $x \in X$.

4. Additive (ρ_1, ρ_2)-functional inequality (0.2): a fixed point method

In this section, we solve and investigate the additive (ρ_1, ρ_2)-functional inequality (0.2) in complex Banach spaces.

From now on, assume that $\rho_1| > 1$.

Lemma 4.1. *If a mapping $f : X \to Y$ satisfies $f(0) = 0$ and*

$$\begin{aligned} \|f(x + y - z) - f(x) - f(y) + f(z)\| &\geq \|\rho_1(f(x + y + z) - f(x) - f(y) - f(z))\| \\ &\quad + \|\rho_2 (f(x - y + z) - f(x) + f(y) - f(z))\| \end{aligned} \tag{4.1}$$

for all $x, y, z \in X$, then $f : X \to Y$ is additive.

Proof. Assume that $f : X \to Y$ satisfies (4.1).

Letting $z = 0$ in (4.1), we get

$$(1 - |\rho_1|)\|f(x + y) - f(x) - f(y)\| \geq |\rho_2|\|f(x - y) - f(x) + f(y)\|$$

for all $x, y \in X$. So $f(x + y) = f(x) + f(y)$ for all $x, y \in X$, since $|\rho_1| > 1$. So f is additive. □

Using the fixed point method, we prove the Hyers-Ulam stability of the additive (ρ_1, ρ_2)-functional inequality (4.1) in complex Banach spaces.

Theorem 4.2. *Let $\varphi : X^3 \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi\left(\frac{x}{2}, \frac{y}{2}, \frac{z}{2}\right) \leq \frac{L}{2} \varphi(x, y, z) \tag{4.2}$$

for all $x, y, z \in X$. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and

$$\begin{aligned} &\|\rho_1(f(x + y + z) - f(x) - f(y) - f(z))\| + \|\rho_2 (f(x - y + z) - f(x) + f(y) - f(z))\| \\ &\leq \|f(x + y - z) - f(x) - f(y) + f(z)\| + \varphi(x, y, z) \end{aligned} \tag{4.3}$$

for all $x, y, z \in X$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \leq \frac{L}{2(1-L)(|\rho_1|-1)} \varphi(x, x, 0)$$

for all $x \in X$.

Proof. Letting $y = x$ and $z = 0$ in (4.3), we get

$$\|f(2x) - 2f(x)\| \leq \frac{1}{|\rho_1|-1} \varphi(x, x, 0) \tag{4.4}$$

for all $x \in X$.

Consider the set

$$S := \{h : X \to Y, \quad h(0) = 0\}$$

and introduce the generalized metric on S:

$$d(g, h) = \inf\{\mu \in \mathbb{R}_+ : \|g(x) - h(x)\| \leq \mu\varphi(x, x, 0), \ \forall x \in X\},$$

where, as usual, $\inf \phi = +\infty$. It is easy to show that (S, d) is complete (see [16]).

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := 2g\left(\frac{x}{2}\right)$$

for all $x \in X$.

Let $g, h \in S$ be given such that $d(g, h) = \varepsilon$. Then

$$\|g(x) - h(x)\| \leq \varepsilon\varphi(x, x, 0)$$

for all $x \in X$. Hence

$$\begin{aligned} \|Jg(x) - Jh(x)\| &= \left\|2g\left(\frac{x}{2}\right) - 2h\left(\frac{x}{2}\right)\right\| \leq 2\varepsilon\varphi\left(\frac{x}{2}, \frac{x}{2}, 0\right) \\ &\leq 2\varepsilon\frac{L}{2}\varphi(x, x, 0) = L\varepsilon\varphi(x, x, 0) \end{aligned}$$

for all $x \in X$. So $d(g, h) = \varepsilon$ implies that $d(Jg, Jh) \leq L\varepsilon$. This means that

$$d(Jg, Jh) \leq Ld(g, h)$$

for all $g, h \in S$.

It follows from (4.4) that

$$\left\|f(x) - 2f\left(\frac{x}{2}\right)\right\| \leq \frac{1}{|\rho_1|-1}\varphi\left(\frac{x}{2}, \frac{x}{2}, 0\right) \leq \frac{L}{2(|\rho_1|-1)}\varphi(x, x, 0)$$

for all $x \in X$ So $d(f, Jf) \leq \frac{L}{2(|\rho_1|-1)}$.

By Theorem 1.1, there exists a mapping $A : X \to Y$ satisfying the following:

(1) A is a fixed point of J, i.e.,

$$A(x) = 2A\left(\frac{x}{2}\right) \tag{4.5}$$

for all $x \in X$. The mapping A is a unique fixed point of J in the set

$$M = \{g \in S : d(f, g) < \infty\}.$$

This implies that A is a unique mapping satisfying (4.5) such that there exists a $\mu \in (0, \infty)$ satisfying

$$\|f(x) - A(x)\| \leq \mu\varphi(x, x, 0)$$

for all $x \in X$;

(2) $d(J^l f, A) \to 0$ as $l \to \infty$. This implies the equality

$$\lim_{l\to\infty} 2^n f\left(\frac{x}{2^n}\right) = A(x)$$

for all $x \in X$;

(3) $d(f, A) \le \frac{1}{1-L} d(f, Jf)$, which implies

$$\|f(x) - A(x)\| \le \frac{L}{2(1-L)(|\rho_1|-1)} \varphi(x, x, 0)$$

for all $x \in X$.

It follows from (4.2) and (4.3) that

$$\begin{aligned}
&\|A(x+y-z) - A(x) - A(y) + A(z)\| \\
&= \lim_{n\to\infty} 2^n \left\| f\left(\frac{x+y-z}{2^n}\right) - f\left(\frac{x}{2^n}\right) - f\left(\frac{y}{2^n}\right) + f\left(\frac{z}{2^n}\right)\right\| + \lim_{n\to\infty} 2^n \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}, \frac{z}{2^n}\right) \\
&\ge \lim_{n\to\infty} 2^n |\rho_1| \left\| f\left(\frac{x+y+z}{2^n}\right) - f\left(\frac{x}{2^n}\right) - f\left(\frac{y}{2^n}\right) - f\left(\frac{z}{2^n}\right)\right\| \\
&+ \lim_{n\to\infty} 2^n |\rho_2| \left\| f\left(\frac{x-y+z}{2^n}\right) - f\left(\frac{x}{2^n}\right) + f\left(\frac{y}{2^n}\right) - f\left(\frac{z}{2^n}\right)\right\| \\
&= \|\rho_1(A(x+y+z) - A(x) - A(y) - A(z))\| \\
&\quad + \|\rho_2(A(x-y+z) - A(x) + A(y) - A(z))\|
\end{aligned}$$

for all $x, y, z \in X$. So

$$\begin{aligned}
\|A(x+y-z) - A(x) - A(y) + A(z)\| &\ge \|\rho_1(A(x+y+z) - A(x) - A(y) - A(z))\| \\
&+ \|\rho_2(A(x-y+z) - A(x) + A(y) - A(z))\|
\end{aligned}$$

for all $x, y, z \in X$. By Lemma 4.1, the mapping $A : X \to Y$ is additive. □

Corollary 4.3. *Let $r > 1$ and θ be nonnegative real numbers, and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and*

$$\begin{aligned}
&\|\rho_1(f(x+y+z) - f(x) - f(y) - f(z))\| + \|\rho_2(f(x-y+z) - f(x) + f(y) - f(z))\| \\
&\le \|f(x+y-z) - f(x) - f(y) + f(z)\| + \theta(\|x\|^r + \|y\|^r + \|z\|^r) \qquad (4.6)
\end{aligned}$$

for all $x, y, z \in X$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \le \frac{2\theta}{(2^r-2)(|\rho_1|-1)} \|x\|^r$$

for all $x \in X$.

Proof. The proof follows from Theorem 4.2 by taking $\varphi(x, y, z) = \theta(\|x\|^r + \|y\|^r + \|z\|^r)$ for all $x, y, z \in X$. Choosing $L = 2^{1-r}$, we obtain the desired result. □

Theorem 4.4. *Let $\varphi : X^3 \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x, y, z) \le 2L\varphi\left(\frac{x}{2}, \frac{y}{2}, \frac{z}{2}\right) \qquad (4.7)$$

for all $x, y, z \in X$. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and (4.3). Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \le \frac{1}{2(1-L)(|\rho_1|-1)} \varphi(x, x, 0)$$

for all $x \in X$.

Proof. Let (S, d) be the generalized metric space defined in the proof of Theorem 4.2.

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := \frac{1}{2} g\left(2x\right)$$

for all $x \in X$.

It follows from (4.4) that

$$\left\| f(x) - \frac{1}{2} f(2x) \right\| \leq \frac{1}{2(|\rho_1| - 1)} \varphi(x, x, 0)$$

for all $x \in X$.

The rest of the proof is similar to the proof of Theorem 4.2. □

Corollary 4.5. *Let $r < 1$ and θ be positive real numbers, and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and (4.6). Then there exists a unique additive mapping $A : X \to Y$ such that*

$$\|f(x) - A(x)\| \leq \frac{2\theta}{(2 - 2^r)(|\rho_1| - 1)} \|x\|^r$$

for all $x \in X$.

Proof. The proof follows from Theorem 4.4 by taking $\varphi(x, y, z) = \theta(\|x\|^r + \|y\|^r + \|z\|^r)$ for all $x, y, z \in X$. Choosing $L = 2^{r-1}$, we obtain the desired result. □

Remark 4.6. If ρ_1 and ρ_2 are real numbers such that $|\rho_1| > 1$ and Y is a real Banach space, then all the assertions in this section remain valid.

5. Additive (ρ_1, ρ_2)-functional inequality (0.2): a direct method

In this section, we prove the Hyers-Ulam stability of the additive (ρ_1, ρ_2)-functional inequality (4.1) in complex Banach spaces by using the direct method.

Theorem 5.1. *Let $\varphi : X^3 \to [0, \infty)$ be a function such that*

$$\Psi(x, y, z) := \sum_{j=1}^{\infty} 2^j \varphi\left(\frac{x}{2^j}, \frac{y}{2^j}, \frac{z}{2^j}\right) < \infty \tag{5.1}$$

for all $x, y, z \in X$. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and (4.3). Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \leq \frac{1}{2(|\rho_1| - 1)} \Psi(x, x, 0) \tag{5.2}$$

for all $x \in X$.

Proof. Letting $y = x$ and $z = 0$ in (4.3), we get

$$\|f\left(2x\right) - 2f(x)\| \leq \frac{1}{|\rho_1| - 1} \varphi(x, x, 0) \tag{5.3}$$

and so

$$\left\| f\left(x\right) - 2f\left(\frac{x}{2}\right) \right\| \leq \frac{1}{|\rho_1| - 1} \varphi\left(\frac{x}{2}, \frac{x}{2}, 0\right)$$

for all $x \in X$. Thus

$$\begin{aligned} \left\| 2^l f\left(\frac{x}{2^l}\right) - 2^m f\left(\frac{x}{2^m}\right) \right\| &\leq \sum_{j=l}^{m-1} \left\| 2^j f\left(\frac{x}{2^j}\right) - 2^{j+1} f\left(\frac{x}{2^{j+1}}\right) \right\| \\ &\leq \sum_{j=l}^{m-1} \frac{2^j}{|\rho_1| - 1} \varphi\left(\frac{x}{2^{j+1}}, \frac{x}{2^{j+1}}, 0\right) \end{aligned} \tag{5.4}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (5.4) that the sequence $\{2^k f(\frac{x}{2^k})\}$ is Cauchy for all $x \in X$. Since Y is a Banach space, the sequence $\{2^k f(\frac{x}{2^k})\}$ converges. So one can define the mapping $A : X \to Y$ by

$$A(x) := \lim_{k\to\infty} 2^k f\left(\frac{x}{2^k}\right)$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (5.4), we get (5.2).

It follows from (5.4) and (5.1) that

$$\begin{aligned}
&\|A(x+y-z) - A(x) - A(y) + A(z)\| \\
&= \lim_{n\to\infty} 2^n \left\|f\left(\frac{x+y-z}{2^n}\right) - f\left(\frac{x}{2^n}\right) - f\left(\frac{y}{2^n}\right) + f\left(\frac{z}{2^n}\right)\right\| + \lim_{n\to\infty} 2^n \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}, \frac{z}{2^n}\right) \\
&\geq \lim_{n\to\infty} 2^n |\rho_1| \left\|f\left(\frac{x+y+z}{2^n}\right) - f\left(\frac{x}{2^n}\right) - f\left(\frac{y}{2^n}\right) - f\left(\frac{z}{2^n}\right)\right\| \\
&+ \lim_{n\to\infty} 2^n |\rho_2| \left\|f\left(\frac{x-y+z}{2^n}\right) - f\left(\frac{x}{2^n}\right) + f\left(\frac{y}{2^n}\right) - f\left(\frac{z}{2^n}\right)\right\| \\
&= \|\rho_1(A(x+y+z) - A(x) - A(y) - A(z))\| \\
&\quad + \|\rho_2(A(x-y+z) - A(x) + A(y) - A(z))\|
\end{aligned}$$

for all $x, y, z \in X$. So

$$\begin{aligned}
\|A(x+y-z) - A(x) - A(y) + A(z)\| &\geq \|\rho_1(A(x+y+z) - A(x) - A(y) - A(z))\| \\
&\quad + \|\rho_2(A(x-y+z) - A(x) + A(y) - A(z))\|
\end{aligned}$$

for all $x, y, z \in X$. By Lemma 4.1, the mapping $A : X \to Y$ is additive.

Now, let $T : X \to Y$ be another additive mapping satisfying (5.2). Then we have

$$\begin{aligned}
\|A(x) - T(x)\| &= \left\|2^q A\left(\frac{x}{2^q}\right) - 2^q T\left(\frac{x}{2^q}\right)\right\| \\
&\leq \left\|2^q A\left(\frac{x}{2^q}\right) - 2^q f\left(\frac{x}{2^q}\right)\right\| + \left\|2^q T\left(\frac{x}{2^q}\right) - 2^q f\left(\frac{x}{2^q}\right)\right\| \\
&\leq \frac{2^q}{|\rho_1| - 1} \Psi\left(\frac{x}{2^q}, \frac{x}{2^q}, 0\right),
\end{aligned}$$

which tends to zero as $q \to \infty$ for all $x \in X$. So we can conclude that $A(x) = T(x)$ for all $x \in X$. This proves the uniqueness of A. □

Corollary 5.2. *Let $r > 1$ and θ be nonnegative real numbers, and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and (4.6). Then there exists a unique additive mapping $A : X \to Y$ such that*

$$\|f(x) - A(x)\| \leq \frac{2\theta}{(2^r - 2)(|\rho_1| - 1)} \|x\|^r$$

for all $x \in X$.

Theorem 5.3. *Let $\varphi : X^3 \to [0, \infty)$ be a function and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$, (4.3) and*

$$\Psi(x, y, z) := \sum_{j=0}^{\infty} \frac{1}{2^j} \varphi(2^j x, 2^j y, 2^j z) < \infty \tag{5.5}$$

for all $x, y, z \in X$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \leq \frac{1}{2(|\rho_1| - 1)} \Psi(x, x, 0)$$

for all $x \in X$.

Proof. It follows from (5.3) that

$$\left\|f(x)-\frac{1}{2}f(2x)\right\| \le \frac{1}{2(|\rho_1|-1)}\varphi(x,x,0)$$

for all $x \in X$. Hence

$$\begin{aligned}\left\|\frac{1}{2^l}f(2^lx)-\frac{1}{2^m}f(2^mx)\right\| &\le \sum_{j=l}^{m-1}\left\|\frac{1}{2^j}f\left(2^jx\right)-\frac{1}{2^{j+1}}f\left(2^{j+1}x\right)\right\| \\ &\le \sum_{j=l}^{m-1}\frac{1}{2^{j+1}(|\rho_1|-1)}\varphi(2^jx,2^jx,0) \qquad (5.6)\end{aligned}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (5.6) that the sequence $\{\frac{1}{2^n}f(2^nx)\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{\frac{1}{2^n}f(2^nx)\}$ converges. So one can define the mapping $A : X \to Y$ by

$$A(x) := \lim_{n\to\infty}\frac{1}{2^n}f(2^nx)$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (5.6), we get (5.6).

The rest of the proof is similar to the proof of Theorem 5.1. □

Corollary 5.4. *Let $r < 1$ and θ be positive real numbers, and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and (4.6). Then there exists a unique additive mapping $A : X \to Y$ such that*

$$\|f(x)-A(x)\| \le \frac{2\theta}{(2-2^r)(|\rho_1|-1)}\|x\|^r$$

for all $x \in X$.

References

[1] M. Adam, *On the stability of some quadratic functional equation*, J. Nonlinear Sci. Appl. **4** (2011), 50–59.

[2] T. Aoki, *On the stability of the linear transformation in Banach spaces*, J. Math. Soc. Japan **2** (1950), 64–66.

[3] L. Cădariu, L. Găvruta, P. Găvruta, *On the stability of an affine functional equation,* J. Nonlinear Sci. Appl. **6** (2013), 60–67.

[4] L. Cădariu, V. Radu, *Fixed points and the stability of Jensen's functional equation*, J. Inequal. Pure Appl. Math. **4**, no. 1, Art. ID 4 (2003).

[5] L. Cădariu, V. Radu, *On the stability of the Cauchy functional equation: a fixed point approach*, Grazer Math. Ber. **346** (2004), 43–52.

[6] L. Cădariu, V. Radu, *Fixed point methods for the generalized stability of functional equations in a single variable*, Fixed Point Theory Appl. **2008**, Art. ID 749392 (2008).

[7] A. Chahbi, N. Bounader, *On the generalized stability of d'Alembert functional equation*, J. Nonlinear Sci. Appl. **6** (2013), 198–204.

[8] P. W. Cholewa, *Remarks on the stability of functional equations,* Aequationes Math. **27** (1984), 76–86.

[9] J. Diaz, B. Margolis, *A fixed point theorem of the alternative for contractions on a generalized complete metric space*, Bull. Amer. Math. Soc. **74** (1968), 305–309.

[10] N. Eghbali, J. M. Rassias, M. Taheri, *On the stability of a k-cubic functional equation in intuitionistic fuzzy n-normed spaces,* Results Math. **70** (2016), 233–248.

[11] G. Z. Eskandani, P. Găvruta, *Hyers-Ulam-Rassias stability of pexiderized Cauchy functional equation in 2-Banach spaces,* J. Nonlinear Sci. Appl. **5** (2012), 459–465.

[12] P. Găvruta, *A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings*, J. Math. Anal. Appl. **184** (1994), 431–436.

[13] D. H. Hyers, *On the stability of the linear functional equation*, Proc. Natl. Acad. Sci. U.S.A. **27** (1941), 222–224.

[14] G. Isac, Th. M. Rassias, *Stability of ψ-additive mappings: Appications to nonlinear analysis*, Internat. J. Math. Math. Sci. **19** (1996), 219–228.

[15] H. Khodaei, *On the stability of additive, quadratic, cubic and quartic set-valued functional equations,* Results Math. **68** (2015), 1–10.

[16] D. Miheţ, V. Radu, *On the stability of the additive Cauchy functional equation in random normed spaces*, J. Math. Anal. Appl. **343** (2008), 567–572.
[17] C. Park, *Orthogonal stability of a cubic-quartic functional equation*, J. Nonlinear Sci. Appl. **5** (2012), 28–36.
[18] C. Park, *Additive ρ-functional inequalities and equations*, J. Math. Inequal. **9** (2015), 17–26.
[19] C. Park, *Additive ρ-functional inequalities in non-Archimedean normed spaces*, J. Math. Inequal. **9** (2015), 397–407.
[20] C. Park, K. Ghasemi, S. G. Ghaleh, S. Jang, *Approximate n-Jordan $*$-homomorphisms in C^*-algebras,* J. Comput. Anal. Appl. **15** (2013), 365–368.
[21] C. Park, A. Najati, S. Jang, *Fixed points and fuzzy stability of an additive-quadratic functional equation,* J. Comput. Anal. Appl. **15** (2013), 452–462.
[22] V. Radu, *The fixed point alternative and the stability of functional equations*, Fixed Point Theory **4** (2003), 91–96.
[23] Th. M. Rassias, *On the stability of the linear mapping in Banach spaces*, Proc. Amer. Math. Soc. **72** (1978), 297–300.
[24] K. Ravi, E. Thandapani, B. V. Senthil Kumar, *Solution and stability of a reciprocal type functional equation in several variables,* J. Nonlinear Sci. Appl. **7** (2014), 18–27.
[25] S. Shagholi, M. Bavand Savadkouhi, M. Eshaghi Gordji, *Nearly ternary cubic homomorphism in ternary Fréchet algebras,* J. Comput. Anal. Appl. **13** (2011), 1106–1114.
[26] D. Shin, C. Park, Sh. Farhadabadi, *On the superstability of ternary Jordan C^*-homomorphisms,* J. Comput. Anal. Appl. **16** (2014), 964–973.
[27] D. Shin, C. Park, Sh. Farhadabadi, *Stability and superstability of J^*-homomorphisms and J^*-derivations for a generalized Cauchy-Jensen equation,* J. Comput. Anal. Appl. **17** (2014), 125–134.
[28] F. Skof, *Propriet locali e approssimazione di operatori*, Rend. Sem. Mat. Fis. Milano **53** (1983), 113–129.
[29] S. M. Ulam, *A Collection of the Mathematical Problems*, Interscience Publ. New York, 1960.
[30] Z. Wang, *Stability of two types of cubic fuzzy set-valued functional equations,* Results Math. **70** (2016), 1–14.
[31] C. Zaharia, *On the probabilistic stability of the monomial functional equation,* J. Nonlinear Sci. Appl. **6** (2013), 51–59.
[32] S. Zolfaghari, *Approximation of mixed type functional equations in p-Banach spaces*, J. Nonlinear Sci. Appl. **3** (2010), 110–122.

Choonkil Park
Research Institute for Natural Sciences, Hanyang University, Seoul 04763, Republic of Korea
E-mail address: baak@hanyang.ac.kr

Dong Yun Shin
Department of Mathematics, University of Seoul, Seoul 02504, Republic of Korea
E-mail address: dyshin@uos.ac.kr

George A. Anastassiou
Department of Mathematical Sciences, University of Memphis, Memphis, TN 38152, USA
E-mail address: ganastss@memphis.edu

TABLE OF CONTENTS, JOURNAL OF COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 27, NO. 2, 2019

TABLE OF CONTENTS, JOURNAL OF COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 27, NO. 2, 2019

(continued)

Volume 27, Number 3 September 2019
ISSN:1521-1398 PRINT,1572-9206 ONLINE

Journal of Computational Analysis and Applications

EUDOXUS PRESS,LLC

Journal of Computational Analysis and Applications
ISSNno.'s:1521-1398 PRINT,1572-9206 ONLINE

SCOPE OF THE JOURNAL

An international publication of Eudoxus Press, LLC
(fifteen times annually)

Editor in Chief: George Anastassiou
Department of Mathematical Sciences,
University of Memphis, Memphis, TN 38152-3240, U.S.A
ganastss@memphis.edu
http://www.msci.memphis.edu/~ganastss/jocaaa

The main purpose of "J.Computational Analysis and Applications" is to publish high quality research articles from all subareas of Computational Mathematical Analysis and its many potential applications and connections to other areas of Mathematical Sciences. Any paper whose approach and proofs are computational,using methods from Mathematical Analysis in the broadest sense is suitable and welcome for consideration in our journal, except from Applied Numerical Analysis articles. Also plain word articles without formulas and proofs are excluded. The list of possibly connected mathematical areas with this publication includes, but is not restricted to: Applied Analysis, Applied Functional Analysis, Approximation Theory, Asymptotic Analysis, Difference Equations, Differential Equations, Partial Differential Equations, Fourier Analysis, Fractals, Fuzzy Sets, Harmonic Analysis, Inequalities, Integral Equations, Measure Theory, Moment Theory, Neural Networks, Numerical Functional Analysis, Potential Theory, Probability Theory, Real and Complex Analysis, Signal Analysis, Special Functions, Splines, Stochastic Analysis, Stochastic Processes, Summability, Tomography, Wavelets, any combination of the above, e.t.c.

"J.Computational Analysis and Applications" is a peer-reviewed Journal. See the instructions for preparation and submission of articles to JoCAAA. Assistant to the Editor: Dr.Razvan Mezei,mezei_razvan@yahoo.com, Madison,WI,USA.

Journal of Computational Analysis and Applications(JoCAAA) is published by **EUDOXUS PRESS,LLC**,1424 Beaver Trail Drive,Cordova,TN38016,USA,anastassioug@yahoo.com http://www.eudoxuspress.com. **Annual Subscription Prices**:For USA and Canada,Institutional:Print $800, Electronic OPEN ACCESS. Individual:Print $400. For any other part of the world add $160 more(handling and postages) to the above prices for Print. No credit card payments.

JoCAAA is reviewed and abstracted by AMS Mathematical Reviews,MATHSCI,and Zentralblaat MATH.

Editorial Board
Associate Editors of Journal of Computational Analysis and Applications

Dartmouth College
8000 Cummings Hall,
Hanover, NH 03755-8000
603-646-3843 (X 3546 Secr.)
e-mail:george.cybenko@dartmouth.edu
Approximation Theory and Neural Networks

Sever S. Dragomir
School of Computer Science and Mathematics, Victoria University,
PO Box 14428,
Melbourne City,
MC 8001, AUSTRALIA
Tel. +61 3 9688 4437
Fax +61 3 9688 4050
sever.dragomir@vu.edu.au
Inequalities, Functional Analysis, Numerical Analysis, Approximations, Information Theory, Stochastics.

Oktay Duman
TOBB University of Economics and Technology,
Department of Mathematics, TR-06530,
Ankara, Turkey,
oduman@etu.edu.tr
Classical Approximation Theory, Summability Theory, Statistical Convergence and its Applications

Saber N. Elaydi
Department Of Mathematics
Trinity University
715 Stadium Dr.
San Antonio, TX 78212-7200
210-736-8246
e-mail: selaydi@trinity.edu
Ordinary Differential Equations, Difference Equations

J .A. Goldstein
Department of Mathematical Sciences
The University of Memphis
Memphis, TN 38152
901-678-3130
jgoldste@memphis.edu
Partial Differential Equations, Semigroups of Operators

H. H. Gonska
Department of Mathematics
University of Duisburg
Duisburg, D-47048
Germany
011-49-203-379-3542
e-mail: heiner.gonska@uni-due.de
Approximation Theory, Computer Aided Geometric Design

John R. Graef
Department of Mathematics
University of Tennessee at Chattanooga
Chattanooga, TN 37304 USA
John-Graef@utc.edu
Ordinary and functional differential equations, difference equations, impulsive systems, differential inclusions, dynamic equations on time scales, control theory and their applications

Weimin Han
Department of Mathematics
University of Iowa
Iowa City, IA 52242-1419
319-335-0770
e-mail: whan@math.uiowa.edu
Numerical analysis, Finite element method, Numerical PDE, Variational inequalities, Computational mechanics

Tian-Xiao He
Department of Mathematics and Computer Science
P.O. Box 2900, Illinois Wesleyan University
Bloomington, IL 61702-2900, USA
Tel (309)556-3089
Fax (309)556-3864
the@iwu.edu
Approximations, Wavelet, Integration Theory, Numerical Analysis, Analytic Combinatorics

Margareta Heilmann
Faculty of Mathematics and Natural Sciences, University of Wuppertal
Gaußstraße 20
D-42119 Wuppertal, Germany,
heilmann@math.uni-wuppertal.de
Approximation Theory (Positive Linear Operators)

Xing-Biao Hu
Institute of Computational Mathematics
AMSS, Chinese Academy of Sciences
Beijing, 100190, CHINA
hxb@lsec.cc.ac.cn

Computational Mathematics

Jong Kyu Kim
Department of Mathematics
Kyungnam University
Masan Kyungnam,631-701,Korea
Tel 82-(55)-249-2211
Fax 82-(55)-243-8609
jongkyuk@kyungnam.ac.kr
Nonlinear Functional Analysis,
Variational Inequalities, Nonlinear
Ergodic Theory, ODE, PDE,
Functional Equations.

Robert Kozma
Department of Mathematical Sciences
The University of Memphis
Memphis, TN 38152, USA
rkozma@memphis.edu
Neural Networks, Reproducing Kernel
Hilbert Spaces,
Neural Percolation Theory

Mustafa Kulenovic
Department of Mathematics
University of Rhode Island
Kingston, RI 02881,USA
kulenm@math.uri.edu
Differential and Difference
Equations

Irena Lasiecka
Department of Mathematical Sciences
University of Memphis
Memphis, TN 38152
PDE, Control Theory, Functional
Analysis, lasiecka@memphis.edu

Burkhard Lenze
Fachbereich Informatik
Fachhochschule Dortmund
University of Applied Sciences
Postfach 105018
D-44047 Dortmund, Germany
e-mail: lenze@fh-dortmund.de
Real Networks, Fourier Analysis,
Approximation Theory

Hrushikesh N. Mhaskar
Department Of Mathematics
California State University
Los Angeles, CA 90032
626-914-7002
e-mail: hmhaska@gmail.com
Orthogonal Polynomials,
Approximation Theory, Splines,
Wavelets, Neural Networks

Ram N. Mohapatra
Department of Mathematics
University of Central Florida
Orlando, FL 32816-1364
tel.407-823-5080
ram.mohapatra@ucf.edu
Real and Complex Analysis,
Approximation Th., Fourier
Analysis, Fuzzy Sets and Systems

Gaston M. N'Guerekata
Department of Mathematics
Morgan State University
Baltimore, MD 21251, USA
tel: 1-443-885-4373
Fax 1-443-885-8216
Gaston.N'Guerekata@morgan.edu
nguerekata@aol.com
Nonlinear Evolution Equations,
Abstract Harmonic Analysis,
Fractional Differential Equations,
Almost Periodicity & Almost
Automorphy

M.Zuhair Nashed
Department Of Mathematics
University of Central Florida
PO Box 161364
Orlando, FL 32816-1364
e-mail: znashed@mail.ucf.edu
Inverse and Ill-Posed problems,
Numerical Functional Analysis,
Integral Equations, Optimization,
Signal Analysis

Mubenga N. Nkashama
Department OF Mathematics
University of Alabama at Birmingham
Birmingham, AL 35294-1170
205-934-2154
e-mail: nkashama@math.uab.edu
Ordinary Differential Equations,
Partial Differential Equations

Vassilis Papanicolaou
Department of Mathematics
National Technical University of
Athens
Zografou campus, 157 80
Athens, Greece
tel:: +30(210) 772 1722
Fax   +30(210) 772 1775
papanico@math.ntua.gr
Partial Differential Equations,
Probability

Instructions to Contributors
Journal of Computational Analysis and Applications

An international publication of Eudoxus Press, LLC, of TN.

Editor in Chief: George Anastassiou

Department of Mathematical Sciences
University of Memphis
Memphis, TN 38152-3240, U.S.A.

1. Manuscripts files in Latex and PDF and in English, should be submitted via email to the Editor-in-Chief:

Prof.George A. Anastassiou
Department of Mathematical Sciences
The University of Memphis
Memphis,TN 38152, USA.
Tel. 901.678.3144
e-mail: ganastss@memphis.edu

Authors may want to recommend an associate editor the most related to the submission to possibly handle it.

Also authors may want to submit a list of six possible referees, to be used in case we cannot find related referees by ourselves.

2. Manuscripts should be typed using any of TEX,LaTEX,AMS-TEX,or AMS-LaTEX and according to EUDOXUS PRESS, LLC. LATEX STYLE FILE. (Click HERE to save a copy of the style file.)They should be carefully prepared in all respects. Submitted articles should be brightly typed (not dot-matrix), double spaced, in ten point type size and in 8(1/2)x11 inch area per page. Manuscripts should have generous margins on all sides and should not exceed 24 pages.

3. Submission is a representation that the manuscript has not been published previously in this or any other similar form and is not currently under consideration for publication elsewhere. A statement transferring from the authors(or their employers,if they hold the copyright) to Eudoxus Press, LLC, will be required before the manuscript can be accepted for publication.The Editor-in-Chief will supply the necessary forms for this transfer.Such a written transfer of copyright,which previously was assumed to be implicit in the act of submitting a manuscript,is necessary under the U.S.Copyright Law in order for the publisher to carry through the dissemination of research results and reviews as widely and effective as possible.

4. The paper starts with the title of the article, author's name(s) (no titles or degrees), author's affiliation(s) and e-mail addresses. The affiliation should comprise the department, institution (usually university or company), city, state (and/or nation) and mail code.

The following items, 5 and 6, should be on page no. 1 of the paper.

5. An abstract is to be provided, preferably no longer than 150 words.

6. A list of 5 key words is to be provided directly below the abstract. Key words should express the precise content of the manuscript, as they are used for indexing purposes.

The main body of the paper should begin on page no. 1, if possible.

7. All sections should be numbered with Arabic numerals (such as: 1. INTRODUCTION) .
Subsections should be identified with section and subsection numbers (such as 6.1. Second-Value Subheading).
If applicable, an independent single-number system (one for each category) should be used to label all theorems, lemmas, propositions, corollaries, definitions, remarks, examples, etc. The label (such as Lemma 7) should be typed with paragraph indentation, followed by a period and the lemma itself.

8. Mathematical notation must be typeset. Equations should be numbered consecutively with Arabic numerals in parentheses placed flush right, and should be thusly referred to in the text [such as Eqs.(2) and (5)]. The running title must be placed at the top of even numbered pages and the first author's name, et al., must be placed at the top of the odd numbed pages.

9. Illustrations (photographs, drawings, diagrams, and charts) are to be numbered in one consecutive series of Arabic numerals. The captions for illustrations should be typed double space. All illustrations, charts, tables, etc., must be embedded in the body of the manuscript in proper, final, print position. In particular, manuscript, source, and PDF file version must be at camera ready stage for publication or they cannot be considered.

Tables are to be numbered (with Roman numerals) and referred to by number in the text. Center the title above the table, and type explanatory footnotes (indicated by superscript lowercase letters) below the table.

10. List references alphabetically at the end of the paper and number them consecutively. Each must be cited in the text by the appropriate Arabic numeral in square brackets on the baseline.
References should include (in the following order):
initials of first and middle name, last name of author(s)
title of article,

name of publication, volume number, inclusive pages, and year of publication.

Authors should follow these examples:

Journal Article

1. H.H.Gonska,Degree of simultaneous approximation of bivariate functions by Gordon operators, (journal name in italics) *J. Approx. Theory*, 62,170-191(1990).

Book

2. G.G.Lorentz, (title of book in italics) *Bernstein Polynomials* (2nd ed.), Chelsea,New York,1986.

Contribution to a Book

3. M.K.Khan, Approximation properties of beta operators,in(title of book in italics) *Progress in Approximation Theory* (P.Nevai and A.Pinkus,eds.), Academic Press, New York,1991,pp.483-495.

11. All acknowledgements (including those for a grant and financial support) should occur in one paragraph that directly precedes the References section.

12. Footnotes should be avoided. When their use is absolutely necessary, footnotes should be numbered consecutively using Arabic numerals and should be typed at the bottom of the page to which they refer. Place a line above the footnote, so that it is set off from the text. Use the appropriate superscript numeral for citation in the text.

13. After each revision is made please again submit via email Latex and PDF files of the revised manuscript, including the final one.

14. Effective 1 Nov. 2009 for current journal page charges, contact the Editor in Chief. Upon acceptance of the paper an invoice will be sent to the contact author. The fee payment will be due one month from the invoice date. The article will proceed to publication only after the fee is paid. The charges are to be sent, by money order or certified check, in US dollars, payable to Eudoxus Press, LLC, to the address shown on the Eudoxus homepage.

No galleys will be sent and the contact author will receive one (1) electronic copy of the journal issue in which the article appears.

15. This journal will consider for publication only papers that contain proofs for their listed results.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 27, NO.3, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC

Modified Halpern's iteration without assumptions on fixed point set in metric space

Kanyarat Cheawchan, Atid Kangtunyakarn*
Department of Mathematics, Faculty of Science,
King Mongkut's Institute of Technology Ladkrabang,
Bangkok 10520, Thailand
E-mail addresses: Kkanyarat.cheaw@gmail.com; beawrock@hotmail.com

Abstract

By improving Halpern's iteration and studing convergence theorem of [1] and [2] in a complete uniformly convex metric space, we prove convergence theorem of a finite family of nonexpansive mappings without the assumption that " the set of common fixed points of nonexpansive mappings is nonempty". We also introduce a mapping in metric space using a concept of the S-mapping defined by [3] for proving our main results.
Keywords: Convex metric space; Nonexpansive mapping; S-mapping.
Mathematics Subject Classification (2000): 31E05, 54E40, 54E50, 47H09.

1 Introduction

Many researchers have theorized for finding a solution of fixed point problems by taking advantage of iteration process, see for instance [4], [5], [6]. Halpern's iteration is a method which has been very popular for finding a solution to fixed point problem. It was introduced for the first time by Halpern [7] and defined by the vector u, x_0 belonging to a closed convex C subset of Hilbert (Banach) space and

$$x_{n+1} = \alpha_n u + (1 - \alpha_n) T x_n,$$

for all $n \geq 1$, where $T : C \to C$ is a mapping and parameter $\{\alpha_n\} \subseteq [0, 1]$.

It has been developed and improved to fixed point theorem to increase efficiency by several researchers, see example [4], [5], [6]. Although the proof of the theorem has been well developed, but the proof is still under critical conditions below;

*Corresponding author

$i)^*$ $F(T) \neq \emptyset$;

$ii)^*$ $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$.

Can we prove a convergence theorem by developing Halpern iteration and without conditions $i)^*$ and $ii)^*$ in space which is more general than Hilbert and Banach spaces?

Throughout this paper, we assume that (X, d) is a complete metric space and C is a nonempty closed convex subset of (X, d). A point x is called a fixed point of T if $Tx = x$. We use $F(T)$ to denote the set of fixed point of T. Recall the following definitions;

Definition 1.1. *The mapping $T : C \to C$ is said to be nonexpansive if*

$$d(Tx, Ty) \leq d(x, y), \quad \forall x, y \in C.$$

In 1970, Takahashi [8] introduced the following definition:

Definition 1.2. *Let (X, d) be a metric space. A mapping $W : X \times X \times [0, 1] \to X$ is said to be a convex structure on X if for each $(x, y, \lambda) \in X \times X \times [0, 1]$ and for all $u \in X$,*

$$d\big(u, W(x, y, \lambda)\big) \leq \lambda d(u, x) + (1 - \lambda) d(u, y).$$

If the mapping W is defined by $W(x, y, \lambda) = \lambda x + (1 - \lambda)y$, then it is a convex structure on a normed linear space. A metric space (X, d) together with a convex structure W is called a *convex metric space* denoted by (X, d, W). A nonempty subset C of X is said to be *convex* if $W(x, y, \lambda) \in C$ for all $x, y \in C$ and $\lambda \in [0, 1]$.

Definition 1.3. *(See [9]) A convex metric space (X, d, W) is said to be uniformly convex if for any $\epsilon > 0$, there exists $\delta = \delta(\epsilon) > 0$ such that for all $r > 0$ and $x, y, z \in X$ with $d(z, x) < r, d(z, y) < r$ and $d(x, y) \geq r\epsilon$,*

$$d\big(z, W(x, y, \frac{1}{2})\big) \leq (1 - \delta) r.$$

It is well known that Hilbert space is uniformly convex metric space.

Very recently, Hafiz Fukhar-ud-din [1] proved convergence theorem in uniformly convex metric spaces (X, d, W) with convex structure but he still assumed the fixed point set is nonempty as follows;

Theorem 1.1. *Let C be a nonempty, closed and convex subset of a uniformly convex complete metric space X with continuous convex structure W and $S, T : C \to C$ be nonexpansive mappings with $F(S) \cap F(T) \neq \emptyset$. Then the sequence $\{x_n\}$, defined by $x_{n+1} = W\left(Tx_n, W\left(Sx_n, x_n, \frac{\beta_n}{1-\alpha_n}\right), \alpha_n\right)$, Δ-converges to an element of $F(S) \cap F(T)$, where $0 < a \leq \alpha_n, \beta_n \leq b < 1$ with $\alpha_n + \beta_n < 1$.*

2

In 2013, Phuengrattana and Suantai [2] proved convergence theorem in uniformly convex metric space for infinite family of nonexpansive mapping by leveraging the map K_n, see [2] for more details, but still assume that $\bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$ as follows;

Theorem 1.2. *Let C be a nonempty compact convex subset of a complete uniformly convex metric space (X, d, W) with the property (H). Let $\{T_i\}$ be a family of nonexpansive mappings of C into itself such that $\bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$ and let $\lambda_1, \lambda_2, \ldots$ be real numbers such that $0 < \lambda_i < 1$ for every $i \in \mathbb{N}$ with $\sum_{i=1}^{\infty} \lambda_i < \infty$. Let K_n be K-mapping generated by $T_1, T_2, \ldots$ and $\lambda_1, \lambda_2, \ldots$. Assume that $x_1 \in C$ and the sequence $\{x_n\}$ is generated by*

$$x_{n+1} = W(x_n, K_n x_n, \alpha_n),$$

for all $n \geq 1$ where $\{\alpha_n\}$ is a sequence in $[0, 1]$ with $\sum_{n=1}^{\infty} \alpha_n (1 - \alpha_n) = \infty$. Then sequence $\{x_n\}$ converges to an element of $\bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$.

Inspired by Theorem 1.1 and 1.2 and improved process of Halpern's iteration, we prove convergence theorem in uniformly convex metric space for a finite family of nonexpansive mappings without using the conditions $i)^*$ and $ii)^*$.

2 Preliminaries

In this section, in order to prove our main theorem, we provide definitions, lemma and also prove the importance lemma to be used as a tool to prove the main theorem:

Lemma 2.1. *(See [8], [10]) Let (X, d, W) be a convex metric space. For each $x, y \in X$ and $\lambda, \lambda_1, \lambda_2 \in [0, 1]$, we have the following.*
(i) $W(x, x, \lambda) = x$, $W(x, y, 0) = y$ and $W(x, y, 1) = x$.
(ii) $d\big(x, W(x, y, \lambda)\big) = (1 - \lambda)d(x, y)$ and $d\big(y, W(x, y, \lambda)\big) = \lambda d(x, y)$.
(iii) $d(x, y) = d\big(x, W(x, y, \lambda)\big) + d\big(W(x, y, \lambda), y\big)$.
(iv) $|\lambda_1 - \lambda_2| d(x, y) \leq d\big(W(x, y, \lambda_1), W(x, y, \lambda_2)\big)$.

We say that a convex metric space (X, d, W) has the following properties:
(C) if $W(x, y, \lambda) = W(y, x, 1 - \lambda)$ for all $x, y \in X$ and $\lambda \in [0, 1]$,
(I) if $d\big(W(x, y, \lambda_1), W(x, y, \lambda_2)\big) \leq |\lambda_1 - \lambda_2| d(x, y)$ for all $x, y \in X$ and $\lambda_1, \lambda_2 \in [0, 1]$,
(H) if $d\big(W(x, y, \lambda), W(x, z, \lambda)\big) \leq (1 - \lambda)d(y, z)$ for all $x, y, z \in X$ and $\lambda \in [0, 1]$,
(S) if $d\big(W(x, y, \lambda), W(z, w, \lambda)\big) \leq \lambda d(x, z) + (1 - \lambda)d(y, w)$ for all $x, y, z, w \in X$ and $\lambda \in [0, 1]$.

Remark 2.2. It is easy to see that the property (C) and (H) imply continuity of a convex structure $W : X \times X \times [0, 1] \to X$ and the property (S) implies the property (H). In 2005, Aoyama et al. [10] proved that a convex metric space with property (C) and (H) has the property (S).

In 2011, Phuengrattana and Suantai [2] proved the following lemma as follows;

Lemma 2.3. *(See [2]) Property (C) holds in uniformly convex metric space.*

Remark 2.4. (See [2]) From Lemma 2.3, a uniformly convex metric space (X, d, W) with the property (H) has the property S and the convex structure W is also continuous.

Lemma 2.5. *(See [11]) Let (X, d, W) be a uniformly convex metric space with continuous convex structure. Then for arbitrary positive number ϵ, there exists $\eta = \eta(\epsilon) > 0$ such that*

$$d\big(z, W(x, y, \lambda)\big) \leq (1 - 2\min\{\lambda, 1-\lambda\}\eta)r,$$

for all $r > 0$ and $x, y, z \in X, d(z, x) \leq r, d(z, y) \leq r, d(x, y) \geq r\varepsilon$ and $\lambda \in [0, 1]$.

We introduce the following definition to use in the next section.

Definition 2.1. *Let (X, d, W) be a complete convex metric space and C be a nonempty closed convex subset of (X, d, W). Let $\{T_i\}_{i=1}^N$ be a finite family of mappings of C into C. For each $j = 1, 2, \cdots, N$, let $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j)$ where $\alpha_1^j, \alpha_2^j, \alpha_3^j \in [0, 1]$ and $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$. For every $x \in C$, we define the mapping $S : C \times C \times [0, 1] \to C$ as follows;*

$$\begin{aligned}
U_0x &= x,\\
U_1x &= W\big(T_1U_0x, W(U_0x, x, \frac{\alpha_2^1}{1-\alpha_1^1}), \alpha_1^1\big),\\
U_2x &= W\big(T_2U_1x, W(U_1x, x, \frac{\alpha_2^2}{1-\alpha_1^2}), \alpha_1^2\big),\\
&\vdots\\
U_{N-1}x &= W\big(T_{N-1}U_{N-2}x, W(U_{N-2}x, x, \frac{\alpha_2^{N-1}}{1-\alpha_1^{N-1}}), \alpha_1^{N-1}\big),\\
Sx &= U_Nx = W\big(T_NU_{N-1}x, W(U_{N-1}x, x, \frac{\alpha_2^N}{1-\alpha_1^N}), \alpha_1^N\big).
\end{aligned}$$

This mapping is called $S-$mapping generated by $T_1, T_2, \ldots, T_N$ and $\alpha_1, \alpha_2, \ldots, \alpha_N$.

Lemma 2.6. *Let C be a nonempty closed convex subset of a complete uniformly convex metric space (X, d, W) with property (H). Let $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive mappings of C into itself with $\bigcap_{i=1}^N F(T_i) \neq \emptyset$ and let $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I$, $j = 1, 2, \ldots, N$, where $I = [0, 1]$, $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$, $\alpha_1^j \in (0, 1)$ for all $j = 1, 2, ..., N-1$, $\alpha_1^N \in (0, 1]$ α_2^j, $\alpha_3^j \in [0, 1)$ for all $j = 1, 2, ..., N$. Let S be the mapping generated by $T_1, T_2, ..., T_N$ and $\alpha_1, \alpha_2, ..., \alpha_N$. Then $F(S) = \bigcap_{i=1}^N F(T_i)$.*

Proof. From Lemma 2.1 and definition of $S-$mapping, it is easy to see that $\bigcap_{i=1}^{N} F(T_i) \subseteq F(S)$. Next, we show that $F(S) \subseteq \bigcap_{i=1}^{N} F(T_i)$. To show this let $x_0 \in F(S)$ and $q \in \bigcap_{i=1}^{N} F(T_i)$, we have

$$\begin{aligned}
d(q, Sx_0) &= d\Big(q, W\big(T_N U_{N-1}x_0, W(U_{N-1}x_0, x_0, \frac{\alpha_2^N}{1-\alpha_1^N}), \alpha_1^N\big)\Big)\\
&\leq \alpha_1^N d(q, T_N U_{N-1}x_0) + (1-\alpha_1^N) d\Big(q, W(U_{N-1}x_0, x_0, \frac{\alpha_2^N}{1-\alpha_1^N})\Big)\\
&\leq \alpha_1^N d(q, T_N U_{N-1}x_0) + (1-\alpha_1^N)\Big(\frac{\alpha_2^N}{1-\alpha_1^N} d(q, U_{N-1}x_0)\\
&+ (1-\frac{\alpha_2^N}{1-\alpha_1^N}) d(q, x_0)\Big)\\
&= \alpha_1^N d(q, T_N U_{N-1}x_0) + \alpha_2^N d(q, U_{N-1}x_0) + \alpha_3^N d(q, x_0)\\
&\leq (1-\alpha_3^N) d(q, U_{N-1}x_0) + \alpha_3^N d(q, x_0)\\
&\leq (1-\alpha_3^N)\Big((1-\alpha_3^{N-1}) d(q, U_{N-2}x_0) + \alpha_3^{N-1} d(q, x_0)\Big)\\
&\quad +\alpha_3^N d(q, x_0)\\
&= (1-\alpha_3^N)(1-\alpha_3^{N-1}) d(q, U_{N-2}x_0) + \alpha_3^{N-1}(1-\alpha_3^N) d(q, x_0)\\
&\quad +\alpha_3^N d(q, x_0)\\
&= \Pi_{j=N-1}^{N}(1-\alpha_3^j) d(q, U_{N-2}x_0) + \big(1-\Pi_{j=N-1}^{N}(1-\alpha_3^j)\big) d(q, x_0)\\
&\vdots\\
&\leq \Pi_{j=3}^{N}(1-\alpha_3^j) d(q, U_2x_0) + \big(1-\Pi_{j=3}^{N}(1-\alpha_3^j)\big) d(q, x_0)\\
&= \Pi_{j=3}^{N}(1-\alpha_3^j) d\big(q, W\big(T_2U_1x_0, W(U_1x_0, x_0, \frac{\alpha_2^2}{1-\alpha_1^2}), \alpha_1^2\big)\big)\\
&\quad +\big(1-\Pi_{j=3}^{N}(1-\alpha_3^j)\big) d(q, x_0)\\
&\leq \Pi_{j=3}^{N}(1-\alpha_3^j)\Big(\alpha_1^2 d(q, T_2U_1x_0) + (1-\alpha_1^2) d\big(q, W(U_1x_0, x_0, \frac{\alpha_2^2}{1-\alpha_1^2})\big)\Big)\\
&\quad +\big(1-\Pi_{j=3}^{N}(1-\alpha_3^j)\big) d(q, x_0)\\
&\leq \Pi_{j=3}^{N}(1-\alpha_3^j)\Big(\alpha_1^2 d(q, T_2U_1x_0) + (1-\alpha_1^2)\big(\frac{\alpha_2^2}{1-\alpha_1^2} d(q, U_1x_0)\\
&\quad +(1-\frac{\alpha_2^2}{1-\alpha_1^2}) d(q, x_0)\big)\Big) + \big(1-\Pi_{j=3}^{N}(1-\alpha_3^j)\big) d(q, x_0)\\
&= \Pi_{j=3}^{N}(1-\alpha_3^j)\Big(\alpha_1^2 d(q, T_2U_1x_0) + \alpha_2^2 d(q, U_1x_0) + \alpha_3^2 d(q, x_0)\Big)\\
&\quad +\big(1-\Pi_{j=3}^{N}(1-\alpha_3^j)\big) d(q, x_0)\\
&\leq \Pi_{j=3}^{N}(1-\alpha_3^j)\Big((1-\alpha_3^2) d(q, U_1x_0) + \alpha_3^2 d(q, x_0)\big)\Big)\\
&\quad +\big(1-\Pi_{j=3}^{N}(1-\alpha_3^j)\big) d(q, x_0)
\end{aligned}$$

$$
\begin{aligned}
&= \Pi_{j=2}^{N}(1-\alpha_3^j)d(q,U_1x_0) + \big(1-\Pi_{j=2}^{N}(1-\alpha_3^j)\big)d(q,x_0) \\
&= \Pi_{j=2}^{N}(1-\alpha_3^j)d\big(q,W\big(T_1U_0x_0,W(U_0x_0,x_0,\frac{\alpha_2^1}{1-\alpha_1^1}),\alpha_1^1\big)\big) \\
&\quad +\big(1-\Pi_{j=2}^{N}(1-\alpha_3^j)\big)d(q,x_0) \\
&= \Pi_{j=2}^{N}(1-\alpha_3^j)d\big(q,W\big(T_1x_0,x_0,\alpha_1^1\big)\big) + \big(1-\Pi_{j=2}^{N}(1-\alpha_3^j)\big)d(q,x_0) \\
&\leq \Pi_{j=2}^{N}(1-\alpha_3^j)\big(\alpha_1^1 d(q,T_1x_0) + (1-\alpha_1^1)d(q,x_0)\big) + \big(1-\Pi_{j=2}^{N}(1-\alpha_3^j)\big)d(q,x_0) \\
&\leq \Pi_{j=2}^{N}(1-\alpha_3^j)d(q,x_0) + \big(1-\Pi_{j=2}^{N}(1-\alpha_3^j)\big)d(q,x_0) \\
&= d(q,x_0). \qquad (2.1)
\end{aligned}
$$

From (2.1), we have

$$d(q,U_1x_0) = d\big(q,W(T_1x_0,x_0,\alpha_1^1)\big) = d(q,x_0) \text{ and } d(q,T_1x_0) = d(q,x_0).$$

Suppose $x_0 \neq T_1x_0$, we have $d(x_0,T_1x_0) > 0$. Choose $r = d(q,x_0) > 0$ and $\epsilon = \frac{d(x_0,T_1x_0)}{r}$, we have $d(q,T_1x_0) \leq d(q,x_0) = r$, $d(q,x_0) \leq r$ and $d(x_0,T_1x_0) \geq r\epsilon$. From Lemma 2.5, we have

$$d\big(q,W(T_1x_0,x_0,\alpha_1^1)\big) < d(q,x_0) \text{ for } \alpha_1^1 \in (0,1).$$

This is a contradiction, we have $x_0 = T_1x_0$ that is $x_0 \in F(T_1)$. Since $x_0 = T_1x_0$ definition of U_1 and Lemma 2.1, we have $U_1x_0 = x_0$ that is $x_0 \in F(U_1)$. From (2.1) and $x_0 = U_1x_0$, we have

$$d(q,U_2x_0) = d\big(q,W\big(T_2x_0,x_0,\alpha_1^2\big)\big) = d(q,x_0) \text{ and } d(q,T_2x_0) = d(q,x_0).$$

Suppose $x_0 \neq T_2x_0$, we have $d(x_0,T_2x_0) > 0$. Choose $r_1 = d(q,x_0) > 0$ and $\epsilon = \frac{d(x_0,T_2x_0)}{r_1}$, we have $d(q,T_2x_0) \leq d(q,x_0) = r_1$, $d(q,x_0) \leq r_1$ and $d(x_0,T_2x_0) \geq r_1\epsilon$. From Lemma 2.5, we have

$$d\big(q,W(T_2x_0,x_0,\alpha_1^2)\big) < d(q,x_0) \text{ for } \alpha_1^2 \in (0,1).$$

This is a contradiction, we have $x_0 = T_2x_0$ that is $x_0 \in F(T_2)$. Since $x_0 = T_2x_0$ definition of U_2 and Lemma 2.1, we have $U_2x_0 = x_0$ that is $x_0 \in F(U_2)$.
By continuing on this way, we can conclude that $x_0 \in F(T_i)$ and $x_0 \in F(U_i)$ for all $i = 1,2,\ldots,N-1$.
Finally, we show that $x_0 \in F(T_N)$. From definition of S and Lemma 2.1, we have

$$Sx_0 = W\big(T_NU_{N-1}x_0,W(U_{N-1}x_0,x_0,\frac{\alpha_2^N}{1-\alpha_1^N}),\alpha_1^N\big) = W(T_Nx_0,x_0,\alpha_1^N).$$

Since

$$0 = d(x_0,Sx_0) = d(x_0,W(T_Nx_0,x_0,\alpha_1^N)) = \alpha_1^N d(T_Nx_0,x_0),$$

we have $x_0 = T_Nx_0$, that is, $x_0 \in F(T_N)$. Hence $F(S) \subseteq \bigcap_{i=1}^{N} F(T_i)$. $\square$

Remark 2.7. From Theorem 2.6, we have the mapping S is nonexpansive. To show this, let $x, y \in C$. By Remark 2.4, we have

$$\begin{aligned}
d(Sx, Sy) &= d\Big(W\big(T_N U_{N-1}x, W(U_{N-1}x, x, \frac{\alpha_2^N}{1-\alpha_1^N}), \alpha_1^N\big), \\
&\quad W\big(T_N U_{N-1}y, W(U_{N-1}y, y, \frac{\alpha_2^N}{1-\alpha_1^N}), \alpha_1^N\big)\Big) \\
&\leq \alpha_1^N d(T_N U_{N-1}x, T_N U_{N-1}y) \\
&\quad +(1-\alpha_1^N) d\big(W(U_{N-1}x, x, \frac{\alpha_2^N}{1-\alpha_1^N}), W(U_{N-1}y, y, \frac{\alpha_2^N}{1-\alpha_1^N})\big) \\
&\leq \alpha_1^N d(T_N U_{N-1}x, T_N U_{N-1}y) \\
&\quad +(1-\alpha_1^N)\Big(\frac{\alpha_2^N}{1-\alpha_1^N} d(U_{N-1}x, U_{N-1}y) + \big(1-\frac{\alpha_2^N}{1-\alpha_1^N}\big) d(x,y)\Big) \\
&\leq \alpha_1^N d(U_{N-1}x, U_{N-1}y) + \alpha_2^N d(U_{N-1}x, U_{N-1}y) + \alpha_3^N d(x,y) \\
&= (1-\alpha_3^N) d(U_{N-1}x, U_{N-1}y) + \alpha_3^N d(x,y) \\
&\leq (1-\alpha_3^N)\big((1-\alpha_3^{N-1}) d(U_{N-2}x, U_{N-2}y) + \alpha_3^{N-1} d(x,y)\big) + \alpha_3^N d(x,y) \\
&= \Pi_{j=N-1}^N (1-\alpha_3^j) d(U_{N-2}x, U_{N-2}y) + \big(1 - \Pi_{j=N-1}^N (1-\alpha_3^j)\big) d(x,y) \\
&\leq \\
&\vdots \\
&= \Pi_{j=1}^N (1-\alpha_3^j) d(U_0 x, U_0 y) + \big(1 - \Pi_{j=1}^N (1-\alpha_3^j)\big) d(x,y) \\
&= d(x,y).
\end{aligned}$$

Example 2.8. Let the metric $d : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$d(x,y) = \max\{|x_1 - y_1|, |x_2 - y_2|\},$$

for all $x = (x_1, x_2), y = (y_1, y_2) \in \mathbb{R}^2$.

Let the mapping $W : \mathbb{R}^2 \times \mathbb{R}^2 \times [0,1] \to \mathbb{R}^2$ be defined by

$$W(x,y,\lambda) = \lambda x + (1-\lambda) y = (\lambda x_1 + (1-\lambda) y_1, \lambda x_2 + (1-\lambda) y_2),$$

for all $x = (x_1, x_2), y = (y_1, y_2) \in \mathbb{R}^2$.

For every $i = 1, 2, \ldots, N$, let the mapping $T_i : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by

$$T_i x = \frac{ix}{i+1},$$

for all $x = (x_1, x_2) \in \mathbb{R}^2$. Let S be the mapping generated by $T_1, T_2, ..., T_N$ and $\alpha_1, \alpha_2, ..., \alpha_N$, where $\alpha_j = \left(\alpha_1^j, \alpha_2^j, \alpha_3^j\right) = \left(\frac{1}{2^j}, \frac{2^j - 1}{2^i(2+j)}, \frac{2^j-1}{2^j} \cdot \left(\frac{j+1}{j+2}\right)\right)$ for all $j = 1, 2, \ldots, N$. Then $F(S) = \bigcap_{i=1}^N F(T_i)$.

Solution. From the properties of $d, W, \mathbb{R}^2$, $\left(\mathbb{R}^2, d, W\right)$ is a complete uniformly

7

convex metric space. Next, we show that $\left(\mathbb{R}^2, d, W\right)$ has a property (H). Let $x = (x_1, x_2), y = (y_1, y_2), z = (z_1, z_2) \in \mathbb{R}^2$ and $a \in [0, 1]$, then

$$d\left(W\left(x, y, a\right), W\left(x, z, a\right)\right) = \max\left\{(1-a)\left|y_1 - z_1\right|, (1-a)\left|y_2 - z_2\right|\right\}.$$

Since $d\left(y, z\right) = \max\left\{\left|y_1 - z_1\right|, \left|y_2 - z_2\right|\right\}$, we get

$$d\left(W\left(x, y, a\right), W\left(x, z, a\right)\right) \leq (1-a)\, d\left(y, z\right).$$

Then $\left(\mathbb{R}^2, d, W\right)$ has a property (H).

It is clear that T_i is a nonexpansive mapping for all $i = 1, 2, \ldots, N$ and $\bigcap_{i=1}^{N} F\left(T_i\right) = \{0\}$, due to the properties of T_i. From Lemma 2.6, we have $F\left(S\right) = \bigcap_{i=1}^{N} F\left(T_i\right)$.

Remark 2.9. Lemma 2.8 in [3] is a spacial case of Lemma 2.6.

3 Main Results

Theorem 3.1. *Let C be a nonempty closed convex subset of a complete uniformly convex metric space (X, d, W) with property (H). Let $\{T_i\}_{i=1}^{N}$ be a finite family of nonexpansive mappings of C into itself and let $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I$, $j = 1, 2, 3, ..., N$, where $I = [0, 1]$ $, \alpha_1^j + \alpha_2^j + \alpha_3^j = 1$, $\alpha_1^j \in (0, 1)$ for all $j = 1, 2, ..., N-1$, $\alpha_1^N \in (0, 1]$ α_2^j, $\alpha_3^j \in [0, 1)$ for all $j = 1, 2, ..., N$. Let S be the mapping generated by $T_1, T_2, ..., T_N$ and $\alpha_1, \alpha_2, ..., \alpha_N$. Let $\{x_n\}$ be a sequence generated by $x_1, u \in C$ and*

$$x_{n+1} = W\left(u, Sx_n, \alpha\right) \tag{3.1}$$

for all $n \geq 1$ and $\alpha \in [0, 1]$. Then the following statements are equivalent:

i) The sequence $\{x_n\}$ converges to $z \in \bigcap_{i=1}^{N} F\left(T_i\right)$,

ii) $\lim_{n\to\infty} d\left(x_n, T_i x_n\right) = 0$ for all $i = 1, 2, \ldots, N$.

Proof. $i) \Rightarrow ii)$. Since $\{x_n\}$ converges to $z \in \bigcap_{i=1}^{N} F\left(T_i\right)$ and

$$d\left(x_n, T_i x_n\right) \leq d\left(x_n, z\right) + d\left(T_i x_n, z\right) \leq 2d\left(x_n, z\right)$$

for all $i = 1, 2, \ldots, N$, so we can prove that ii) is true.

For the next result, we prove $ii) \Rightarrow i)$. For every $n \in \mathbb{N}$ and remark (S property), we have

$$\begin{aligned}
d\left(x_{n+1}, x_n\right) &\leq d\left(W\left(u, Sx_n, \alpha\right), W\left(u, Sx_{n-1}, \alpha\right)\right) \\
&\leq (1-\alpha)\, d\left(x_n, x_{n-1}\right) \\
&\leq (1-\alpha)^2\, d\left(x_{n-1}, x_{n-2}\right) \\
&\vdots \\
&\leq (1-\alpha)^n\, d\left(x_1, x_0\right).
\end{aligned}$$

Using the benefits from the inequality above, we have

$$\begin{aligned} d\left(x_{n+k}, x_n\right) &\leq \sum_{j=n}^{n+k-1} d\left(x_{j+1}, x_j\right) \\ &\leq \sum_{j=n}^{n+k-1} (1-\alpha)^j\, d\left(x_1, x_0\right) \\ &\leq \frac{(1-\alpha)^n}{\alpha} \cdot d\left(x_1, x_0\right), \end{aligned}$$

for all $k \in \mathbb{N}$. Since $\lim_{n\to\infty}(1-\alpha)^n = 0$, we get the sequence $\{x_n\}$ is Cauchy. Then there exists $z \in C$ such that $\lim_{n\to\infty} x_n = z$.

From the condition $ii)$ and

$$d\left(z, T_i z\right) \leq d\left(x_n, z\right) + d\left(x_n, T_i x_n\right) + d\left(T_i x_n, T_i z\right) \leq 2d\left(x_n, z\right) + d\left(x_n, T_i x_n\right),$$

for all $i = 1, 2, \ldots, N$, we have $d\left(z, T_i z\right) = 0$. We can conclude that $z \in \bigcap_{i=1}^N F\left(T_i\right)$. Hence the sequence $\{x_n\}$ converges to $z \in \bigcap_{i=1}^N F\left(T_i\right)$. □

Theorem 3.2. *Let C be a nonempty closed convex subset of a complete uniformly convex metric space (X, d, W) with property (H). Let $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive mappings of C into itself with $\lim_{n\to\infty} d\left(x_n, T_i x_n\right) = 0$ for all $i = 1, 2, \ldots, N$ and let $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I$, $j = 1, 2, 3, ..., N$, where $I = [0, 1]$, $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$, $\alpha_1^j \in (0, 1)$ for all $j = 1, 2, ..., N-1$, $\alpha_1^N \in (0, 1]$, α_2^j, $\alpha_3^j \in [0, 1)$ for all $j = 1, 2, ..., N$. Let S be the mapping generated by $T_1, T_2, ..., T_N$ and $\alpha_1, \alpha_2, ..., \alpha_N$. Let $\{x_n\}$ be a sequence generated by $x_1, u \in C$ and*

$$x_{n+1} = W\left(u, Sx_n, \alpha\right) \tag{3.2}$$

for all $n \geq 1$ and $\alpha \in [0, 1]$. Then the sequence $\{x_n\}$ converges to $z \in F(S)$.

Proof. The sequence $\{x_n\}$ is a Cauchy by using the same method of Theorem 3.1. Then there exists $z \in C$ such that $\lim_{n\to\infty} x_n = z$.

Since $\lim_{n\to\infty} d\left(x_n, T_i x_n\right) = 0$ for all $i = 1, 2, \ldots, N$ and

$$d\left(z, T_i z\right) \leq 2d\left(x_n, z\right) + d\left(x_n, T_i x_n\right),$$

for all $i = 1, 2, \ldots, N$, we have $z \in \bigcap_{i=1}^N F\left(T_i\right)$. From Lemma 2.6, we have $z \in F(S)$. Hence the sequence $\{x_n\}$ converges to $z \in F(S)$. □

If the condition ii) in Theorem 3.1 and 3.2 are replaced by
"$\liminf_{n\to\infty} d\left(x_n, \bigcap_{i=1}^N F\left(T_i\right)\right) = 0$" where $d\left(x_n, \bigcap_{i=1}^N F\left(T_i\right)\right)$
$= \inf_{v \in \bigcap_{i=1}^N F(T_i)} d\left(x_n, v\right)$". Then, the following theorems are still true.

Theorem 3.3. *Let C be a nonempty closed convex subset of a complete uniformly convex metric space (X, d, W) with property (H). Let $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive mappings of C into itself and let $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in$*

$I \times I \times I$, $j = 1, 2, 3, ..., N$, where $I = [0,1]$, $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$, $\alpha_1^j \in (0,1)$ for all $j = 1, 2, ..., N-1$, $\alpha_1^N \in (0,1]$, α_2^j, $\alpha_3^j \in [0,1)$ for all $j = 1, 2, ..., N$. Let S be the mapping generated by $T_1, T_2, ..., T_N$ and $\alpha_1, \alpha_2, ..., \alpha_N$. Let $\{x_n\}$ be a sequence generated by $x_1, u \in C$ and

$$x_{n+1} = W(u, Sx_n, \alpha) \tag{3.3}$$

for all $n \geq 1$ and $\alpha \in [0,1]$. Then the following statements are equivalent:
i) The sequence $\{x_n\}$ converges to $z \in \bigcap_{i=1}^N F(T_i)$.
ii) $\liminf_{n\to\infty} d\left(x_n, \bigcap_{i=1}^N F(T_i)\right) = 0$ where $d\left(x_n, \bigcap_{i=1}^N F(T_i)\right)$
$= \inf_{v \in \bigcap_{i=1}^N F(T_i)} d(x_n, v)$.

Proof. It is very clear that case $i) \Rightarrow ii)$. Next, we show that case $ii) \Rightarrow i)$. Using the same method in Theorem 3.1, we obtain that the sequence $\{x_n\}$ is a Cauchy sequence. Then, there exists $z \in C$ such that $\lim_{n\to\infty} x_n = z$.

For every $\varepsilon > 0$, there exists $N_0 \in \mathbb{N}$ such that

$$d\left(x_n, \cap_{j=1}^N F(T_i)\right) < \frac{\varepsilon}{2}$$

and

$$d(x_n, z) < \frac{\varepsilon}{2},$$

for all $n \geq N_0$.

From the above inequality, there exists $p \in \bigcap_{i=1}^N F(T_i)$ such that $d(x_n, p) < \frac{\varepsilon}{2}$.

Since

$$d(p, z) \leq d(x_n, p) + d(x_n, z) < \varepsilon$$

and ε is arbitrary, we have $d(p, z) = 0$. Hence $z = p$. Therefore, the sequence $\{x_n\}$ converges to $z \in \bigcap_{i=1}^N F(T_i)$. □

Theorem 3.4. *Let C be a nonempty closed convex subset of a complete uniformly convex metric space (X, d, W) with property (H). Let $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive mappings of C into itself with $\liminf_{n\to\infty} d\left(x_n, \bigcap_{i=1}^N F(T_i)\right) = 0$ where $d\left(x_n, \bigcap_{i=1}^N F(T_i)\right) = \inf_{v \in \bigcap_{i=1}^N F(T_i)} d(x_n, v)$ and let $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I$, $j = 1, 2, 3, ..., N$, where $I = [0,1]$, $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$, $\alpha_1^j \in (0,1)$ for all $j = 1, 2, ..., N-1$, $\alpha_1^N \in (0,1]$, α_2^j, $\alpha_3^j \in [0,1)$ for all $j = 1, 2, ..., N$. Let S be the mapping generated by $T_1, T_2, ..., T_N$ and $\alpha_1, \alpha_2, ..., \alpha_N$. Let $\{x_n\}$ be a sequence generated by $x_1, u \in C$ and*

$$x_{n+1} = W(u, Sx_n, \alpha) \tag{3.4}$$

for all $n \geq 1$ and $\alpha \in [0,1]$. Then the sequence $\{x_n\}$ converges $z \in F(S)$.

Proof. Applying the method of Theorem 3.2 and 3.3, we can obtain the desired result. □

Acknowledgements This paper was supported by the Royal Golden Jubilee (RGJ) Ph.D. Programme, the Thailand Research Fund (TRF), under Grant No. PHD/0082/2558 and the Research and Innovation Services of King Mongkuts Institute of Technology Ladkrabang.

References

[1] H. Fukhar-ud-din, Convergence of Ishikawa type iteration process for three quasi-nonexpansive mappings in a convex metric space 23(2), 83-92 2015.5

[2] W. Phuengrattana and S. Suantai, Strong Convergence Theorems for a Countable Family of Nonexpansive Mappings in Convex Metric Spaces, Abstract and Applied Analysis 2011, Article ID 929037, 18 pages (2011).9

[3] A. Kangtunyakarn and S. Suantai, Strong convergence of a new iterative scheme for a finite family of strict pseudo-contractions, Comput. Math. Appl. 60, 680-694 (2010).11

[4] S. Takahashi and W. Takahashi, Strong convergence theorem for a generalized equilibrium problem and a nonexpansive mapping in a Hilbert space, Nonlinear Analysis 69, 1025-1033 (2008).2

[5] G. Cai and S. Bu, Strong convergence theorems for variational inequality problems and fixed point problems in uniformly smooth and uniformly convex Banach spaces, J Glob Optim 56, 1529-1542 (2013).3

[6] C. Tian and Y. Song, Strong convergence of a regularization method for Rockafellars proximal point algorithm, J Glob Optim 55, 831-837 (2013).4

[7] B. Halpern, Fixed points of nonexpanding maps, Bull. Amer. Math. Soc. 73, 957-961 (1967).1

[8] W. Takahashi, A convexity in metric space and nonexpansive mappings. Kodai Mathematical Seminar Reports 22, 142-149 (1970).10

[9] T. Shimizu and W. Takahashi, Fixed points of multivalued mappings in certain convex metric spaces, Topological Methods in Nonlinear Analysis 8, 197-203 (1996).8

[10] K. Aoyama, K. Eshita, and W. Takahashi, Iteration processes for nonexpansive mappings in convex metric spaces, Proceedings of the 4th International Conference on Nonlinear Analysis and Convex Analysis, Yokohama Publishers, 31-39 (2007).6

[11] T. Shimizu, A convergence theorem to common fixed points of families of nonexpansive mappings in convex metric spaces, Proceedings of the 4th International Conference on Nonlinear Analysis and Convex Analysis, Yokohama Publishers, 575-585 (2007).7

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 27, NO.3, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC

CONVERGENCE OF DOUBLE ACTING ITERATIVE SCHEME FOR A FAMILY OF GENERALIZED φ-WEAK CONTRACTION MAPPINGS IN $CAT(0)$ SPACES

Kyung Soo Kim

Graduate School of Education, Mathematics Education
Kyungnam University, Changwon, Gyeongnam, 51767, Republic of Korea
e-mail: kksmj@kyungnam.ac.kr

Abstract. The purpose of this paper, we discuss the convergence theorems for the double acting iterative scheme to a common fixed point for a family of generalized φ-weak contraction mappings in $CAT(0)$ spaces.

1. Introduction and preliminaries

Let (X, d) be a metric space. A mapping $T : X \to X$ is a *contraction* if there exists a constant $\alpha \in (0, 1)$ such that

$$d(Tx, Ty) \leq \alpha \cdot d(x, y), \quad \forall\, x, y \in X,$$

holds. A mapping $T : X \to X$ is a *φ-weak contraction* if there exists a continuous and nondecreasing function $\varphi : [0, \infty) \to [0, \infty)$ with $\varphi(0) = 0$ such that

$$d(Tx, Ty) \leq d(x, y) - \varphi(d(x, y)), \quad \forall\, x, y \in X, \tag{1.1}$$

holds.

The concept of the φ-weak contraction was introduced by Alber and Guerre-Delabriere [1] in 1997, who proved the existence of fixed points in Hilbert spaces. Later Rhoades [15] in 2001, who extended the results of [1] to metric spaces.

Theorem 1.1. ([15]) *Let (X, d) be a complete metric space, $T : X \to X$ be a φ-weak contractive self-map on X. The T has a unique fixed point p in X.*

Remark 1.1. Theorem 1.1 is one of generalizations of the Banach contraction principle because it takes $\varphi(t) = (1 - \alpha)t$ for $\alpha \in (0, 1)$, then φ-weak contraction contains contraction as special cases.

In 2016, Xue [18] introduced a new contraction type mapping as follows.

---

[0]2010 Mathematics Subject Classification: 47H09, 47H10, 47J25, 41A65.
[0]Keywords: generalized φ-weak contraction mapping, common fixed point, double acting iterative scheme, CAT(0) space.

Definition 1.1. ([18]) A mapping $T : X \to X$ is a *generalized φ-weak contraction* if there exists a continuous and nondecreasing function $\varphi : [0, \infty) \to [0, \infty)$ with $\varphi(0) = 0$ such that

$$d(Tx, Ty) \leq d(x, y) - \varphi(d(Tx, Ty)), \quad \forall\, x, y \in X \tag{1.2}$$

holds.

We notice immediately that if $T : X \to X$ is φ-weak contraction, then T satisfies the following inequality

$$d(Tx, Ty) \leq d(x, y) - \varphi(d(Tx, Ty)), \quad \forall\, x, y \in X.$$

However, the converse is not true in general.

Example 1.1. Let $X = (-\infty, +\infty)$ be endowed with the Euclidean metric $d(x, y) = |x-y|$ and let $Tx = \frac{2}{5}x$ for each $x \in X$. Define $\varphi : [0, +\infty) \to [0, +\infty)$ by $\varphi(t) = \frac{4}{3}t$. Then T satisfies (1.2), but T does not satisfy inequality (1.1). Indeed,

$$\begin{aligned} d(Tx, Ty) &= \left|\frac{2}{5}x - \frac{2}{5}y\right| \\ &\leq |x-y| - \frac{4}{3} \cdot \frac{2}{5}|x-y| \\ &= d(x, y) - \varphi(d(Tx, Ty)) \end{aligned}$$

and

$$\begin{aligned} d(Tx, Ty) &= \left|\frac{2}{5}x - \frac{2}{5}y\right| \\ &\geq |x-y| - \frac{4}{3}|x-y| \\ &= d(x, y) - \varphi(d(x, y)) \end{aligned}$$

for all $x, y \in X$.

Example 1.2. ([18]) Let $X = (-1, +\infty)$ be endowed by $d(x, y) = |x - y|$ and let $Tx = \frac{x}{1+x}$ for each $x \in X$. Define $\varphi : [0, +\infty) \to [0, +\infty)$ by $\varphi(t) = \frac{t^2}{1+t}$. Then

$$\begin{aligned} d(Tx, Ty) &= \left|\frac{x}{1+x} - \frac{y}{1+y}\right| = \frac{|x-y|}{(1+x)(1+y)} \\ &\leq \frac{|x-y|}{1+|x-y|} = |x-y| - \frac{|x-y|^2}{1+|x-y|} \\ &= d(x, y) - \varphi(d(x, y)) \end{aligned}$$

holds for all $x, y \in X$. So T is a φ-weak contraction. However T is not a contraction.

Remark 1.2. The above examples show that the class of generalized φ-weak contractions properly includes the class of φ-weak contractions and the class of φ-weak contractions properly includes the class of contractions. In fact, let $T : X \to X$ be a contraction, there exists $\alpha \in (0,1)$ such that

$$d(Tx,Ty) \le \alpha \cdot d(x,y), \ \ \forall\, x,y \in X.$$

Then

$$\begin{aligned} d(Tx,Ty) &\le \alpha \cdot d(x,y) = d(x,y) - (1-\alpha)d(x,y) \\ &= d(x,y) - \varphi(d(x,y)), \end{aligned}$$

where, $\varphi(d(x,y)) = (1-\alpha)d(x,y)$. So, T is a φ-weak contraction. Moreover, let T be a φ-weak contraction, from property of φ, we have $d(Tx,Ty) \le d(x,y)$ and

$$\varphi(d(Tx,Ty)) \le \varphi(d(x,y)).$$

From (1.1),

$$\begin{aligned} d(Tx,Ty) &\le d(x,y) - \varphi(d(x,y)) \\ &\le d(x,y) - \varphi(d(Tx,Ty)), \quad \forall\, x,y \in X. \end{aligned}$$

Therefore, T is a generalized φ-weak contraction.

In the meantime, if T is a φ-weak contractive self mapping for one mapping φ so we do not expect that the φ-weak contractivity should be satisfied with the same function φ. Let us suppose that T is a φ-weak contractive self mapping and consider

$$\tilde{\varphi}(x) = \min\{\varphi(x/2); x/2\}.$$

Then, if $d(Tx,Ty) > \frac{1}{2}d(x,y)$, we have

$$d(Tx,Ty) \le d(x,y) - \varphi(d(Tx,Ty)) \le d(x,y) - \varphi\left(\frac{1}{2}d(x,y)\right)$$

on account of monotonocity of φ and finally

$$d(Tx,Ty) \le d(x,y) - \tilde{\varphi}(d(x,y)).$$

On the other hand, if $d(Tx,Ty) < \frac{1}{2}d(x,y)$, we get

$$d(Tx,Ty) < d(x,y) - \frac{1}{2}d(x,y) \le d(x,y) - \tilde{\varphi}(d(x,y)).$$

So T is just thr $\tilde{\varphi}$-weak contractive mapping. The continuity and monotonocity of $\tilde{\varphi}$ follows directly from properties of min function, φ and the metric.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 27, NO.3, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC

One of the most interesting aspects of metric fixed point theory is to extend a linear version of known result to the nonlinear case in metric spaces. To achieve this, Takahashi [16] introduced a convex structure in a metric space (X, d). A mapping $W : X \times X \times [0, 1] \to X$ is a *convex structure* in X if

$$d(u, W(x, y, \lambda)) \le \lambda d(u, x) + (1 - \lambda)d(u, y)$$

for all $x, y \in X$ and $\lambda \in [0, 1]$. A metric space with a convex structure W is known as a convex metric space which denoted by (X, d, W). A nonempty subset K of a convex metric space is said to be *convex* if

$$W(x, y, \lambda) \in K$$

for all $x, y \in K$ and $\lambda \in [0, 1]$. In fact, every normed linear space and its convex subsets are convex metric spaces but the converse is not true, in general (see, [16]).

Example 1.3. ([9]) Let $X = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 > 0, x_2 > 0\}$. For all $x = (x_1, x_2)$, $y = (y_1, y_2) \in X$ and $\lambda \in [0, 1]$. We define a mapping $W : X \times X \times [0, 1] \to X$ by

$$W(x, y, \lambda) = \left(\lambda x_1 + (1 - \lambda)y_1, \frac{\lambda x_1 x_2 + (1 - \lambda)y_1 y_2}{\lambda x_1 + (1 - \lambda)y_1}\right)$$

and define a metric $d : X \times X \to [0, \infty)$ by

$$d(x, y) = |x_1 - y_1| + |x_1 x_2 - y_1 y_2|.$$

Then we can show that (X, d, W) is a convex metric space but not a normed linear space.

A metric space X is a $CAT(0)$ space. This term is due to M. Gromov [6] and it is an acronym for E. Cartan, A.D. Aleksandrov and V.A. Toponogov. If X is geodesically connected, and if every geodesic triangle in X is at least as 'thin' as its comparison triangle in the Euclidean plane(see, *e.g.*, [2, p.159]). It is well known that any complete, simply connected Riemannian manifold nonpositive sectional curvature is a $CAT(0)$ space. The precise definition is given below. For a thorough discussion of these spaces and of the fundamental role they play in various branches of mathematics, see Bridson and Haefliger [2] or Burago *et al.* [4].

Let (X, d) be a metric space. A *geodesic path* joining $x \in X$ to $y \in X$ (or, more briefly, a *geodesic* from x to y) is a mapping c from a closed interval $[0, l] \subset \mathbb{R}$ to X such that $c(0) = x, c(l) = y$, and $d(c(t), c(t')) = |t - t'|$ for all $t, t' \in [0, l]$. In particular, c is an isometry and $d(x, y) = l$. The image α of c is called a *geodesic* (or, *metric*) *segment* joining x and y. When it is unique, this geodesic is denoted by $[x, y]$. The space (X, d) is said to be a *geodesic space* if every two points of X are joined by a geodesic, and X is said to be *uniquely*

geodesic if there is exactly one geodesic joining x and y for each $x, y \in X$. A subset $Y \subseteq X$ is said to be *convex* if Y includes every geodesic segment joining any two of its points.

A *geodesic triangle* $\triangle(x_1, x_2, x_3)$ is a geodesic metric space (X, d) consists of three points $x_1, x_2, x_3 \in X$ (the *vertices* of $\triangle$) and a geodesic segment between each pair of vertices (the *edges* of $\triangle$). A *comparison triangle* for the geodesic triangle $\triangle(x_1, x_2, x_3)$ in (X, d) is a triangle $\bar{\triangle}(x_1, x_2, x_3) = \triangle(\bar{x_1}, \bar{x_2}, \bar{x_3})$ in $\mathbb{R}^2$ such that $d_{\mathbb{R}^2}(\bar{x_i}, \bar{x_j}) = d(x_i, x_j)$ for $i, j \in \{1, 2, 3\}$. Such a triangle always exists(see, [2]).

A geodesic metric space is said to be a $CAT(0)$ *space* if all geodesic triangles of appropriate size satisfy the following $CAT(0)$ comparison axiom.

> Let $\triangle$ be a geodesic triangle in X and let $\bar{\triangle} \subset \mathbb{R}^2$ be a comparison triangle for $\triangle$. Then $\triangle$ is said to satisfy the $CAT(0)$ *inequality* if for all $x, y \in \triangle$ and all comparison points $\bar{x}, \bar{y} \in \bar{\triangle}$,
>
> $$d(x, y) \leq d(\bar{x}, \bar{y}).$$

Complete $CAT(0)$ spaces are often called *Hadamard spaces*(see, [11]). If x, y_1, y_2 are points of a $CAT(0)$ space and if y_0 is the midpoint of the segment $[y_1, y_2]$, which we will denote by $\frac{y_1 \oplus y_2}{2}$, then the $CAT(0)$ inequality implies

$$d^2\left(x, \frac{y_1 \oplus y_2}{2}\right) \leq \frac{1}{2}d^2(x, y_1) + \frac{1}{2}d^2(x, y_2) - \frac{1}{4}d^2(y_1, y_2).$$

This inequality is the (CN) inequality of Bruhat and Tits [3]. In fact, a geodesic space is a $CAT(0)$ space if and only if satisfies the (CN) inequality (*cf.* [2, p.163]). The above inequality has been extended by [5] as

$$\begin{aligned} &d^2(z, \alpha x \oplus (1-\alpha)y) \\ &\leq \alpha d^2(z, x) + (1-\alpha)d^2(z, y) - \alpha(1-\alpha)d^2(x, y), \end{aligned} \tag{CN*}$$

for any $\alpha \in [0, 1]$ and $x, y, z \in X$.

Let us recall that a geodesic metric space is a $CAT(0)$ space if and only if it satisfies the (CN) inequality(see, [2, p.163]). Moreover, if X is a $CAT(0)$ metric space and $x, y \in X$, then for any $\alpha \in [0, 1]$, there exists a unique point $\alpha x \oplus (1-\alpha)y \in [x, y]$ such that

$$d(z, \alpha x \oplus (1-\alpha)y) \leq \alpha d(z, x) + (1-\alpha)d(z, y) \tag{1.3}$$

for any $z \in X$ and $[x, y] = \{\alpha x \oplus (1-\alpha)y : \alpha \in [0, 1]\}$. In view of the above inequality, $CAT(0)$ space have Takahashi's convex structure

$$W(x, y, \alpha) = \alpha x \oplus (1-\alpha)y.$$

It is easy to see that for any $x, y \in X$ and $\lambda \in [0,1]$,

$$d(x, (1-\lambda)x \oplus \lambda y) = \lambda d(x,y),$$
$$d(y, (1-\lambda)x \oplus \lambda y) = (1-\lambda) d(x,y).$$

As a consequence,

$$1 \cdot x \oplus 0 \cdot y = x,$$
$$(1-\lambda)x \oplus \lambda x = \lambda x \oplus (1-\lambda)x = x.$$

Moreover, a subset K of $CAT(0)$ space X is convex if for any $x, y \in K$, we have $[x,y] \subset K$(see, [1, 10, 13]).

The purpose of this paper, we discuss the convergence theorems for the double acting iterative scheme to a common fixed point for a family of generalized φ-weak contraction mappings in $CAT(0)$ spaces.

2. Convergence theorems of double acting iterative schemes

Xue [18] proved the following very intersting fixed point theorem in complete metric space.

Theorem 2.1. ([18]) *Let (X,d) be a complete metric space and let $T : X \to X$ be a generalized φ-weak contraction. Then the Picard iterative scheme* ([14])

$$x_{n+1} = Tx_n$$

converges to the unique fixed point.

Theorem 2.2. *Let T be a generalized φ-weak contractive self mapping of a closed convex subset K of a Banach space X. Then the Picard iterative scheme*

$$x_{n+1} = Tx_n$$

converges strongly to the fixed point p with the following error estimate:

$$\|x_{n+1} - p\| \le \Phi^{-1}(\Phi(\|x_1 - p\| - n)),$$

where Φ is defined by the antiderivative

$$\Phi(t) = \int \frac{1}{\varphi(t)}\, dt, \quad \Phi(0) = 0$$

and Φ^{-1} is the inverse of Φ.

Proof. The proof is similar as Rhoades ([15], Theorem 2). However, for completeness, we give a sketch of the proof. We can obtain convergence follows from Theorem 2.1. To establish the error estimete, from (1.2) with $\lambda_n = \|x_n - p\|$,

$$\begin{aligned}\lambda_{n+1} &= \|x_{n+1} - p\| = \|Tx_n - p\| \\ &\leq \|x_n - p\| - \varphi(\|x_{n+1} - p\|) \\ &= \lambda_n - \varphi(\lambda_{n+1}),\end{aligned}$$

so, we have

$$\varphi(\lambda_{n+1}) \leq \lambda_n - \lambda_{n+1}. \tag{2.1}$$

Thus

$$\Phi(\lambda_n) - \Phi(\lambda_{n+1}) = \int_{\lambda_{n+1}}^{\lambda_n} \frac{1}{\varphi(t)}\, dt = \frac{\lambda_n - \lambda_{n+1}}{\varphi(\mu_n)},$$

for some $\lambda_{n+1} < \mu_n < \lambda_n$. Since φ is nondecreasing, from (2.1),

$$\Phi(\lambda_n) - \Phi(\lambda_{n+1}) = \frac{\lambda_n - \lambda_{n+1}}{\varphi(\mu_n)} \geq \frac{\lambda_n - \lambda_{n+1}}{\varphi(\lambda_n)} \geq 1.$$

Thus

$$\Phi(\lambda_{n+1}) \leq \Phi(\lambda_n) - 1 \leq \cdots \leq \Phi(\lambda_1) - n.$$

This completes the proof of Theorem 2.2. □

In this section, we will use $I = \{1, 2, \cdots, r\}$, where $r \geq 1$. Let $\{T_i : i \in I\}$ be a family of generalized φ-weak contraction self mappings on K. The scheme introduced in [8] is

$$x_1 \in K, \quad x_{n+1} = U_{n(r)}x_n, \quad n \geq 1, \tag{2.2}$$

where

$$\begin{aligned}U_{n(0)} &= I_d \ \ (: \text{ the identity mapping}), \\ U_{n(1)}x &= \alpha_{n(1)}x \oplus (1 - \alpha_{n(1)})T_1^n U_{n(0)}x, \\ U_{n(2)}x &= \alpha_{n(2)}x \oplus (1 - \alpha_{n(2)})T_2^n U_{n(1)}x, \\ &\vdots \\ U_{n(r-1)}x &= \alpha_{n(r-1)}x \oplus (1 - \alpha_{n(r-1)})T_{r-1}^n U_{n(r-2)}x, \\ U_{n(r)}x &= \alpha_{n(r)}x \oplus (1 - \alpha_{n(r)})T_r^n U_{n(r-1)}x,\end{aligned}$$

where $\alpha_{n(i)} \in [0, 1]$ for each $i \in I$.

After this, the we called the iterative scheme (2.2) is *double acting iterative scheme*.

The existence of fixed (or common fixed) points of one mapping (or two mappings or a family of mappigs) is not known in many situations. So the

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 27, NO.3, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC

approximation of fixed (or common fixed) points of one or more mappings by various iterations have been extensively studied in many other spaces.

In the sequel, it is assumed that

$$\mathcal{F}=\bigcap_{i=1}^{r} F(T_i)\neq\emptyset,$$

where $F(T_i)=\{x\in K : T_i x = x,\ i\in I\}$.

Now, we shall investigate the convergence of double acting iterative scheme applied to $\{T_i : i\in I\}$.

Theorem 2.3. *Let (X,d) be a complete $CAT(0)$ space, K be a closed convex subset of X, $\{T_i : i\in I\}$ be a family of generalized φ-weak contraction self mappings of K. Then the double acting iterative scheme (2.2) satisfies*

(i) $0\le\alpha_{n(i)}\le 1,\ i\in I,$
(ii) $\sum_{n=1}^{\infty}(1-\alpha_{n(1)})(1-\alpha_{n(2)})\cdots(1-\alpha_{n(r)})=\infty$

converges to commom fixed point $p\in\mathcal{F}$.

Proof. For $p\in\mathcal{F}$, using (2.2) and (1.3),

$$\begin{aligned}
&d(U_{n(1)}x_n,p)\\
&=d(\alpha_{n(1)}x_n\oplus(1-\alpha_{n(1)})T_1^nU_{n(0)}x_n,p)\\
&\le\alpha_{n(1)}d(x_n,p)+(1-\alpha_{n(1)})d(T_1^nx_n,p)\\
&\le\alpha_{n(1)}d(x_n,p)+(1-\alpha_{n(1)})[d(x_n,p)-\varphi(d(T_1^nx_n,p))]\\
&\le d(x_n,p)-(1-\alpha_{n(1)})\varphi(d(T_1^nx_n,p)).
\end{aligned}\tag{2.3}$$

Using (2.3), we get

$$\begin{aligned}
&d(U_{n(2)}x_n,p)\\
&=d(\alpha_{n(2)}x_n\oplus(1-\alpha_{n(2)})T_2^nU_{n(1)}x_n,p)\\
&\le\alpha_{n(2)}d(x_n,p)+(1-\alpha_{n(2)})d(T_2^nU_{n(1)}x_n,p)\\
&\le\alpha_{n(2)}d(x_n,p)+(1-\alpha_{n(2)})[d(U_{n(1)}x_n,p)-\varphi(d(T_2^nU_{n(1)}x_n,p))]\\
&\le\alpha_{n(2)}d(x_n,p)+(1-\alpha_{n(2)})[d(x_n,p)-(1-\alpha_{n(1)})\varphi(d(T_1^nx_n,p))]\\
&\quad-(1-\alpha_{n(2)})\varphi(d(T_2^nU_{n(1)}x_n,p))\\
&\le d(x_n,p)-(1-\alpha_{n(1)})(1-\alpha_{n(2)})\varphi(d(T_1^nx_n,p))\\
&\quad-(1-\alpha_{n(1)})(1-\alpha_{n(2)})\varphi(d(T_2^nU_{n(1)}x_n,p))
\end{aligned}$$

and

$$\begin{aligned}
&d(U_{n(3)}x_n, p)\\
&= d(\alpha_{n(3)}x_n \oplus (1-\alpha_{n(3)})T_2^nU_{n(2)}x_n, p)\\
&\le d(x_n, p) - (1-\alpha_{n(1)})(1-\alpha_{n(2)})(1-\alpha_{n(3)})\varphi(d(T_1^nx_n, p))\\
&\quad - (1-\alpha_{n(1)})(1-\alpha_{n(2)})(1-\alpha_{n(3)})\varphi(d(T_2^nU_{n(1)}x_n, p))\\
&\quad - (1-\alpha_{n(1)})(1-\alpha_{n(2)})(1-\alpha_{n(3)})\varphi(d(T_3^nU_{n(2)}x_n, p)).
\end{aligned}$$

Continue this processing, we obtain

$$\begin{aligned}
&d(U_{n(r)}x_n, p)\\
&= d(\alpha_{n(r)}x_n \oplus (1-\alpha_{n(r)})T_r^nU_{n(r-1)}x_n, p)\\
&\le d(x_n, p) - (1-\alpha_{n(1)})(1-\alpha_{n(2)})\cdots(1-\alpha_{n(r)})\varphi(d(T_1^nx_n, p))\\
&\quad - (1-\alpha_{n(1)})(1-\alpha_{n(2)})\cdots(1-\alpha_{n(r)})\varphi(d(T_2^nU_{n(1)}x_n, p))\\
&\quad\ \vdots\\
&\quad - (1-\alpha_{n(1)})(1-\alpha_{n(2)})\cdots(1-\alpha_{n(r)})\varphi(d(T_r^nU_{n(r-1)}x_n, p))\\
&\le d(x_n, p) - (1-\alpha_{n(1)})(1-\alpha_{n(2)})\cdots(1-\alpha_{n(r)})\varphi(d(T_i^nU_{n(i-1)}x_n, p)), \quad (2.4)
\end{aligned}$$

for each $i \in I$. From property of φ, we conclude

$$d(U_{n(r)}x_n, p) \le d(x_n, p),$$

that is

$$d(x_{n+1}, p) \le d(x_n, p).$$

Therefore, $\{d(x_n, p)\}$ is a nonnegative nonincreasing sequence, which converges to a limit $L \ge 0$.

(I) Most of all, we want to show that

$$d(T_i^nU_{n(i-1)}x_n, p) \ge L, \quad \forall\, n \ge 1,\ i \in I. \tag{2.5}$$

To show (2.5), it is sufficient to show that there exists $k \in \mathbb{N}$ such that

$$d(x_k, p) \le d(T_i^nU_{n(i-1)}x_n, p), \quad n \ge 1,\ i \in I.$$

To verify (2.5), suppose that $d(T_i^nU_{n(i-1)}x_n, p) < L$. Then

$$d(x_k, p) > d(T_i^nU_{n(i-1)}x_n, p), \quad \forall\, k \in \mathbb{N}, \tag{2.6}$$

for $n \ge 1$, $i \in I$. Since $\{d(x_n, p)\}$ is a nonincreasing sequence, we have

$$d(x_n, p) \ge d(x_{n+1}, p) \ge \cdots \ge L, \quad \forall\, n \ge 1. \tag{2.7}$$

Let

$$\frac{\varepsilon}{2n} = L - d(T_i^nU_{n(i-1)}x_n, p) > 0. \tag{2.8}$$

Since $\lim_{n\to\infty} d(x_n, p) = L$ and (2.6), there exists $N \in \mathbb{N}$ with

$$d(x_N, p) < d(T_i^n U_{n(i-1)} x_n, p) + \frac{\varepsilon}{4n} \tag{2.9}$$

such that

$$\begin{aligned}
|d(x_n, p) - L| &\le |L - d(T_i^n U_{n(i-1)} x_n, p)| + |d(T_i^n U_{n(i-1)} x_n, p) - d(x_n, p)| \\
&= L - d(T_i^n U_{n(i-1)} x_n, p) + d(x_n, p) - d(T_i^n U_{n(i-1)} x_n, p) \\
&\le \frac{\varepsilon}{2n} + d(x_N, p) - d(T_i^n U_{n(i-1)} x_n, p) \qquad \text{(from (2.7))} \\
&< \frac{\varepsilon}{2n} + \frac{\varepsilon}{4n} < \varepsilon, \quad \forall\, n \ge N.
\end{aligned}$$

On the other hand, from (2.9), (2.8) and (2.6), we obtain

$$\begin{aligned}
d(x_N, p) &< d(T_i^n U_{n(i-1)} x_n, p) + \frac{\varepsilon}{4n} \\
&= d(T_i^n U_{n(i-1)} x_n, p) + \frac{1}{2}(L - d(T_i^n U_{n(i-1)} x_n, p)) \\
&= \frac{1}{2}(L + d(T_i^n U_{n(i-1)} x_n, p)) \\
&< \frac{1}{2}(L + d(x_N, p)),
\end{aligned}$$

i.e.,

$$d(x_N, p) < L.$$

This is a contradiction to (2.7). Therefore, (2.5) holds. That is

$$d(T_i^n U_{n(i-1)} x_n, p) \ge L, \quad \forall\, n \ge 1,\ i \in I.$$

(II) We claim that $L = 0$. Suppose that $L > 0$. It follows that, from (2.4) and (2.5), for any fixed integer $N \in \mathbb{N}$ and $i \in I$

$$\begin{aligned}
&\sum_{n=N}^{\infty} (1 - \alpha_{n(1)})(1 - \alpha_{n(2)}) \cdots (1 - \alpha_{n(r)}) \varphi(L) \\
&\le \sum_{n=N}^{\infty} (1 - \alpha_{n(1)})(1 - \alpha_{n(2)}) \cdots (1 - \alpha_{n(r)}) \varphi(d(T_i^n U_{n(i-1)} x_n, p)) \\
&\le \sum_{n=N}^{\infty} (d(x_n, p) - d(x_{n+1}, p)) \\
&\le d(x_N, p).
\end{aligned}$$

This is a contradiction to the condition (ii). Therefore, $L \le 0$. Thus

$$\lim_{n\to\infty} d(x_n, p) = L = 0.$$

This completes the proof of Theorem 2.3. $\square$

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 27, NO.3, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC

Remark 2.1. The author does not apply the real $CAT(0)$ properties of a space such as for example (CN*) inequality,

$$\begin{aligned} &d^2(\alpha x \oplus (1-\alpha)y, z) \\ &\leq \alpha d^2(x,z) + (1-\alpha)d^2(y,z) - \alpha(1-\alpha)d^2(x,y), \end{aligned} \tag{CN*}$$

but only the fact that

$$d(\alpha x \oplus (1-\alpha)y, z) \leq \alpha d(x,z) + (1-\alpha)d(y,z),$$

i.e., the convexity of the metric.

Corollary 2.1. *Let (X,d) be a complete $CAT(0)$ space, K be a closed convex subset of X, T be a generalized φ-weak contraction self mapping of K. Then the Noor iterative scheme* ([17])

$$\begin{aligned} x_{n+1} &= \alpha_n x_n \oplus (1-\alpha_n)Ty_n, \\ y_n &= \beta_n x_n \oplus (1-\beta_n)Tz_n, \\ z_n &= \gamma_n x_n \oplus (1-\gamma_n)Tx_n \end{aligned}$$

satisfies

(i) $0 \leq \alpha_n, \beta_n, \gamma_n \leq 1$,
(ii) $\sum_{n=1}^{\infty}(1-\alpha_n)(1-\beta_n)(1-\gamma_n) = \infty$

converges to fixed point $p \in F(T)$.

Proof. In the double acting iterative scheme (2.2), if $r = 3$ and $T_1 = T_2 = T_3 = T$, then it reduces to the Noor iterative scheme. So the proof is similar to that of Theorem 2.3, and will be omitted. □

Corollary 2.2. *Let (X,d) be a complete $CAT(0)$ space, K be a closed convex subset of X, T be a generalized φ-weak contraction self mapping of K. Then the Ishikawa iterative scheme* ([7])

$$\begin{aligned} x_{n+1} &= \alpha_n x_n \oplus (1-\alpha_n)Ty_n, \\ y_n &= \beta_n x_n \oplus (1-\beta_n)Tx_n \end{aligned}$$

satisfies

(i) $0 \leq \alpha_n, \beta_n \leq 1$,
(ii) $\sum_{n=1}^{\infty}(1-\alpha_n)(1-\beta_n) = \infty$

converges to fixed point $p \in F(T)$.

Proof. In the double acting iterative scheme (2.2), if $r = 2$ and $T_1 = T_2 = T$, then it reduces to the Ishikawa iterative scheme. So the proof is similar to that of Theorem 2.3, and will be omitted. □

Corollary 2.3. *Let (X, d) be a complete $CAT(0)$ space, K be a closed convex subset of X, T be a generalized φ-weak contraction self mapping of K. Then the Mann iterative scheme* ([12])

$$x_{n+1} = \alpha_n x_n \oplus (1 - \alpha_n)Tx_n,$$

satisfies

(i) $0 \leq \alpha_n \leq 1$,
(ii) $\sum_{n=1}^{\infty}(1 - \alpha_n) = \infty$

converges to fixed point $p \in F(T)$.

Proof. In the double acting iterative scheme (2.2), if $r =$ and $T_1 = T$, then it reduces to the Mann iterative scheme. So the proof is similar to that of Theorem 2.3, and will be omitted. □

Competing interests

The authors declares that there is no conflict of interest regarding the publication of this paper.

Acknowledgments

This work was supported by Kyungnam University Research Fund, 2017.

References

[1] Y.I. Alber and S. Guerre-Delabriere, *Principle of weakly contractive maps in Hilbert spaces*, in: I. Gohberg, Yu. Lyubich(Eds.), New Results in Operator Theory, in: Advances and Appl., vol. 98, Birkhäuser, Basel, 1997, 7–22.

[2] M. Bridson and A. Haefliger, Metric spaces of Non-Positive Curvature, Springer-Verlag, Berlin, Heidelberg, 1999.

[3] F. Bruhat and J. Tits, *Groups réductifss sur un corps local. I. Données radicielles valuées*, Publ. Math. Inst. Hautes Études Sci., **41** (1972), 5–251.

[4] D. Burago, Y. Burago and S. Ivanov, A course in metric Geometry, in:Graduate studies in Math., 33, Amer. Math. Soc., Providence, Rhode Island, 2001.

[5] S. Dhompongsa and B. Panyanak, *On triangle-convergence theorems in $CAT(0)$ spaces*, Comput. Math. Anal., **56** (2008), 2572–2579.

[6] M. Gromov, Hyperbolic groups, Essays in group theory, Math. Sci. Res. Inst. Publ. **8**. Springer, New York, 1987.

[7] S. Ishikawa, *Fixed point by a new iteration*, Proc. Amer. Math. Soc., **44** (1974), 147–150.

[8] A.R. Khan, M.A. Khamsi and H. Fukhar-ud-din, *Strong convergence of a general iteration scheme in $CAT(0)$ spaces*, Nonlinear Anal., **74**(3) (2011), 783-791.

[9] J.K. Kim, K.S. Kim and S.M. Kim, *Convergence theorems of implicit iteration process for a finite family of asymptotically quasi-nonexpansive mappings in convex metric spaces*, Proc. of RIMS Kokyuroku, Kyoto Univ., **1484** (2006), 40–51.

[10] K.S. Kim, *Some convergence theorems for contractive type mappings in $CAT(0)$ spaces*, Abstract and Applied Analysis, 2013, Article ID 381715, 9 pages, http://dx.doi.org/10.1155/2013/381715

[11] W.A. Kirk, *A fixed point theorem in $CAT(0)$ spaces and $\mathbb{R}$-trees*, Fixed Point Theory Appl., **2004(4)** (2004), 309–316.

[12] W.R. Mann, *Mean value methods in iteration*, Proc. Amer. Math. Soc., **4** (1953), 506–510.

[13] A. Nicolae, *Asymptotic behavior of averaged and firmly nonexpansive mappings in geodesic spaces*, Nonlinear Analysis, 2013, **87**, 102–115

[14] E. Picard, *Sur les groupes de transformation des équations différentielles linéaires*, Comptes Rendus Acad. Sci. Paris, **96** (1883), 1131–1134.

[15] B.E. Rhoades, *Some theorems on weakly contractive maps*, Nonlinear Anal., **47** (2001), 2683–2693.

[16] W. Takahashi, *A convexity in metric spaces and nonexpansive mappings*, Kodai Math. Sem. Rep., **22** (1970), 142–149.

[17] B.L. Xu and M.A. Noor, *Fixed-point iterations for asymptotically nonexpansive mappings in Banach spaces*, J. Math. Anal. Appl., **267** (2002), 444–453.

[18] Z. Xue, *The convergence of fixed point for a kind of weak contraction*, Nonlinear Func. Anal. Appl., **21**(3) (2016), 497–500.

On solution of a system of differential equations via fixed point theorem

Muhammad Nazam[1], Muhammad Arshad[1], Choonkil Park[2*], Özlem Acar[3], Sungsik Yun[4*], George A. Anastassiou[5]

[1]Department of Mathematics and Statistics, International Islamic University, H-10, Islamabad, Pakistan
e-mail: nazim254.butt@gmail.com; marshadzia@iiu.edu.pk

[2]Research Institute for Natural Sciences, Hanyang University, Seoul 04763, Korea
e-mail: baak@hanyang.ac.kr

[3]Department of Mathematics, Faculty of Science and Arts, Mersin University, 33343, Yenişehir, Mersin, Turkey
e-mail: ozlemacar@mersin.edu.tr

[4]Department of Financial Mathematics, Hanshin University, Gyeonggi-do 18101, Korea
e-mail: ssyun@hs.ac.kr

[5]Department of Mathematical Sciences, University of Memphis, Memphis, TN 38152, USA
e-mail: ganastss@@memphis.edu

Abstract. The purpose of the present paper is to study the existence of solution of a system of differential equations using fixed point technique. In this regard, in the first part of this article, along with some properties of partial b-metric topology, we prove a common fixed point theorem for generalized Geraghty type contraction mappings in a complete partial b-metric spaces. Then in second part we apply this result to show the existence of the solution of a system of ordinary differential equations.

1. Introduction and preliminaries

One of the most important results in fixed point theory is the Banach contraction principle introduced by Banach [4]. There were many authors who have studied and proved the results for fixed point theory by generalizing the Banach contraction principle in several directions (see [1, 5–7, 18, 22, 24]).

Czerwik [9] introduced the notion of b-metric to generalize the concept of a distance. The analog of the famous Banach fixed point theorem was proved by Czerwik in the frame of complete b-metric spaces. Following these initial papers, the existence and the uniqueness of (common) fixed points for the classes of both singlevalued and multivalued operators in the setting of (generalized) b-metric spaces have been investigated extensively (see [2, 3, 10, 13, 15, 16, 20, 23, 26–28] and related references therein).

Shukla [29] introduced the concept of partial b-metric space and established some fixed point theorems. Shukla, in fact, generalized Matthews partial metric to partial b-metric. Recently, Mustafa *et al.* [20], Latif *et al.* [19] and Piri *et al.* [21] have established some fixed point results in complete partial b-metric spaces.

In this paper, we introduce the notion of generalized Geraghty type contraction mappings and develop new common fixed point theorems for such mappings in complete partial b-metric spaces and properties of partial b-metric topology. Examples are given to support the usability of our results. In the last section of this paper, we utilize our results to present an application on existence of a solution of a pair of ordinary differential equations. We also study well-posedness of common fixed point problem for generalized Geraghty type contraction mappings.

First of all, we recall some definitions and properties of partial b-metric spaces.

Definition 1. [29] Let X be a nonempty set and $s \geq 1$ be a real number. A function $p_b : X \times X \to [0, \infty)$ is said to be a partial b-metric if for all $x, y, z \in X$, we have

---

[0]2010 Mathematics Subject Classification: 47H10; 54H25

[0]Keywords: complete partial b-metric space; generalized Geraghty type contraction mapping; differential equation; well posed.

*Corresponding authors.

(p_b1) $x = y$ if and only if $p_b(x,y) = p_b(x,x) = p_b(y,y)$,
(p_b2) $p_b(x,x) \leq p_b(x,y)$,
(p_b3) $p_b(x,y) = p_b(y,x)$,
(p_b4) $p_b(x,y) \leq s\,[p_b(x,z) + p_b(z,y)] - p_b(z,z)$.

In this case, the pair (X, p_b) is called a partial b-metric space (with constant s).

It is clear that every partial metric space is a partial b-metric space with coefficient $s = 1$ and every b-metric space is a partial b-metric space with the same coefficient and zero self-distance. However, the converse of this fact need not to hold. The self distance $p_b(x,x)$, referred to as the size or weight of x, is a feature used to describe the amount of information contained in x.

Definition 2. Let (X, p_b) be a partial b-metric space. The distance function $d_{p_b} : X \times X \to \mathbb{R}_0^+$, defined by

$$d_{p_b}(x,y) = 2p_b(x,y) - p_b(x,x) - p_b(y,y), \text{ for all } x, y \in X,$$

defines a metric on X called an induced metric.

Example 1. [29] Let $X = \mathbb{R}^+$ and $l > 1$. Then the functional $p_b : X \times X \to \mathbb{R}^+$, defined by

$$p_b(x,y) = \left\{(\max\{x,y\})^l + |x-y|^l\right\}, \text{ for all } x, y \in X,$$

is a partial b-metric.

Example 2. [29] Let X be a nonempty set such that p is a partial metric and d is a b-metric with coefficient $s > 1$ on X. Then the function $p_b : X \times X \to \mathbb{R}^+$, defined by $p_b(x,y) = p(x,y) + d(x,y)$ for all $x, y \in X$, is a partial b-metric on X and (X, p_b) is a partial b-metric space.

Example 3. [29] Let X be a nonempty set and p be a partial metric defined on X. The functional $p_b : X \times X \to \mathbb{R}^+$, defined by

$$p_b(x,y) = [p(x,y)]^q \text{ for all } x, y \in X \text{ and } q > 1,$$

defines a partial b-metric.

For a partial b-metric space (X, p_b), we immediately have a natural definition for the open balls:

$$B_\epsilon(x; p_b) = \{y \in X | p_b(x,y) < p_b(x,x) + \epsilon\} \text{ for each } x \in X \text{ and } \epsilon > 0.$$

Proposition 1. *The set $\{B_\epsilon(x; p_b) | x \in X,\ \epsilon > 0\}$ of open balls forms the basis for partial b-metric topology denoted by $\mathcal{T}[p_b]$.*

Proof. It is obvious that

$$X = \cup_{x \in X} B_\epsilon(x; p_b)$$

and for any two open balls $B_\epsilon(x; p_b)$, $B_\delta(y; p_b)$ we note that

$$B_\epsilon(x; p_b) \cap B_\delta(y; p_b) = \cup \{B_\kappa(c; p_b) |\, c \in B_\epsilon(x; p_b) \cap B_\delta(y; p_b)\}$$
$$\text{where, } \kappa = p_b(c,c) + \min\{\epsilon - p_b(x,c), \delta - p_b(y,c)\},$$

as desired. □

Proposition 2. *Each partial b-metric topology is T_0 topology but not T_1.*

Proof. Suppose $p_b : X \times X \to \mathbb{R}_0^+$ is a partial b-metric and $x \neq y$. Then without loss of generality, we have $p_b(x,x) < p_b(x,y)$ for all $x, y \in X$. Choose $\epsilon = p_b(x,y) - p_b(x,x)$. Since

$$p_b(x,x) < p_b(x,x) + \epsilon = p_b(x,y),$$

$x \in B_\epsilon(x; p_b)$ and $y \notin B_\epsilon(x; p_b)$. Otherwise we obtain an absurdity $(p_b(x,y) < p_b(x,y))$. It is obvious that for $x \neq v$,

$$x \in B_\delta(x; p_b) \subseteq B_\epsilon(v; p_b),$$

which contradicts T_1 axiom. □

The following definition and lemma describe the convergence criteria established by Shukla in [29].

Definition 3. [29] Let (X, p_b) be a partial b-metric space.

(1) A sequence $\{x_n\}_{n\in\mathbb{N}}$ in (X, p_b) is called a Cauchy sequence if $\lim_{n,m\to\infty} p_b(x_n, x_m)$ exists and is finite.

(2) A partial b-metric space (X, p_b) is said to be complete if every Cauchy sequence $\{x_n\}_{n\in\mathbb{N}}$ in X converges, with respect to $\mathcal{T}[p_b]$, to a point $\upsilon \in X$ such that
$$p_b(x, x) = \lim_{n,m\to\infty} p_b(x_n, x_m).$$

Lemma 1. [29] Let (X, p_b) be a partial b-metric space.

(1) Every Cauchy sequence in (X, d_{p_b}) is also a Cauchy sequence in (X, p_b).

(2) A partial b-metric (X, p_b) is complete if and only if the metric space (X, d_{p_b}) is complete.

(3) A sequence $\{x_n\}_{n\in\mathbb{N}}$ in X converges to a point $\upsilon \in X$ with respect to $\mathcal{T}[(d_{p_b})]$ if and only if
$$\lim_{n\to\infty} p_b(\upsilon, x_n) = p_b(\upsilon, \upsilon) = \lim_{n\to\infty} p_b(x_n, x_m).$$

(4) If $\lim_{n\to\infty} x_n = \upsilon$ such that $p_b(\upsilon, \upsilon) = 0$, then $\lim_{n\to\infty} p_b(x_n, k) = p_b(\upsilon, k)$ for every $k \in X$.

The following important lemma is useful in the sequel.

Lemma 2. [20] Let (X, p_b) be a partial b-metric space with coefficient $s > 1$. Suppose that the sequences $\{x_n\}$,$\{y_n\}$ converge to x, y, respectively. Then we have
$$\begin{aligned}\frac{1}{s^2}p_b(x,y) - \frac{1}{s}p_b(x,x) - p_b(y,y) &\le \lim_{n\to\infty}\inf p_b(x_n,y_n) \le \lim_{n\to\infty}\sup p_b(x_n,y_n)\\ &\le sp_b(x,x) + s^2p_b(y,y) + s^2p_b(x,y).\end{aligned}$$
If $p_b(x, y) = 0$ then we have $\lim_{n\to\infty} p_b(x_n, y_n) = 0$. Moreover, for each $x^* \in X$ we obtain
$$\begin{aligned}\frac{1}{s}p_b(x,x^*) - p_b(x,x) &\le \lim_{n\to\infty}\inf p_b(x_n,x^*) \le \lim_{n\to\infty}\sup p_b(x_n,x^*)\\ &\le sp_b(x,x^*) + sp_b(x,x).\end{aligned}$$
If $p_b(x, x) = 0$, then we have
$$\frac{1}{s}p_b(x,x^*) \le \lim_{n\to\infty}\inf p_b(x_n,x^*) \le \lim_{n\to\infty}\sup p_b(x_n,x^*) \le sp_b(x,x^*).$$

Let Ω denote to the class of all functions $\beta : [0, +\infty) \to [0, 1)$ such that for any bounded sequence $\{t_n\}$ of positive reals, $\beta(t_n) \to 1$ implies $t_n \to 0$. Geraghty [11] presented a very important generalization of Banach Contraction Principle as follows:

Theorem 1. [11] Let (X, d) be a metric space. Let $S : X \to X$ be a self-mapping. Suppose that there exists $\beta \in \Omega$ such that for all $x, y \in X$,
$$d(Sx, Sy) \le \beta(d(x, y))\, d(x, y).$$
Then S has a unique fixed point $x^* \in X$ and $\{S^n x\}$ converges to x^* for each $x \in X$.

Following [8], we let Ψ denote to the class of functions $\psi : [0, \infty) \to [0, \infty)$ satisfying the following conditions:
(1) ψ is nondecreasing,
(2) ψ is continuous,
(3) $\psi(t) = 0$ if and only if $t = 0$.

Definition 4. Let $S, T : X \to X$ be two self-mappings and $F(S)$ and $F(T)$ denote the set of fixed points of S and T, respectively. Then a fixed point problem for S and T is well posed if for any sequence $\{x_n\}$ in X and $x^* \in F(S) \cap F(T)$, $\lim_{n\to\infty} p_b(x_n, S(x_n)) = 0$ or $\lim_{n\to\infty} p_b(x_n, T(x_n)) = 0$ implies $\lim_{n\to\infty} p_b(x_n, x^*) = p_b(x^*, x^*)$.

2. Fixed point results

We begin with the introduction of the concept of generalized Geraghty type contraction mappings as follows:

Definition 5. Let (X, p_b) be a partial b-metric space. The pair $S, T : X \to X$ of self-mappings is called a generalized Geraghty type contraction mapping if there exist $\beta \in \Omega$ and $\psi \in \Psi$ such that for $x, y \in X$, the pair (S, T) satisfies the following inequality:

$$\psi\left(s^3 p_b\left(Sx, Ty\right)\right) \leq \beta\left(\psi\left(\mathcal{M}\left(x, y\right)\right)\right) \cdot \psi\left(\mathcal{M}\left(x, y\right)\right) \tag{2.1}$$

where

$$\mathcal{M}\left(x, y\right) = \max\left\{p_b\left(x, y\right), p_b\left(x, Sx\right), p_b\left(y, Ty\right), \frac{p_b\left(x, Ty\right) + p_b\left(y, Sx\right)}{2s}\right\}.$$

The main result of this section is the following.

Theorem 2. *Let (X, p_b) be a complete partial b-metric space and $S, T : X \to X$ be two self-mappings satisfying the following conditions:*

1. *(S, T) is a pair of generalized Geraghty type contraction mappings;*
2. *S or T is a continuous mapping.*

Then S and T have a common fixed point $x^* \in X$.

Proof. First, we suppose that $s > 1$. Let $x_0 \in X$ and choose $x_1 = S(x_0)$, $x_2 = T(x_1)$. Continuing in the same way we construct a sequence $\{x_n\}$ in X such that $x_{2i+1} = S(x_{2i})$ and $x_{2i+2} = T(x_{2i+1})$, $i = 0, 1, 2, \ldots$. Without loss of generality, we can assume that $\mathcal{M}(x, y) > 0$ for $x \neq y$. Now, for $i \in \mathbb{N}$, we have

$$\begin{aligned} 0 < \psi\left(p_b\left(x_{2i+1}, x_{2i+2}\right)\right) &\leq \psi\left(s^3 p_b\left(Sx_{2i}, Tx_{2i+1}\right)\right) \\ &\leq \beta\left(\psi\left(\mathcal{M}\left(x_{2i}, x_{2i+1}\right)\right)\right) . \psi\left(\mathcal{M}\left(x_{2i}, x_{2i+1}\right)\right), \end{aligned} \tag{2.2}$$

where

$$\begin{aligned} \mathcal{M}\left(x_{2i}, x_{2i+1}\right) &= \max\left\{\begin{array}{c} p_b\left(x_{2i}, x_{2i+1}\right), p_b\left(x_{2i}, Sx_{2i}\right), p_b\left(x_{2i+1}, Tx_{2i+1}\right), \\ \frac{p_b\left(x_{2i}, Tx_{2i+1}\right) + p_b\left(x_{2i+1}, Sx_{2i}\right)}{2s} \end{array}\right\} \\ &= \max\left\{\begin{array}{c} p_b\left(x_{2i}, x_{2i+1}\right), p_b\left(x_{2i}, x_{2i+1}\right), p_b\left(x_{2i+1}, x_{2i+2}\right), \\ \dfrac{p_b\left(x_{2i}, x_{2i+2}\right) + p_b\left(x_{2i+1}, x_{2i+1}\right)}{2s} \end{array}\right\} \\ &\leq \max\left\{\begin{array}{c} p_b\left(x_{2i}, x_{2i+1}\right), p_b\left(x_{2i}, x_{2i+1}\right), p_b\left(x_{2i+1}, x_{2i+2}\right), \\ \dfrac{p_b\left(x_{2i}, x_{2i+1}\right) + p_b\left(x_{2i+1}, x_{2i+2}\right)}{2s} \end{array}\right\} \\ &= \max\left\{p_b\left(x_{2i}, x_{2i+1}\right), p_b\left(x_{2i+1}, x_{2i+2}\right)\right\}. \end{aligned}$$

If $\max\left\{p_b\left(x_{2i}, x_{2i+1}\right), p_b\left(x_{2i+1}, x_{2i+2}\right)\right\} = p_b\left(x_{2i+1}, x_{2i+2}\right)$, then from (2.2) we have

$$\begin{aligned} \psi\left(p_b\left(x_{2i+1}, x_{2i+2}\right)\right) &\leq \beta\left(\psi\left(p_b\left(x_{2i+1}, x_{2i+2}\right)\right)\right) . \psi\left(p_b\left(x_{2i+1}, x_{2i+2}\right)\right) \\ &< \psi\left(p_b\left(x_{2i+1}, x_{2i+2}\right)\right), \end{aligned}$$

which is a contradiction. Thus we conclude that

$$\max\left\{p_b\left(x_{2i}, x_{2i+1}\right), p_b\left(x_{2i+1}, x_{2i+2}\right)\right\} = p_b\left(x_{2i}, x_{2i+1}\right).$$

By (2.2), we get that $\psi\left(p_b\left(x_{2i+1}, x_{2i+2}\right)\right) < \psi\left(p_b\left(x_{2i}, x_{2i+1}\right)\right)$. Since ψ is nondecreasing, we have

$$p_b\left(x_{2i+1}, x_{2i+2}\right) < p_b\left(x_{2i}, x_{2i+1}\right).$$

This implies that

$$p_b\left(x_{n+1}, x_{n+2}\right) < p_b\left(x_n, x_{n+1}\right), \text{ for all } n \in \mathbb{N}.$$

Hence we deduce that the sequence $\{p_b\left(x_n, x_{n+1}\right)\}$ is nonincreasing. Therefore, there exists $r \geq 0$ such that $\lim_{n\to\infty} p_b\left(x_n, x_{n+1}\right) = r$.

Now we shall prove that $r = 0$. Suppose that $r > 0$. From (2.1), we have

$$\begin{aligned}\psi\left(p_b\left(x_{n+1}, x_{n+2}\right)\right) &\leq \psi\left(s^3 p_b\left(Sx_n, Tx_{n+1}\right)\right)\\ &\leq \beta\left(\psi\left(\mathcal{M}\left(x_n, x_{n+1}\right)\right)\right).\psi\left(\mathcal{M}\left(x_n, x_{n+1}\right)\right),\end{aligned}$$

which implies

$$\psi\left(p_b\left(x_{n+1}, x_{n+2}\right)\right) \leq \beta\left(\psi\left(p_b\left(x_n, x_{n+1}\right)\right)\right).\psi\left(p_b\left(x_n, x_{n+1}\right)\right).$$

Hence

$$\frac{\psi\left(p_b\left(x_{n+1}, x_{n+2}\right)\right)}{\psi\left(p_b\left(x_n, x_{n+1}\right)\right)} \leq \beta\left(\psi\left(p_b\left(x_n, x_{n+1}\right)\right)\right) < 1.$$

This implies that $\lim_{n\to\infty} \beta\left(\psi\left(p_b\left(x_n, x_{n+1}\right)\right)\right) = 1$. Since $\beta \in \Omega$, we have

$$\lim_{n\to\infty} \psi\left(p_b\left(x_n, x_{n+1}\right)\right) = 0,$$

which yields

$$r = \lim_{n\to\infty} p_b\left(x_n, x_{n+1}\right) = 0, \tag{2.3}$$

which is a contradiction.

Now we will show that $\{x_n\}$ is a Cauchy sequence. For this purpose, we use Lemma 1. Suppose that there exists $\varepsilon > 0$ such that for all $k \in \mathbb{N}$, there exists $m(k) > n(k) > k$ with $d_{p_b}\left(x_{n(k)}, x_{m(k)}\right) \geq \varepsilon$. Let $m(k)$ be the smallest number satisfying the condition above. Then we have $d_{p_b}\left(x_{n(k)}, x_{m(k)-1}\right) < \varepsilon$. Therefore,

$$\begin{aligned}\varepsilon \leq d_{p_b}\left(x_{n(k)}, x_{m(k)}\right) &\leq s\left[d_{p_b}\left(x_{n(k)}, x_{m(k)-1}\right) + dp_b\left(x_{m(k)-1}, x_{m(k)}\right)\right] \\ &< s\left[\varepsilon + d_{p_b}\left(x_{m(k)-1}, x_{m(k)}\right)\right].\end{aligned} \tag{2.4}$$

By taking the upper limit as $k \to \infty$ in (2.4) and using (2.3), we get

$$\varepsilon \leq \lim_{k\to\infty} \sup d_{p_b}\left(x_{n(k)}, x_{m(k)}\right) < s\varepsilon. \tag{2.5}$$

From the triangular inequality, we have

$$d_{p_b}\left(x_{n(k)}, x_{m(k)}\right) \leq s[d_{p_b}\left(x_{n(k)}, x_{n(k)+1}\right) + d_{p_b}\left(x_{n(k)+1}, x_{m(k)}\right)] \tag{2.6}$$

and

$$d_{p_b}\left(x_{n(k)+1}, x_{m(k)}\right) \leq s[d_{p_b}\left(x_{n(k)+1}, x_{n(k)}\right) + d_{p_b}\left(x_{n(k)}, x_{m(k)}\right)]. \tag{2.7}$$

By taking upper limit as $k \to \infty$ in (2.6) and applying (2.3) and (2.5),

$$\varepsilon \leq \lim_{k\to\infty} \sup d_{p_b}\left(x_{n(k)}, x_{m(k)}\right) \leq s\left(\lim_{k\to\infty} \sup d_{p_b}\left(x_{n(k)+1}, x_{m(k)}\right)\right).$$

Again, by taking the upper limit as $k \to \infty$ in (2.7), we get

$$\lim_{k\to\infty} \sup d_{p_b}\left(x_{n(k)+1}, x_{m(k)}\right) \leq s\left(\lim_{k\to\infty} \sup d_{p_b}\left(x_{n(k)}, x_{m(k)}\right)\right) \leq s.s\varepsilon = s^2\varepsilon.$$

Thus

$$\frac{\varepsilon}{s} \leq \lim_{k\to\infty} \sup d_{p_b}\left(x_{n(k)+1}, x_{m(k)}\right) \leq s^2\varepsilon. \tag{2.8}$$

Similarly

$$\frac{\varepsilon}{s} \leq \lim_{k\to\infty} \sup d_{p_b}\left(x_{n(k)}, x_{m(k)+1}\right) = \lim_{k\to\infty} \sup d_{p_b}\left(x_{n(k)+1}, x_{m(k)+2}\right) \leq s^2\varepsilon. \tag{2.9}$$

By the triangular inequality, we have

$$d_{p_b}\left(x_{n(k)+1}, x_{m(k)}\right) \leq s[d_{p_b}\left(x_{n(k)+1}, x_{m(k)+1}\right) + d_{p_b}\left(x_{m(k)+1}, x_{m(k)}\right)]. \tag{2.10}$$

Letting $k \to \infty$ in (2.10) and using (2.3) and (2.8), we get

$$\frac{\varepsilon}{s^2} \leq \lim_{k\to\infty} \sup d_{p_b}\left(x_{n(k)+1}, x_{m(k)+1}\right). \tag{2.11}$$

Following the above process, we find

$$\lim_{k\to\infty} \sup d_{p_b}\left(x_{n(k)+1}, x_{m(k)+1}\right) \leq s^3\varepsilon. \tag{2.12}$$

From (2.11) and (2.12), we get

$$\frac{\varepsilon}{s^2} \leq \lim_{k\to\infty} \sup d_{p_b}\left(x_{n(k)+1}, x_{m(k)+1}\right) \leq s^3\varepsilon.$$

Since $x_{n(k)} \neq x_{m(k)+1}$, we get

$$\begin{aligned}
& \psi\left(d_{p_b}\left(x_{n(k)+1}, x_{m(k)+2}\right)\right) \\
\leq\ & \psi\left(s^3 d_{p_b}\left(Sx_{n(k)}, Tx_{m(k)+1}\right)\right) \\
\leq\ & \beta\left(\psi\left(\mathcal{M}\left(x_{n(k)}, x_{m(k)+1}\right)\right)\right)\cdot\psi\left(\mathcal{M}\left(x_{n(k)}, x_{m(k)+1}\right)\right) \\
\leq\ & \beta\left(\psi\left(\mathcal{M}\left(x_{n(k)}, x_{m(k)+1}\right)\right)\right)\cdot\psi\left(\mathcal{M}\left(x_{n(k)}, x_{m(k)+1}\right)\right),
\end{aligned}$$

where

$$\begin{aligned}
M\left(x_{n(k)}, x_{m(k)+1}\right) &= \max\left\{\begin{array}{l} d_{p_b}\left(x_{n(k)}, x_{m(k)+1}\right), d_{p_b}\left(x_{n(k)}, Sx_{n(k)}\right), \\ d_{p_b}\left(x_{m(k)+1}, Tx_{m(k)+1}\right), \\ \frac{d_{p_b}\left(x_{n(k)}, Tx_{m(k)+1}\right)+d_{p_b}\left(x_{m(k)+1}, Sx_{n(k)}\right)}{2s} \end{array}\right\} \\
&= \max\left\{\begin{array}{l} d_{p_b}\left(x_{n(k)}, x_{m(k)-1}\right), d_{p_b}\left(x_{n(k)}, x_{n(k)+1}\right), \\ d_{p_b}\left(x_{m(k)+1}, x_{m(k)+2}\right), \\ \frac{d_{p_b}\left(x_{n(k)}, x_{m(k)+2}\right)+d_{p_b}\left(x_{m(k)+1}, x_{n(k)+1}\right)}{2s} \end{array}\right\}.
\end{aligned}$$

Taking the limit as $k\to\infty$ and using $(2.3), (2.5), (2.8)$ and (2.9), we get

$$\frac{\varepsilon}{s} = \max\left\{\frac{\varepsilon}{s}, \frac{s\varepsilon}{4}\right\} \leq \lim_{k\to\infty}\sup\mathcal{M}\left(x_{n(k)}, x_{m(k)+1}\right) \leq \max\left\{s^2\varepsilon, \frac{s^2\varepsilon}{4}\right\} = s^2\varepsilon.$$

Similarly, we can show that

$$\frac{\varepsilon}{s} = \max\left\{\frac{\varepsilon}{s}, \frac{s\varepsilon}{4}\right\} \leq \lim_{k\to\infty}\inf\mathcal{M}\left(x_{n(k)}, x_{m(k)+1}\right) \leq \max\left\{s^2\varepsilon, \frac{s^2\varepsilon}{4}\right\} = s^2\varepsilon.$$

From (2.9), we have

$$\begin{aligned}
\psi\left(s^2\varepsilon\right) &= \psi\left(s^3\left(\frac{\varepsilon}{s}\right)\right) \leq \psi\left(s^3 \lim_{k\to\infty}\sup d_{p_b}\left(x_{n(k)+1}, x_{m(k)+2}\right)\right) \\
&\leq \beta\left(\psi\left(\lim_{k\to\infty}\sup\mathcal{M}\left(x_{n(k)}, x_{m(k)+1}\right)\right)\right)\cdot\psi\left(\lim_{k\to\infty}\sup\mathcal{M}\left(x_{n(k)}, x_{m(k)+1}\right)\right) + 0 \\
&\leq \beta\left(\psi\left(s^2\varepsilon\right)\right)\psi\left(s^2\varepsilon\right) \\
&< \psi\left(s^2\varepsilon\right),
\end{aligned}$$

which is a contradiction. Thus $\{x_n\}$ is a Cauchy sequence in (X, d_{p_b}). Since (X, p_b) is a complete partial b-metric space, from Lemma 1, (X, d_{p_b}) is a complete b-metric space. Therefore, the sequence $\{x_n\}$ converges to some $x^* \in (X, d_{p_b})$. From Lemma 1, there exists $x^* \in X$ such that $\lim_{n\to\infty} d_{p_b}(x_n, x^*) = 0$ if and only if

$$\lim_{n\to\infty} p_b(x^*, x_n) = p_b(x^*, x^*) = \lim_{n,m\to\infty} p_b(x_n, x_m). \tag{2.13}$$

Since $d_{p_b}(x,y) = 2p_b(x,y) - p_b(x,x) - p_b(y,y)$, considering (2.3) and the axiom (p_b2) with $\lim_{n\to\infty} d_{p_b}(x_n, x^*) = 0$, we conclude that

$$\lim_{n\to\infty} p_b(x_n, x_m) = 0. \tag{2.14}$$

Combining (2.13) and (2.14), we have

$$\lim_{n\to\infty} p_b(x^*, x_n) = p_b(x^*, x^*) = \lim_{n,m\to\infty} p_b(x_n, x_m) = 0.$$

Now $\lim_{n\to\infty} p_b(x^*, x_n) = 0$ implies that $\lim_{i\to\infty} p_b(x_{2i+1}, x^*) = 0$ and $\lim_{i\to\infty} p_b(x_{2i+2}, x^*) = 0$. As one of S and T is a continuous mapping, so we $\lim_{i\to\infty} p_b(Sx_{2i+1}, Sx^*) = 0$. Thus

$$p_b(x^*, Sx^*) = \lim_{i\to\infty} p_b(x_{2i+2}, Sx^*) \leq \lim_{i\to\infty} p_b(Sx_{2i+1}, Sx^*) = 0,$$

and so $x^* = Sx^*$. By (2.1), we have

$$\begin{aligned}\psi\left(s^3 p_b\left(x^*, T(x^*)\right)\right) &= \psi\left(s^3 p_b\left(S(x^*), T(x^*)\right)\right) \\ &\leq \beta\left(\psi\left(\mathcal{M}\left(x^*, x^*\right)\right)\right).\psi\left(\mathcal{M}\left(x^*, x^*\right)\right) \\ &\leq \beta\left(\psi\left(p_b\left(x^*, T(x^*)\right)\right)\right).\psi\left(p_b\left(x^*, T(x^*)\right)\right).\end{aligned}$$

Due to the definitions of β and ψ, we deduce that $x^* = Tx^*$. Therefore, S and T have a common fixed point $x^* \in X$. It is easy to check that x^* is unique. □

Remark 1. *We note that Theorem 2 is more general than the results established in [11, 12, 14, 25, 26, 29].*

Example 4. Let $X = [0, 1]$. Define a function $p_b : X \times X \to [0, +\infty)$ by $p_b(x, y) = (\max\{x, y\})^2 + (x - y)^2$. Clearly, (X, p_b) is a complete partial b-metric space with the constant $s = 2$. Let β be a function on $[0, +\infty)$ defined by $\beta(t) = \frac{1}{1+t}$ for all $t > 0$ and $\beta(0) = 0$. Then $\beta \in \Omega$. Let ψ be a function on $[0, +\infty)$ defined by $\psi(t) = t$. Then $\psi \in \Psi$. Define the mappings $S, T : X \to X$ by

$$T(x) = \begin{cases} \frac{2}{245}x, & \text{if } x \in \left[0, \frac{1}{2}\right) \\ 1, & \text{if } x \in \left[\frac{1}{2}, 1\right] \end{cases} \quad \text{and } S(x) = 0.$$

If $\{x_n\}$ is a Cauchy sequence such that $\{x_n\} \subseteq \left[0, \frac{1}{2}\right)$. Since $\left(\left[0, \frac{1}{2}\right), p_b\right)$ is a complete partial b-metric space, the sequence $\{x_n\}$ converges in $\left[0, \frac{1}{2}\right) \subseteq X$. Thus (X, p_b) is a complete partial b-metric space. We note that $x, y, Sy, Ty \in \left[0, \frac{1}{2}\right)$ and S is continuous. It is easy to check that for all $x, y \in \left[0, \frac{1}{2}\right)$, the following inequality is true

$$\psi\left(s^3 p_b\left(Sx, Ty\right)\right) \leq \beta\left(\psi\left(\mathcal{M}\left(x, y\right)\right)\right) \cdot \psi\left(\mathcal{M}\left(x, y\right)\right),$$

Thus all the conditions of Theorem 2 are satisfied. Hence S and T have a common fixed point $(x = 0)$.

3. Derived results

In Theorem 2, if we set $S = T$ and

$$\mathcal{M}\left(x, y\right) = \max\left\{p_b\left(x, y\right), p_b\left(x, Sx\right), p_b\left(y, Sy\right), \frac{p_b\left(x, Sy\right) + p_b\left(y, Sx\right)}{2s}\right\},$$

then we obtain the following result.

Corollary 1. Let (X, p_b) be a complete partial b-metric space. Suppose that $S : X \to X$ is a self-mapping satisfying the following conditions:

(1) S is a generalized Geraghty type contraction mapping;
(2) S is a continuous mapping.

Then S has a fixed point $x^* \in X$.

In Theorem 2, if $\psi(t) = t$, then we obtain the following corollary.

Corollary 2. Let (X, p_b) be a complete partial b-metric space. Suppose that $S, T : X \to X$ are two self-mappings such that

(1) there exists $\beta \in \Omega$ such that for $x, y \in X$, the pair (S, T) satisfies the following inequality

$$s^3 p_b\left(Sx, Ty\right) \leq \beta\left(\left(\mathcal{M}\left(x, y\right)\right)\right).\left(\mathcal{M}\left(x, y\right)\right),$$

where

$$\mathcal{M}\left(x, y\right) = \max\left\{p_b\left(x, y\right), p_b\left(x, Sx\right), p_b\left(y, Ty\right), \frac{p_b\left(x, Ty\right) + p_b\left(y, Sx\right)}{2s}\right\}.$$

(2) S or T is a continuous mapping

Then S and T have a common fixed point $x^* \in X$.

In particular, if $p_b(x,x)=0$ for all $x\in X$, then the following result can easily be obtained from Theorem 2.

Corollary 3. Let (X,d) be a b-metric space. Suppose that $S,T:X\to X$ are two self-mappings satisfying the following conditions:

(1) (S,T) is a pair of Geraghty type contraction mappings;
(2) S or T is a continuous mapping.

Then S and T have a common fixed point $x^*\in X$.

In the following, we see that the problem stated in Theorem 2 is well posed.

Theorem 3. *Let (X,p_b) be a complete partial b-metric space. Let $S,T\colon X\to X$ be two self-mappings as in Theorem 2 with $\psi(t)=t$. Then the fixed point problem for S and T is well posed.*

Proof. Let $\{x_n\}$ be a sequence in X and $x^*\in F(S)\cap F(T)$. Suppose that $\lim_{n\to\infty}p_b(x_n,S(x_n))=0$. If $\lim_{n\to\infty}p_b(x_n,x^*)=0$, then we are done. Assume that $\lim_{n\to\infty}p_b(x_n,x^*)=r>0$. Using (p_b3), we have

$$\begin{aligned}
s^3p_b(x_n,x^*) &\leq s^4[p_b(x_n,S(x_n))+p_b(S(x_n),x^*)-p_b(S(x_n),S(x_n))],\\
s^2p_b(x_n,x^*) &\leq s^3p_b(x_n,S(x_n))+s^3p_b(S(x_n),T(x^*))\\
&\leq s^3p_b(x_n,S(x_n))+\beta\left(\mathcal{M}\left(x_n,x^*\right)\right)\cdot\mathcal{M}\left(x_n,x^*\right),\\
\frac{1}{s}\lim_{n\to\infty}p_b(x_n,x^*) &\leq s^3\lim_{n\to\infty}p_b(x_n,S(x_n))+\lim_{n\to\infty}\beta(p_b(x_n,x^*))\cdot p_b(x_n,x^*),\\
\frac{r}{s} &\leq 0+\frac{r}{s^3}\beta(r),\ \text{a contradiction due to the definition of }\beta.
\end{aligned}$$

Similarly, we obtain $\lim_{n\to\infty}x_n=x^*$ if we assume $\lim_{n\to\infty}d(x_n,T(x_n))=0$. $\square$

4. Application

In this section, we present an application on existence of a solution of a pair of ordinary differential equations. In particular, inspired from [17] and using Theorem 2, we consider the following pair of differential equations:

$$\left\{\begin{array}{ll}-\frac{d^2x}{dt^2}=f\left(t,x\left(t\right)\right), & t\in[0,1]\\ x(0)=x\left(1\right)=0\end{array}\right. \quad\text{and}\quad \left\{\begin{array}{ll}-\frac{d^2y}{dt^2}=K\left(t,y\left(t\right)\right), & t\in[0,1]\\ y(0)=y\left(1\right)=0\end{array}\right. \tag{4.1}$$

where $f,K:[0,1]\times\mathbb{R}\to\mathbb{R}$ are continuous functions. The Green function associated to (4.1) is defined by

$$G\left(t,s\right)=\left\{\begin{array}{ll}t\left(1-s\right), & 0\leq t\leq s\leq 1\\ s\left(1-t\right), & 0\leq t\leq s\leq 1.\end{array}\right.$$

Let $C\left(I\right)$ be the space of all continuous functions defined on I, where $I=[0,1]$. Suppose that

$$p_b(x,y)=\left(\sup_{t\in I}|x(t)-y(t)|\right)^2+(\max\{x(t),y(t)\})^2.$$

It is known that $\left(C\left(I\right),p_b\right)$ is a complete partial b-metric space with constant $s=2$. Now, define the operators $S,T:C\left(I\right)\to C\left(I\right)$ by

$$Sx(t)=\int_0^1G(t,s)f(s,x(s))ds\text{ and }Tx(t)=\int_0^1G(t,s)K(s,y(s))ds$$

for all $t\in I$. Note that (4.1) has a solution if and only if the operators S and T have a common fixed point.

The main result is the following.

Theorem 4. *Assume that*

(1) *there exist continuous functions $f,K:[0,1]\times\mathbb{R}\to\mathbb{R}$ such that for all $a,b,\rho\in\mathbb{R}$, we have*

$$|f(t,a)-K(t,b)|^2\leq 64\ln\left(\frac{\mathcal{M}(a,b)+1}{\rho}\right)\ \textit{for all } t\in I,$$

where

$$\mathcal{M}(a,b)=\max\left\{p_b(a,b),p_b(a,S(a)),p_b(b,T(b)),\frac{p_b(a,S(b))+p_b(b,T(a))}{2s}\right\}>\rho;$$

(2) *the operators S,T are such that*

$$(\max\{Sx(t),Ty(t)\})^2\leq\ln(\rho)\left(\sup_{t\in I}\int_0^1 G(t,s)\,ds\right)^2.$$

Then the system of ordinary differential equations (4.1) has a solution.

Proof. It is well known that $x^*\in C^2(I)$ is a solution of (4.1) if and only if $x^*\in C(I)$ is a solution of the integral equation (see [17]). Define the mappings $S,T:C(I)\to C(I)$ by

$$Sx(t)=\int_0^1 G(t,s)f(s,x(s))ds \text{ and } Tx(t)=\int_0^1 G(t,s)K(s,y(s))ds.$$

Hence the solution of (4.1) is equivalent to find $x^*\in C(I)$, that is, a fixed point of T. By (1), we get

$$\begin{aligned}
p_b(Sx,Ty) &= \left(\sup_{t\in I}|Sx(t)-Ty(t)|\right)^2+(\max\{Sx(t),Ty(t)\})^2\\
&\leq \left[\sup_{t\in I}\left|\int_0^1 G(t,s)\left[f(s,x(s))-K(s,y(s))\right]ds\right|\right]^2+\ln(\rho)\left(\sup_{t\in I}\int_0^1 G(t,s)\,ds\right)^2\\
&\leq \left[\left(\sup_{t\in I}\int_0^1 G(t,s)\,ds\right)^2|f(s,x(s))-K(s,y(s))|^2\right]+\ln(\rho)\left(\sup_{t\in I}\int_0^1 G(t,s)\,ds\right)^2\\
&\leq \left[64\left(\sup_{t\in I}\int_0^1 G(t,s)\,ds\right)^2\ln\left(\frac{\mathcal{M}(a,b)+1}{\rho}\right)\right]+\ln(\rho)\left(\sup_{t\in I}\int_0^1 G(t,s)\,ds\right)^2\\
&= \left[8^2\ln\left(\frac{\mathcal{M}(a,b)+1}{\rho}\right)+\ln(\rho)\right]\left(\sup_{t\in I}\left[\int_0^1 G(t,s)ds\right]^2\right).
\end{aligned}$$

Since $\int G(t,s)ds=-\frac{t^2}{2}+\frac{t}{2}$ for all $t\in I$, we have $\left(\sup_{t\in I}\left[\int_0^1 G(t,s)ds\right]^2\right)=\frac{1}{8^2}$. Therefore,

$$p_b(Sx,Ty)\leq\ln\left(\mathcal{M}(a,b)+1\right),$$

which implies that

$$\begin{aligned}
\ln\left(p_b(Sx,Ty)+1\right) &\leq \ln\left(\ln\left(\mathcal{M}(x,y)+1\right)+1\right)\\
&= \frac{\ln\left(\ln\left(\mathcal{M}(x,y)+1\right)+1\right)}{\ln\left(\mathcal{M}(x,y)+1\right)}\ln\left(\mathcal{M}(x,y)+1\right).
\end{aligned}$$

Define the functions $\psi:[0,\infty)\to[0,\infty)$ and $\beta:[0,\infty)\to[0,1)$ by

$$\psi(x)=\ln(x+1) \text{ and } \beta(x)=\begin{cases}\frac{\psi(x)}{x}, & if\ x\neq 0\\ 0, & \text{otherwise.}\end{cases}$$

Note that $\psi:[0,\infty)\to[0,\infty)$ is continuous, nondecreasing, positive in $(0,\infty)$, $\psi(0)=0$ and $\psi(x)<x$. Hence $\beta\in\Omega$, $\psi\in\Psi$ and

$$\psi\left(s^3p_b(Sx,Ty)\right)\leq\beta\left(\psi\left(\mathcal{M}(x,y)\right)\right).\psi\left(\mathcal{M}(x,y)\right)$$

for all $x,y\in C(I)$. Therefore, all the assumptions of Theorem 2 are satisfied. Hence S and T have a common fixed point $x^*\in C(I)$, that is, $Sx^*=Tx^*=x^*$, which is a solution of (4.1). □

References

[1] G. A. Anastassiou, I. K. Argyros, *Approximating fixed points with applications in fractional calculus*, J. Comput. Anal. Appl. **21** (2016), 1225–1242.

[2] H. Aydi, M. F. Bota, E. Karapinar, S. Moradi, *A common fixed point for weak ϕ-contractions on b-metric spaces*, Fixed Point Theory **13** (2012), 337–346.

[3] H. Aydi, A. Felhi, S. Sahmim, *Common fixed points in rectangular b-metric spaces using $(E:A)$ property*, J. Adv. Math. Studies **8** (2015), 159–169.

[4] S. Banach, *Sur les opérations dans les ensembles abstraits et leur application aux equations itegrales*, Fund. Math. **3** (1922), 133–181.

[5] A. Batool, T. Kamran, S. Jang, C. Park, *Generalized φ-weak contractive fuzzy mappings and related fixed point results on complete metric space*, J. Comput. Anal. Appl. **21** (2016), 729–737.

[6] M. Berzig, E. Karapınar, *On modified α-ψ-contractive mappings with application,* Thai J. Math. **13** (2015), 147–152.

[7] S. H. Cho, J. S. Bae, E. Karapinar, *Fixed point theorems for α-Geraghty contraction type maps in metric spaces*, Fixed Point Theory Appl. **2013**, 2013:329.

[8] P. Chuadchawna, A. Kaewcharoen, S. Plubtieng, *Fixed point theorems for generalized α-η-ψ-Geraghty contraction type mappings in α-η-complete metric spaces*, J. Nonlinear Sci. App. **9** (2016), 471–485.

[9] S. Czerwik, *Contraction mappings in b-metric spaces.* Acta Math. Inf. Univ. Ostrav. **1** (1993), 5–11.

[10] A. Felhi, S. Sahmim, H. Aydi, *Ulam-Hyers stability and well-posedness of fixed point problems for α-λ-contractions on quasi b-metric spaces*, Fixed Point Theory Appl. **2016**, 2016:1.

[11] M. Geraghty, *On contractive mappings*, Proc. Amer. Math. Soc. **40** (1973), 604–608.

[12] V. Gupta, W. Shatanawi, N. Mani, *Fixed point theorems for* (Ψ, β)*-Geraghty contraction type maps in ordered metric spaces and some applications to integral and ordinary differential equations*, J. Fixed Point Theory Appl. **19** (2017). 1251–1267.

[13] H. Huang, S. Xu, *Fixed point theorems of contractive mappings in cone b-metric spaces and applications*, Fixed Point Theory Appl. **2013**, 2013:112.

[14] N. Hussain, M. A. Kutbi, P. Salimi, *Fixed point theory in α-complete metric space with applications*, Abstr. Appl. Anal. **2014**, Art. ID 280817 (2014).

[15] N. Hussain, M. H. Shah, *KKM mappings in cone b-metric spaces*, Comput. Math. Appl. **62** (2011), 1677–1684.

[16] M. Jovanovic, Z. Kadelburg, S. Radenovic, *Common fixed point results in metric type spaces*, Fixed Point Theory Appl. **2010** , Art. ID 978121 (2010).

[17] E. Karapinar, *α-ψ-Geraghty contraction type mappings and some related fixed point results*, Filomat **28** (2014), 37–48.

[18] E. Karapınar, P. Kumam, P. Salimi, *On α-ψ-Meir-Keeler contractive mappings*, Fixed Point Theory Appl. **2013**, 2013:94.

[19] A. Latif, J. R. Roshan, V. Parvaneh, N. Hussain, *Fixed point results via α-admissible mappings and cyclic contractive mappings in partial b-metric spaces*, J. Inequal. Appl. **2014**, 2014:345.

[20] Z. Mustafa, J. R. Roshan, V. Parvaneh, Z. Kadelburg, *Some common fixed point results in ordered partial b-metric spaces*, J. Inequal. Appl. **2013**, 2013:562.

[21] H. Piri, H. Afshari, *Some fixed point theorems in complete partial b-metric spaces*, Adv. Fixed Point Theory **4** (2014), 444–461.

[22] O. Popescu, *Some new fixed point theorems for α-Geraghty contraction type maps in metric spaces*, Fixed Point Theory Appl. **2014**, 2014:190.

[23] J. R. Roshan, V. Parvaneh, Sh. Sedghi, N. Shobkolaei, W. Shatanawi, *Common fixed points of almost generalized $(\psi$-$\varphi)_s$-contraction mappings in ordered b-metric spaces.* Fixed Point Theory Appl. **2013**, 2013:159.

[24] P. Salimi, A. Latif, N. Hussain, *Modified α-ψ-contractive mappings with applications*, Fixed Point Theory Appl. **2013**, 2013:151.

[25] B. Samet, C. Vetro, P. Vetro, *Fixed point theorems for α-ψ-contractive type mappings*, Nonlinear Anal. **75** (2012), 2154–2165.

[26] R. J. Shahkoohi, A. Razani, *Some fixed point theorems for rational Geraghty contractive mappings in ordered b-metric spaces*, J. Inequal. Appl. **2014**, 2014:373.

[27] W. Shatanawi, M. B. Hani, *A fixed point theorem in b-metric spaces with nonlinear contractive condition*, Far East J. Math. Sci. **100** (2016), 1901–1908.

[28] L. Shi, S. Xu, *Common fixed point theorems for two weakly compatible self-mappings in cone b-metric spaces*, Fixed Point Theory Appl. **2013**, 2013:120.

[29] S. Shukla, *Partial b-metric spaces and fixed point theorems*, Mediterr. J. Math. **11** (2014), 703–711.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 27, NO.3, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC

SOME EQUALITIES AND INEQUALITIES FOR K-G-FRAMES

ZHONG-QI XIANG† AND YIN-SUO JIA

Abstract. In this paper we establish some equalities and inequalities for K-g-frames. Our results generalize the remarkable results obtained by Balan et al. and Găvruţa. We also give several new inequalities for K-g-frames by using operator theory methods, which differ in structure from those for frames.

1. Introduction

Throughout this paper, $\mathscr{H}$ and $\mathscr{K}$ are separable Hilbert spaces, $\{\mathscr{K}_j\}_{j\in\mathbb{J}}$ is a sequence of closed subspaces of $\mathscr{K}$, where $\mathbb{J}$ is a finite or countable index set. For any $\mathbb{I}\subset\mathbb{J}$, we denote $\mathbb{I}^c=\mathbb{J}\backslash\mathbb{I}$. The notation $B(\mathscr{H},\mathscr{K})$ is reserved for the set of all linear bounded operators from $\mathscr{H}$ to $\mathscr{K}$, and $B(\mathscr{H},\mathscr{H})$ is abbreviated to $B(\mathscr{H})$; $K\in B(\mathscr{H})$.

Frames for Hilbert spaces, appeared first in the early 1950's, have now been applied in a variety of fields because of their redundancy and flexibility. For more information on frame theory and its applications, the interested reader can consult [4–8, 16, 19]. G-frames, proposed by Sun in [17], generalize the concept of frames extensively and possess some distinct properties though they share many similar properties with frames, see [15, 18].

A K-frame is a generalization of a frame, which was put forward by Găvruţa in [10] to investigate the atomic systems associated with a linear bounded operator K. When K is an orthogonal projection, a K-frame is just an atom system for subspace which was introduced by Feichtinger and Werther in [9]. It should be remarked that the properties of K-frames are quite different from those of frames as shown in [1, 12, 20, 22], though the definition of a K-frame is similar to a frame in form. Recently, Xiao et al. [23] applied Găvruţa's idea to the case of g-frames, thereby leading to the notion of K-g-frames, which have attracted much attention, see [2, 13].

Balan et al. [3] found a surprising identity for Parseval frames when they devoted to the study of efficient algorithms for signal reconstruction, given below.

Theorem 1.1. *Let $\{f_j\}_{j\in\mathbb{J}}$ be a Parseval frame for $\mathscr{H}$, then for every $\mathbb{I}\subset\mathbb{J}$ and every $f\in\mathscr{H}$, we have*

$$\sum_{j\in\mathbb{I}}|\langle f,f_j\rangle|^2-\Big\|\sum_{j\in\mathbb{I}}\langle f,f_j\rangle f_j\Big\|^2=\sum_{j\in\mathbb{I}^c}|\langle f,f_j\rangle|^2-\Big\|\sum_{j\in\mathbb{I}^c}\langle f,f_j\rangle f_j\Big\|^2. \tag{1.1}$$

In [3], the following inequality was also obtained.

Theorem 1.2. *Let $\{f_j\}_{j\in\mathbb{J}}$ be a Parseval frame for $\mathscr{H}$, then for every $\mathbb{I}\subset\mathbb{J}$ and every $f\in\mathscr{H}$, we have*

$$\sum_{j\in\mathbb{I}}|\langle f,f_j\rangle|^2+\Big\|\sum_{j\in\mathbb{I}^c}\langle f,f_j\rangle f_j\Big\|^2\geq\frac{3}{4}\|f\|^2. \tag{1.2}$$

† Corresponding author.

2010 *Mathematics Subject Classification*. Primary 42C15; Secondary 42C40.

Key words and phrases. Parseval K-g-frame; K-dual g-frame; operator; pseudo-inverse.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 27, NO.3, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC

Later on, Găvruţa [11] extended Theorems 1.1 and 1.2 to alternate dual frames:

Theorem 1.3. *Let $\{f_j\}_{j\in\mathbb{J}}$ be a frame for $\mathscr{H}$ and $\{g_j\}_{j\in\mathbb{J}}$ be an alternate dual frames of $\{f_j\}_{j\in\mathbb{J}}$. Then for all $\mathbb{I}\subset\mathbb{J}$ and all $f\in\mathscr{H}$, we have*

$$\begin{aligned}&\mathrm{Re}\sum_{j\in\mathbb{I}}\langle f,g_j\rangle\overline{\langle f,f_j\rangle}+\Big\|\sum_{j\in\mathbb{I}^c}\langle f,g_j\rangle f_j\Big\|^2\\ &=\mathrm{Re}\sum_{j\in\mathbb{I}^c}\langle f,g_j\rangle\overline{\langle f,f_j\rangle}+\Big\|\sum_{j\in\mathbb{I}}\langle f,g_j\rangle f_j\Big\|^2\geq\frac{3}{4}\|f\|^2.\end{aligned}\tag{1.3}$$

In fact, Theorem 1.3 is a particular case of the following result, given in [11].

Theorem 1.4. *Let $\{f_j\}_{j\in\mathbb{J}}$ be a frame for $\mathscr{H}$ and $\{g_j\}_{j\in\mathbb{J}}$ be an alternate dual frame of $\{f_j\}_{j\in\mathbb{J}}$. Then for all bounded sequence $\{\omega_j\}_{j\in\mathbb{J}}$ and all $f\in\mathscr{H}$, we have*

$$\begin{aligned}&\mathrm{Re}\sum_{j\in\mathbb{J}}\omega_j\langle f,g_j\rangle\overline{\langle f,f_j\rangle}+\Big\|\sum_{j\in\mathbb{J}}(1-\omega_j)\langle f,g_j\rangle f_j\Big\|^2\\ &=\mathrm{Re}\sum_{j\in\mathbb{J}}(1-\omega_j)\langle f,g_j\rangle\overline{\langle f,f_j\rangle}+\Big\|\sum_{j\in\mathbb{J}}\omega_j\langle f,g_j\rangle f_j\Big\|^2\geq\frac{3}{4}\|f\|^2.\end{aligned}\tag{1.4}$$

In this paper we generalize the equalities and inequalities (1.1), (1.2) and (1.4) to K-g-frames. Since g-frames can be considered as a class of K-g-frames, Theorem 2.2 in [21] and Theorem 4.1 in [24] which are a generalization of Theorems 1.1 and 1.2, and Theorem 1.4 respectively, can be obtained as a special case of the results we establish on K-g-frames. We also present some new inequalities for K-g-frames by using operator theory methods, which are different in structure from those in (1.2)–(1.4).

2. Preliminaries

In the following we briefly recall some definitions and basic properties of operators.

Definition 2.1. We call a sequence $\{\Lambda_j\in B(\mathscr{H},\mathscr{K}_j)\}_{j\in\mathbb{J}}$ a K-g-frame for $\mathscr{H}$ with respect to $\{\mathscr{K}_j\}_{j\in\mathbb{J}}$, if there exist $0<C\leq D<\infty$ such that

$$C\|K^*f\|^2\leq\sum_{j\in\mathbb{J}}\|\Lambda_jf\|^2\leq D\|f\|^2,\quad\forall f\in\mathscr{H}.\tag{2.1}$$

If we only require the right-hand inequality of (2.1), then $\{\Lambda_j\}_{j\in\mathbb{J}}$ is said to be a g-Bessel sequence for $\mathscr{H}$ with respect to $\{\mathscr{K}_j\}_{j\in\mathbb{J}}$ with g-Bessel bound D.

Remark 2.2. If $K=\mathrm{I}_{\mathscr{H}}$, the identity operator on $\mathscr{H}$, then a K-g-frame is just a g-frame.

A K-g-frame $\{\Lambda_j\in B(\mathscr{H},\mathscr{K}_j)\}_{j\in\mathbb{J}}$ for $\mathscr{H}$ is said to be Parseval if

$$\|K^*f\|^2=\sum_{j\in\mathbb{J}}\|\Lambda_jf\|^2,\quad\forall f\in\mathscr{H}.\tag{2.2}$$

Let $\{\Lambda_j\}_{j\in\mathbb{J}}$ be a Parseval K-g-frame for $\mathscr{H}$ with respect to $\{\mathscr{K}_j\}_{j\in\mathbb{J}}$. Then it is easy to check that

$$KK^*f=\sum_{j\in\mathbb{J}}\Lambda_j^*\Lambda_jf,\quad\forall f\in\mathscr{H}.\tag{2.3}$$

Definition 2.3. Let $\{\Lambda_j\}_{j\in\mathbb{J}}$ be a K-g-frame for $\mathscr{H}$ with respect to $\{\mathscr{K}_j\}_{j\in\mathbb{J}}$. A g-Bessel sequence $\{\Gamma_j\}_{j\in\mathbb{J}}$ for $\mathscr{H}$ with respect to $\{\mathscr{K}_j\}_{j\in\mathbb{J}}$ is called a K-dual g-frame of $\{\Lambda_j\}_{j\in\mathbb{J}}$, if

$$Kf = \sum_{j\in\mathbb{J}} \Lambda_j^*\Gamma_j f, \quad \forall f \in \mathscr{H}. \tag{2.4}$$

To prove the main results, we need the following lemmas.

Lemma 2.4. *(see [6]) Suppose that $\mathscr{T} \in B(\mathscr{H})$ has closed range, then there exists a unique operator $\mathscr{T}^\dagger \in B(\mathscr{H})$, called the pseudo-inverse of $\mathscr{T}$, satisfying*

$$\mathscr{T}\mathscr{T}^\dagger\mathscr{T} = \mathscr{T}, \quad \mathscr{T}^\dagger\mathscr{T}\mathscr{T}^\dagger = \mathscr{T}^\dagger, \quad (\mathscr{T}^\dagger\mathscr{T})^* = \mathscr{T}^\dagger\mathscr{T}, \quad (\mathscr{T}^\dagger)^* = (\mathscr{T}^*)^\dagger.$$

In the following, the notation $\Theta^\dagger$ is reserved to denote the pseudo-inverse of the linear bounded operator Θ (if it exists).

Lemma 2.5. *(see [14]) Suppose that $U, V, \mathscr{T} \in B(\mathscr{H})$, that $U + V = \mathscr{T}$, and that $\mathscr{T}$ has closed range. Then we have*

$$\mathscr{T}^*\mathscr{T}^\dagger U + V^*\mathscr{T}^\dagger V = V^*\mathscr{T}^\dagger\mathscr{T} + U^*\mathscr{T}^\dagger U.$$

Lemma 2.6. *If $U, V, K \in B(\mathscr{H})$ satisfy $U + V = K$, then*

$$U^*U + \frac{1}{2}(V^*K + K^*V) \geq \frac{3}{4}K^*K.$$

Proof. We have

$$\begin{aligned} U^*U + \frac{1}{2}(V^*K + K^*V) &= (K^* - V^*)(K - V) + \frac{1}{2}(V^*K + K^*V)\\ &= V^*V - (K^*V + V^*K) + K^*K + \frac{1}{2}(V^*K + K^*V)\\ &= V^*V - \frac{1}{2}(V^*K + K^*V) + K^*K\\ &= \left(V - \frac{1}{2}K\right)^*\left(V - \frac{1}{2}K\right) + \frac{3}{4}K^*K\\ &\geq \frac{3}{4}K^*K. \end{aligned}$$

□

3. Main results and their proofs

We begin with several equalities and inequalities for Parseval K-g-frames and K-dual g-frames.

Theorem 3.1. *Let $\{\Lambda_j\}_{j\in\mathbb{J}}$ be a Parseval K-g-frame for $\mathscr{H}$ with respect to $\{\mathscr{K}_j\}_{j\in\mathbb{J}}$. Then for every $\mathbb{I} \subset \mathbb{J}$ and every $f \in \mathscr{H}$ we have*

$$\sum_{j\in\mathbb{I}}\langle\Lambda_j f, \Lambda_j KK^* f\rangle - \left\|\sum_{j\in\mathbb{I}}\Lambda_j^*\Lambda_j f\right\|^2 = \overline{\sum_{j\in\mathbb{I}^c}\langle\Lambda_j f, \Lambda_j KK^* f\rangle} - \left\|\sum_{j\in\mathbb{I}^c}\Lambda_j^*\Lambda_j f\right\|^2. \tag{3.1}$$

Proof. For every $\mathbb{I} \subset \mathbb{J}$, one can easily check that the operators $S_\mathbb{I}$ and $S_{\mathbb{I}^c}$ defined by

$$S_\mathbb{I} f = \sum_{j\in\mathbb{I}}\Lambda_j^*\Lambda_j f, \quad S_{\mathbb{I}^c} f = \sum_{j\in\mathbb{I}^c}\Lambda_j^*\Lambda_j f$$

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 27, NO.3, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC

are positive, linear bounded and self-adjoint. Moreover, the definition of a Parseval K-g-frame gives $S_{\mathbb{I}} + S_{\mathbb{I}^c} = KK^*$. Hence for each $f \in \mathscr{H}$,

$$\begin{aligned}\sum_{j\in\mathbb{I}}\langle\Lambda_j f, \Lambda_j KK^* f\rangle - \Big\|\sum_{j\in\mathbb{I}}\Lambda_j^*\Lambda_j f\Big\|^2 &= \langle S_{\mathbb{I}} f, KK^* f\rangle - \|S_{\mathbb{I}} f\|^2\\ &= \langle (KK^* - S_{\mathbb{I}})S_{\mathbb{I}} f, f\rangle\\ &= \langle S_{\mathbb{I}^c}(KK^* - S_{\mathbb{I}^c}) f, f\rangle\\ &= \langle S_{\mathbb{I}^c}KK^* f, f\rangle - \langle S_{\mathbb{I}^c}^2 f, f\rangle\\ &= \overline{\langle S_{\mathbb{I}^c} f, KK^* f\rangle} - \|S_{\mathbb{I}^c} f\|^2\\ &= \overline{\sum_{j\in\mathbb{I}^c}\langle\Lambda_j f, \Lambda_j KK^* f\rangle} - \Big\|\sum_{j\in\mathbb{I}^c}\Lambda_j^*\Lambda_j f\Big\|^2.\end{aligned}$$

□

A version of the equality obtained in Theorem 3.1 for overlapping divisions is derived in the following result.

Theorem 3.2. *Let $\{\Lambda_j\}_{j\in\mathbb{J}}$ be a Parseval K-g-frame for $\mathscr{H}$ with respect to $\{\mathscr{K}_j\}_{j\in\mathbb{J}}$. Then for every $\mathbb{I}\subset\mathbb{J}$, every $\mathbb{E}\subset\mathbb{I}^c$, and every $f\in\mathscr{H}$, we have*

$$\begin{aligned}&\Big\|\sum_{j\in\mathbb{I}\cup\mathbb{E}}\Lambda_j^*\Lambda_j f\Big\|^2 - \Big\|\sum_{j\in\mathbb{I}^c\setminus\mathbb{E}}\Lambda_j^*\Lambda_j f\Big\|^2\\ &\quad= \Big\|\sum_{j\in\mathbb{I}}\Lambda_j^*\Lambda_j f\Big\|^2 - \Big\|\sum_{j\in\mathbb{I}^c}\Lambda_j^*\Lambda_j f\Big\|^2 + 2\mathrm{Re}\sum_{j\in\mathbb{E}}\langle\Lambda_j f, \Lambda_j KK^* f\rangle.\end{aligned}$$

Proof. Applying Theorem 3.1 twice yields

$$\begin{aligned}&\Big\|\sum_{j\in\mathbb{I}\cup\mathbb{E}}\Lambda_j^*\Lambda_j f\Big\|^2 - \Big\|\sum_{j\in\mathbb{I}^c\setminus\mathbb{E}}\Lambda_j^*\Lambda_j f\Big\|^2\\ &\quad= \sum_{j\in\mathbb{I}\cup\mathbb{E}}\langle\Lambda_j f, \Lambda_j KK^* f\rangle - \overline{\sum_{j\in\mathbb{I}^c\setminus\mathbb{E}}\langle\Lambda_j f, \Lambda_j KK^* f\rangle}\\ &\quad= \sum_{j\in\mathbb{I}}\langle\Lambda_j f, \Lambda_j KK^* f\rangle - \overline{\sum_{j\in\mathbb{I}^c}\langle\Lambda_j f, \Lambda_j KK^* f\rangle} + 2\mathrm{Re}\sum_{j\in\mathbb{E}}\langle\Lambda_j f, \Lambda_j KK^* f\rangle\\ &\quad= \Big\|\sum_{j\in\mathbb{I}}\Lambda_j^*\Lambda_j f\Big\|^2 - \Big\|\sum_{j\in\mathbb{I}^c}\Lambda_j^*\Lambda_j f\Big\|^2 + 2\mathrm{Re}\sum_{j\in\mathbb{E}}\langle\Lambda_j f, \Lambda_j KK^* f\rangle.\end{aligned}$$

□

Theorem 3.3. *Let $\{\Lambda_j\}_{j\in\mathbb{J}}$ be a K-g-frame for $\mathscr{H}$ with respect to $\{\mathscr{K}_j\}_{j\in\mathbb{J}}$ and $\{\Gamma_j\}_{j\in\mathbb{J}}$ be a K-dual g-frame of $\{\Lambda_j\}_{j\in\mathbb{J}}$. Then for every $\{\alpha_j\}_{j\in\mathbb{J}}\in\ell^\infty(\mathbb{J})$ and every $f\in\mathscr{H}$ we have*

$$\begin{aligned}&\sum_{j\in\mathbb{J}}(1-\alpha_j)\langle\Gamma_j f, \Lambda_j K f\rangle + \Big\|\sum_{j\in\mathbb{J}}\alpha_j\Lambda_j^*\Gamma_j f\Big\|^2\\ &\quad= \overline{\sum_{j\in\mathbb{J}}\alpha_j\langle\Gamma_j f, \Lambda_j K f\rangle} + \Big\|\sum_{j\in\mathbb{J}}(1-\alpha_j)\Lambda_j^*\Gamma_j f\Big\|^2. \qquad (3.2)\end{aligned}$$

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 27, NO.3, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC

Proof. For any $\{\alpha_j\}_{j\in\mathbb{J}} \in \ell^\infty(\mathbb{J})$ and any $f \in \mathscr{H}$, we let

$$S_\alpha f = \sum_{j\in\mathbb{J}} \alpha_j \Lambda_j^* \Gamma_j f, \quad S_{1-\alpha} f = \sum_{j\in\mathbb{J}} (1-\alpha_j)\Lambda_j^*\Gamma_j f.$$

Denote by D_1 and D_2 the g-Bessel bounds of $\{\Lambda_j\}_{j\in\mathbb{J}}$ and $\{\Gamma_j\}_{j\in\mathbb{J}}$ respectively. Then

$$\begin{aligned}
\Big\|\sum_{j\in\mathbb{J}} \alpha_j\Lambda_j^*\Gamma_j f\Big\|^2 &= \sup_{g\in\mathscr{H},\|g\|=1} \Big|\Big\langle \sum_{j\in\mathbb{J}} \alpha_j\Lambda_j^*\Gamma_j f, g\Big\rangle\Big|^2 \\
&= \sup_{g\in\mathscr{H},\|g\|=1} \Big|\sum_{j\in\mathbb{J}} \alpha_j \langle \Gamma_j f, \Lambda_j g\rangle\Big|^2 \\
&\le \sup_{g\in\mathscr{H},\|g\|=1} \Big(\sum_{j\in\mathbb{J}} |\alpha_j|\,|\langle \Gamma_j f, \Lambda_j g\rangle|\Big)^2 \\
&\le \sup_{g\in\mathscr{H},\|g\|=1} \|\{\alpha_j\}_{j\in\mathbb{J}}\|^2 \sum_{j\in\mathbb{J}} \|\Gamma_j f\|^2 \cdot \sum_{j\in\mathbb{J}} \|\Lambda_j g\|^2 \\
&\le D_1 D_2 \|\{\alpha_j\}_{j\in\mathbb{J}}\|^2 \|f\|^2.
\end{aligned}$$

It follows that S_α is well defined and bounded. By the same way we can show that $S_{1-\alpha}$ is also well defined and bounded. Since $\{\Gamma_j\}_{j\in\mathbb{J}}$ is a K-dual g-frame of $\{\Lambda_j\}_{j\in\mathbb{J}}$, we have

$$S_\alpha f + S_{1-\alpha} f = \sum_{j\in\mathbb{J}} \alpha_j\Lambda_j^*\Gamma_j f + \sum_{j\in\mathbb{J}} (1-\alpha_j)\Lambda_j^*\Gamma_j f = \sum_{j\in\mathbb{J}} \Lambda_j^*\Gamma_j f = Kf$$

for each $f \in \mathscr{H}$. It follows that

$$\begin{aligned}
\sum_{j\in\mathbb{J}} (1-\alpha_j)\langle \Gamma_j f, \Lambda_j K f\rangle + \Big\|\sum_{j\in\mathbb{J}} \alpha_j\Lambda_j^*\Gamma_j f\Big\|^2 &= \langle S_{1-\alpha} f, Kf\rangle + \langle S_\alpha f, S_\alpha f\rangle \\
&= \langle (K - S_\alpha) f, Kf\rangle + \langle S_\alpha f, S_\alpha f\rangle \\
&= \langle Kf, Kf\rangle - \langle S_\alpha f, Kf\rangle + \langle S_\alpha f, S_\alpha f\rangle, \qquad (3.3)
\end{aligned}$$

and

$$\begin{aligned}
&\overline{\sum_{j\in\mathbb{J}} \alpha_j\langle \Gamma_j f, \Lambda_j K f\rangle} + \Big\|\sum_{j\in\mathbb{J}} (1-\alpha_j)\Lambda_j^*\Gamma_j f\Big\|^2 \\
&\quad = \overline{\langle S_\alpha f, Kf\rangle} + \langle S_{1-\alpha} f, S_{1-\alpha} f\rangle \\
&\quad = \langle Kf, S_\alpha f\rangle + \langle (K - S_\alpha) f, (K - S_\alpha) f\rangle \\
&\quad = \langle Kf, S_\alpha f\rangle + \langle Kf, Kf\rangle + \langle S_\alpha f, S_\alpha f\rangle - \langle S_\alpha f, Kf\rangle - \langle Kf, S_\alpha f\rangle \\
&\quad = \langle Kf, Kf\rangle - \langle S_\alpha f, Kf\rangle + \langle S_\alpha f, S_\alpha f\rangle. \qquad (3.4)
\end{aligned}$$

Combination of (3.3) and (3.4) yields (3.2). □

Corollary 3.4. Let $\{\Lambda_j\}_{j\in\mathbb{J}}$ be a K-g-frame for $\mathscr{H}$ with respect to $\{\mathscr{K}_j\}_{j\in\mathbb{J}}$ and $\{\Gamma_j\}_{j\in\mathbb{J}}$ be a K-dual g-frame of $\{\Lambda_j\}_{j\in\mathbb{J}}$. Then for every $\mathbb{I} \subset \mathbb{J}$ and every $f \in \mathscr{H}$ we have

$$\sum_{j\in\mathbb{I}^c} \langle \Gamma_j f, \Lambda_j K f\rangle + \Big\|\sum_{j\in\mathbb{I}} \Lambda_j^*\Gamma_j f\Big\|^2 = \overline{\sum_{j\in\mathbb{I}} \langle \Gamma_j f, \Lambda_j K f\rangle} + \Big\|\sum_{j\in\mathbb{I}^c} \Lambda_j^*\Gamma_j f\Big\|^2.$$

Proof. The result follows directly from Theorem 3.3 if we take $\mathbb{I} \subset \mathbb{J}$ and

$$\alpha_j = \begin{cases} 1, & j \in \mathbb{I}, \\ 0, & j \in \mathbb{I}^c. \end{cases}$$

□

Theorem 3.5. *Let $\{\Lambda_j\}_{j\in\mathbb{J}}$ be a K-g-frame for $\mathscr{H}$ with respect to $\{\mathscr{K}_j\}_{j\in\mathbb{J}}$ and $\{\Gamma_j\}_{j\in\mathbb{J}}$ be a K-dual g-frame of $\{\Lambda_j\}_{j\in\mathbb{J}}$. Then for every $\{\alpha_j\}_{j\in\mathbb{J}}\in\ell^\infty(\mathbb{J})$ and every $f\in\mathscr{H}$ we have*

$$\begin{aligned}&\Big\|\sum_{j\in\mathbb{J}}\alpha_j\Lambda_j^*\Gamma_j f\Big\|^2+\mathrm{Re}\Big(\sum_{j\in\mathbb{J}}(1-\alpha_j)\langle\Gamma_j f,\Lambda_j Kf\rangle\Big)\\ &=\Big\|\sum_{j\in\mathbb{J}}(1-\alpha_j)\Lambda_j^*\Gamma_j f\Big\|^2+\mathrm{Re}\Big(\sum_{j\in\mathbb{J}}\alpha_j\langle\Gamma_j f,\Lambda_j Kf\rangle\Big)\geq\frac{3}{4}\|Kf\|^2.\end{aligned}\tag{3.5}$$

Proof. The equality is obtained immediately if we take the real part on both sides of (3.2). For the inequality, taking

$$Uf=\sum_{j\in\mathbb{J}}\alpha_j\Lambda_j^*\Gamma_j f\quad\text{and}\quad Vf=\sum_{j\in\mathbb{J}}(1-\alpha_j)\Lambda_j^*\Gamma_j f$$

for each $f\in\mathscr{H}$ in Lemma 2.6, then we have

$$\begin{aligned}&\Big\|\sum_{j\in\mathbb{J}}\alpha_j\Lambda_j^*\Gamma_j f\Big\|^2+\mathrm{Re}\Big(\sum_{j\in\mathbb{J}}(1-\alpha_j)\langle\Gamma_j f,\Lambda_j Kf\rangle\Big)\\ &=\|Uf\|^2+\mathrm{Re}\langle Vf,Kf\rangle=\langle Uf,Uf\rangle+\frac{1}{2}(\langle Vf,Kf\rangle+\langle Kf,Vf\rangle)\\ &=\Big\langle\Big(U^*U+\frac{1}{2}(V^*K+K^*V)\Big)f,f\Big\rangle\\ &\geq\frac{3}{4}\langle K^*Kf,f\rangle=\frac{3}{4}\|Kf\|^2.\end{aligned}$$

□

Theorem 3.6. *Let $\{\Lambda_j\}_{j\in\mathbb{J}}$ be a Parseval K-g-frame for $\mathscr{H}$ with respect to $\{\mathscr{K}_j\}_{j\in\mathbb{J}}$. Then for every $\mathbb{I}\subset\mathbb{J}$ and every $f\in\mathscr{H}$ we have*

$$\begin{aligned}&\Big\|\sum_{j\in\mathbb{I}}\Lambda_j^*\Lambda_j f\Big\|^2+\mathrm{Re}\sum_{j\in\mathbb{I}^c}\langle\Lambda_j f,\Lambda_j KK^*f\rangle\\ &=\Big\|\sum_{j\in\mathbb{I}^c}\Lambda_j^*\Lambda_j f\Big\|^2+\mathrm{Re}\sum_{j\in\mathbb{I}}\langle\Lambda_j f,\Lambda_j KK^*f\rangle\geq\frac{3}{4}\|KK^*f\|^2.\end{aligned}\tag{3.6}$$

Proof. The equality follows if we take the real part on both sides of (3.1). It remains to prove the inequality. Since $S_{\mathbb{I}}+S_{\mathbb{I}^c}=KK^*$, it follows that

$$\begin{aligned}S_{\mathbb{I}}^2+S_{\mathbb{I}^c}^2&=S_{\mathbb{I}}^2+(KK^*-S_{\mathbb{I}})^2\\ &=2S_{\mathbb{I}}^2+(KK^*)^2-KK^*S_{\mathbb{I}}-S_{\mathbb{I}}KK^*\\ &=2\Big(\frac{KK^*}{2}-S_{\mathbb{I}}\Big)^2+\frac{(KK^*)^2}{2}.\end{aligned}\tag{3.7}$$

Therefore,

$$\begin{aligned}KK^*S_{\mathbb{I}^c}+S_{\mathbb{I}^c}KK^*+2S_{\mathbb{I}}^2&=KK^*(S_{\mathbb{I}}+S_{\mathbb{I}^c})-KK^*S_{\mathbb{I}}+S_{\mathbb{I}^c}KK^*+2S_{\mathbb{I}}^2\\ &=(KK^*)^2-(S_{\mathbb{I}}+S_{\mathbb{I}^c})S_{\mathbb{I}}+S_{\mathbb{I}^c}(S_{\mathbb{I}}+S_{\mathbb{I}^c})+2S_{\mathbb{I}}^2\\ &=(KK^*)^2-S_{\mathbb{I}}^2-S_{\mathbb{I}^c}S_{\mathbb{I}}+S_{\mathbb{I}^c}S_{\mathbb{I}}+S_{\mathbb{I}^c}^2+2S_{\mathbb{I}}^2\\ &=(KK^*)^2+S_{\mathbb{I}}^2+S_{\mathbb{I}^c}^2\geq\frac{3}{2}(KK^*)^2.\end{aligned}$$

Thus for every $f \in \mathscr{H}$ we have

$$\begin{aligned}
&\Big\|\sum_{j\in\mathbb{I}} \Lambda_j^*\Lambda_j f\Big\|^2 + \mathrm{Re}\sum_{j\in\mathbb{I}^c}\langle \Lambda_j f, \Lambda_j KK^* f\rangle \\
&\quad = \|S_{\mathbb{I}} f\|^2 + \frac{1}{2}(\langle S_{\mathbb{I}^c} f, KK^* f\rangle + \langle KK^* f, S_{\mathbb{I}^c} f\rangle) \\
&\quad = \frac{1}{2}(2\langle S_{\mathbb{I}}^2 f, f\rangle + \langle S_{\mathbb{I}^c} f, KK^* f\rangle + \langle KK^* f, S_{\mathbb{I}^c} f\rangle) \\
&\quad = \frac{1}{2}\langle (KK^* S_{\mathbb{I}^c} + S_{\mathbb{I}^c} KK^* + 2S_{\mathbb{I}}^2) f, f\rangle \geq \frac{3}{4}\langle (KK^*)^2 f, f\rangle = \frac{3}{4}\|KK^* f\|^2.
\end{aligned}$$

□

We give an upper bound condition for the left-hand-side of the equality in (3.6) under the condition that K has closed range.

Theorem 3.7. *Suppose that K has closed range and that $\{\Lambda_j\}_{j\in\mathbb{J}}$ is a Parseval K-g-frame for $\mathscr{H}$ with respect to $\{\mathscr{K}_j\}_{j\in\mathbb{J}}$. Then for every $\mathbb{I} \subset \mathbb{J}$ and every $f \in \mathscr{H}$ we obtain*

$$\Big\|\sum_{j\in\mathbb{I}} \Lambda_j^*\Lambda_j f\Big\|^2 + \mathrm{Re}\sum_{j\in\mathbb{I}^c}\langle \Lambda_j f, \Lambda_j KK^* f\rangle \leq \|K\|\,\|K^\dagger\|(1 + \|K\|\,\|K^\dagger\|)\|KK^* f\|^2.$$

Proof. For each $f \in \mathscr{H}$, by Lemma 2.4 we have

$$\begin{aligned}
\Big\|\sum_{j\in\mathbb{I}} \Lambda_j^*\Lambda_j f\Big\|^2 &\leq \|S_{\mathbb{I}}\| \sum_{j\in\mathbb{I}} \|\Lambda_j f\|^2 \leq \|S_{\mathbb{I}}\| \sum_{j\in\mathbb{J}} \|\Lambda_j f\|^2 \\
&\leq \|K\|^2\|K^* f\|^2 = \|K\|^2\|K^*(K^*)^\dagger K^* f\|^2 \\
&= \|K\|^2\|K^\dagger KK^* f\|^2 \leq \|K\|^2\|K^\dagger\|^2\|KK^* f\|^2, \qquad (3.8)
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{Re}\sum_{j\in\mathbb{I}^c}\langle \Lambda_j f, \Lambda_j KK^* f\rangle &\leq \Big(\sum_{j\in\mathbb{J}} \|\Lambda_j f\|^2\Big)^{\frac{1}{2}}\Big(\sum_{j\in\mathbb{J}} \|\Lambda_j KK^* f\|^2\Big)^{\frac{1}{2}} \\
&= \|K^* f\|\,\|K^*KK^* f\| \\
&= \|K^\dagger KK^* f\|\,\|K^*KK^* f\| \\
&\leq \|K\|\,\|K^\dagger\|\,\|KK^* f\|^2. \qquad (3.9)
\end{aligned}$$

Now, the result follows by combining (3.8) and (3.9). □

In the following we give some new inequalities for K-g-frames, which possess different structure comparing with the inequalities for frames shown in Theorems 1.2, 1.3 and 1.4.

Theorem 3.8. *Let $\{\Lambda_j\}_{j\in\mathbb{J}}$ be a K-g-frame for $\mathscr{H}$ with respect to $\{\mathscr{K}_j\}_{j\in\mathbb{J}}$ and $\{\Gamma_j\}_{j\in\mathbb{J}}$ be a K-dual g-frame of $\{\Lambda_j\}_{j\in\mathbb{J}}$. Then for every $\mathbb{I} \subset \mathbb{J}$ and every $f \in \mathscr{H}$ we have*

$$\frac{3}{4}\|Kf\|^2 \leq \Big\|\sum_{j\in\mathbb{I}} \Lambda_j^*\Gamma_j f\Big\|^2 + \mathrm{Re}\sum_{j\in\mathbb{I}^c}\langle \Gamma_j f, \Lambda_j Kf\rangle \leq \frac{3\|K\|^2 + \|F_{\mathbb{I}} - F_{\mathbb{I}^c}\|^2}{4}\|f\|^2, \qquad (3.10)$$

where the operators $F_{\mathbb{I}}$ is defined by $F_{\mathbb{I}} f = \sum_{j\in\mathbb{I}} \Lambda_j^*\Gamma_j f$.

Proof. The left-hand inequality follows from Theorem 3.5 if we consider $\mathbb{I} \subset \mathbb{J}$ and

$$\alpha_j = \begin{cases} 1, & j \in \mathbb{I}, \\ 0, & j \in \mathbb{I}^c. \end{cases}$$

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 27, NO.3, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC

We now prove the right-hand inequality of (3.10). For any $f \in \mathscr{H}$ we get

$$\begin{aligned}
&\Big\|\sum_{j\in\mathbb{I}} \Lambda_j^*\Gamma_j f\Big\|^2 + \mathrm{Re}\sum_{j\in\mathbb{I}^c}\langle \Gamma_j f, \Lambda_j K f\rangle \\
&= \langle F_{\mathbb{I}} f, F_{\mathbb{I}} f\rangle + \mathrm{Re}\langle K f, F_{\mathbb{I}^c} f\rangle \\
&= \langle F_{\mathbb{I}} f, F_{\mathbb{I}} f\rangle + \mathrm{Re}\langle K f, (K - F_{\mathbb{I}}) f\rangle \\
&= \langle F_{\mathbb{I}} f, F_{\mathbb{I}} f\rangle + \langle K f, K f\rangle - \mathrm{Re}\langle K f, F_{\mathbb{I}} f\rangle \\
&= \langle K f, K f\rangle - \mathrm{Re}(\langle (K - F_{\mathbb{I}}) f, F_{\mathbb{I}} f\rangle) \\
&= \langle K f, K f\rangle - \mathrm{Re}\langle F_{\mathbb{I}^c} f, F_{\mathbb{I}} f\rangle \\
&= \langle K f, K f\rangle - \frac{1}{2}\langle F_{\mathbb{I}} f, F_{\mathbb{I}^c} f\rangle - \frac{1}{2}\langle F_{\mathbb{I}^c} f, F_{\mathbb{I}} f\rangle \\
&= \frac{3}{4}\|K f\|^2 + \frac{1}{4}\langle F_{\mathbb{I}} f + F_{\mathbb{I}^c} f, F_{\mathbb{I}} f + F_{\mathbb{I}^c} f\rangle - \frac{1}{2}\langle F_{\mathbb{I}} f, F_{\mathbb{I}^c} f\rangle - \frac{1}{2}\langle F_{\mathbb{I}^c} f, F_{\mathbb{I}} f\rangle \\
&= \frac{3}{4}\|K f\|^2 + \frac{1}{4}\langle (F_{\mathbb{I}} - F_{\mathbb{I}^c}) f, (F_{\mathbb{I}} - F_{\mathbb{I}^c}) f\rangle \\
&\le \frac{3}{4}\|K\|^2\|f\|^2 + \frac{1}{4}\|F_{\mathbb{I}} - F_{\mathbb{I}^c}\|^2\|f\|^2 \\
&= \frac{3\|K\|^2 + \|F_{\mathbb{I}} - F_{\mathbb{I}^c}\|^2}{4}\|f\|^2.
\end{aligned}$$

This completes the proof. □

Theorem 3.9. *Suppose that K is positive and that it has closed range. Let $\{\Lambda_j\}_{j\in\mathbb{J}}$ be a K-g-frame for $\mathscr{H}$ with respect to $\{\mathscr{K}_j\}_{j\in\mathbb{J}}$ and $\{\Gamma_j\}_{j\in\mathbb{J}}$ be a K-dual g-frame of $\{\Lambda_j\}_{j\in\mathbb{J}}$. Then for every $\mathbb{I}\subset\mathbb{J}$ and every $f\in\mathscr{H}$ we have*

$$\begin{aligned}
&\mathrm{Re}\sum_{j\in\mathbb{I}}\langle \Gamma_j f, \Lambda_j K^\dagger K f\rangle + \Big\langle \sum_{j\in\mathbb{I}^c} K^\dagger\Lambda_j^*\Gamma_j f, \sum_{j\in\mathbb{I}^c}\Lambda_j^*\Gamma_j f\Big\rangle \\
&\quad= \mathrm{Re}\sum_{j\in\mathbb{I}^c}\langle \Gamma_j f, \Lambda_j K^\dagger K f\rangle + \Big\langle \sum_{j\in\mathbb{I}} K^\dagger\Lambda_j^*\Gamma_j f, \sum_{j\in\mathbb{I}}\Lambda_j^*\Gamma_j f\Big\rangle \ge \frac{3}{4}\|K^{\frac{1}{2}} f\|^2.
\end{aligned}$$

Proof. Since K is positive, it is self-adjoint and thus by Lemma 2.4, $(K^\dagger)^* = (K^*)^\dagger = K^\dagger$. Hence, $\langle K^\dagger F_{\mathbb{I}} f, F_{\mathbb{I}} f\rangle$ and $\langle K^\dagger F_{\mathbb{I}^c} f, F_{\mathbb{I}^c} f\rangle$ are real numbers for every $f\in\mathscr{H}$. From Lemma 2.5 it follows that

$$\begin{aligned}
&\mathrm{Re}\sum_{j\in\mathbb{I}}\langle \Gamma_j f, \Lambda_j K^\dagger K f\rangle + \Big\langle \sum_{j\in\mathbb{I}^c} K^\dagger\Lambda_j^*\Gamma_j f, \sum_{j\in\mathbb{I}^c}\Lambda_j^*\Gamma_j f\Big\rangle \\
&= \mathrm{Re}\langle F_{\mathbb{I}} f, K^\dagger K f\rangle + \langle K^\dagger F_{\mathbb{I}^c} f, F_{\mathbb{I}^c} f\rangle \\
&= \mathrm{Re}\langle K K^\dagger F_{\mathbb{I}} f, f\rangle + \langle F_{\mathbb{I}^c}^* K^\dagger F_{\mathbb{I}^c} f, f\rangle \\
&= \mathrm{Re}\langle (K K^\dagger F_{\mathbb{I}} + F_{\mathbb{I}^c}^* K^\dagger F_{\mathbb{I}^c}) f, f\rangle \\
&= \mathrm{Re}\langle (F_{\mathbb{I}^c}^* K^\dagger K + F_{\mathbb{I}}^* K^\dagger F_{\mathbb{I}}) f, f\rangle \\
&= \mathrm{Re}(\langle F_{\mathbb{I}^c}^* K^\dagger K f, f\rangle + \langle F_{\mathbb{I}}^* K^\dagger F_{\mathbb{I}} f, f\rangle) \\
&= \mathrm{Re}(\langle K^\dagger K f, F_{\mathbb{I}^c} f\rangle + \langle K^\dagger F_{\mathbb{I}} f, F_{\mathbb{I}} f\rangle) \\
&= \mathrm{Re}\langle F_{\mathbb{I}^c} f, K^\dagger K f\rangle + \langle K^\dagger F_{\mathbb{I}} f, F_{\mathbb{I}} f\rangle \\
&= \mathrm{Re}\sum_{j\in\mathbb{I}^c}\langle \Gamma_j f, \Lambda_j K^\dagger K f\rangle + \Big\langle \sum_{j\in\mathbb{I}} K^\dagger\Lambda_j^*\Gamma_j f, \sum_{j\in\mathbb{I}}\Lambda_j^*\Gamma_j f\Big\rangle.
\end{aligned}$$

Again by Lemma 2.4 we have

$$\begin{aligned}
&\mathrm{Re}\sum_{j\in\mathbb{I}^c}\langle\Gamma_j f,\Lambda_j K^\dagger K f\rangle+\Big\langle\sum_{j\in\mathbb{I}}K^\dagger\Lambda_j^*\Gamma_j f,\sum_{j\in\mathbb{I}}\Lambda_j^*\Gamma_j f\Big\rangle\\
&\quad=\mathrm{Re}\langle(KK^\dagger F_{\mathbb{I}}+F_{\mathbb{I}^c}^*K^\dagger F_{\mathbb{I}^c})f,f\rangle\\
&\quad=\mathrm{Re}\langle(KK^\dagger(K-F_{\mathbb{I}^c})+F_{\mathbb{I}^c}^*K^\dagger F_{\mathbb{I}^c})f,f\rangle\\
&\quad=\langle Kf,f\rangle-\mathrm{Re}\langle KK^\dagger F_{\mathbb{I}^c}f,f\rangle+\langle F_{\mathbb{I}^c}^*K^\dagger F_{\mathbb{I}^c}f,f\rangle\\
&\quad=\langle K^{\frac12}f,K^{\frac12}f\rangle-\mathrm{Re}\langle K^{\frac12}K^{\frac12}K^\dagger F_{\mathbb{I}^c}f,f\rangle+\langle(K^{\frac12}K^\dagger F_{\mathbb{I}^c})^*(K^{\frac12}K^\dagger F_{\mathbb{I}^c})f,f\rangle\\
&\quad=\frac34\|K^{\frac12}f\|^2+\Big\langle\frac12K^{\frac12}f-K^{\frac12}K^\dagger F_{\mathbb{I}^c}f,\frac12K^{\frac12}f-K^{\frac12}K^\dagger F_{\mathbb{I}^c}f\Big\rangle\geq\frac34\|K^{\frac12}f\|^2
\end{aligned}$$

for each $f\in\mathscr{H}$ and the proof is finished. □

Theorem 3.10. *Let $\{\Lambda_j\}_{j\in\mathbb{J}}$ be a Parseval K-g-frame for $\mathscr{H}$ with respect to $\{\mathscr{K}_j\}_{j\in\mathbb{J}}$. If $S_{\mathbb{I}}$ commutes with $S_{\mathbb{I}^c}$ for every $\mathbb{I}\subset\mathbb{J}$, then for every $f\in\mathscr{H}$ we have*

$$\frac12\|KK^*f\|^2\leq\Big\|\sum_{j\in\mathbb{I}}\Lambda_j^*\Lambda_j f\Big\|^2+\Big\|\sum_{j\in\mathbb{I}^c}\Lambda_j^*\Lambda_j f\Big\|^2\leq\|KK^*f\|^2. \tag{3.11}$$

$$0\leq\sum_{j\in\mathbb{I}}\langle\Lambda_j KK^*f,\Lambda_j f\rangle-\Big\|\sum_{j\in\mathbb{I}}\Lambda_j^*\Lambda_j f\Big\|^2\leq\frac14\|KK^*f\|^2. \tag{3.12}$$

Proof. From (3.7) it follows that

$$\begin{aligned}
\Big\|\sum_{j\in\mathbb{J}}\Lambda_j^*\Lambda_j f\Big\|^2+\Big\|\sum_{j\in\mathbb{I}^c}\Lambda_j^*\Lambda_j f\Big\|^2&=\|S_{\mathbb{I}}f\|^2+\|S_{\mathbb{I}^c}f\|^2=\langle S_{\mathbb{I}}f,S_{\mathbb{I}}f\rangle+\langle S_{\mathbb{I}^c}f,S_{\mathbb{I}^c}f\rangle\\
&=\langle(S_{\mathbb{I}}^2+S_{\mathbb{I}^c}^2)f,f\rangle\geq\frac12\langle(KK^*)^2f,f\rangle=\frac12\|KK^*f\|^2
\end{aligned}\tag{3.13}$$

for every $f\in\mathscr{H}$. Since $S_{\mathbb{I}}$ commutes with $S_{\mathbb{I}^c}$, $S_{\mathbb{I}^c}S_{\mathbb{I}}\geq0$ and

$$0\leq S_{\mathbb{I}}S_{\mathbb{I}^c}=S_{\mathbb{I}}(KK^*-S_{\mathbb{I}})=S_{\mathbb{I}}KK^*-S_{\mathbb{I}}^2. \tag{3.14}$$

It follows that

$$\begin{aligned}
S_{\mathbb{I}}^2+S_{\mathbb{I}^c}^2&=S_{\mathbb{I}}^2+(KK^*)^2-KK^*S_{\mathbb{I}}-S_{\mathbb{I}}KK^*+S_{\mathbb{I}}^2\\
&=(KK^*)^2+(S_{\mathbb{I}}^2-S_{\mathbb{I}}KK^*)+(S_{\mathbb{I}}^2-KK^*S_{\mathbb{I}})\\
&=(KK^*)^2-(S_{\mathbb{I}}KK^*-S_{\mathbb{I}}^2)-S_{\mathbb{I}^c}S_{\mathbb{I}}\leq(KK^*)^2.
\end{aligned}$$

Hence for every $f\in\mathscr{H}$ we have

$$\Big\|\sum_{j\in\mathbb{J}}\Lambda_j^*\Lambda_j f\Big\|^2+\Big\|\sum_{j\in\mathbb{I}^c}\Lambda_j^*\Lambda_j f\Big\|^2=\langle(S_{\mathbb{I}}^2+S_{\mathbb{I}^c}^2)f,f\rangle\leq\langle(KK^*)^2f,f\rangle=\|KK^*f\|^2.$$

This together with (3.13) gives (3.11). We next prove (3.12). Using formula (3.14) we get

$$\begin{aligned}
\sum_{j\in\mathbb{I}}\langle\Lambda_j KK^*f,\Lambda_j f\rangle-\Big\|\sum_{j\in\mathbb{I}}\Lambda_j^*\Lambda_j f\Big\|^2&=\langle S_{\mathbb{I}}KK^*f,f\rangle-\|S_{\mathbb{I}}f\|^2=\langle(S_{\mathbb{I}}KK^*-S_{\mathbb{I}}^2)f,f\rangle\\
&=\langle S_{\mathbb{I}}(KK^*-S_{\mathbb{I}})f,f\rangle=\langle S_{\mathbb{I}}S_{\mathbb{I}^c}f,f\rangle\geq0
\end{aligned}$$

for every $f \in \mathscr{H}$. On the other hand we obtain

$$\begin{aligned}\sum_{j\in\mathbb{I}}\langle\Lambda_j KK^* f, \Lambda_j f\rangle - \left\|\sum_{j\in\mathbb{I}}\Lambda_j^*\Lambda_j f\right\|^2 &= \langle(S_{\mathbb{I}}KK^* - S_{\mathbb{I}}^2)f, f\rangle \\ &= \left\langle -\left(S_{\mathbb{I}} - \frac{KK^*}{2}\right)^2 f + \frac{(KK^*)^2}{4}f, f\right\rangle \\ &\le \left\langle \frac{(KK^*)^2}{4}f, f\right\rangle \\ &= \frac{1}{4}\|KK^* f\|^2.\end{aligned}$$

This completes the proof. □

Acknowledgements

The research was partially supported by the National Natural Science Foundation of China (Grant Nos. 11761057 and 11561057), the Natural Science Foundation of Jiangxi Province (Grant No. 20151BAB201007), and the Science Foundation of Jiangxi Education Department (Grant No. GJJ151061).

References

[1] F. Arabyani Neyshaburi and A. Arefijamaal, Some constructions of K-frames and their duals, to appear in *Rocky Mountain J. Math*.
[2] M.S. Asgari and H. Rahimi, Generalized frames for operators in Hilbert spaces, *Infin. Dimens. Anal. Quantum. Probab. Relat. Top*. 17, 1450013, 20 pp (2014).
[3] R. Balan, P.G. Casazza, D. Edidin, and G. Kutyniok, A new identity for Parseval frames, *Proc. Amer. Math. Soc*. 135, 1007–1015 (2007).
[4] J.J. Benedetto, A.M. Powell, and O. Yilmaz, Sigma-Delta ($\Sigma\Delta$) quantization and finite frames, *IEEE Trans. Inform. Theory* 52, 1990–2005 (2006).
[5] P.G. Casazza, The art of frame theory, *Taiwanese J. Math*. 4, 129–201 (2000).
[6] O. Christensen, *An Introduction to Frames and Riesz Bases*, Birkhäuser, Boston, 2003.
[7] I. Daubechies, A. Grossmann, and Y. Meyer, Painless nonorthogonal expansions, *J. Math. Phys*. 27, 1271–1283 (1986).
[8] R.J. Duffin and A.C. Schaeffer, A class of nonharmonic Fourier series, *Trans. Amer. Math. Soc*. 72, 341–366 (1952).
[9] H.G. Feichtinger and T. Werther, Atomic systems for subspaces, in *Proceedings SampTA 2001* (L. Zayed, eds.), Orlando, FL, 2001, pp. 163–165.
[10] L. Găvruţa, Frames for operators, *Appl. Comput. Harmon. Anal*. 32, 139–144 (2012).
[11] P. Găvruţa, On some identities and inequalities for frames in Hilbert spaces, *J. Math. Anal. Appl*. 321, 469–478 (2006).
[12] X.X. Guo, Canonical dual K-Bessel sequences and dual K-Bessel generators for unitary systems of Hilbert spaces, *J. Math. Anal. Appl*. 444, 598–609 (2016).
[13] D.L. Hua and Y.D. Huang, K-g-frames and stability of K-g-frames in Hilbert spaces, *J. Korean Math. Soc*. 53, 1331-1345 (2016).
[14] J.Z. Li and Y.C. Zhu, Some equalities and inequalities for g-Bessel sequences in Hilbert spaces, *Appl. Math. Lett*. 25, 1601–1607 (2012).
[15] J.Z. Li and Y.C. Zhu, Exact g-frames in Hilbert spaces, *J. Math. Anal. Appl*. 374, 201–209 (2011).
[16] T. Strohmer and R. Heath, Grassmannian frames with applications to coding and communication, *Appl. Comput. Harmon. Anal*. 14, 257–275 (2003).
[17] W. Sun, G-frames and g-Riesz bases, *J. Math. Anal. Appl*. 322, 437–452 (2006).
[18] W. Sun, Stability of g-frames, *J. Math. Anal. Appl*. 326, 858–868 (2007).
[19] W. Sun, Asymptotic properties of Gabor frame operators as sampling density tends to infinity, *J. Funct. Anal*. 258, 913–932 (2010).
[20] Z.Q. Xiang and Y.M. Li, Frame sequences and dual frames for operators, *ScienceAsia* 42, 222–230 (2016).

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 27, NO.3, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC

[21] X.C. Xiao, Y.C. Zhu, and X.M. Zeng, Some properties of g-Parseval frames in Hilbert spaces, *Acta Math. Sin. (Chin. Ser.)* 51, 1143–1150 (2008).

[22] X.C. Xiao, Y.C. Zhu, and L. Găvruţa, Some properties of K-frames in Hilbert spaces, *Results Math*. 63, 1243–1255 (2013).

[23] X.C. Xiao, Y.C. Zhu, Z.B. Shu, and M.L. Ding, G-frames with bounded linear operators, *Rocky Mountain J. Math*. 45, 675–693 (2015).

[24] X.H. Yang and D.F. Li, Some new equalities and inequalities for g-frames and their dual frames, *Acta Math. Sin. (Chin. Ser.)* 52, 1033–1040 (2009).

College of Mathematics and Computer Science, Shangrao Normal University, Shangrao, Jiangxi 334001, P.R. China

E-mail address: lxsy20110927@163.com; jiayinsuo2002@sohu.com.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 27, NO.3, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC

AQ-functional equation in matrix non-Archimedean fuzzy normed spaces

Jung-Rye Lee[1], George A. Anastassiou[2], Choonkil Park[3*], Murali Ramdoss[4*] and Vithya Veeramani[5]

[1]Department of Mathematics, Daejin University, Kyunggi 11159, Korea
[2]Department of Mathematical Sciences, University of Memphis, Memphis, TN 38152, USA
[3]Research Institute for Natural Sciences Hanyang University, Seoul 04763, Korea
[4,5]Department of Mathematics, Sacred Heart College, Tirupattur - 635 601, TamilNadu, India
e-mail: rlee@daejin.ac.kr; ganastss@memphis.edu; baak@hanyang.ac.kr; shcrmurali@yahoo.co.in; viprutha26@gmail.com

Abstract. Using the fixed point method, we establish some stability results concerning the following new mixed type AQ-functional equation

$$f(-x+2y)+2[f(3x-2y)+f(2x+y)-f(y)-f(y-x)]=3[f(x+y)+f(x-y)+f(-x)]+4f(2x-y)$$

in matrix non-Archimedean fuzzy normed spaces.

1. Introduction and preliminaries

A basic question in the theory of functional equations is as follows: "When is it true that a function, which approximately satisfies a functional equation must be close to an act solution of the equation? If the problem accepts a solution, we say the equation is stable. The first stability problem concerning group homomorphisms was raised by Ulam [29] in 1940 and affirmatively solved by Hyers [7]. The result of Hyers was generalized by Aoki [1] for approximate additive mappings and by Rassias [24] for approximate linear mappings by allowing the difference Cauchy equation $\|f(x+y)-f(x)-f(y)\|$ to be controlled by $\epsilon(\|x\|^p+\|y\|^p)$. In 1994, a generalization of the Rassias theorem was obtained by Gavruta [6] who replaced $\epsilon(\|x\|^p+\|y\|^p)$ by a general control function $\chi(x,y)$. In addition, Rassias [20]–[23] generalized the Hyers-Ulam stability result by introducing two weaker conditions controlled by a product of different powers of norms and a mixed product sum of powers of norms, respectively. applied to the cases of other functional equations in various spaces [2, 5, 13, 15, 16, 26, 27]. In particular Mirmostafafe and Moslehian [14] introduced a notation of non-Archimedean fuzzy normed spaces. They presented the interdisciplinary relation between the theory of fuzzy spaces, the theory of non-Archimedean spaces and the theory of functional equations. Many authors [8, 11, 12, 14, 19, 25, 32] investigated the Hyers-Ulam stability in non-Archimedean fuzzy normed spaces.

Definition 1. [8, 32] Let X be a linear space over a non-Archimedean field $\mathbb{K}$. A function $N : X \times R \to [0,1]$ is said to be a non-Archimedean fuzzy norm on X if for all $x, y \in X$ and all $s, t \in R$
(N1) $N(x,c) = 0$ for $c \leq 0$;
(N2) $x = 0 \Leftrightarrow N(x,c) = 1$ for all $c > 0$;
(N3) $N(cx,t)= N(x, \frac{t}{|c|})$ if $c \neq 0$;
(N4) $N(x+y, \max\{s+t\}) \geq \min\{N(x,s), N(y,t)\}$;
(N5) $\lim_{t\to\infty} N(x,t) = 1$.
The pair (X,N) is called a non-Archimedean fuzzy normed space. Clearly, if $(N4)$ holds then so does
(N6) $N(x+y, s+t) \geq \min\{N(x,s), N(y,t)\}$. A classical vector space over a complex or real field satisfying $(N1)$ and $(N5)$ is called fuzzy normed space. It is easy to see that $(N4)$ is equivalent to the following condition
(N7) $N(x+y,t) \geq \min\{N(x,t), N(y,t)\} \quad (x, y \in X; t \in R)$.

---

[0]2010 Mathematics Subject Classification: 46S40, 46S50, 47L25, 47H10, 54C30, 54E70.
[0]Keywords: Hyers-Ulam stability, fixed point, mixed type additive-quadratic functional equation, matrix non-Archemedian fuzzy normed space.
*Corresponding authors.

Definition 2. Let (X, N) be a non-Archimedean fuzzy normed space. A sequence $\{x_n\}$ in X is said to be convergent or converge if there exists an $x \in X$ such that $\lim_{n\to\infty} N(x_n - x, t) = 1$ for all $t > 0$. In this case, x is called the limit of the sequence $\{x_n\}$ and we denote it by $N - \lim_{n\to\infty} x_n = x$

Definition 3. Let (X, N) be a non-Archimedean fuzzy normed space. A sequence $\{x_n\}$ in X is said to be Cauchy if for each $\epsilon > 0$ and each $t > 0$ there exists an $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ and all $p > 0$, we have $N(x_{n+p} - x_n, t) > 1 - \epsilon$. Due to $N(x_{n+p} - x_n, t) \geq \min\ \{N(x_{n+p} - x_{n+p-1}, t), ..., N(x_{n+1} - x_n, t)\}$ the sequence $\{x_n\}$ is Cauchy if for each $\epsilon > 0$ and each $t > 0$ there exists an $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ we have $N(x_{n+1} - x_n, t) > 1 - \epsilon$.

It is well known that every convergent sequence in a (non-Archimedean) fuzzy normed space is Cauchy. If each Cauchy sequence is convergent, then the fuzzy norm is said to be complete and the (non-Archimedean) fuzzy normed space is called a (non-Archimedean) fuzzy Banach space.

The abstract characterization given for linear spaces of bounded Hilbert space operators in terms of matricially normed spaces [4] implies that quotients, mapping spaces and various tensor products of operator spaces may again be regarded as operator spaces. Owing in part to this result, the theory of operator spaces is having an increasingly significant effect on operator algebra theory [18]. Recently, Lee *et al.* [9] researched the Hyers-Ulam stability of the Cauchy functional equation and the quadratic functional equation in matrix normed spaces. This terminology may also be applied to the cases of other functional equations [3, 10, 28, 30, 31].

We will use the following notations:

$M_n(X)$ is the set of all $n \times n$-matrices in X;

$e_j \in M_{1,n}(\mathbb{C})$ is that jth component is 1 and the other components are zero;

$E_{ij} \in M_n(\mathbb{C})$ is that (i,j)-component is 1 and the other components are zero;

$E_{ij} \otimes x \in M_n(X)$ is that (i,j)-component is x and the other components are zero. For $x \in M_n(X), y \in M_k(X)$,

$$x \oplus y := \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}.$$

Note that $(X, \{\|.\|_n\})$ is a matrix normed space if and only if $(M_n(X), \|.\|_n)$ is a normed space for each positive integer n and $\|AxB\|_k \leq \|A\| \|B\| \|x\|_n$ holds for $A \in M_{k,n}(\mathbb{C})$, $B \in M_{n,k}(\mathbb{C})$ and $x = (x_{ij}) \in M_n(X)$, and that $(X, \{\|.\|_n\})$ is a matrix Banach space if and only if X is a Banach space and $(X, \{\|.\|_n\})$ is a matrix normed space. A matrix normed space $(X, \{\|.\|_n\})$ is called an L^∞ -matrix normed space if $\|x \otimes y\|_{n+k} = max\left\{\|x\|_n, \|y\|_k\right\}$ holds for all $x \in M_n(X)$ and all $y \in M_k(X)$. Let E, F be vector spaces. For a given mapping $h : E \to F$ and a given positive integer n, define $h_n : M_n(E) \to M_n(F)$ by $h_n([x_{ij}]) = [h(x_{ij})]$ for all $[x_{ij}] \in M_n(E)$.

We introduce the concept of matrix non-Archimedean fuzzy normed space.

Definition 4. Let (X, N) be a non-Archimedean fuzzy normed space.
(i) $(X, \{N_n\})$ is called a matrix non-Archimedean fuzzy normed space if for each positive integer n, $(M_n(X), N_n)$ is a non-Archimedean fuzzy normed space and $N_k(AxB, t) \geq N_n\left(x, \frac{t}{\|A\|\cdot\|B\|}\right)$ for all $t > 0$, $A \in M_{k,n}(\mathbb{R})$, $B \in M_{n,k}(\mathbb{R})$ and $x = [x_{ij}] \in M_n(X)$ with $\|A\| \cdot \|B\| \neq 0$.
(ii) $(X, \{N_n\})$ is called a complete matrix non-Archimedean fuzzy normed space if (X, N) is a non-Archimedean fuzzy Banach space and $(X, \{N_n\})$ is a matrix non-Archimedean fuzzy normed space.

Example 5. Let $(X, \{\|.\|_n\})$ is a matrix normed space. Let $N_n(x, t) = \frac{t}{t+\|x\|_n}$ for all $t > 0$ and $x = [x_{ij}] \in M_n(X)$. Then

$$N_k(AxB, t) = \frac{t}{t+\|AxB\|_k} \geq \frac{t}{t+\|A\|\cdot\|x\|_n\cdot\|B\|} = \frac{\frac{t}{\|A\|\cdot\|B\|}}{\frac{t}{\|A\|\|B\|+\|x\|_n}}$$

for all $t > 0$, $A \in M_{k,n}(\mathbb{R})$, $B \in M_{n,k}(\mathbb{R})$ and $x = [x_{ij}] \in M_n(X)$ with $\|A\| \cdot \|B\| \neq 0$. So $(X, \{N_n\})$ is a matrix non-Archimeaden fuzzy normed space.

Definition 6. Let X be a set. A function $d : X \times X \to [0, \infty]$ is called a generalized metric on X if d satisfies

(1) $d(x, y) = 0$ if and only if $x = y$;
(2) $d(x, y) = d(y, x)$ forall $x, y \in X$;
(3) $d(x, z) \leq d(x, y) + d(y, z)$ forall $x, y, z \in X$.

Theorem 7. [17] Let (x, d) be a complete generalized metric space and let $J : X \to Y$ be a strictly contractive mapping with a Lipschitz constant $\alpha < 1$.Then, for each given element $x \in X$, either $d(J^n x, J^{n+1}x) = \infty$. for all nonnegative integers n or there exists a positive integer n_0 such that

(1) $d(J^n x, J^{n+1}x) < \infty$ for all $n \geq n_0$;
(2) the sequence $\{J^n x\}$converges to a fixed point $y*$ of J;
(3) $y*$ is the unique fixed pointof J;
(4) $d(y, y*) \leq \frac{1}{1-\alpha} d(y, Jy)$ forall $y \in Y$.

In this paper, we establish some stability results concerning the following new mixed type AQ-functional equation

$$\begin{aligned} f(-x+2y) + 2[f(3x-2y) + f(2x+y) - f(y) - f(y-x)] \\ = 3[f(x+y) + f(x-y) + f(-x)] + 4f(2x-y) \end{aligned} \tag{1.1}$$

in matrix non-Archimedean fuzzy normed spaces by using the fixed point method.

Theorem 8. Let $\mathcal{A}$ and $\mathcal{B}$ be real vector spaces. If an odd mapping $f : \mathcal{A} \to \mathcal{B}$ satisfies (1.1), then f is additive.

Proof. Suppose that f is an odd mapping. Then (1.1) is equivalent to

$$-f(x-2y) + 2[f(3x-2y) + f(2x+y) - f(y)] = 3[f(x+y) - f(x)] + f(x-y) + 4f(2x-y) \tag{1.2}$$

for all $x, y \in \mathcal{A}$. Replacing x by $x + y$ in (1.2), we obtain

$$-f(x-y) + 2[f(3x+y) + f(2x+3y) - f(y)] = 3[f(x+2y) - f(x+y)] + f(x) + 4f(2x+y) \tag{1.3}$$

for all $x, y \in \mathcal{A}$. Replacing (x, y) by $(x + y, -y)$ in (1.3), we obtain

$$-f(x+2y) + 2[f(3x+2y) + f(2x-y) + f(y)] = 3[f(x-y) - f(x)] + f(x+y) + 4f(2x+y) \tag{1.4}$$

for all $x, y \in \mathcal{A}$. Subtracting (1.3) from (1.4) and then dividing the resulting equation by 2, we get

$$\begin{aligned} -f(x+2y) + f(x-y) + f(2x+3y) - f(3x+2y) + f(3x+y) - f(2x-y) \\ = -2f(x+y) + 2f(x) + 2f(y) \end{aligned} \tag{1.5}$$

for all $x, y \in \mathcal{A}$. Interchanging x and y in (1.5) and then adding the resulting equation to (1.5), we get

$$\begin{aligned} -f(x+2y) - f(2x+y) + f(3x+y) + f(x+3y) - f(2x-y) + f(x-2y) \\ = -4f(x+y) + 4f(x) + 4f(y) \end{aligned} \tag{1.6}$$

for all $x, y \in \mathcal{A}$. Replacing x by $x - y$ in (1.6), we obtain

$$\begin{aligned} -f(x+y) - f(2x-y) + f(3x-2y) + f(x+2y) - f(2x-3y) + f(x-3y) \\ = -4f(x) + 4f(x-y) + 4f(y) \end{aligned} \tag{1.7}$$

for all $x, y \in \mathcal{A}$. Replacing y by $-y$ in (1.7), we obtain

$$\begin{aligned} -f(x-y) - f(2x+y) + f(3x+2y) + f(x-2y) - f(2x+3y) + f(x+3y) \\ = -4f(x) + 4f(x+y) - 4f(y) \end{aligned} \tag{1.8}$$

for all $x, y \in \mathcal{A}$. Adding (1.7) to (1.8), we get

$$\begin{aligned} -f(x+2y) - f(2x+y) + f(3x+y) + f(x+3y) - f(2x-y) + f(x-2y) \\ = -2f(x) + 2f(x+y) - 2f(y) \end{aligned} \tag{1.9}$$

for all $x, y \in \mathcal{A}$. Subtracting (1.9) from (1.6) and then dividing the resulting equation by 6, we get

$$f(x+y) = f(x) + f(y)$$

for all $x, y \in \mathcal{A}$, as desired. □

Theorem 9. Let $\mathcal{A}$ and $\mathcal{B}$ be real vector spaces. If an even mapping $f : \mathcal{A} \to \mathcal{B}$ satisfies (1.1), then f is quadratic.

Proof. Suppose that f is an even mapping. Then (1.1) is equivalent to

$$\begin{aligned} f(x-2y)+2[f(3x-2y)+f(2x+y)-f(y)] \\ =3[f(x+y)+f(x)]+5f(x-y)+4f(2x-y) \end{aligned} \tag{1.10}$$

for all $x, y \in \mathcal{A}$. Replacing x by $x+y$ and y by $x+y$ in (1.10) and then comparing the two resulting equations, we get

$$\begin{aligned} 2f(x)+4f(x+2y)-2f(2x+3y) \\ =-f(x+y)+5f(x-y)+3f(y)-f(2x+y)-2f(x-2y) \end{aligned} \tag{1.11}$$

for all $x, y \in \mathcal{A}$. Interchanging x and y in (1.11), we obtain

$$\begin{aligned} 2f(y)+4f(2x+y)-2f(3x+2y) \\ =-f(x+y)+5f(x-y)+3f(x)-f(x+2y)-2f(2x-y) \end{aligned} \tag{1.12}$$

for all $x, y \in \mathcal{A}$. Replacing y by $-y$ in (1.12), we get

$$\begin{aligned} 2f(y)+4f(2x-y)-2f(3x-2y) \\ =-f(x-y)+5f(x+y)+3f(x)-f(x-2y)-2f(2x+y) \end{aligned} \tag{1.13}$$

for all $x, y \in \mathcal{A}$. Subtracting (1.13) from (1.10) and then dividing the resulting equation by 2, we get

$$2f(y)+4f(2x-y)-2f(3x-2y)=-3f(x-y)+f(x+y) \tag{1.14}$$

for all $x, y \in \mathcal{A}$. Replacing x by $x+y$ in (1.14), we get

$$f(x+2y)+2[f(3x+y)-f(y)]=3f(x)+4f(2x+y) \tag{1.15}$$

for all $x, y \in \mathcal{A}$. Replacing y by $y-x$ in (1.15), we get

$$f(-x+2y)+2[f(2x+y)-f(y-x)]=3f(x)+4f(x+y) \tag{1.16}$$

for all $x, y \in \mathcal{A}$. Replacing y by $-y$ in (1.16), we obtain

$$f(x+2y)+2[f(2x-y)-f(x+y)]=3f(x)+4f(x-y) \tag{1.17}$$

for all $x, y \in \mathcal{A}$. Replacing x by y and y by x in (1.16), we obtain that

$$f(2x-y)+2[f(x+2y)-f(x-y)]=3f(y)+4f(x+y) \tag{1.18}$$

for all $x, y \in \mathcal{A}$. Adding (1.17) to (1.18) and then dividing the resulting equation by 3, we get

$$f(x+2y)+f(2x-y)=f(x)+f(y)+2f(x+y)+2f(x-y) \tag{1.19}$$

for all $x, y \in \mathcal{A}$. Subtracting (1.17) from (1.18) and then adding the resulting equation to (1.19), we get

$$f(x+2y)+f(x)=2f(y)+2f(x+y) \tag{1.20}$$

for all $x, y \in \mathcal{A}$. Replacing x by $x-y$ in (1.20), we obtain $f(x+y)+f(x-y)=2f(x)+2f(y)$ for all $x, y \in \mathcal{A}$. This completes the proof. □

2. Hyers-Ulam stability of the additive-quadratic functional equation (1.1)

Throughout this paper, we assume that $\mathbb{K}$ is a non-Archimedean field, X is a vector space over $\mathbb{K}$ and $(Y, \mathcal{N}_n)$ is a complete matrix non-Archimedean fuzzy normed space over $\mathbb{K}$, and $(Z, \mathcal{N}')$ is (an Archimedean or a non-Archimedean fuzzy) normed space.
For a mapping $f : X \to Y$, define $\mathcal{G}\, f : X^2 \to Y$ and $\mathcal{G}\, f_n : M_n(X^2) \to M_n(Y)$ by

$$\begin{aligned} \mathcal{G}f(a,b) &= f(-a+2b)+2[f(3a-2b)+f(2a+b)-f(b)-f(b-a)] \\ &\quad -3[f(a+b)+f(a-b)+f(-a)]-4f(2a-b), \\ \mathcal{G}f_n([x_{ij}],[y_{ij}]) &= f_n([-x_{ij}+2y_{ij}])+2[f_n([3x_{ij}-2y_{ij}])+f_n([2x_{ij}+y_{ij}])-f_n([y_{ij}])-f_n([y_{ij}-x_{ij}])] \\ &\quad -3[f_n([x_{ij}+y_{ij}])+f_n([x_{ij}-y_{ij}])+f_n([-x_{ij}])]-4f_n([2x_{ij}-y_{ij}]) \end{aligned}$$

for all $a, b \in X$ and all $x = [x_{ij}], y = [y_{ij}] \in M_n(X)$.

In this section, we investigate the Hyers-Ulam stability for the functional equation (1.1) in matrix non-Archimedean fuzzy normed spaces by using the fixed point method.

Theorem 10. Let $q = \pm 1$ be fixed and let $\psi : X \times X \to Z$ be a mapping such that for some $\eta \neq 2$ with $\left(\frac{\eta}{2}\right)^q < 1$

$$\mathcal{N}'(\psi(2^q a, 2^q b) \geq \mathcal{N}'\left(\psi(a,b), \eta^{-q} t\right) \tag{2.1}$$

for all $a, b \in X$ and $t > 0$, and

$$\lim_{k\to\infty} \mathcal{N}(2^{-kq}\mathcal{G}f(2^{kq}a, 2^{kq}b), t) = 1$$

for all $a, b \in X$ and $t > 0$. Suppose that an odd mapping $f : X \to Y$ satisfies the inequality

$$\mathcal{N}(\mathcal{G}f_n([x_{ij}],[y_{ij}]),t) \geq \mathcal{N}'\left(\sum_{i,j=1}^{n} \psi(x_{ij}, y_{ij}), t\right) \quad \forall\ x = [x_{ij}], y = [y_{ij}] \in M_n(X), \text{ and } t > 0. \tag{2.2}$$

Then there exists a unique additive mapping $\mathcal{A} : X \to Y$ such that

$$\mathcal{N}_n(f_n([x_{ij}]) - \mathcal{A}_n([x_{ij}]), t) \geq min\left\{\mathcal{N}'(\psi(x_{ij}, 0), |\eta - 2|\, n^{-2} t) : i, j = 1, 2, \cdots, n\right\} \tag{2.3}$$

for all $x = [x_{ij}] \in M_n(X)$ and $t > 0$.

Proof. For the cases $q = 1$ and $q = -1$, we consider $\eta < 2$ and $\eta > 2$, respectively. Letting $n = 1$ in (2.2), we obtain

$$\mathcal{N}(\mathcal{G}f(a,b),t) \geq \mathcal{N}'(\psi(a,b),t) \tag{2.4}$$

for all $a, b \in X$ and $t > 0$. Replacing (a, b) by $(0, a)$ in (2.4), we get

$$\mathcal{N}(f(2a) - 2f(a), t) \geq \mathcal{N}'(\psi(0,a),t)$$

for all $a \in X$ and $t > 0$. Thus

$$\mathcal{N}\left(f(a) - \frac{1}{2^q} f(2^q a), \frac{\eta^{\frac{q-1}{2}}}{|2|^{(\frac{1+q}{2})}} t\right) \geq \mathcal{N}'(\psi(0,a),t) \qquad \forall\ a \in X \text{ and } t > 0. \tag{2.5}$$

Consider the set $\mathcal{M} = \{f : X \to Y\}$ and introduce the generalized metric ρ on $\mathcal{M}$ as follows:

$$\rho(f,g) =\in \rho(f,g) = f\left\{\mu \in \mathbb{R}_+ : \mathcal{N}(f(a) - g(a), \mu t) \geq \mathcal{N}'(\psi(0,a),t), \forall a \in X, t > 0\right\}$$

We will prove that $(\mathcal{M}, \rho)$ is a complete generalized metric, First we will prove that ρ is a generalized metric on $\mathcal{M}$. Let $\rho(f,g) = \mu_1$ and $\rho(g,h) = \mu_2$. Then $\mathcal{N}(f(a) - g(a), \mu_1 t) \geq \mathcal{N}'(\psi(0,a),t)$ and $\mathcal{N}(g(a) - h(a), \mu_2 t) \geq \mathcal{N}'(\psi(0,a),t)$ for all $a \in X$ and $t > 0$. Therefore, $\mathcal{N}(f(a) - h(a), (\mu_1 + \mu_2)t) \geq \mathcal{N}'(\psi(0,a),t)$. By definition of ρ, $\rho(f,h) \leq \mu_1 + \mu_2 = \rho(f,g) + \rho(g,h)$. which means that ρ satisfies the triangle inequality. One can show that other properties are satisfied. So ρ is a generalized metric on $\mathcal{M}$.

Next we will prove that $(\mathcal{M}, \rho)$ is a complete generalized metric.
Suppose that $\{f_n\}$ is ρ-Cauchy, i.e., for any $\tau > 0$, there exist $n_0, n > m \geq n_0$, such that $\rho(f_n, f_m) < \tau$.
By definition of ρ, there exists $0 < \mu_0 < \tau$, which satisfies

$$\mathcal{N}(f_n(a) - f_m(a), \tau t) \geq \mathcal{N}'(\psi(0,a),t)$$

for all $a \in X$ and $t > 0, n > m \geq n_0$, i.e., $\{f_n(a)\}$ is a Cauchy sequence in Y. Since Y is complete, there exists $\{f_0(a)\} \subseteq Y$ and $\{f_n(a)\} \to \{f_0(a)\}$. Taking the limit as $m \to \infty$, we obtain

$$\mathcal{N}(f_n(a) - f_0(a), \tau t) \geq \mathcal{N}'(\psi(0,a),t)$$

for all $a \in X$ and $t > 0, n \geq n_0$. Therefore,

$$\rho(f_n, f_0) = inf\left\{\mu \in \mathbb{R}_+ : \mathcal{N}(f_n(a) - f_0(a), \mu t) \geq \mathcal{N}'(\psi(0,a),t)\right\} < \tau.$$

for all $n \geq n_0$, so that $\{f_n\}$ is ρ-convergent, i.e., $(\mathcal{M}, \rho)$ is a complete generalized metric.

Now consider the mapping $\mathcal{P} : \mathcal{M} \to \mathcal{M}$ by

$$\mathcal{P}f(a) = \frac{1}{2^q} f(2^q a) \qquad \forall\ f \in \mathcal{M} \text{ and } a \in X.$$

Let $f, g \in \mathcal{M}$ and ν be an arbitrary constant with $\rho(f,g) \leq \nu$. Then

$$\mathcal{N}(f(a) - g(a), \nu t) \geq \mathcal{N}'(\psi(0,a),t) \qquad \text{for all } a \in X \text{ and } t > 0.$$

Therefore, using (2.1), we get

$$\mathcal{N}\left(\mathcal{P}f(a) - \mathcal{P}g(a), 2^{-q}\nu t\right) = \mathcal{N}(f(2^q a) - g(2^q a), \nu t), \geq \mathcal{N}'\left(\psi(0,a), \eta^{-q} t\right)$$

for all $a \in X$ and $t > 0$. Hence by definition $\rho(\mathcal{P}f, \mathcal{P}g) \leq \left(\frac{\eta}{2}\right)^q \nu$, that is, $\rho(\mathcal{P}f, \mathcal{P}g) \leq L\rho(f, g)$ for all $f, g \in \mathcal{M}$. This means that $\mathcal{P}$ is a contractive mapping with Lipschitz constant $L = \left(\frac{\eta}{2}\right)^q < 1$.

It follows from (2.5) that $\rho(f, \mathcal{P}f) \leq \dfrac{\eta^{\left(\frac{q-1}{2}\right)}}{|2|^{\left(\frac{1+q}{2}\right)}}$. Therefore according to Theorem 7, there exists a mapping $\mathcal{A} : X \to Y$ which satisfies

(1) $\mathcal{A}$ is a unique fixed point of $\mathcal{P}$ in the set $\mathcal{S} = \{g \in \mathcal{M} : \rho(f, g) < \infty\}$, which satisfies

$$\mathcal{A}(2^q a) = 2^q \mathcal{A}(a) \qquad \forall\ a \in X.$$

In other words, there exists a $\mu > 0$ satisfying

$$\mathcal{N}\left(f(a) - g(a), \mu t\right) \geq \mathcal{N}'\left(\psi(0, a), t\right) \qquad \forall\ a \in X \text{ and } t > 0.$$

(2) $\rho(\mathcal{P}^k f, \mathcal{V}_\mathcal{U}) \to 0$ as $k \to \infty$. This implies the equality

$$\lim_{k\to\infty} \frac{1}{2^{kq}} f(2^{kq} a) = \mathcal{A}(a) \qquad \forall\ a \in X.$$

(3) $\rho(f, \mathcal{A}) \leq \dfrac{1}{1-\eta}\rho(f, \mathcal{P}f)$, which implies the inequality $\rho(f, \mathcal{A}) \leq \dfrac{1}{|2-\eta|}$. So

$$\mathcal{N}\left(f(a) - \mathcal{A}(a), \frac{1}{|2-\eta|} t\right) \geq \mathcal{N}'\left(\psi(0, a), t\right) \qquad \forall\ a \in X \text{ and } t > 0. \tag{2.6}$$

By (2.4),

$$\mathcal{N}(\mathcal{G}\mathcal{A}(a, b), t) = \lim_{k\to\infty} \mathcal{N}(2^{-kq}\mathcal{G}f(2^{kq}a, 2^{kq}b), t) \geq \lim_{k\to\infty} \mathcal{N}'(2^{-kq}\psi(2^{kq}a, 2^{kq}b), t) = 1.$$

Hence by (N2), $\mathcal{G}\mathcal{A}(a, b) = 0$. Thus $\mathcal{A}$ is additive.

We note that $e_j \in M_{1,n}(\mathbb{R})$ means that the j-th component is 1 and the others are zero, $E_{ij} \in M_n(X)$ means that (i, j)-component is 1 and the others are zero, and $E_{ij} \otimes x \in M_n(X)$ means that (i, j)-component is x and the others are zero. Since $N(E_{kl} \otimes x, t) = N(x, t)$, we have

$$\begin{aligned} N_n([x_{ij}], t) &= N_n\left(\sum_{i,j=1}^{n} E_{ij} \otimes x_{ij}, t\right) \geq min\left\{N_n(E_{ij} \otimes x_{ij}, t_{ij}) : i, j = 1, 2, ..., n\right\} \\ &= min\left\{N(x_{ij}, t_{ij}) : i, j = 1, 2, ..., n\right\}, \end{aligned}$$

where $t = \sum_{i,j=1}^{n} t_{ij}$. So $N_n([x_{ij}], t) \geq \mathcal{T}\left\{N(x_{ij}, \frac{t}{n^2}) : i, j = 1, 2, \cdots, n\right\}$.

By (2.6),

$$\begin{aligned} \mathcal{N}\left(f_n([x_{ij}]) - \mathcal{A}_n([x_{ij}]), t\right) &\geq min\left\{\mathcal{N}\left(f(x_{ij}) - \mathcal{A}(x_{ij}), \frac{t}{n^2}\right) : i, j = 1, 2, \cdots, n.\right\} \\ &\geq min\left\{\mathcal{N}'\left(\psi(0, x_{ij}), |2-\eta|\, n^{-2} t\right) : i, j = 1, 2, ..., n\right\} \end{aligned}$$

for all $x = [x_{ij}] \in M_n(X)$ and $t > 0$. Thus $\mathcal{A} : X \to Y$ is a unique additive mapping satisfying (2.3). □

Corollary 1. Let $q = \pm 1$ be fixed and let p be a nonnegative real number with $p \neq 1$ and $\Upsilon \in Z$. Let $f : X \to Y$ be an odd mapping such that

$$\mathcal{N}_n(\mathcal{G}f_n([x_{ij}], [y_{ij}]), t) \geq \sum_{i,j=1}^{n} \mathcal{N}'(\Upsilon(\|x_{ij}\|^p + \|y_{ij}\|^p), t)$$

for all $x = [x_{ij}], y = [y_{ij}] \in M_n(X)$ and $t > 0$. Then there exists a unique additive mapping $\mathcal{A} : X \to Y$ such that

$$\mathcal{N}\left(f_n([x_{ij}]) - \mathcal{A}_n([x_{ij}]), t\right) \geq min\left\{\mathcal{N}'(\|x\|^p\, \Upsilon, |2 - 2^p|\, n^{-2} t) : i, j = 1, 2, \cdots, n\right\} \tag{2.7}$$

for all $x = [x_{ij}] \in M_n(X)$ and $t > 0$.

Proof. The proof follows from Theorem 10 by taking $\psi(a, b) = \Upsilon(\|a\|^p + \|b\|^p)$ for all $a, b \in X$. Then we can choose $\eta = 2^{q(p-1)}$, and we can obtain the required result. □

The following corollary gives the Hyers-Ulam stability for the additive functional equation (1.1).

Corollary 2. Let $q = \pm 1$ be fixed and let p be a nonnegative real number with $p = v + w \neq 1$ and $\Upsilon \in Z$. Let $f : X \to Y$ be an odd mapping such that

$$\mathcal{N}_n(\mathcal{G}f_n([x_{ij}], [y_{ij}]), t) \geq \sum_{i,j=1}^{n} \mathcal{N}'(\Upsilon(\|x_{ij}\|^v \cdot \|y_{ij}\|^w + \|x_{ij}\|^{v+w} + \|y_{ij}\|^{v+w}), t)$$

for all $x = [x_{ij}], y = [y_{ij}] \in M_n(X)$ and $t > 0$. Then there exists a unique additive mapping $\mathcal{A} : X \to Y$ satisfying (2.7).

Proof. The proof follows from Theorem 10 by taking $\psi(a, b) = \Upsilon(\|a\|^v \cdot \|b\|^w + \|a\|^p + \|b\|^p)$ for all $a, b \in X$. Then we can choose $\eta = 2^{q(p-1)}$, and we can obtain the required result. □

Theorem 11. Let $q = \pm 1$ be fixed and let $\psi : X \times X \to Z$ be a mapping such that for some $\eta \neq 4$ with $\left(\frac{\eta}{4}\right)^q < 1$

$$\mathcal{N}'(\psi(2^q a, 2^q b) \geq \mathcal{N}'\left(\psi(a, b), \eta^{-q} t\right) \tag{2.8}$$

for all $a, b \in X$ and $t > 0$, and $\lim_{k\to\infty} \mathcal{N}(4^{-kq}\mathcal{G}f(2^{kq}a, 2^{kq}b), t) = 1$ for all $a, b \in X$ and $t > 0$. Suppose that an even mapping $f : X \to Y$ with $f(0) = 0$ satisfies the inequality

$$\mathcal{N}(\mathcal{G}f_n([x_{ij}], [y_{ij}]), t) \geq \mathcal{N}'\left(\sum_{i,j=1}^{n} \psi(x_{ij}, y_{ij}), t\right) \qquad \forall\ x = [x_{ij}], y = [y_{ij}] \in M_n(X), \text{ and } t > 0. \tag{2.9}$$

Then there exists a unique quadratic mapping $\mathcal{Q} : \mathcal{X} \to \mathcal{Y}$ such that

$$\mathcal{N}_n(f_n([x_{ij}]) - \mathcal{Q}_n([x_{ij}]), t) \geq min\left\{\mathcal{N}'(\psi(0, x_{ij}), |\eta - 4|\, n^{-2} t) : i, j = 1, 2, \cdots, n\right\} \tag{2.10}$$

for all $x = [x_{ij}] \in M_n(X)$ and $t > 0$.

Proof. For the cases $q = 1$ and $q = -1$, we consider $\eta < 4$ and $\eta > 4$, respectively. Letting $n = 1$ in (2.9), we obtain

$$\mathcal{N}(\mathcal{G}f(a, b), t) \geq \mathcal{N}'\left(\psi(a, b), t\right) \tag{2.11}$$

for all $a, b \in X$ and $t > 0$. Replacing (a, b) by $(0, a)$ in (2.11), we get

$$\mathcal{N}\left(f(2a) - 4f(a), t\right) \geq \mathcal{N}'\left(\psi(0, a), t\right) \tag{2.12}$$

for all $a \in X$ and $t > 0$. Thus

$$\mathcal{N}\left(f(a) - \frac{1}{4^q} f(2^q a), \frac{\eta^{\frac{q-1}{2}}}{|4|^{\left(\frac{1+q}{2}\right)}} t\right) \geq \mathcal{N}'\left(\psi(0, a), t\right) \qquad \forall\ a \in X \text{ and } t > 0. \tag{2.13}$$

We consider the set $\mathcal{M} = \{f : X \to Y\}$ and introduce the generalized metric ρ on $\mathcal{M}$ as follows:

$$\rho(f, g) = \inf\left\{\mu \in \mathbb{R}_+ : \mathcal{N}\left(f(a) - g(a), \mu t\right) \geq \mathcal{N}'\left(\psi(0, a), t\right), \forall a \in X, t > 0\right\}.$$

It is easy to check that $(\mathcal{M}, \rho)$ is a complete generalized metric (see also Theorem 10).

Define the mapping $\mathcal{P} : \mathcal{M} \to \mathcal{M}$ by $\mathcal{P}f(a) = \dfrac{1}{4^q} f(2^q a)$ for all $f \in \mathcal{M}$ and $a \in X$.

Let $f, g \in \mathcal{M}$ and ν be an arbitrary constant with $\rho(f, g) \leq \nu$. Then

$$\mathcal{N}\left(f(a) - g(a), \nu t\right) \geq \mathcal{N}'\left(\psi(0, a), t\right)$$

for all $a \in X$ and $t > 0$. Therefore, using (2.8), we get

$$\mathcal{N}\left(\mathcal{P}f(a) - \mathcal{P}g(a), 4^{-q}\nu t\right) = \mathcal{N}\left(f(2^q a) - g(2^q a), \nu t\right), \geq \mathcal{N}'\left(\psi(0, a), \eta^{-q} t\right)$$

for all $a \in X$ and $t > 0$. Hence by definition $\rho(\mathcal{P}f, \mathcal{P}g) \leq \left(\dfrac{\eta}{4}\right)^q \nu$, that is, $\rho(\mathcal{P}f, \mathcal{P}g) \leq L\rho(f, g)$ for all $f, g \in \mathcal{M}$. This means that $\mathcal{P}$ is a contractive mapping with Lipschitz constant $L = \left(\dfrac{\eta}{4}\right)^q < 1$.

It follows from (2.13) that $\rho(f, \mathcal{P}f) \leq \dfrac{\eta^{\left(\frac{q-1}{2}\right)}}{|4|^{\left(\frac{1+q}{2}\right)}}$. Therefore according to Theorem 7, there exists a mapping $\mathcal{Q} : X \to Y$ which satisfies

(1) $\mathcal{Q}$ is a unique fixed point of $\mathcal{P}$, which satisfies $\mathcal{Q}(2^q a) = 4^q \mathcal{Q}(a)$ for all $a \in X$.

(2) $\rho(f,\mathcal{Q}) \le \frac{1}{1-\eta}\rho(f,\mathcal{P}f)$, which implies the inequality $\rho(f,\mathcal{Q}) \le \frac{1}{|4-\eta|}$. So

$$\mathcal{N}\left(f(a)-\mathcal{Q}(a), \frac{1}{|4-\eta|}t\right) \ge \mathcal{N}'\left(\psi(0,a),t\right) \qquad \forall\ a \in X \text{ and } t>0. \tag{2.14}$$

By (2.11), $\mathcal{N}(\mathcal{G}\mathcal{Q}(a,b),t) = \lim_{k\to\infty} \mathcal{N}(4^{-kq}\mathcal{G}f(2^{kq}a,2^{kq}b),t) \ge \lim_{k\to\infty} \mathcal{N}'(4^{-kq}\psi(2^{kq}a,2^{kq}b),t) = 1$. Hence by (N2), $\mathcal{G}\mathcal{Q}(a,b)=0$. Thus $\mathcal{Q}$ is quadratic.

Since $N(E_{kl}\otimes x,t) = N(x,t)$, we have

$$\begin{aligned} N_n([x_{ij}],t) &= N_n\left(\sum_{i,j=1}^{n} E_{ij}\otimes x_{ij}, t\right) \ge min\left\{N_n(E_{ij}\otimes x_{ij},t_{ij}) : i,j=1,2,...,n\right\} \\ &= min\left\{N(x_{ij},t_{ij}) : i,j=1,2,...,n\right\}, \end{aligned}$$

where $t = \sum_{i,j=1}^{n} t_{ij}$. So $N_n([x_{ij}],t) \ge min\left\{N(x_{ij},\frac{t}{n^2}) : i,j=1,2,...,n\right\}$.

By (2.14),

$$\begin{aligned} \mathcal{N}\left(f_n([x_{ij}]) - \mathcal{Q}_n([x_{ij}]),t\right) &\ge min\left\{\mathcal{N}\left(f(x_{ij})-\mathcal{Q}(x_{ij}),\frac{t}{n^2}\right) : i,j=1,2,\cdots,n.\right\} \\ &\ge min\left\{\mathcal{N}'\left(\psi(0,x_{ij}), |4-\eta|\, n^{-2}t\right) : i,j=1,2,\cdots,n\right\} \end{aligned}$$

for all $x=[x_{ij}] \in M_n(X)$ and $t>0$. Thus $\mathcal{Q}:\mathcal{X}\to\mathcal{Y}$ is a unique quadratic mapping satisfying (2.10). □

Corollary 3. Let $q=\pm 1$ be fixed and let p be a nonnegative real number with $p\neq 2$ and $\Upsilon \in Z$. Let $f : X \to Y$ be an even mapping satisfying $f(0)=0$ and

$$\mathcal{N}_n(\mathcal{G}f_n([x_{ij}],[y_{ij}]),t) \ge \mathcal{N}'(\sum_{i,j=1}^{n}\Upsilon(\|x_{ij}\|^p + \|y_{ij}\|^p),t)$$

for all $x=[x_{ij}], y=[y_{ij}] \in M_n(X)$ and $t>0$. Then there exists a unique quadratic mapping $\mathcal{Q}: X\to Y$ such that

$$\mathcal{N}\left(f_n([x_{ij}]) - \mathcal{Q}_n([x_{ij}]),t\right) \ge min\left\{\mathcal{N}'(\|x\|^p\,\Upsilon, |4-2^p|\, n^{-2}t) : i,j=1,2,\cdots,n\right\} \tag{2.15}$$

for all $x=[x_{ij}] \in M_n(X)$ and $t>0$.

Proof. The proof follows from Theorem 11 by taking $\psi(a,b) = \Upsilon(\|a\|^p + \|b\|^p)$ for all $a,b\in X$. Then we can choose $\eta = 2^{q(p-2)}$, and we can obtain the required result. □

The following corollary gives the Hyers-Ulam stability for the quadratic functional equation (1.1).

Corollary 4. Let $q=\pm 1$ be fixed and let p be a nonnegative real number with $p=v+w\neq 2$ and $\Upsilon\in Z$. Let $f : X\to Y$ be an even mapping satisfying $f(0)=0$ and

$$\mathcal{N}_n(\mathcal{G}f_n([x_{ij}],[y_{ij}]),t) \ge \mathcal{N}'(\sum_{i,j=1}^{n}\Upsilon(\|x_{ij}\|^v\cdot\|y_{ij}\|^w + \|x_{ij}\|^{v+w} + \|y_{ij}\|^{v+w}),t)$$

for all $x=[x_{ij}], y=[y_{ij}] \in M_n(X)$ and $t>0$. Then there exists a unique quadratic mapping $\mathcal{Q}: X\to Y$ satisfying (2.15).

Proof. The proof follows from Theorem 11 by taking $\psi(a,b) = \Upsilon(\|a\|^v\cdot\|b\|^w + \|a\|^p + \|b\|^p)$ for all $a,b\in X$. Then we can choose $\eta = 2^{q(p-2)}$, and we can obtain the required result. □

References

[1] T. Aoki , On the stability of the linear transformation in Banach spaces, *J. Math. Soc. Japan* **2** (1950), 64–66.

[2] A. Bodaghi, C. Park, J. M. Rassias, Fundamental stabilities of the nonic functional equation In intuitionistic fuzzy normed spaces , *Commun. Korean Math. Soc.* **31** (2016), 729–743.

[3] A. Ebadian, S. Zolfaghari, S. Ostadbashi, C. Park, Approximation on the reciprocal functional equation in several variables in matrix non-Archimedean random normed spaces, *Adv. Difference Equ.* **2015**, 2015:314.

[4] E. Effros, Z. J. Ruan, On approximation properties for operator spaces, *Int. J. Math.* **1** (1990), 163–187.

[5] Iz. EL-Fassi, S. Kabbaj, Non-Archimedean random stability of σ-quadratic functional equation, *Thai J. Math.* **14)** (2016), 151–165.

[6] P. Găvruta, A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings, *J. Math. Anal. Appl.* **184** (1994), 431–436.

[7] D. H. Hyers, On the stability of the linear functional equation, *Proc. Natl. Acad. Sci. USA*, **27** (1941), 222–224.

[8] A. Kumar, S. Kumar, On stability of cubic functional equation in non-Archimedean fuzzy normed spaces , *Int. J. Math. Archive* **7** (2016), No. 10, 167–174.

[9] J. Lee, C. Park, D. Shin, Functional equations in matrix normed spaces, *Proc. Indian Acad. Sci.* **125** (2015), 399–412.

[10] J. Lee, D. Shin, C. Park, Fuzzy stabilty of functional inequalities in matrix fuzzy normed spaces, *J. Inequal. Appl.* **2013**, 2013:224.

[11] D. Mihet, Fuzzy φ-contractive mapping in non-Archimedean fuzzy metric spaces, *Fuzzy Sets Syst.* **159** (2008), 739–744.

[12] D. Mihet, The stability of the additive Cauchy functional equation in non-Archimedean fuzzy normed spaces, *Fuzzy Sets Syst.* **161** (2010), 2206–2212.

[13] D. Mihet, V. Radu, On the stability of the additive Cauchy functional equation in random normed spaces, *J. Math. Anal. Appl.* **343** (2008), 567–572.

[14] A. K. Mirmostafaee, M. S. Moslehian, Stability of additive mappings in non- Archimedean fuzzy normed spaces, *Fuzzy Sets Syst.* **160** (2009), 1643–1652.

[15] C. Park, K. Ghasemi, S. G. Ghaleh, S. Jang, Approximate n-Jordan $*$-homomorphisms in C^*-algebras, *J. Comput. Anal. Appl.* **15** (2013), 365–368.

[16] C. Park, A. Najati, S. Jang, Fixed points and fuzzy stability of an additive-quadratic functional equation, *J. Comput. Anal. Appl.* **15** (2013), 452–462.

[17] V. Radu, The fixed point alternative and the stability of functional equations, *Fixed Point Theory* **4** (2003), 91–96.

[18] Z. J. Ruan, Subspaces of $C*$-algebras, *J. Funct. Anal.* **76** (1988), 217–230.

[19] C. Renu, Sushma, A fixed point approach to Ulam stability problem for cubic and quartic mapping in non-Archimedean fuzzy normed spaces, *Proceeding of the World Congress on Engineering 2010 vol VIII WCE*, (2010).

[20] J. M. Rassias, On approximation of approximately linear mappings by linear mappings, *J. Funct. Anal.* **46** (1982), 126–130.

[21] J. M. Rassias, On approximation of approximately linear mappings by linear mappings, *Bull. Sci. Math.* **108** (1984), 445–446.

[22] J. M. Rassias, On a new approximation of approximately linear mappings by linear mappings, *Discuss. Math.* **7** (1985), 193–196.

[23] J. M. Rassias, Solution of a problem of Ulam, *J. Approx. Theory* **57** (1989), 268–273.

[24] Th. M. Rassias, On the stability of the linear mapping in Banach spaces, *Proc. Am. Math. Soc.* **72** (1978), 297–300.

[25] B. V. Senthil Kumar, A. Kumar, P. Narasimman, Estimation of approximate nonic functional equation in non-Archimedean fuzzy normed spaces, *Int. J. Pure Appl. Math. Tech.* **1** (2016), No. 2, 18–29.

[26] D. Shin, C. Park, Sh. Farhadabadi, On the superstability of ternary Jordan C^*-homomorphisms, *J. Comput. Anal. Appl.* **16** (2014), 964–973.

[27] D. Shin, C. Park, Sh. Farhadabadi, Stability and superstability of J^*-homomorphisms and J^*-derivations for a generalized Cauchy-Jensen equation, *J. Comput. Anal. Appl.* **17** (2014), 125–134.

[28] A. Song, The Ulam stability of matrix intuitionistic fuzzy normed spaces, *J. Intelligent Fuzzy Syst.* **32** (2017), 629–641.

[29] S. M. Ulam, Problems in Modern Mathematics, *Science Editions*, Wiley, NewYork, 1964.

[30] Z. Wang, P. K. Sahoo, Stability of an ACQ-functional equation in various matrix normed spaces, *J. Nonlinear Sci. Appl.* **8** (2015), 64–85.

[31] Z. Wang, P. K. Sahoo, Stability of the generalized quadratic and quartic type functional equation in non-Archimedean fuzzy normed spaces, *J. Appl. Anal. Comput.* **6** (2016), 917–938.

[32] T. Z. Xu, J. M. Rassias, W. X. Xu, Stability of a general mixed additive-cubic functional equation in non-Archimedean fuzzy normed spaces, *J. Math. Phys.* **51** (2010), 1–19.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 27, NO.3, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC

Existence of continuous selection for some special kind of multivalued mappings

G. Poonguzali[a], Muthiah Marudai[b], George A. Anastassiou[c] and Choonkil Park[d*]

[a]Department of Mathematics, Bharathidasan University, Tiruchirappalli 620 024, Tamil Nadu, India
[b]Department of Mathematics, Bharathidasan University, Tiruchirappalli 620 024, Tamil Nadu, India
[c]Department of Mathematical Sciences, University of Memphis, Memphis, TN 38152, USA
[d]Research Institute for Natural Sciences, Hanyang University, Seoul 04763, Korea
E-mail: poomath09@gmail.com; mmarudai@yahoo.co.in; ganastss@memphis.edu; baak@hanyang.ac.kr

Abstract

This paper deals with the existence of continuous selection of a multivalued mapping in product space. Many authors provided existence of continuous map for lower semicontinuous. We provide continuous selection for weakly lower semicontinuous. Rhybinski [9] proved the existence for contraction type mapping. We prove the existence for some general type of mapping different from contraction mapping.

1 Introduction and preliminaries

Let X be a normed linear space. Then $B = \{x \in X : \|x\| \leq 1\}$ represents the closed unit ball and $B^0 = \{x \in X : \|x\| < 1\}$ represents the open unit ball in X. First we quote some notations and basic facts that are used in the sequal

$$\begin{aligned}
\mathcal{P}(X) &= \{A \subset X : A \neq \emptyset\},\\
\mathcal{P}_{cl}(X) &= \{A \in \mathcal{P}(X) : A \text{ } is\ closed\},\\
\mathcal{P}_{cv}(X) &= \{A \in \mathcal{P}(X) : A \text{ } is\ convex\},\\
\mathcal{P}_{cl,cv}(X) &= \{A \in \mathcal{P}(X) : A \text{ } is\ closed,\ convex\}.
\end{aligned}$$

For $x \in X,\ A, B \in \mathcal{P}(X)$,

$$\delta(A, B) = \sup\{d(x, B) : x \in A\}.$$

$\mathcal{H}(A, B) = \max\{\delta(A, B), \delta(B, A)\}$.

Let us consider the mapping $T : X \times Y \to \mathcal{P}_{cl,cv}(Y)$. Then the fixed point set is defined as $P_T(x) := \{y \in Y : y \in T(x, y)\}$. See [1, 2] for more information on fixed point theory.

Definition 1.1. *A multivalued mapping $F : X \to \mathcal{P}(Y)$ is called lower semicontinuous (l.s.c.) at $x_0 \in X$ if and only if for every $\epsilon > 0$ and $z \in F(x_0)$ there exists a neighborhood U_z containing x_0 with the property that*

$$z \in \cap\{F(x) + \epsilon B^0 : x \in U_z\}.$$

2010 Mathematics Subject Classification: 47H04, 47H10.

Keywords: weakly lower semicontinuous map, continuous selection, paracompact space, perfectly normal space.

*Corresponding author: Choonkil Park (email: baak@hanyang.ac.kr, office: +82-2-2220-0892).

Definition 1.2. *A multivalued mapping F is said to be weakly lower semicontinuous (w.l.s.c.) at $x_0 \in X$ if ond only if for every $\epsilon > 0$ and for every neighborhood V containing x_0, there exists a point $x_1 \in V$ so that for every $z \in F(x_1)$ there is a neighborhood U_z containing x_0 satifying the condition that*

$$z \in \cap\{F(x) + \epsilon B^0 : x \in U_z\}.$$

It is well known that F is l.s.c. (w.l.s.c.) if and only if F is l.s.c. (w.l.s.c.) at every $x \in X$. Also, it is easy to see that f is l.s.c. implies that F is w.l.s.c., but the converse is not true [8].

A topological space X is said to be paracompact if every open cover of X has a locally finite refinement. A cover $\{U_\beta\}_{\beta\in J}$ is called a refinement of $\{W_\alpha\}_{\alpha\in I}$ if for all $\beta \in J$, there exists $\alpha \in I$ such that $U_\beta \subset W_\alpha$. Also, a collection $\{A_i : i \in I\}$ of subsets of X is locally finite if and only if for each $x \in X$ there is an open $U \ni x$ with $|\{i \in I : A_i \cap U \neq \emptyset\}| < \infty$. A topological space X is said to be perfectly normal if it is normal and every closed subset is a G_δ subset. A multivalued mapping $T : X \times Y \to \mathcal{P}_{cl,cv}(Y)$ is said to satisfy condition $\mathcal{C}$ if there exists $K < 1$ such that

$$\mathcal{H}(T(x, y_1), T(x, y_2)) \leq K\|y_1 - y_2\| \ for \ x \in X, \ y_1, y_2 \in Y.$$

In a similar way, a multivalued mapping $H : X \times Y \to \mathcal{P}_{cl,cv}(Y)$ is said to satisfy condition $\mathcal{N}$ if it satisfies

$$\mathcal{H}(T(x, y_1), T(x, y_2)) \leq \|y_1 - y_2\| \ for \ x \in X, \ y_1, y_2 \in Y.$$

In 1956, Michael [6] was the first person to study about continuous selection for a given multivalued mapping under some suitable conditions. The following theorem is due to Michael.

Theorem 1.3. [6] *In a paracompact space X, the lower semi-continuous multivalued mapping $F : X \to \mathcal{P}_{cl,cv}(Y)$ has a continuous selection, where Y is a Banach space.*

The importance of the above theorem was first noticed by Browder [4], who used the theorem to prove Fan Browder theorem. Later, many researchers established results on continuous selections with applications (see [3, 5, 7, 10, 11]). Further, in [8], Przeslawski and Rybinski has generalized Michael selection theorem for weakly lower semicontinuous mapping. They proved the existence of continuous selection for w.l.s.c. which is weaker than l.s.c. Rybinski [9] proved the following theorem.

Theorem 1.4. *Let X be a paracompact and perfectly normal topological space and Y be a closed subset of a Banach space $(Z, \|\cdot\|)$. Assume that $T : X \times Y \to \mathcal{P}_{cl,cv}(Y)$ satisfies condition $\mathcal{C}$ and also, satisfies the condition that for every $y \in Y$ the multivalued mapping $T(\cdot, y)$ is w.l.s.c. Then there exists a continuous mapping $h : X \times Y \to Y$ such that $h(x, y) \in P_T(x)$ for every $(x, y) \in X \times Y$.*

In this direction, we study the existence of continuous selection for multivalued mapping with certain conditions. For that, we need the following lemma and theorem.

Lemma 1.5. [8] *Let X and Y be any topological spaces. If $T : X \to \mathcal{P}_{cl,cv}(Y)$ is a w.l.s.c. multivalued mapping and $f : X \to Y$ is a continuous and open mapping, then $T \circ f$ is w.l.s.c.*

Theorem 1.6. [8] *If X is a paracompact topological space, Y is a normed linear space and $F : X \to \mathcal{P}_{cl,cv}(Y)$ is w.l.s.c., then F has a continuous selection.*

2 Existence of continuous selections

In this section, we provide continuous selection for some general type of mapping.

Theorem 2.1. *Let $F : X_1 \times X_2 \to X_2$ be any multivalued mapping with the property that*

1. *$F(.,x_2)$ is w.l.s.c for every $x_2 \in X_2$,*

2. *F satisfies property $(\mathcal{N})$.*

Then, for a given continuous mapping $\alpha : X_1 \times X_2 \to X_2$, the new mapping $(x_1, x_2) \to F(x_1, \alpha(x_1, x_2))$ is w.l.s.c.

Proof. Let us define $S := X_1 \times X_2$ and define $G : S \times X_2 \to \mathcal{P}_{cl,cv}(X_2)$ by $G(s,u) = F(P_X(s), u) = F(x_1, u)$ for $s \in S$ and $u \in X_2$. Now our aim is to show that the mapping $s \to G(s, \alpha(s))$ is w.l.s.c. By Lemma 1.5, it is clear that $G(., u)$ is w.l.s.c. for every $u \in X_2$.

Step 1: For an $s_0 \in S$ and an $\epsilon > 0$ and a neighborhood $O \ni s_0$, by continuity of α, we can choose a neighborhood $V \subset O$ of s_0 with the property that

$$\|\alpha(s) - \alpha(s_0)\| < \frac{\epsilon}{3}$$

for each $s \in V$.

Step 2: Since $G(., u)$ is w.l.s.c., by applying the definition of weakly lower semicontinuity for this V, we can find a point s_1 so that for any $v \in G(s_1, \alpha(s_0))$, there exists a neighborhood U_v of s_0 with

$$v \in \cap\{G(s, \alpha(s_0)) + \frac{\epsilon}{3} B^0 : s \in U_v\}. \tag{1}$$

Step 3: Let $v_1 \in G(s_1, \alpha(s_1))$. Since F satisfies property $(\mathcal{N})$, we have

$$H(G(s, u_1), G(s, u_2)) \leq \|u_1 - u_2\|.$$

Using the above, we can find $v \in G(s_1, \alpha(s_0))$ such that

$$\|v - v_1\| \leq \|\alpha(s_1) - \alpha(s_0)\| < \frac{\epsilon}{3}.$$

For such v, applying Step 2, we get U_v which satisfies (1). Observe that $G(s, \alpha(s_0)) \subseteq G(s, \alpha(s)) + \frac{\epsilon}{3} B^0$. Hence $G(s, \alpha(s_0)) + \frac{\epsilon}{3} \subseteq G(s, \alpha(s)) + 2\frac{\epsilon}{3} B^0$, and so $v \in \cap\{G(s, \alpha(s)) + 2\frac{\epsilon}{3} B^0 : s \in U_v \cap V\}$. Thus $v_1 \in \cap\{G(s, \alpha(s) + \epsilon B^0 : s \in U_v \cap V)\}$, which gives our claim. □

Lemma 2.2. *Let $H : X \to \mathcal{P}_{cl}(Y)$ be w.l.s.c. and $h : X \to Y$ is continuous. Then for every continuous function $d : X \to [0, \infty)$ such that $H(x) \cap (h(x) + d(x)B) \neq \emptyset$, the multivalued mapping $S : X \to \mathcal{P}_{cl}(Y)$ defined by $S(x) = H(x) \cap (h(x) + d(x)B)$ is w.l.s.c.*

Proof. Fix any $x_0 \in X$. If $d(x_0) = 0$, then nothing to prove. Suppose $d(x_0) > 0$. Fix $\epsilon > 0$ and any neighborhood V of x_0. Then choose $\delta > 0$ such that $\delta < min\{d(x_0), \epsilon\}$. Now, choose a neighborhood W of x_0, $W \subseteq V$, such that for $x_1, x_2 \in W$,

$$\begin{aligned} |d(x_1) - d(x_2)| &< \frac{\delta}{2}, \\ \|f(x_1) - f(x_2)\| &< \frac{\delta}{2}. \end{aligned}$$

Choose a point x' in W such that for every $z \in H(x')$, there exists U_z of x_0 such that

$$z \in \cap\{H(x) + \delta B^0 : x \in U_z\}. \tag{2}$$

Now our claim is that this x' is the required point. To see this, take any arbitrary

$$z' \in H(x') \cap (h(x') + d(x')B).$$

Since $z' \in H(x')$, by (2), there exists $U_{z'} \ni x_0$ such that $z' \in \cap\{H(x) + \delta B^0 : x \in U_{z'}\}$. Let us define $W_{z'} := W \cap U_{z'}$. Then this is our required neighborhood for each z'.

$$\begin{aligned}\|z' - h(x)\| &\le \|z' - h(x')\| + \|f(x') - h(x)\| \\ &\le d(x') + \frac{\delta}{2} \\ &< d(x) + \delta.\end{aligned}$$

It follows that

$$\begin{aligned}&z' \in \cap\{(H(x) + \delta B^0) \cap (h(x) + d(x)B + \delta B^0) : x \in W_{z'}\}, \\ &z' \in \cap\{(H(x) \cap (h(x) + d(x)B) + \delta B^0 : x \in W_{z'}\}, \\ &z' \in \cap\{G(x) + \delta B^0 : x \in W_{z'}\}.\end{aligned}$$

Hence $z' \in \cap\{G(x) + \epsilon B^0 : x \in W_{z'}\}$. □

Theorem 2.3. *Let α_1, $\alpha_2 : X_1 \times X_2 \to X_2$ be mappings such that α_2 is a selection of the multivalued mapping $(x_1, x_2) \to F(x_1, \alpha(x_1, x_2))$. Then there exists a continuous selection α_3 of the multivalued mapping $(x_1, x_2) \to F(x_1, \alpha_2(x_1, x_2))$ such that*

$$\begin{aligned}\|\alpha_1(x_1, x_2) - \alpha_2(x_1, x_2)\| &\le \frac{\lambda}{1-\lambda}\|\alpha_2(x_1, x_2) - \alpha_1(x_1, x_2)\|, \\ d(\alpha_3(x_1, x_2), F(x_1, \alpha_3(x_1, x_2))) &\le \frac{\lambda}{1-\lambda}\|\alpha_2(x_1, x_2) - \alpha_1(x_1, x_2)\|\end{aligned}$$

for all $(x_1, x_2) \in X_1 \times X_2$.

Proof. By hypothesis, we have $\alpha_2(x_1, x_2) \in F(x_1, \alpha_1(x_1, x_2))$. Then

$$\begin{aligned}d(\alpha_2(x_1, x_2), F(x_1, \alpha_2(x_1, x_2))) &\le H(F(x_1, \alpha_1(x_1, x_2)), F(x_1, \alpha_2(x_1, x_2))) \\ &\le \lambda[d(\alpha_1(x_1, x_2), F(x_1, \alpha_1(x_1, x_2))) \\ &\quad + d(\alpha_2(x_1, x_2), F(x_1, \alpha_2(x_1, x_2)))],\end{aligned}$$

$$(1-\lambda)d(\alpha_2(x_1, x_2), F(x_1, \alpha_2(x_1, x_2))) \le \lambda d(\alpha_1(x_1, x_2), F(x_1, \alpha_1(x_1, x_2))),$$

$$\begin{aligned}d(\alpha_2(x_1, x_2), F(x_1, \alpha_2(x_1, x_2))) &\le \frac{\lambda}{1-\lambda} d(\alpha_1(x_1, x_2), F(x_1, \alpha_1(x_1, x_2))) \\ &\le \frac{\lambda}{1-\lambda}\|\alpha_1(x_1, x_2) - \alpha_2(x_1, x_2)\|.\end{aligned}$$

Now, define a new mapping $G : X_1 \times X_2 \to \mathcal{P}_{cl,cv}(X_2)$ by $G(x_1, x_2) := F(x_1, \alpha_2(x_1, x_2)) \cap (\alpha_2(x_1, x_2)) + \frac{\lambda}{1-\lambda}\|\alpha_1(x_1, x_2) - \alpha_2(x_1, x_2)\|$. Then, clearly, G is well defined and by Lemma 2.2 G is w.l.s.c. By Theorem 1.6, G has a continuous selection $\alpha_3 : X_1 \times X_2 \to X$,. which is our required mapping. □

Theorem 2.4. *Let X_1 be a paracompact and perfectly normal topological space and X_2 be a Banach space. Assume that*

1. *$F : X_1 \times X_2 \to \mathcal{P}_{cl,cv}(X_2)$ satisfies property $(\mathcal{N})$,*
2. *for a given $x \in X$, the mapping F satifies $\mathcal{H}(F(x, v_1), F(x, v_2)) \le \lambda[d(v_1, F(x, v_1)) + d(v_2, F(x, v_2))]$, where $\lambda < \frac{1}{2}$,*

3. for each $x_2 \in X_2$, the mapping $F(.,x_2)$ is w.l.s.c.

Then there exists a continuous mapping $f : X_1 \times X_2 \to X_2$ such that $f(x_1,x_2) \in P_H(x_1)$ for every $(x_1,x_2) \in X_1 \times X_2$.

Proof. Choose $\alpha_0 : X_1 \times X_2 \to X_2$ by $\alpha_0(x_1,x_2) = x_2$. Then α_0 is continuous. Now using Theorem 2.1, we get $(x_1,x_2) \to F(x_1,\alpha_0(x_1,x_2))$ is w.l.s.c. Applying Theorem 1.6, we get a continuous selection $\alpha_1 : X_1 \times X_2 \to X_2$ for the mapping $(x_1,x_2) \to F(x_1,\alpha_0(x_1,x_2))$.

By Theorem 2.3, there exists a continuous selection $\alpha_2 : X_1 \times X_2 \to X_2$ for $(x_1,x_2) \to F(x_1,\alpha_1(x_1,x_2))$ satisfying the following two conditions

$$\|\alpha_2(x_1,x_2) - \alpha_1(x_1,x_2)\| \leq \frac{\lambda}{1-\lambda}\|\alpha_1(x_1,x_2) - \alpha_0(x_1,x_2)\|,$$
$$d(\alpha_2(x_1,x_2), F(x_1,\alpha_2(x_1,x_2))) \leq \frac{\lambda}{1-\lambda}\|\alpha_1(x_1,x_2) - \alpha_0(x_1,x_2)\|$$

for every $(x_1,x_2) \in X_1 \times X_2$.

By proceeding the above process, we get a sequence of continuous functions $\alpha_n : X_1 \times X_2 \to X_2$ with the following properties:

$$\|\alpha_n(x_1,x_2) - \alpha_{n-1}(x_1,x_2)\| \leq \frac{\lambda}{1-\lambda}\|\alpha_{n-1}(x_1,x_2) - \alpha_{n-2}(x_1,x_2)\|,$$
$$d(\alpha_n(x_1,x_2), F(x_1,\alpha_n(x_1,x_2))) \leq \frac{\lambda}{1-\lambda}\|\alpha_n(x_1,x_2) - \alpha_{n-1}(x_1,x_2)\|$$

for $n = 1,2,\ldots$, $(x_1,x_2) \in X_1 \times X_2$. For a fixed pair (x_1,x_2), the sequence $(\alpha_n(x_1,x_2))$ is a Cauchy sequence. To see this, using the following inequality

$$\|\alpha_n(x_1,x_2) - \alpha_{n-1}(x_1,x_2)\| \leq (\frac{\lambda}{1-\lambda})^{n-1}\|\alpha_1(x_1,x_2) - \alpha_0(x_1,x_2)\|.,$$

we show that $(\alpha_n(x_1,x_2))$ is a Cauchy sequence. Since X_2 is complete, this Cauchy sequence converges.

Now, define $f : X_1 \times X_2 \to X_2$ by $f(x_1,x_2) = \lim_{n\to\infty} \alpha_n(x_1,x_2)$. It is clear that f is well-defined.

Next, our aim is to claim that f is continuous. For that, fix any $(x_1',x_2') \in X_1 \times X_2$. Then, consider

$$\begin{aligned}\|f(x_1,x_2) - f(x_1',x_2')\| \leq &\|f(x_1,x_2) - \alpha_n(x_1,x_2)\| \\ &+ \|\alpha_n(x_1,x_2) - \alpha_n(x_1',x_2')\| \\ &+ \|\alpha_n(x_1',x_2') - f(x_1,x_2)\|.\end{aligned}$$

Since $\alpha_n's$ are continuous and $\alpha_n(x_1,x_2)$ is convergent for every $(x_1,x_2) \in X_1 \times X_2$, applying all these in the above inequality, we can conclude f is continuous.

Next, consider

$$\begin{aligned}d(f(x_1,x_2), F(x_1,f(x_1,x_2))) &\leq \|f(x_1,x_2) - \alpha_n(x_1,x_2)\| \\ &+ d(\alpha_n(x_1,x_2), F(x_1,f(x_1,x_2))) \\ &\leq \sum_{m=n}^{\infty}(\frac{\lambda}{1-\lambda})^m\|\alpha_1(x_1,x_2) - \alpha_0(x_1,x_2)\| \\ &+ \frac{\lambda}{1-\lambda}\|\alpha_m(x_1,x_2) - \alpha_{m-1}(x_1,x_2)\| \\ &\leq \sum_{m=n}^{\infty}(\frac{\lambda}{1-\lambda})^m\|\alpha_1(x_1,x_2) - \alpha_0(x_1,x_2)\| \\ &+ \sum_{m=n}^{\infty}(\frac{\lambda}{1-\lambda})^n\|\alpha_1(x_1,x_2) - \alpha_0(x_1,x_2)\|.\end{aligned}$$

Hence $f(x_1, x_2) \in F(x_1, f(x_1, x_2))$. □

Acknowledgments

G. Poonguzali was funded by Human Resource Development Group and Council of Scientific and Industrial Research, Sanction No. 09/475(0198)/2016-EMR-I dated 15.11.2016. C. Park was supported by Basic Science Research Program through the National Research Foundation of Korea funded by the Ministry of Education, Science and Technology (NRF-2017R1D1A1B04032937).

References

[1] G.A. Anastassiou, I.K. Argyros, *Approximating fixed points with applications in fractional calculus*, J. Comput. Anal. Appl. **21** (2016), 1225–1242.

[2] A. Batool, T. Kamran, S. Jang, C. Park, *Generalized φ-weak contractive fuzzy mappings and related fixed point results on complete metric space*, J. Comput. Anal. Appl. **21** (2016), 729–737.

[3] F. E. Browder, *A new generation of the Schauder fixed point theorem*, Math. Ann. **174** (1967), 285–290.

[4] F. E. Browder, *The fixed point theory of multi-valued mappings in topological vector spaces,* Math. Ann. **177** (1968), 283–301.

[5] C. D. Horvath,it Extension and selection theorems in topological vector spaces with a generalized convexity structure, Ann. Fac. Sci. **2** (1993), 253–269.

[6] E. Michael, *Continuous selections I*. Ann. Math. **63** (1956), 361–382.

[7] S. Park, *Continuous selection theorems in generalized convex spaces,* Numer. Funct. Anal. Optim. **25** (1999), 567–583.

[8] K. Przeslawski, L. E. Rybinski, *Michael selection theorem under weak lower semicontinuity assumption*. Proc. Amer. Math. Soc. **109** (1990), 537–543.

[9] L. Rybinski, *An application of the continuous selection theorem to the study of the fixed points of multivalued mappings*, J. Math. Anal. Appl. **153** (1990), 391–396.

[10] N. C. Yannelis, N. D. Prabhakar, *Existence of maximal elements and equilibria in linear topological spaces*, J. Math. Economics **12** (1983), 233–245.

[11] E. Zeidler, *Nonlinear Functional Analysis and its Applications I: Fixed-point theorems*, Springer-Verlag, New York, 1986.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 27, NO.3, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC

REFINED STABILITY OF SET-VALUED FUNCTIONAL EQUATIONS

HONG-MEI LIANG, HARK-MAHN KIM, AND HWAN-YONG SHIN

Abstract. Recently, stability results of set-valued functional equations on domain of cones in Banach spaces are obtained by several authors. In this paper, we present the refined stability results of set-valued functional equations which is stable in the sense of Aoki, Rassias and Găvruţă on domain of cones.

1. Introduction

The Hyers–Ulam stability problem was originated by S. M. Ulam [16] in 1940 as follows: *Let G_1 be a group and let G_2 be a metric group with the metric $d(\cdot,\cdot)$. Given $\varepsilon > 0$, does there exist a $\delta > 0$ such that if a function $h : G_1 \to G_2$ satisfies the inequality $d(h(xy), h(x)h(y)) \le \delta$ for all $x, y \in G_1$, then there exists a homomorphism $H : G_1 \to G_2$ with $d(h(x), H(x)) < \varepsilon$ for all $x \in G_1$.*

Ulam's question was partially solved by D. H. Hyers [6] in the case of approximately additive functions and when the groups in the question are Banach spaces. In fact, Hyers proved that each solution of the inequality $\|f(x+y)-f(x)-f(y)\| \le \varepsilon$ for all x and y can be approximated by an exact solution, say an additive function. In this case, it is said that the Cauchy additive functional equation $f(x+y) = f(x)+f(y)$ satisfies Hyers–Ulam stability or that the equation is stable in the sense of Hyers–Ulam.

Many mathematicians attempted to moderate the condition for the bound of the norm of the Cauchy difference. First, T. Aoki [1] proved the stability of Cauchy functional equations by changing the bound of Cauchy difference as follows

$$\|f(x+y) - f(x) - f(y)\| \le \varepsilon(\|x\|^p + \|y\|^p),$$

where $p \in (0, 1)$, and Rassias [14] obtained additional linear properties of this results. Furthermore, the control function of Cauchy difference with some regularity conditions has been employed by Găvruţă [5] as follows

$$\|f(x+y) - f(x) - f(y)\| \le \varphi(x, y).$$

1991 *Mathematics Subject Classification.* 39B52, 39B82, 54C65.

Key words and phrases. set-valued functional equation; generalized Hyers–Ulam stability; Cantor intersection theorem; cone subset in Banach spaces.

† Corresponding author. hyshin31@cnu.ac.kr.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 27, NO.3, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC

Recently, as the development of non-convex analysis, cone sets were investigated by many authors and were applied for various regions of optimization theory and mathematical physics [10, 11]. Let X be a real Banach space and P a subset of X. P is called a *cone* [15] if

(i) P is closed, non-empty and $P \neq \{0\}$,
(ii) $ax + by \in P$ for all $x, y \in P$ and non-negative real numbers a, b,
(iii) $P \cap (-P) = \{0\}$.

Set-valued functions in Banach spaces also have received a lot of attention in the literature [2]. Functional inclusion is a tool for defining many notions of set-valued analysis, e.g., linear, affine, convex, concave, subadditive, superadditive set-valued maps. Finding a selection of such set-valued maps, with some special properties, is one of the main problems of set-valued analysis (see [2]). The stability theory of functional equations leads in some cases to such problems and solving them provides Hyers–Ulam stability results [3, 4, 7]. In setting domain of set-valued functions as a cone, some stability results of set-valued functional equations were obtained by several authors [9, 13].

In this sequel, we introduce a result concerning with stability of set-valued functional equations under cone domain. Let Y be a Banach space and P be a cone. We define the following families of sets :

$$\mathcal{P}_0(Y) := \{A \subseteq Y : \text{A is nonempty set}\}$$
$$cl(Y) := \{A \in \mathcal{P}_0(A) : \text{A is closed set}\}$$
$$cz(Y) := \{A \in \mathcal{P}_0(A) : \text{A is closed set containing zero}\}.$$

Theorem 1.1. (*C. Park, D. O'Regan, R. Saadati,* [13]) *If $F : P \to cz(Y)$ is a set-valued mapping satisfying $F(0) = \{0\}$,*

$$F(x) + F(y) \subseteq 2F\Big(\frac{x+y}{2}\Big) \tag{1.1}$$

and

$$sup\{diam(F(x)) : x \in P\} < +\infty$$

for all $x, y \in P$, then there exists a unique additive mapping $g : P \to Y$ such that $g(x) \in F(x)$ for all $x \in P$.

In view of Theorem 1.1, if $diam(F(x)) = \varepsilon$, then $\sup\{diam(F(x)) : x \in P\} < +\infty$. So if, in addition, F satisfies (1.1), we may understand Theorem 1.1 works good in the sense of Hyers–Ulam. On the other hand, if $diam(F(x)) = \|x\|^p, p \neq 0$, we confirm that $sup\{diam(F(x)) : x \in P\} = \infty$, and so Theorem 1.1 cannot be favorably applied in this case.

Thus, in this paper, we are devoted to investigate refined stability results of Theorem 1.1, and also we present alternative new stability theorems and examples to provide refined stability theorems of Theorem 1.1.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 27, NO.3, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC

Let A, B be nonempty subsets of a real vector space X and λ a real number. We define

$$A + B = \{x \in X : x = a + b,\ a \in A,\ b \in B\}$$
$$\lambda A = \{x \in X : x = \lambda a,\ a \in A\}.$$

Lemma 1.2. [12] *Let λ and μ be real numbers. If A and B are empty subsets of a real vector space X, then*

$$\lambda(A + B) = \lambda A + \lambda B$$
$$(\lambda + \mu)A \subseteq \lambda A + \mu A.$$

Moreover, if A is convex in X and $\lambda\mu \geq 0$, then we have

$$(\lambda + \mu)A = \lambda A + \mu A.$$

Lemma 1.3. *If A_n and B_n are non-empty subsets of a real vector space X for all nonnegative positive integer n, then*

$$\bigcap_{n=0}^{l} A_n + \bigcap_{n=0}^{l} B_n = \bigcap_{n=0}^{l} (A_n + B_n)$$

for any given $l \in \mathbb{N}$.

The following famous theorem is a crucial tool to prove our main theorems.

Theorem 1.4. (*Cantor Intersection Theorem,* [8]) *Suppose (X, d) is a non-empty complete metric space, and $\{C_n\}_{n\geq 0}$ closed subsets of X which satisfies*

$$C_1 \supseteq C_2 \supseteq \cdots \supseteq C_n \supseteq C_{n+1} \supseteq \cdots .$$

If $\lim_{n\to\infty} diam(C_n) = 0$, where $diam(C_n)$ is defined by $diam(C_n) = \sup\{d(x, y)|x, y \in C_n\}$, then $\bigcap_{n=1}^{\infty} C_n$ consists of a single point.

From now on, let P be a cone for a Banach space Y. We present a main theorem, which is an extended Hyers–Ulam stability of a set-valued functional equations on the domain of cones.

Theorem 1.5. *If $F : P \to cl(Y)$ is a set-valued mapping satisfying*

$$\sum_{i=1}^{m} F(x_i) \subseteq mF\Big(\frac{\sum_{j=1}^{m} x_j}{m}\Big) \tag{1.2}$$

and

$$\lim_{n\to\infty} \frac{diam(F(m^n x))}{m^n} = 0 \tag{1.3}$$

for all $x_1, \cdots, x_m, x \in P$, where $m > 1$ is a positive integer, then there exists a unique additive mapping $g : P \to cl(Y)$ such that $g(x) \subseteq F(x) + (-1)F(0)$ for all $x \in P$.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 27, NO.3, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC

Proof. Since $F(0) \in cl(Y)$, $F(0)$ has at least an element, say $p \in F(0)$.

Letting $x_1 = x$ and $x_k = 0$ for all $k \neq 1$, in (1.2), we have

$$F(x) + (m-1)\{p\} \subseteq F(x) + (m-1)F(0) \subseteq mF(\frac{x}{m}) \tag{1.4}$$

and so

$$F(x) + (-1)\{p\} \subseteq m\big(F(\frac{x}{m}) + (-1)\{p\}\big) \tag{1.5}$$

for all $x \in P$. Replacing x by $m^{n+1}x$ in (1.5), then we obtain

$$F(m^{n+1}x) + (-1)\{p\} \subseteq m\big(F(m^n x) + (-1)\{p\}\big)$$

and hence

$$\frac{F(m^{n+1}x) + (-1)\{p\}}{m^{n+1}} \subseteq \frac{F(m^n x) + (-1)\{p\}}{m^n}$$

for all $x \in P$ and all $n \in \mathbb{N} \cup \{0\}$. Denoting $F_n(x) := \dfrac{F(m^n x) + (-1)\{p\}}{m^n}$ for all $x \in P$ and all $n \in \mathbb{N} \cup \{0\}$, it results that $\{F_n(x)\}_{\geq 0}$ is a decreasing sequence of closed subsets of the Banach space Y. We have also

$$diam(F_n(x)) = \frac{1}{m^n} diam(F(m^n x) + (-1)\{p\}) = \frac{1}{m^n} diam(F(m^n x)).$$

By (1.3), we get $\lim_{n\to\infty} diam(F_n(x)) = 0$ for all $x \in P$. Using the Cantor Intersection Theorem for the sequence $\{F_n(x)\}_{n\geq 0}$, the intersection $\bigcap_{n\geq 0} F_n(x)$ is a singleton and we denote this intersection by $g(x)$ for all $x \in P$. Thus we obtain a mapping $g : P \to cl(Y)$, defined as $g(x) := \bigcap_{n\geq 0} F_n(x)$, which is a singleton from F because $g(x) \subseteq F_0(x) = F(x) + (-1)\{p\} \subseteq F(x) + (-1)F(0)$ for all $x \in P$.

Now, we show that g is additive. It follows from the definition of g and Lemma 1.3 that

$$\sum_{i=1}^{m} g(x_i) \subseteq \sum_{i=1}^{m} \bigcap_{n=0}^{l} F_n(x_i) = \bigcap_{n=0}^{l} \sum_{i=1}^{m} F_n(x_i) \subseteq \bigcap_{n=0}^{l} \Big(mF_n\Big(\frac{\sum_{j=1}^m x_j}{m}\Big)\Big)$$

for any $l \in \mathbb{N} \cup \{0\}$, thus

$$\sum_{i=1}^{m} g(x_i) \subseteq \bigcap_{n=0}^{\infty} \Big(mF_n\Big(\frac{\sum_{j=1}^m x_j}{m}\Big)\Big)$$

for all $x_1, \cdots, x_m \in P$. On the other hand, one obtains vacuously

$$mg\Big(\frac{\sum_{j=1}^m x_j}{m}\Big) \subseteq \bigcap_{n=0}^{\infty} \Big(mF_n\Big(\frac{\sum_{j=1}^m x_j}{m}\Big)\Big)$$

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 27, NO.3, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC

for all $x_1, \cdots, x_m \in P$. Thus, since $\bigcap_{n=0}^{\infty} \Big(mF_n\Big(\frac{\sum_{j=1}^m x_j}{m}\Big)\Big)$ is a singleton, we arrive at

$$\sum_{i=1}^{m} g(x_i) = mg\Big(\frac{\sum_{j=1}^m x_j}{m}\Big)$$

for all $x_1, \cdots, x_m \in P$. Thus g is additive since $g(0) = \{0\}$. Therefore, we conclude that there exists an additive mapping $g : P \to cl(Y)$ such that $g(x) \subseteq F_0(x) \subseteq F(x) + (-1)F(0)$ for all $x \in P$.

Next, we will finalize the proof by proving the uniqueness of g. Suppose that $g' : P \to cl(Y)$ is another additive mapping such that $g'(x) \subseteq F(x) + (-1)F(0)$ for all $x \in P$. Then we have

$$m^n g(x) = g(m^n x) \subseteq F(m^n x) + (-1)F(0)$$
$$m^n g'(x) = g'(m^n x) \subseteq F(m^n x) + (-1)F(0)$$

for all $n \in \mathbb{N} \cup \{0\}$ and all $x \in P$. Thus, we get

$$\begin{aligned} m^n diam(g(x) - g'(x)) &= diam(m^n g(x) - m^n g'(x)) \\ &= diam(g(m^n x) - g'(m^n x)) \\ &\leq diam(F(m^n x) + (-1)F(0)) \\ &= diam(F(m^n x)) + diam((-1)F(0)) \end{aligned}$$

which implies

$$diam(g(x) - g'(x)) \leq \frac{1}{m^n}\big[diam(F(m^n x)) + diam((-1)F(0))\big]$$

for all $n \in \mathbb{N} \cup \{0\}$ and all $x \in P$. Therefore, it follows from $\lim_{n\to\infty} \frac{diam(F(m^n x))}{m^n} = 0$ that $g(x) = g'(x)$ for all $x \in P$, as desired. □

The following corollary is a refined stability result of Theorem 1.1, if we take $m = 2$.

Corollary 1.6. *If $F : P \to cl(Y)$ is a set-valued mapping satisfying*

$$\sum_{i=1}^{m} F(x_i) \subseteq mF\big(\frac{\sum_{j=1}^m x_j}{m}\big)$$

and

$$\sup\{diam(F(x)) : x \in X\} < +\infty$$

for all $x_1, \cdots, x_m, x \in P$, then there exists a unique additive mapping $g : P \to cl(Y)$ such that $g(x) \subseteq F(x) + (-1)F(0)$ for all $x \in P$.

Proof. Since $\sup\{diam(F(x)) : x \in P\} < +\infty$, $\lim_{n\to\infty} \dfrac{diamF(m^n x)}{m^n} = 0$ for all $x \in P$. Applying Theorem 1.5, we complete the proof. □

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 27, NO.3, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC

Now, let us consider the following example with nontrivial set-valued function at zero.

Example 1.7. Let $F:[0,\infty)\to cl(\mathbb{R})$ be defined by

$$F(x)=\begin{cases}[ax, ax+bx^p], & \text{if} \quad x\neq 0,\\ [0,c], & \text{if} \quad x=0,\end{cases}$$

where a,b are positive real numbers, $c\geq 0$ and $p\in(-\infty,0)\cup(0,1)$. It is easy to see that

$$F(x)+F(y)\subseteq 2F\big(\frac{x+y}{2}\big),\quad \lim_{n\to\infty}\frac{diam(F(2^n x))}{2^n}=0$$

for all $x,y\in[0,\infty)$. Also, we can check that

$$\bigcap_{n=0}^{\infty}\frac{F(2^n x)+(-1)F(0)}{2^n}=\{ax\}$$

for all $x\in[0,\infty)$. Therefore, there exists additive mapping $g:[0,\infty)\to cl(\mathbb{R})$ defined by $g(x)=\{ax\}$ such that $g(x)\subseteq F(x)+(-1)F(0)=[ax-c, ax+bx^p]$ for all $x\in[0,\infty)$. This result can be found by applying Theorem 1.5.

However, it is noted that we cannot apply Theorem 1.1 to this example because

$$\sup\{diam(F(x)) : x\in[0,\infty)\}=+\infty.$$

Next, we provide an alternative main theorem of Theorem 1.5.

Theorem 1.8. *If $F:P\to cl(Y)$ is a set-valued mapping satisfying*

$$mF\big(\frac{\sum_{j=1}^m x_j}{m}\big)\subseteq\sum_{i=1}^m F(x_i) \tag{1.6}$$

and

$$\lim_{n\to\infty} m^n diam(F(\frac{x}{m^n}))=0 \tag{1.7}$$

for all $x_1,\cdots,x_m,x\in P$, then there exists a unique additive mapping $g:P\to cl(Y)$ such that $g(x)\subseteq F(x)+(-1)F(0)$ for all $x\in P$.

Proof. By assumption (1.7), one has

$$\lim_{n\to\infty} m^n diam(F(0))=0$$

and so $F(0)$ is a singleton, say $F(0)=\{p\}$. Taking $x_1=x$ and $x_k=0$ for all $k\neq 0$ in (1.6), we obtain

$$m\big(F\big(\frac{x}{m}\big)+(-1)\{p\}\big)\subseteq F(x)+(-1)\{p\}. \tag{1.8}$$

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 27, NO.3, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC

for all $x \in P$. And if we replace x by $\dfrac{x}{m^n}$ in (1.8), then we obtain

$$m\big(F(\frac{x}{m^{n+1}}) + (-1)\{p\}\big) \subseteq F\big(\frac{x}{m^n}\big) + (-1)\{p\}$$

and so

$$m^{n+1}\big(F\big(\frac{x}{m^{n+1}}\big) + (-1)\{p\}\big) \subseteq m^n\big(F\big(\frac{x}{m^n}\big) + (-1)\{p\}\big)$$

for all $x \in P$ and all $n \in \mathbb{N} \cup \{0\}$. Defining $F_n(x) = m^n\big(F(\frac{x}{m^n}) + (-1)\{p\}\big)$ for all $x \in P$ and all $n \in \mathbb{N} \cup \{0\}$, we obtain that $\{F_n(x)\}_{n \geq 0}$ is a decreasing sequence of closed subsets of the Banach space Y. It is noted that

$$diam(F_n(x)) = diam\big(m^n\big(F\big(\frac{x}{m^n}\big) + (-1)\{p\}\big)\big) = m^n diam\big(F\big(\frac{x}{m^n}\big)\big),$$

which implies $\lim_{n\to\infty} diam(F_n(x)) = 0$ for all $x \in P$ by (1.7).

Employing the Cantor Intersection Theorem to the sequence $\{F_n(x)\}_{n\geq 0}$, $\bigcap_{n\geq 0} F_n(x)$ is a singleton set and so we may define a mapping $g : P \to cl(Y)$ by $g(x) := \bigcap_{n\geq 0} F_n(x), x \in P$, which satisfies $g(x) \subseteq F_0(x) = F(x) + (-1)\{p\} \subseteq F(x) + (-1)F(0)$ for all $x \in P$.

Now, we show that g is additive. It follows from Lemma 1.2 that

$$\begin{aligned} mF_n\Big(\frac{\sum_{j=1}^m x_j}{m}\Big) &= m \cdot m^n\Big(F\big(\sum_{j=1}^m \frac{x_j}{m^n \cdot m}\big) + (-1)\{p\}\Big) \\ &\subseteq m^n \sum_{i=1}^m \big(F\big(\frac{x_i}{m^n}\big) + (-1)\{p\}\big) = \sum_{i=1}^m F_n(x_i) \end{aligned}$$

for all $x_1, \cdots, x_m \in P$. By the definition of g, we can get

$$mg\Big(\frac{\sum_{j=1}^m x_j}{m}\Big) \subseteq \bigcap_{n=0}^{l} mF_n\Big(\frac{\sum_{j=1}^m x_j}{m}\Big) \subseteq \bigcap_{n=0}^{l} \sum_{i=1}^m F_n(x_i)$$

for any $l \in \mathbb{N} \cup \{0\}$ and all $x_1, \cdots, x_m \in P$, which yields

$$mg\Big(\frac{\sum_{j=1}^m x_j}{m}\Big) \subseteq \bigcap_{n=0}^{\infty} \sum_{i=1}^m F_n(x_i). \tag{1.9}$$

Moreover, it is easy to show that, for all $x_1, \cdots, x_m \in P$,

$$\sum_{i=1}^m g(x_i) \subseteq \sum_{i=1}^m F_n(x_i), \forall n \in \mathbb{N} \cup \{0\}$$

and so

$$\sum_{i=1}^m g(x_i) \subseteq \bigcap_{n=0}^{\infty} \sum_{i=1}^m F_n(x_i). \tag{1.10}$$

Therefore, it follows from (1.9) and (1.10) that

$$mg\Big(\frac{\sum_{j=1}^m x_j}{m}\Big) = \sum_{i=1}^m g(x_i),$$

that is, g is additive because $g(0) = \{0\}$.

Finally, let us prove the uniqueness of g. Suppose that $g' : P \to cl(Y)$ is an additive mapping such that $g'(x) \subseteq F(x) + (-1)F(0)$ for all $x \in P$. Then we have

$$\frac{1}{m^n} g(x) = g\big(\frac{x}{m^n}\big) \subseteq F\big(\frac{x}{m^n}\big) + (-1)F(0),$$
$$\frac{1}{m^n} g'(x) = g'\big(\frac{x}{m^n}\big) \subseteq F\big(\frac{x}{m^n}\big) + (-1)F(0)$$

for all $n \in \mathbb{N} \cup \{0\}$ and all $x \in P$. Thus, noting singleton $F(0)$, we get

$$\begin{aligned}\frac{1}{m^n} diam(g(x) - g'(x)) &= diam(g(\frac{x}{m^n}) - g'(\frac{x}{m^n})) \\ &\le diam\Big(F(\frac{x}{m^n}) + (-1)F(0)\Big) = diam\Big(F(\frac{x}{m^n})\Big)\end{aligned}$$

for all $x \in P$ and all $n \in \mathbb{N} \cup \{0\}$. It follows from (1.7) that $g(x) = g'(x)$ for all $x \in P$, as desired. □

Corollary 1.9. *If $F : P \to cl(Y)$ is a set-valued mapping satisfying $F(0) = \{0\}$,*

$$mF\big(\frac{\sum_{j=1}^m x_j}{m}\big) \subseteq \sum_{i=1}^m F(x_i)$$

and

$$\lim_{n\to\infty} m^n diam(F(\frac{x}{m^n})) = 0$$

for all $x_1, \cdots, x_m, x \in P$, then there exists a unique additive mapping $g : P \to cl(Y)$ such that $g(x) \subseteq F(x)$ for all $x \in P$.

Example 1.10. Let $F : [0, \infty) \to cl(\mathbb{R})$ be defined by $F(x) = [ax, ax + bx^p]$, where a, b are positive real numbers and $p > 1$. Then, since the function x^p is convex, it is easily checked that $2F\big(\frac{x+y}{2}\big) \subseteq F(x) + F(y)$ and $\lim_{n\to\infty} 2^n diam(F(\frac{x}{2^n})) = 0$ for all $x, y \in [0, \infty)$. Thus, there exists an additive mapping $g : [0, \infty) \to cl(\mathbb{R})$ such that $g(x) = \{ax\} \subseteq F(x)$ for all $x \in [0, \infty)$ by Corollary 1.9.

Example 1.11. Finally, let $H : [0, \infty) \to cl(\mathbb{R})$ be defined by $H(x) = [ax, ax + bx]$, where a, b are positive real numbers. Then, it follows easily that $2H\big(\frac{x+y}{2}\big) = H(x) + H(y)$ for all $x, y \in [0, \infty)$. However, there are two different additive mappings $g_1(x) := \{ax\}, g_2(x) := \{(a + b)x\}$ such that $g_1(x), g_2(x) \subseteq H(x)$ for all $x \in [0, \infty)$. In fact, one notes that either

(1.3) or (1.7) is not satisfied for the function H, and so one cannot apply Theorems 1.5 and 1.8 to this example.

Thus, we remark that the set-valued function $F(x) = [ax, ax + bx]$ has no Hyers–Ulam stability property for the set-valued Cauchy–Jensen additive functional equation.

Acknowledgements

This research was supported by Basic Science Research Program through the National Research Foundation of Korea(NRF) funded by the Ministry of Education(2016R1D1A3B03930971). Correspondence should be addressed to Hark-Mahn Kim(hmkim@cnu.ac.kr) and Hwan-Yong Shin(hyshin31@cnu.ac.kr).

References

[1] T. Aoki : On the stability of the linear transformation in Banach spaces. J. Math. Soc. Japan **2**, 64–66 (1950)

[2] J.P. Aubin, H. Frankowska, : Set-valued analysis, in Modern Birkhäuser Classics. Birkhäuser, Boston (2008)

[3] H.-Y. Chu, S. K. Yoo, : On the Stability of the Generalized Quadratic Set-Valued Functional Equation. Journal of Computational Analysis and Applicartions. **20**, 1007-1020 (2016)

[4] J. Brzdęk, D. Popa, B. Xu,: Selections of set-valued maps satisfying a linear inclusion in a single variable. Nonlinear Analysis: Theory, Methods and Applications **74**, 324–330 (2011)

[5] P. Găvruta,: A generalization of the Hyers–Ulam–Rassias stability of approximately additive mappings. J. Math. Anal. Appl. **184**, 431–436 (1994)

[6] D.H. Hyers,: On the stability of the linear functional equation. Proc. Nat. Acad. Sci. **27**, 222–224 (1941)

[7] D. Kang,: Stabiltiy of generalized cubic set-valued functional equations Journal of Computational Analysis and Applicartions. **20**, 296-306 (2016)

[8] J. Lewin,: An Interactive Introduction to Mathematical Analysis. Cambridge University Press (2003)

[9] G. Lu, C. Park,: Hyers–Ulam stability of additive set-valued functional equations. Appl. Math. Lett. **24**, 1312–1316 (2011)

[10] H. Mohebi,: Topical functions and their properties in a class of ordered Banach spaces, in Continuous Optimization. Applied Optimization, Springer **99**, 343–361 (2005)

[11] H. Mohebi, H. Sadeghi, A.M. Rubinov, : Best approximation in a class of normed spaces with star-shaped cone. Current Numer. Funct. Anal. Optim. **27**, 411–436 (2006)

[12] K. Nikodem,: K-convex and K-concave set-valued functions. Z. K. Nr. 559 (1989)

[13] C. Park, D. O'Regon, R. Saadati,: Stability of some set-valued functional equations. Appl. Math. Lett. **24**, 1910–1914 (2011)

[14] T.M. Rassias,: On the stability of the linear mapping in Banach spaces. Proc. Amer. Math. Soc. **72**, 297–300 (1978)

[15] S. Rezapour, R. Hamlbarani,: Some notes on the paper "Cone metric spaces and fixed point theorems of contractive mappings". J. Math. Anal. Appl. **345**, 719–724 (2008)

[16] S.M. Ulam,: Problems in Modern Mathematics. Chapter 6 Wiley Interscience, New York (1964)

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 27, NO.3, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC

Hong-Mei Liang, Department of Mathematics, Qiqihar University, Qiqihar, 161006, China; Department of Mathematics, Chungnam National University,99 Daehangno, Yuseong-gu, Daejeon 34134, Korea

E-mail address: hmliang124@126.com

Hark-Mahn Kim, Department of Mathematics, Chungnam National University,99 Daehangno, Yuseong-gu, Daejeon 34134, Korea

E-mail address: hmkim@cnu.ac.kr

Department of Mathematics, Chungnam National University,99 Daehangno, Yuseong-gu, Daejeon 34134, Korea

E-mail address: hyshin31@cnu.ac.kr

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 27, NO.3, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC

APPROXIMATE CAUCHY-JENSEN AND BI-QUADRATIC MAPPINGS IN 2-BANACH SPACES

WON-GIL PARK AND JAE-HYEONG BAE

ABSTRACT. In this paper, we obtain the stability of the Cauchy-Jensen and bi-quadratic functional equation

$$2f\left(x+y,\frac{z+w}{2}\right)=f(x,z)+f(x,w)+f(y,z)+f(y,w),$$

$$\begin{aligned}f(x+y,z+w)+f(x+y,z-w)+f(x-y,z+w)+f(x-y,z-w)\\=4[f(x,z)+f(x,w)+f(y,z)+f(y,w)],\end{aligned}$$

respectively, in 2-Banach spaces.

1. Introduction

In 1940, Ulam [7] suggested the stability problem of functional equations concerning the stability of group homomorphisms: Let a group G and a metric group H with the metric ρ be given. For each $\varepsilon > 0$, the question is whether or not there is a $\delta > 0$ such that if $f : G \to H$ satisfies $\rho\big(f(xy), f(x)f(y)\big) < \delta$ for all $x, y \in G$, then there exists a group homomorphism $h : G \to H$ satisfying $\rho(f(x), h(x)) < \varepsilon$ for all $x \in G$.

We introduce some definitions on 2-Banach spaces [2], [3].

Definition 1. Let X be a real linear space with $\dim X \geq 2$ and $\|\cdot,\cdot\| : X^2 \to \mathbb{R}$ be a function. Then $(X, \|\cdot,\cdot\|)$ is called a *linear 2-normed space* if the following conditions hold:

(a) $\|x, y\| = 0$ if and only if x and y are linearly dependent,

(b) $\|x, y\| = \|y, x\|$,

(c) $\|\alpha x, y\| = |\alpha|\|x, y\|$,

(d) $\|x, y + z\| \leq \|x, y\| + \|x, z\|$

for all $\alpha \in \mathbb{R}$ and $x, y, z \in X$. In this case, the function $\|\cdot,\cdot\|$ is called a *2-norm* on X.

Definition 2. Let $\{x_n\}$ be a sequence in a linear 2-normed space X. The sequence $\{x_n\}$ is said to *convergent* in X if there exits an element $x \in X$ such that

$$\lim_{n\to\infty} \|x_n - x, y\| = 0$$

1991 *Mathematics Subject Classification.* 39B52, 39B72.

Key words and phrases. linear 2-normed space, Cauchy-Jensen mapping, bi-quadratic mapping.

This research was supported by Basic Science Research Program through the National Research Foundation of Korea(NRF) funded by the Ministry of Education, Science and Technology(grant number 2017028238).

Competing interests. The authors declare that they have no competing interests.

for all $y \in X$. In this case, we say that a sequence $\{x_n\}$ converges to the limit x, simply dented by $\lim_{n\to\infty} x_n = x$.

Definition 3. A sequence $\{x_n\}$ in a linear 2-normed space X is called a *Cauchy sequence* if for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $m, n \geq N$, $\|x_m - x_n, y\| < \varepsilon$ for all $y \in X$. For convenience, we will write $\lim_{m,n\to\infty} \|x_n - x_m, y\| = 0$ for a Cauchy sequence $\{x_n\}$. A 2-*Banach space* is defined to be a linear 2-normed space in which every Cauchy sequence is convergent.

In the following lemma, we obtain some basic properties in a linear 2-normed space which will be used to prove the stability results.

Lemma 4. ([1]) Let $(X, \|\cdot,\cdot\|)$ be a linear 2-normed space and $x \in X$.
(a) If $\|x, y\| = 0$ for all $y \in X$, then $x = 0$.
(b) $\big|\|x, z\| - \|y, z\|\big| \leq \|x - y, z\|$ for all $x, y, z \in X$.
(c) If a sequence $\{x_n\}$ is convergent in X, then $\lim_{n\to\infty} \|x_n, y\| = \|\lim_{n\to\infty} x_n, y\|$ for all $y \in X$.

Throughout this paper, let X be a normed space and Y a 2-Banach space. We introduce the definitions of Cauchy-Jensen and bi-quadratic mappings.

Definition 5. A mapping $f : X \times X \to Y$ is called a *Cauchy-Jensen mapping* if f satisfies the system of equations

$$\begin{aligned} f(x+y,z) &= f(x,z) + f(y,z), \\ 2f\big(x, \tfrac{y+z}{2}\big) &= f(x,y) + f(x,z). \end{aligned} \tag{1}$$

Definition 6. A mapping $f : X \times X \to Y$ is called *bi-quadratic* if f satisfies the system of equations

$$\begin{aligned} f(x+y,z) + f(x-y,z) &= 2f(x,z) + 2f(y,z), \\ f(x,y+z) + f(x,y-z) &= 2f(x,y) + 2f(x,z). \end{aligned} \tag{2}$$

For a mapping $f : X \times X \to Y$, consider the functional equations:

$$2f\left(x+y, \frac{z+w}{2}\right) = f(x,z) + f(x,w) + f(y,z) + f(y,w) \tag{3}$$

and

$$\begin{aligned} f(x+y,z+w) + f(x+y,z-w) + f(x-y,z+w) + f(x-y,z-w) \\ = 4[f(x,z) + f(x,w) + f(y,z) + f(y,w)]. \end{aligned} \tag{4}$$

When $X = Y = \mathbb{R}$, the function $f : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ given by $f(x,y) := axy + bx$ and $f(x,y) := ax^2y^2$ are solutions of (3) and (4), respectively.

In 2011, W.-G. Park [4] investigate approximate additive, Jensen and quadratic mappings in 2-Banach spaces. In this papaer, we also investigate Cauchy-Jensen and bi-quadratic mappings in 2-Banach spaces with different assumptions from [4].

2. Approximate Cauchy-Jensen mappings

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 27, NO.3, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC

Let $\varphi : X^5 \to [0, \infty)$ be a function satisfying

$$\begin{aligned}\tilde{\varphi}(x, y, z, w, s) := \sum_{j=0}^{\infty} \frac{1}{6^{j+1}} \Big[&\varphi(2^j x, 2^j y, 3^j z, 3^j w, s) + 2\varphi(2^j x, 2^j y, -3^j z, 3^j w, s) \\ &+ \varphi(2^j x, 2^j y, -3^j z, 3^{j+1} w, s) + \frac{1}{2}\varphi(2^j x, 2^j y, 3^{j+1} z, 3^{j+1} w, s) \\ &+ 3\varphi(2^j x, 2^j y, 3^j z, -3^j w, s) + 2\|f(2^{j+1} x, 0), t\| + 5\|f(x, 0), t\| \Big] < \infty \end{aligned} \tag{5}$$

for all $x, y, z, w, s \in X$, where $t = f(s)$.

Theorem 7. *Suppose that $f : X \times X \to Y$ is a surjective mapping such that*

$$\left\| 2f\left(x + y, \frac{z + w}{2}\right) - f(x, z) - f(x, w) - f(y, z) - f(y, w), t \right\| \le \varphi(x, y, z, w, s) \tag{6}$$

for all $x, y, z, w, s \in X$, where $t = f(s)$. Then there exists a unique Cauchy-Jensen mapping $F : X \times X \to Y$ such that

$$\|f(x, y) - f(x, 0) - F(x, y), t\| \le \tilde{\varphi}(x, x, y, y, s) \tag{7}$$

for all $x, y, s \in X$, where $t = f(s)$.

Proof. Let $t = f(s)$. Letting $y = x$ in (6), we gain

$$\left\| 2f\left(2x, \frac{z + w}{2}\right) - 2f(x, z) - 2f(x, w), t \right\| \le \varphi(x, x, z, w, s) \tag{8}$$

for all $x, z, w, s \in X$. Putting $w = -z$ in (8), we get

$$\| - 2f(2x, 0) + 2f(x, z) + 2f(x, -z), t\| \le \varphi(x, x, z, -z, s) \tag{9}$$

for all $x, z, s \in X$. Replacing z by $-z$ and w by $-z$ in (8), we have

$$\|f(2x, -z) - 2f(x, -z), t\| \le \frac{1}{2}\varphi(x, x, -z, -z, s) \tag{10}$$

for all $x, z, s \in X$. By (9) and (10),

$$\|f(2x, -z) + 2f(x, z) - 2f(2x, 0), t\| \le \frac{1}{2}\varphi(x, x, -z, -z, s) + \varphi(x, x, z, -z, s) \tag{11}$$

for all $x, z, s \in X$. Setting $w = -3z$ in (8),

$$\|2f(2x, -z) - 2f(x, z) - 2f(x, -3z), t\| \le \varphi(x, x, z, -3z, s)$$

for all $x, z, s \in X$. By (11) and the above inequality,

$$\|6f(x, z) + 2f(x, -3z) - 4f(2x, 0), t\| \le \varphi(x, x, -z, -z, s) + 2\varphi(x, x, z, -z, s) + \varphi(x, x, z, -3z, s) \tag{12}$$

for all $x, z, s \in X$. Replacing z by $3z$ in (10),

$$\|f(2x, -3z) - 2f(x, -3z), t\| \le \frac{1}{2}\varphi(x, x, -3z, -3z, s)$$

for all $x, z, s \in X$. By (12) and the above inequality,

$$\begin{aligned}&\|6f(x,z)+f(2x,-3z)-4f(2x,0),t\|\\&\le \varphi(x,x,-z,-z,s)+2\varphi(x,x,z,-z,s)+\varphi(x,x,z,-3z,s)+\frac{1}{2}\varphi(x,x,-3z,-3z,s)\end{aligned}$$

for all $x, z, s \in X$. Replacing z by $-z$ in the above inequality,

$$\begin{aligned}&\|6f(x,-z)+f(2x,3z)-4f(2x,0),t\|\\&\le \varphi(x,x,z,z,s)+2\varphi(x,x,-z,z,s)+\varphi(x,x,-z,3z,s)+\frac{1}{2}\varphi(x,x,3z,3z,s)\end{aligned}$$

for all $x, z, s \in X$. By (9) and the above inequality,

$$\begin{aligned}&\|6f(x,z)-f(2x,3z)-2f(2x,0),t\|\\&\le \varphi(x,x,z,z,s)+2\varphi(x,x,-z,z,s)+\varphi(x,x,-z,3z,s)+\frac{1}{2}\varphi(x,x,3z,3z,s)+3\varphi(x,x,z,-z,s)\end{aligned}$$

for all $x, z, s \in X$. Replacing x by $2^j x$ and z by $3^j y$ in the above inequality and dividing 6^{j+1},

$$\begin{aligned}&\left\|\frac{1}{6^j}f(2^jx,3^jy)-\frac{1}{6^{j+1}}f(2^{j+1}x,3^{j+1}y)-\frac{2}{6^{j+1}}f(2^{j+1}x,0),t\right\|\\&\le \frac{1}{6^{j+1}}\Big[\varphi(2^jx,2^jx,3^jy,3^jy,s)+2\varphi(2^jx,2^jx,-3^jy,3^jy,s)\\&\quad+\varphi(2^jx,2^jx,-3^jy,3^{j+1}y,s)+\frac{1}{2}\varphi(2^jx,2^jx,3^{j+1}y,3^{j+1}y,s)+3\varphi(2^jx,2^jx,3^jy,-3^jy,s)\Big]\end{aligned}$$

for all $x, y, s \in X$. For given integers $l, m (0 \le l < m)$,

$$\begin{aligned}&\left\|\frac{1}{6^l}f(2^lx,3^ly)-\frac{1}{6^m}f(2^mx,3^my)-\sum_{j=l}^{m-1}\frac{2}{6^{j+1}}f(2^{j+1}x,0),t\right\|\\&\le \sum_{j=l}^{m-1}\frac{1}{6^{j+1}}\Big[\varphi(2^jx,2^jx,3^jy,3^jy,s)+2\varphi(2^jx,2^jx,-3^jy,3^jy,s)\\&\quad+\varphi(2^jx,2^jx,-3^jy,3^{j+1}y,s)+\frac{1}{2}\varphi(2^jx,2^jx,3^{j+1}y,3^{j+1}y,s)+3\varphi(2^jx,2^jx,3^jy,-3^jy,s)\Big]\end{aligned} \tag{13}$$

for all $x, y, s \in X$. By (14) and (13), the sequence $\{\frac{1}{6^j}f(2^jx,3^jy)\}$ is a Cauchy sequence for all $x, y \in X$. Since Y is complete, the sequence $\{\frac{1}{6^j}f(2^jx,3^jy)\}$ converges for all $x, y \in X$. Define $F : X \times X \to Y$ by

$$F(x,y) := \lim_{j\to\infty}\frac{1}{6^j}f(2^jx,3^jy)$$

for all $x, y \in X$.

By (6),

$$\begin{aligned}\Big\| \frac{1}{6^j} f\left(2^j(x+y), \frac{3^j(z+w)}{2}\right) - \frac{1}{6^j} f(2^jx, 3^jz) - \frac{1}{6^j} f(2^jx, 3^jw) \\ - \frac{1}{6^j} f(2^jy, 3^jz) - \frac{1}{6^j} f(2^jy, 3^jw), t\Big\| \le \frac{1}{6^j}\varphi(2^jx, 2^jy, 3^jz, 3^jw, s)\end{aligned}$$

for all $x, y, z, w, s \in X$. Letting $j \to \infty$ and using (14), F satisfies (3). By Theorem 4 in [6], F is a Cauchy-Jensen mapping. Setting $l = 0$ and taking $m \to \infty$ in (13), one can obtain the inequality (7). If $G : X \times X \to Y$ is another Cauchy-Jensen mapping satisfying (7),

$$\begin{aligned}\|F(x,y) - G(x,y), t\| &= \frac{1}{6^n}\|F(2^nx, 3^ny) - G(2^nx, 3^ny), t\| \\ &\le \frac{1}{6^n}\|F(2^nx, 3^ny) - f(2^nx, 0) - f(2^nx, 3^ny), t\| \\ &\quad + \frac{1}{6^n}\|f(2^nx, 2^ny) + f(2^nx, 0) - G(2^nx, 3^ny), t\| \\ &\le \frac{2}{6^n}\tilde{\varphi}(2^nx, 2^nx, 3^ny, 3^ny, s) \to 0 \text{ as } n \to \infty\end{aligned}$$

for all $x, y, s \in X$. Hence the mapping F is the unique Cauchy-Jensen mapping, as desired. □

Corollary 8. *Let $\varepsilon > 0$. Suppose that $f : X \times X \to Y$ is a surjective mapping satisfying*

$$\left\|2f\left(x+y, \frac{z+w}{2}\right) - f(x,z) - f(x,w) - f(y,z) - f(y,w), t\right\| \le \varepsilon,$$

for all $x, y, z, w, s \in X$, where $t = f(s)$ and $\varphi_\varepsilon(x,s) := \frac{3}{2}\varepsilon + \|f(x,0), t\| + \sum_{j=0}^\infty \frac{2}{6^{j+1}}\|f(2^{j+1}x, 0), t\| < \infty$ for all $x, s \in X$, where $t = f(s)$. Then there exists a unique Cauchy-Jensen mapping $F : X \times X \to Y$ such that

$$\|f(x,y) - f(x,0) - F(x,y), t\| \le \varphi_\varepsilon(x,s)$$

for all $x, y, s \in X$, where $t = f(s)$.

Proof. Taking $\varphi(x, y, z, w, s) := \varepsilon$ in Theorem 7, we have

$$\tilde{\varphi}(x, x, y, y, s) = \frac{3}{2}\varepsilon + \|f(x,0), t\| + \sum_{j=0}^\infty \frac{2}{6^{j+1}}\|f(2^{j+1}x, 0), t\| = \varphi_\varepsilon(x,s)$$

for all $x, y, z, w, s \in X$, where $t = f(s)$. □

3. Approximate bi-quadratic mappings

From now on, let $\varphi : X^5 \to [0, \infty)$ be a function satisfying

$$\tilde{\varphi}(x, y, z, w, s) := \sum_{j=0}^\infty \frac{1}{16^{j+1}}\varphi(2^jx, 2^jy, 2^jz, 2^jw, s) < \infty \tag{14}$$

for all $x, y, z, w, s \in X$.

Theorem 9. *Let $f : X \times X \to Y$ be a surjective mapping such that*

$$\begin{aligned}&\|f(x+y,z+w)+f(x+y,z-w)+f(x-y,z+w)+f(x-y,z-w)\\ &-4[f(x,z)-f(x,w)-f(y,z)-f(y,w)],t\| \le \varphi(x,y,z,w,s)\end{aligned} \tag{15}$$

and let $f(x,0)=0$ and $f(0,y)=0$ for all $x,y,z,w,s \in X$, where $t=f(s)$. Then there exists a unique bi-quadratic mapping $F : X \times X \to Y$ such that

$$\|f(x,y)-F(x,y),t\| \le \tilde{\varphi}(x,x,y,y,s) \tag{16}$$

for all $x,y,s \in X$, where $t=f(s)$.

Proof. Let $t=f(s)$. Putting $y=x$ and $w=z$ in (15), we have

$$\left\|f(x,z)-\frac{1}{16}f(2x,2z),t\right\| \le \frac{1}{16}\varphi(x,x,z,z,s)$$

for all $x,z,s \in X$. Thus we obtain

$$\left\|\frac{1}{16^j}f(2^jx,2^jz)-\frac{1}{16^{j+1}}f(2^{j+1}x,2^{j+1}z),t\right\| \le \frac{1}{16^{j+1}}\varphi(2^jx,2^jx,2^jz,2^jz,s)$$

for all $x,z,s \in X$ and all j. Replacing z by y in the above inequality, we see that

$$\left\|\frac{1}{16^j}f(2^jx,2^jy)-\frac{1}{16^{j+1}}f(2^{j+1}x,2^{j+1}y),t\right\| \le \frac{1}{16^{j+1}}\varphi(2^jx,2^jx,2^jy,2^jy,s)$$

for all $x,y,s \in X$ and all j. For given integers $l,m(0 \le l < m)$, we get

$$\left\|\frac{1}{16^l}f(2^lx,2^ly)-\frac{1}{16^m}f(2^mx,2^my),t\right\| \le \sum_{j=l}^{m-1}\frac{1}{16^{j+1}}\varphi(2^jx,2^jx,2^jy,2^jy,s) \tag{17}$$

for all $x,y,s \in X$. By (17), the sequence $\{\frac{1}{16^j}f(2^jx,2^jy)\}$ is a Cauchy sequence for all $x,y \in X$. Since Y is complete, the sequence $\{\frac{1}{16^j}f(2^jx,2^jy)\}$ converges for all $x,y \in X$. Define $F : X \times X \to Y$ by

$$F(x,y) := \lim_{j\to\infty}\frac{1}{16^j}f(2^jx,2^jy)$$

for all $x,y \in X$. By (15), we have

$$\begin{aligned}&\Big\|\frac{1}{16^j}f\big(2^j(x+y),2^j(z+w)\big)+\frac{1}{16^j}f\big(2^j(x+y),2^j(z-w)\big)\\ &+\frac{1}{16^j}f\big(2^j(x-y),2^j(z+w)\big)+\frac{1}{16^j}f\big(2^j(x-y),2^j(z-w)\big)\\ &-\frac{4}{16^j}f(2^jx,2^jz)-\frac{4}{16^j}f(2^jx,2^jw)-\frac{4}{16^j}f(2^jy,2^jz)-\frac{4}{16^j}f(2^jy,2^jw),t\Big\|\\ &\le \frac{1}{16^j}\varphi(2^jx,2^jy,2^jz,2^jw,s)\end{aligned}$$

for all $x,y,z,w,s \in X$ and all j. Letting $j \to \infty$ and using (14), we see that F satisfies (4). By Theorem 4 in [5], we obtain that F is bi-quadratic. Setting $l=0$ and taking $m \to \infty$ in (17), one

can obtain the inequality (16). If $G : X \times X \to Y$ is another bi-quadratic mapping satisfying (16), we obtain

$$\begin{aligned}
&\|F(x,y) - G(x,y), t\| \\
&= \frac{1}{16^n}\|F(2^n x, 2^n y) - G(2^n x, 2^n y), t\| \\
&\leq \frac{1}{16^n}\|F(2^n x, 2^n y) - f(2^n x, 2^n y), t\| + \frac{1}{16^n}\|f(2^n x, 2^n y) - G(2^n x, 2^n y), t\| \\
&\leq \frac{2}{16^n}\tilde{\varphi}(2^n x, 2^n x, 2^n y, 2^n y, s) \\
&\to 0 \text{ as } n \to \infty
\end{aligned}$$

for all $x, y, s \in X$. Hence the mapping F is the unique bi-quadratic mapping, as desired. □

Corollary 10. *Let $\varepsilon > 0$. Suppose that $f : X \times X \to Y$ is a surjective mapping satisfying*

$$\begin{aligned}
&\|f(x+y, z+w) + f(x+y, z-w) + f(x-y, z+w) + f(x-y, z-w) \\
&\quad -4[f(x,z) - f(x,w) - f(y,z) - f(y,w)], t\| \leq \varepsilon,
\end{aligned}$$

for all $x, y, z, w, s \in X$, where $t = f(s)$. Then there exists a unique bi-quadratic mapping $F : X \times X \to Y$ such that

$$\|f(x,y) - F(x,y), t\| \leq \frac{1}{15}\varepsilon$$

for all $x, y, s \in X$, where $t = f(s)$.

Proof. Taking $\varphi(x, y, z, w, s) := \varepsilon$ in Theorem 9, we have $\tilde{\varphi}(x, x, y, y, s) = \frac{1}{15}\varepsilon$ for all $x, y, z, w, s \in X$, where $t = f(s)$. □

References

[1] H.-Y. Chu, A. Kim and J. Park, *On the Hyers-Ulam stabilities of functional equations on n-Banach spaces*, Math. Nachr. **289** (2016), 1177–1188.

[2] S. Gähler, *2-metrische Räume und ihre topologische Struktur*, Math. Nachr. **26** (1963), 115–148.

[3] S. Gähler, *Lineare 2-normierte Räumen*, Math. Nachr. **28** (1964), 1–43.

[4] W.-G. Park, *Approximate additive mappings in 2-Banach spaces and related topics*, J. Math. Anal. Appl. **376** (2011), 193–202.

[5] W.-G. Park and J.-H. Bae, *On a bi-quadratic functional equation and its stability*, Nonlinear Anal. **62** (2005), 643–654.

[6] W.-G. Park and J.-H. Bae, *On a Cauchy-Jensen functional equation and its stability*, J. Math. Anal. Appl. **323** (2006), 634–643.

[7] S. M. Ulam, *A Collection of Mathematical Problems*, Interscience Publishers, New York, 1960.

Won-Gil Park, Department of Mathematics Education, College of Education, Mokwon University, Daejeon 35349, Republic of Korea

E-mail address: wgpark@mokwon.ac.kr

Jae-Hyeong Bae, Humanitas College, Kyung Hee University, Yongin 17104, Republic of Korea

E-mail address: jhbae@khu.ac.kr

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 27, NO.3, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC

Birkhoff Normal Forms, KAM theory and continua of periodic points for certain planar system

M. R. S. Kulenović†[1,2] E. Pilav‡ and N. Mujić‡

†Department of Mathematics
University of Rhode Island, Kingston, Rhode Island 02881-0816, USA

‡Department of Mathematics
University of Sarajevo, Sarajevo, Bosnia and Herzegovina

Abstract. By using the KAM theory and time reversal symmetries we investigate the stability of the equilibrium solutions of the system:

$$\begin{cases} x_{n+1} &= \frac{a}{x_n+y_n} \\ y_{n+1} &= \frac{x_n}{y_n} \end{cases}, \quad n=0,1,2,\ldots,$$

where the parameter $a>0$, and initial conditions x_0 and y_0 are positive numbers. We obtain the Birkhoff normal form for this system and prove the existence of periodic points with arbitrarily large periods in every neighborhood of the unique positive equilibrium. We also use the time reversal symmetry method to find effectively some feasible periods and the corresponding periodic orbits. Finally, we give computational procedure for finding an infinite number of periodic solutions with the given period. The second order difference equation obtained by eliminating x_n from this system is an equation of the type $y_{n+1}=f(y_n,y_{n-1})$, where f is decreasing in both variables. Such equation can be embedded into fifth order difference equation which is increasing in all its arguments and it exhibits chaotic behavior.

Keywords. area preserving map, Birkhoff normal form, difference equation, KAM theory, periodic solutions, symmetry, time reversal, competitive map, global stable manifold, monotonicity, period-two solution.

AMS 2010 Mathematics Subject Classification: 37E40, 37J40, 37N25, 39A28, 39A30

1 Introduction

In this paper we consider the following rational system of difference equations

$$\begin{cases} x_{n+1} &= \frac{a}{x_n+y_n} \\ y_{n+1} &= \frac{x_n}{y_n} \end{cases}, \quad n=0,1,2,\ldots, \tag{1}$$

and the corresponding equation

$$y_{n+1}=\frac{a}{y_n y_{n-1}(1+y_n)}, \quad n=0,1,2,\ldots, \tag{2}$$

where the parameter $a>0$, and initial conditions x_0 and y_0 are positive numbers. System (1) was first considered in [6], where boundedness of all its solutions was proved using the invariant. We will use this invariant in Section 3 to prove the stability of the unique equilibrium. Equation (2) gives an example of second order difference equation where transition function decreases in both variables and yet equation exhibits complicated dynamics. First such example was given in [5]. We will use similar techniques as in [5] with the addition of the new computational procedure from [7], which uses an invariant of the system to find effectively continua of periodic solutions of certain feasible periods.

We will show that the corresponding map can be transformed into an area preserving map and using Birkhoff Normal form we will apply the KAM theorem to prove stability of the unique positive equilibrium and the existence of periodic points with arbitrarily large period in every neighborhood of the unique positive equilibrium. In addition, we will prove that the corresponding map is conjugate to its inverse map through the involution map and then use this conjugacy to find some feasible periods of this map. The method of invariants for proving stability of the equilibrium solution for all values of parameter a will be used along with Morse's lemma to prove that the level sets of the invariants are diffeomorphic with circles. This method was used successfully in [11, 12] and the KAM theory was used for the same objective in [8, 10, 13, 14].

Let T be the map associated to the system (1), i.e.,

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{a}{x+y} \\ \frac{x}{y} \end{pmatrix}. \tag{3}$$

[1]Corresponding author, *e-mail: mkulenovic@uri.edu*

[2]Partially supported by Maitland P. Simmons Foundation

The map (3) has the unique fixed point $(\bar{y}^2, \bar{y})$ in the positive quadrant, where

$$\bar{y}^3(\bar{y}+1) = a.$$

An invertible mapping T is area preserving if the area of $T(A)$ coincides with the area of A for all measurable subsets A [9, 12, 18]. We claim that in logarithmic coordinates, i.e., $u = \ln(x/\bar{y}^2)$, $v = \ln(y/\bar{y})$ the map (3) is area preserving.

Lemma 1 *The map* (3) *is area preserving in the logarithmic coordinates.*

Proof. The Jacobian matrix of the corresponding transformation T is

$$J_T(x,y) = \begin{pmatrix} -\frac{a}{(x+y)^2} & -\frac{a}{(x+y)^2} \\ \frac{1}{y} & -\frac{x}{y^2} \end{pmatrix} \tag{4}$$

with

$$det J_T(x,y) = \frac{a}{y^2(x+y)}.$$

We substitute $u = \ln(x/\bar{y}^2)$, $v = \ln(y/\bar{y})$ and rewrite the map in (u, v) coordinates to obtain the transformation

$$\begin{pmatrix} u \\ v \end{pmatrix} \to \begin{pmatrix} \ln a - 3\ln\bar{y} - \ln(\bar{y}e^u + e^v) \\ u - v \end{pmatrix} \tag{5}$$

The Jacobian of this transformation is

$$J(u,v) = \begin{pmatrix} -\frac{e^u\bar{y}}{e^u\bar{y}+e^v} & -\frac{e^v}{e^u\bar{y}+e^v} \\ 1 & -1 \end{pmatrix} \tag{6}$$

It is easy to see that $det J(u, v) = 1$. □

A point $(\bar{x}, \bar{y})$ is a fixed point of T if $T(\bar{x}, \bar{y}) = (\bar{x}, \bar{y})$. A fixed point is elliptic if the eigenvalues of $J_T(\bar{x}, \bar{y})$ form a complex conjugate pair $\lambda, \bar{\lambda}$ on the unit circle and is hyperbolic if the the modulus of the eigenvalues is different from 1, see [9, 12].

Lemma 2 *The map T in the (x, y) coordinates has elliptic fixed point $(\bar{y}^2, \bar{y})$. In the logarithmic coordinates, the corresponding fixed points is* $(0, 0)$.

Proof. For the fixed points in (x, y) coordinates, solving $a/(\bar{x}+\bar{y}) = x$ and $\bar{x}/\bar{y} = \bar{y}$ yields the fixed points $(\bar{y}^2, \bar{y})$ where $\bar{y}^3(\bar{y}+1) = a$. Evaluating the Jacobian matrix (4) of T at $(\bar{y}^2, \bar{y})$ gives

$$J_T(\bar{y}^2, \bar{y}) = \begin{pmatrix} -\frac{a}{(\bar{y}^2+\bar{y})^2} & -\frac{a}{(\bar{y}^2+\bar{y})^2} \\ \frac{1}{\bar{y}} & -1 \end{pmatrix} \tag{7}$$

By using $a = \bar{y}^3(1+\bar{y})$ we obtain that the eigenvalues of $J_T(\bar{y}^2, \bar{y})$ are λ and $\bar{\lambda}$ where

$$\lambda = \frac{-1 - 2\bar{y} + i\sqrt{4\bar{y}+3}}{2\bar{y}+2}. \tag{8}$$

It is easy to see that $|\lambda| = 1$ and so $(\bar{y}^2, \bar{y})$ is an elliptic fixed point.

Under the logarithmic coordinate change $(x, y) \to (u, v)$, the fixed point $(\bar{y}^2, \bar{y})$ becomes $(0, 0)$. Evaluating the Jacobian matrix (6) of T at $(0, 0)$ gives

$$J(0,0) = \begin{pmatrix} -\frac{\bar{y}}{\bar{y}+1} & -\frac{1}{\bar{y}+1} \\ 1 & -1 \end{pmatrix} \tag{9}$$

with eigenvalues which are given by (8). □

This paper is organized as follows. In section 2 the KAM theorem is explained in some detail and Birkhoff normal form for map T is derived. By using the KAM theory stability of the unique equilibrium and existence of infinite number of periodic solution is proven except for a single value of the parameter a. Section 3 uses the invariant of the equation (2) in proving stability for all values of a. In section 4 by using symmetries it is shown that the map T is conjugate to its inverse through an involution. Then by using time reversal symmetry method some feasible periods and corresponding orbits of the map T are found. Finally in Section 5 we use the recent method of Gasull and *al.* [7] to find continua of p-periodic points lying on the level sets of the invariant I. The method is based on use of resultants and is implemented by *Mathematica*. The special attention is given to period-seven solution.

2 The KAM theory and Birkhoff normal form

The KAM Theorem asserts that in any sufficiently small neighborhood of a non degenerate elliptic fixed point of a smooth area-preserving map there exists many invariant closed curves.We explain this theorem in some detail. Consider a smooth, area-preserving mapping $(x, y) \to T(x, y)$ of the plane that has $(0, 0)$ as an elliptic fixed point. After a linear transformation one can put the map in the form

$$z \to \lambda z + g(z, \bar{z})$$

where λ is the eigenvalue of the elliptic fixed point, $z = x + iy$ and $\bar{z} = x - iy$ are complex variables, and g vanishes with its derivative at $z = 0$. Assume that the eigenvalue λ of the elliptic fixed point satisfies the non-resonance condition $\lambda^k \neq 1$ for $k = 1, \ldots, q$, for some $q \geq 4$. Then Birkhoff showed that there exist new, canonical complex coordinates $(\zeta, \bar{\zeta})$ relative to which the mapping takes the normal form

$$\zeta \to \lambda \zeta e^{i\tau(\zeta\bar{\zeta})} + h(\zeta, \bar{\zeta})$$

in a neighborhood of the elliptic fixed point, where $\tau(\zeta\bar{\zeta}) = \tau_1|\zeta|^2 + \ldots + \tau_s|\zeta|^{2s}$ is a real polynomial, $s = [(q-2)/2]$, and h vanishes with its derivatives up to order $q-1$. The numbers $\tau_1, \ldots, \tau_s$ are called twist coefficients. Consider an invariant annulus $\epsilon < |\zeta| < 2\epsilon$ in a neighborhood of the elliptic fixed point, for ϵ a very small positive number. Note that under the neglect of the remainder h, the normal form approximation $\zeta \to \lambda\zeta e^{i\tau(\zeta\bar{\zeta})}$ leaves invariant all circles $|\zeta|^2 = const.$ The motion restricted to each of these circles is a rotation by some angle. Also note that if at least one of the twist coefficients τ_j is nonzero, the angle of rotation will vary from circle to circle. A radial line through the fixed point will undergo twisting under the mapping. The KAM theorem (Moser's twist theorem) says that, under the addition of the remainder term, most of these invariant circles will survive as invariant closed curves under the full map.

Theorem 1 *Assuming that $\tau(\zeta\bar{\zeta})$ is not identically zero and ϵ is sufficiently small, then the map T has a set of invariant closed curves of positive Lebesque measure close to the original invariant circles. Moreover the relative measure of the set of surviving invariant curves approaches full measure as ϵ approaches 0. The surviving invariant closed curves are filled with dense irrational orbits.*

The KAM theorem requires that the elliptic fixed point be non-resonant and non degenerate. Note that for $q = 4$ the non-resonance condition $\lambda^k \neq 1$ requires that $\lambda \neq \pm 1$ or $\lambda \neq \pm i$. The above normal form yields the approximation

$$\zeta \to \lambda\zeta + c_1\zeta^2\bar{\zeta} + O(|\zeta|^4)$$

with $c_1 = i\lambda\tau_1$ and τ_1 being the first twist coefficient. We will call an elliptic fixed point non-degenerate if $\tau_1 \neq 0$.

Consider a general map T that has a fixed point at the origin with complex eigenvalues λ and $\bar{\lambda}$ satisfying $|\lambda| = 1$ and $Im(\lambda) \neq 0$. By putting the linear part of such a map into Jordan Canonical form, we may assume T to have the following form near the origin

$$T\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} Re(\lambda) & -Im(\lambda) \\ Im(\lambda) & Re(\lambda) \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} g_1(x_1, x_2) \\ g_2(x_1, x_2) \end{pmatrix} \tag{10}$$

One can now switch to the complex coordinates $z = x_1 + ix_2$ to obtain the complex form of the system

$$z \to \lambda z + \xi_{20}z^2 + \xi_{11}z\bar{z} + \xi_{02}\bar{z}^2 + \xi_{30}z^3 + \xi_{21}z^2\bar{z} + \xi_{12}z\bar{z}^2 + \xi_{03}\bar{z}^3 + O(|z|^4)$$

The coefficient c_1 can be computed directly using the formula below derived by Wan in the context of Hopf bifurcation theory [19]. In [16] it is shown that when one uses area-preserving coordinate changes this formula by Wan yields the twist coefficient τ_1 that is used to verify the non-degeneracy condition necessary to apply the KAM theorem. We use the formula:

$$c_1 = \frac{\xi_{20}\xi_{11}(\bar{\lambda} + 2\lambda - 3)}{(\lambda^2 - \lambda)(\bar{\lambda} - 1)} + \frac{|\xi_{11}|^2}{1 - \bar{\lambda}} + \frac{2|\xi_{02}|^2}{\lambda^2 - \bar{\lambda}} + \xi_{21} \tag{11}$$

where

$$\xi_{20} = \frac{1}{8}\left\{(g_1)_{x_1x_1} - (g_1)_{x_2x_2} + 2(g_2)_{x_1x_2} + i\left[(g_2)_{x_1x_1} - (g_2)_{x_2x_2} - 2(g_1)_{x_1x_2}\right]\right\},$$

$$\xi_{11} = \frac{1}{4}\left\{(g_1)_{x_1x_1} + (g_1)_{x_2x_2} + i\left[(g_2)_{x_1x_1} + (g_2)_{x_2x_2}\right]\right\},$$

$$\xi_{02} = \frac{1}{8}\left\{(g_1)_{x_1x_1} - (g_1)_{x_2x_2} - 2(g_2)_{x_1x_2} + i\left[(g_2)_{x_1x_1} - (g_2)_{x_2x_2} + 2(g_1)_{x_1x_2}\right]\right\},$$

$$\xi_{21} = \frac{1}{16}\left\{(g_1)_{x_1x_1x_1} + (g_1)_{x_1x_2x_2} + (g_2)_{x_1x_1x_2} + (g_2)_{x_2x_2x_2} + i\left[(g_2)_{x_1x_1x_1} + (g_2)_{x_1x_2x_2} - (g_1)_{x_1x_1x_2} - (g_1)_{x_2x_2x_2}\right]\right\}.$$

Theorem 2 *The elliptic fixed point $(0,0)$, in the (u,v) coordinates, is non-degenerate for $a \neq \frac{3}{16}$ and non-resonant for $a > 0$.*

Proof.
Let F be the function defined by

$$F\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \ln a - 3\ln\bar{y} - \ln(\bar{y}e^u + e^v) \\ u - v \end{pmatrix}. \tag{12}$$

Then F has the unique elliptic fixed point $(0,0)$. The Jacobian matrix of F is given by

$$J_F(u,v) = \begin{pmatrix} -\frac{e^u\bar{y}}{e^u\bar{y}+e^v} & -\frac{e^v}{e^u\bar{y}+e^v} \\ 1 & -1 \end{pmatrix}. \tag{13}$$

At $(0,0)$, $J_F(u,v)$ has the form

$$J_0 = J_F(0,0) = \begin{pmatrix} -\frac{\bar{y}}{\bar{y}+1} & -\frac{1}{\bar{y}+1} \\ 1 & -1 \end{pmatrix}. \tag{14}$$

The eigenvalues of (14) are λ and $\bar{\lambda}$ where

$$\lambda = \frac{-1-2y+i\sqrt{4y+3}}{2y+2}. \tag{15}$$

One can prove that

$$\begin{aligned} &|\lambda| = 1, \\ \lambda^2 =& \frac{2\bar{y}^2-1}{2(\bar{y}+1)^2} - \frac{i(2\bar{y}+1)\sqrt{4\bar{y}+3}}{2(\bar{y}+1)^2}, \\ \lambda^3 =& \frac{\bar{y}((3-2\bar{y})\bar{y}+6)+2}{2(\bar{y}+1)^3} + \frac{i\bar{y}\sqrt{4\bar{y}+3}(3\bar{y}+2)}{2(\bar{y}+1)^3}, \\ \lambda^4 =& \frac{2\bar{y}(\bar{y}((\bar{y}-4)\bar{y}-8)-4)-1}{2(\bar{y}+1)^4} - \frac{i(2\bar{y}+1)\sqrt{4\bar{y}+3}(2\bar{y}^2-1)}{2(\bar{y}+1)^4}, \end{aligned} \tag{16}$$

from which follows that $\lambda^k \neq 1$ for $k = 1,2,3,4$ and $a > 0$.
Then we have that

$$F\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} -\frac{\bar{y}}{\bar{y}+1} & -\frac{1}{\bar{y}+1} \\ 1 & -1 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} f_1(\delta,u,v) \\ f_2(\delta,u,v) \end{pmatrix}, \tag{17}$$

where

$$\begin{aligned} f_1(\delta,u,v) =& -\ln(e^u\bar{y}+e^v) + \frac{u\bar{y}}{\bar{y}+1} + \frac{v}{\bar{y}+1} - 3\ln\bar{y} + \ln a \\ f_2(\delta,u,v) =& 0. \end{aligned} \tag{18}$$

The system $(u_{n+1}, v_{n+1}) = F(u_n, v_n)$ takes the form

$$\begin{pmatrix} u_{n+1} \\ v_{n+1} \end{pmatrix} = \begin{pmatrix} -\frac{\bar{y}}{\bar{y}+1} & -\frac{1}{\bar{y}+1} \\ 1 & -1 \end{pmatrix}\begin{pmatrix} u_n \\ v_n \end{pmatrix} + \begin{pmatrix} f_1(u_n,v_n) \\ f_2(u_n,v_n) \end{pmatrix}, \tag{19}$$

Let

$$\begin{pmatrix} u_n \\ v_n \end{pmatrix} = P\begin{pmatrix} \tilde{u}_n \\ \tilde{v}_n \end{pmatrix},$$

where

$$P = \frac{1}{\sqrt{D}}\begin{pmatrix} \frac{1}{2\bar{y}+2} & -\frac{\sqrt{4\bar{y}+3}}{2\bar{y}+2} \\ 1 & 0 \end{pmatrix}$$

and

$$P^{-1} = \sqrt{D}\begin{pmatrix} 0 & 1 \\ -\frac{2\bar{y}+2}{\sqrt{4\bar{y}+3}} & \frac{1}{\sqrt{4\bar{y}+3}} \end{pmatrix},$$

with

$$D = \frac{\sqrt{4\bar{y}+3}}{2\bar{y}+2}.$$

Then the system $(u_{n+1}, v_{n+1}) = F(u_n, v_n)$ becomes

$$\begin{pmatrix} \tilde{u}_{n+1} \\ \tilde{v}_{n+1} \end{pmatrix} = \begin{pmatrix} \frac{-2\bar{y}-1}{2\bar{y}+2} & -\frac{\sqrt{4\bar{y}+3}}{2\bar{y}+2} \\ \frac{\sqrt{4\bar{y}+3}}{2\bar{y}+2} & \frac{-2\bar{y}-1}{2\bar{y}+2} \end{pmatrix}\begin{pmatrix} \tilde{u}_n \\ \tilde{v}_n \end{pmatrix} + P^{-1}H\left(P\begin{pmatrix} \tilde{u}_n \\ \tilde{v}_n \end{pmatrix}\right), \tag{20}$$

where

$$H\begin{pmatrix}u\\v\end{pmatrix} := \begin{pmatrix}f_1(u,v)\\f_2(u,v)\end{pmatrix}.$$

Let

$$G\begin{pmatrix}u\\v\end{pmatrix} = \begin{pmatrix}g_1(u,v)\\g_2(u,v)\end{pmatrix} = P^{-1}H\left(P\begin{pmatrix}u\\v\end{pmatrix}\right).$$

The straightforward calculation yields

$$\begin{aligned}
g_1(u,v) =&0\\
g_2(u,v) =& -\frac{1}{\sqrt{D}}\ln\left(\frac{a}{\bar{y}^3}\right) - \frac{\bar{y}\left(u-2Dv\left(\bar{y}+1\right)\right)}{2D\left(\bar{y}+1\right)^2} + \frac{1}{\sqrt{D}}\ln\left(\bar{y}e^{\frac{u-2Dv(\bar{y}+1)}{2\sqrt{D}(\bar{y}+1)}} + e^{\frac{u}{\sqrt{D}}}\right) - \frac{u}{D\left(\bar{y}+1\right)}.
\end{aligned} \tag{21}$$

By straightforward calculation we obtain that

$$\begin{aligned}
\xi_{20}|_{u=v=0} =&\frac{\bar{y}\left(2\bar{y}\left(\sqrt{4\bar{y}+3}+i\bar{y}\right)+\sqrt{4\bar{y}+3}-i\right)}{8\left(\bar{y}+1\right)^3\sqrt{D\left(4\bar{y}+3\right)}},\\
\xi_{11}|_{u=v=0} =&\frac{i\bar{y}}{2\left(\bar{y}+1\right)\sqrt{D\left(4\bar{y}+3\right)}},\\
\xi_{02}|_{u=v=0} =&\frac{i\bar{y}\left(2\bar{y}\left(\bar{y}+i\sqrt{4\bar{y}+3}\right)+i\sqrt{4\bar{y}+3}-1\right)}{8\left(\bar{y}+1\right)^3\sqrt{D\left(4\bar{y}+3\right)}},\\
\xi_{21}|_{u=v=0} =&\frac{\left(\bar{y}-1\right)\bar{y}\left(2i\bar{y}+\sqrt{4\bar{y}+3}+i\right)}{16D\left(\bar{y}+1\right)^3\sqrt{4\bar{y}+3}}.
\end{aligned} \tag{22}$$

Since

$$\begin{aligned}
\xi_{21}\xi_{11} =&\frac{i\bar{y}^2\left(2\bar{y}\left(\sqrt{4\bar{y}+3}+i\bar{y}\right)+\sqrt{4\bar{y}+3}-i\right)}{16D\left(\bar{y}+1\right)^4\left(4\bar{y}+3\right)},\\
\xi_{11}\overline{\xi_{11}} =&\frac{\bar{y}^2}{4D\left(\bar{y}+1\right)^2\left(4\bar{y}+3\right)},\\
\xi_{02}\overline{\xi_{02}} =&\frac{\bar{y}^2}{16D\left(\bar{y}+1\right)^2\left(4\bar{y}+3\right)}.
\end{aligned} \tag{23}$$

the simplification of the expression for c_1 yields

$$\begin{aligned}
c_1 &= \frac{\xi_{20}\xi_{11}(\bar{\lambda}+2\lambda-3)}{(\lambda^2-\lambda)(\bar{\lambda}-1)} + \frac{|\xi_{11}|^2}{1-\bar{\lambda}} + \frac{2|\xi_{02}|^2}{\lambda^2-\bar{\lambda}} + \xi_{21}\\
&= \frac{\bar{y}\left(2\bar{y}-1\right)\left(2\bar{y}+2\right)\left(2i\bar{y}+\sqrt{4\bar{y}+3}+i\right)}{8\left(\bar{y}+1\right)^2\left(4\bar{y}+3\right)^2}
\end{aligned} \tag{24}$$

One can prove that

$$\tau_1 = -i\bar{\lambda}c_1 = -\frac{\bar{y}\left(2\bar{y}-1\right)}{2\left(4\bar{y}+3\right)^2},$$

which implies that $\tau_1 \neq 0$ for $a \neq \frac{3}{16}$ since $\bar{y}^2(1+\bar{y}) = a$.

□

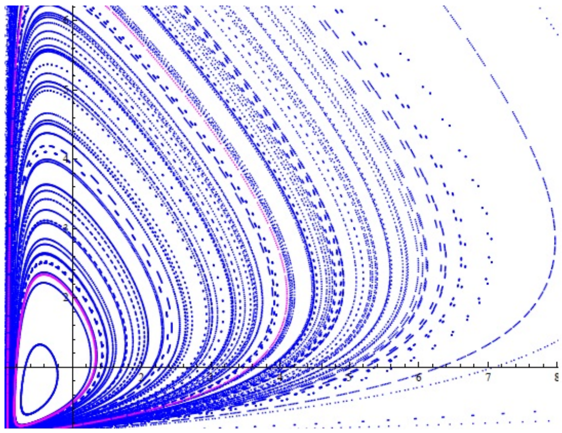
The following result is a consequence of Moser's twist map theorem [8, 15, 17, 18].

Theorem 3 *Let T be a map (3) associated to the system (1), and $(\bar{x}, \bar{y})$ a non-degenerate elliptic fixed point. If $a \neq \frac{3}{16}$ then there exist periodic points with arbitrarily large period in every neighbourhood of $(\bar{x}, \bar{y})$. In adition, $(\bar{x}, \bar{y})$ is a stable fixed point.*

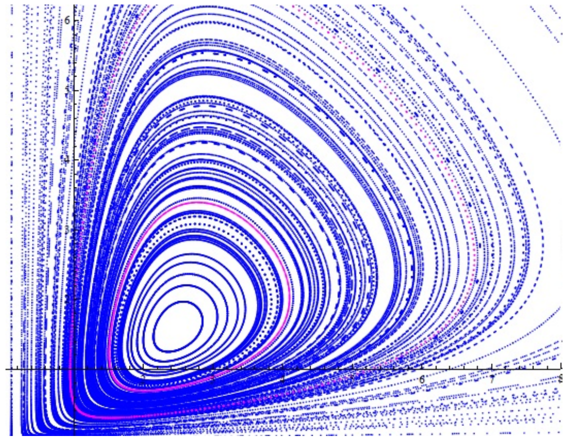
3 Invariant

In this section we prove that the restriction $a \neq \frac{3}{16}$ is not necessary for stability of the equilibrium solution.

The system (1) possesses the invariant given by



(a)



(b)

Figure 1: Some orbits of the map T for (a) $a = 0.5$ and (b) $a = 10.0$

$$I(x_n, y_n) = x_n + y_n + \frac{a}{x_n} + \frac{x_n}{y_n}. \tag{25}$$

Indeed, it is easy to see that I is continuous and that $I(x_{n+1}, y_{n+1}) = I(x_n, y_n)$. In this section we use the invariant I to find a Lyapunov function and prove stability of the equilibrium point for all values of parameter $a > 0$, see[11, 12].

The partial derivatives of the function $I(x, y)$ are given with

$$\begin{aligned} \frac{\partial I}{\partial x} &= -\frac{a}{x^2} + \frac{1}{y} + 1, \\ \frac{\partial I}{\partial y} &= 1 - \frac{x}{y^2}. \end{aligned} \tag{26}$$

The unique positive equilibrium of (1) satisfies that $\bar{x} = \bar{y}^2$ and $\bar{y}^3(\bar{y} + 1) = a$. Equation (26) implies that any critical point (x, y) of (25) satisfies the system

$$\begin{aligned} x &= y^2 \\ y^4 + y^3 &= a. \end{aligned}$$

Hance, $(\bar{y}^2, \bar{y})$ of (1) is the unique positive solution of this system and $(\bar{y}^2, \bar{y})$ is critical point of the invariant (25). Thus the unique equilibrium $(\bar{y}^2, \bar{y})$ is critical point of the invariant (25).

Lemma 3 *The graph of the function $I(x, y)$ associated with* (25) *is a simple closed curve in a neighborhood of the equilibrium point of* (1). *The equilibrium point $(\bar{y}^2, \bar{y})$ is stabile.*

Proof. The Hessian matrix assosiated with $I(x, y)$ is

$$H(x, y) = \begin{pmatrix} \frac{2a}{x^3} & -\frac{1}{y^2} \\ -\frac{1}{y^2} & \frac{2x}{y^3} \end{pmatrix}$$

with determinant

$$\det(H(x, y)) = \frac{4ay - x^2}{x^2 y^4}.$$

For the equilibrium $(\bar{y}^2, \bar{y})$ we have

$$\det(H(\bar{y}^2, \bar{y})) = \frac{4a\bar{y} - \bar{x}^2}{\bar{x}^2 \bar{y}^4} = \frac{4\bar{y}(\bar{y}^4 + \bar{y}^3) - \bar{y}^4}{\bar{y}^8} = \frac{4\bar{y}^5 + 3\bar{y}^4}{\bar{y}^8} = \frac{4\bar{y} + 3}{\bar{y}^4} > 0$$

Thus, in view of Morse's lema, [9], the level sets of the function $I(x, y)$ are diffeomorphic to circles in the neighborhood of $(\bar{x}, \bar{y})$. In adition, the function

$$V(x, y) = I(x, y) - I(\bar{x}, \bar{y})$$

is Lyapunov function, and so the equilibrium point $(\bar{x}, \bar{y})$ is stable, see [11]. □

4 Symmetries

In this section we will show that mat T is conjugate to its inverse map and use this conjugacy to find some feasible periods of T and corresponding periodic orbits. A transformation R of the plane is said to be a time reversal symmetry for T if

$$R^{-1} \circ T \circ R = T^{-1}.$$

If the time reversal symmetry R is an involution, i.e. $R^2 = I$, where I is identity map then the time reversal symmetry condition is equivalent to

$$R \circ T \circ R = T^{-1},$$

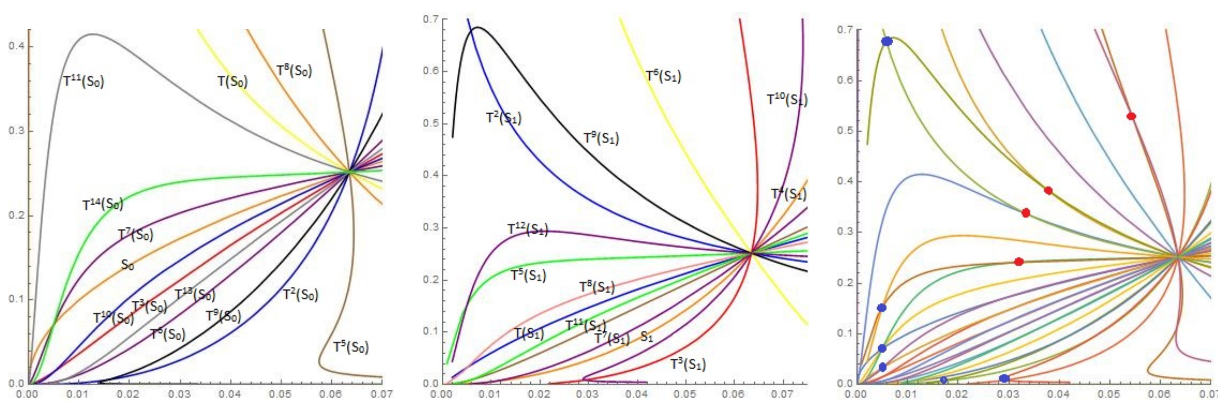
and T can be written as the composition of two involutions $T = I_1 \circ I_0$ where $I_0 = R$ and $I_1 = T \circ R$. Let us note here that if $I_0 = R$ is reversor then so is $I_1 = T \circ R$. Also, the jth involution defined as $I_j = T^j \circ R$ is also a reversor.
The invariant sets of the involution maps

$$S_{0,1} = \{(x, y) | I_{0,1}(x, y) = (x, y)\}$$

are one-dimensional sets called the symmetry lines of the map. When the sets $S_{0,1}$ are known the search for periodic orbits can be reduced to one-dimensional root finding problem using the following result, see [2, 8]

Theorem 4 *If $(x, y) \in S_{0,1}$ then $T^n(x, y) = (x, y)$ if and only if*

$$\begin{cases} T^{n/2}(x, y) \in S_{0,1}, & \text{for } n \text{ even} \\ T^{(n\pm1)/2}(x, y) \in S_{1,0}, & \text{for } n \text{ odd.} \end{cases}$$



(a) (b) (c)

Figure 2: a) The first fourteen iterations of symmetry line S_0 of the map T for $a = 0.02$ (b) The first twelve iterations of symmetry line S_1 of the map T for $a = 0.02$ (c) The periodic orbits of period 14 (blue) and 17 (red)

The inverse map of the map T is

$$T^{-1}(x, y) = \left(\frac{ay}{x(y+1)}, \frac{a}{x(y+1)} \right).$$

The involution $R = \left(x, \frac{x}{y}\right)$ is reversor for T. Indeed,

$$(R \circ T \circ R)(x, y) = (R \circ T)\left(x, \frac{x}{y}\right) = R\left(\frac{ay}{x(y+1)}, y\right) = \left(\frac{ay}{x(y+1)}, \frac{a}{x(y+1)}\right) = T^{-1}(x, y).$$

Thus $T = I_1 \circ I_0$ where $I_0(x, y) = R(x, y)$ and

$$I_1(x, y) = T \circ R = \left(\frac{ay}{x(y+1)}, y\right).$$

The symmetry lines corresponding to I_0 and I_1 are

$$S_0 = \{(x, y) : x = y^2\}, \quad S_1 = \{(x, y) : ay = x^2(y+1)\}.$$

Periodic orbits of different orders can be found at the intersection of the symmetry lines S_j, $j = 1, 2, ...$ associated to the jth involution. So if $(x, y) \in S_j \cap S_k$ then $T^{j-k}(x, y) = (x, y)$. The symmetry lines are also related to each other by the relation

$$S_{2j+i} = T^j(S_i), \quad S_{2j-i} = I_j(S_i), \quad \forall i, j.$$

Now we start with the point $(x_0, \sqrt{x_0}) \in S_0$ in search for periodic orbits on the symmetry line S_0 with even period n and impose that $(x_{n/2}, y_{n/2}) \in S_0$, where

$$(x_{n/2}, y_{n/2}) = T^{n/2}(x_0, \sqrt{x_0}).$$

This reduces to one-dimensional root finding for the equation $x_{n/2} = y_{n/2}^2$, where the unknown is x_0.

Periodic orbits on S_0 with odd period n are obtained by solving for x_0 the equation

$$a y_{(n+1)/2} = x_{(n+1)/2}^2 (y_{(n+1)/2} + 1),$$

where

$$(x_{(n+1)/2}, y_{(n+1)/2}) = T^{(n+1)/2}(x_0, \sqrt{x_0}).$$

For example, for $a = 0.02$ in Figure 2 we have an intersection between the symmetry lines S_0 and $S_{14} = T^7(S_0)$, $S_4 = T^2(S_0)$ and $S_{18} = T^9(S_0)$, $S_5 = T^2(S_1)$ and $S_{19} = T^9(S_1)$, and $S_{11} = T^5(S_1)$ and $S_{25} = T^{12}(S_1)$ of the map T. The intersection points of these lines correspond to the periodic orbits of period 14.

5 Continua of periodic points for map T

In this section we use resultants and technique from [7] for finding continua of p-periodic points lying on the level sets of the invariant I.

Let

$$T^p(x, y) = (T_1^p(x, y), T_2^p(x, y)).$$

The idea is to find the values of h for which the system

$$\begin{aligned} T_1^p(x, y) &= x \\ I(x, y) &= h \end{aligned} \tag{27}$$

has continua of solutions. Let

$$F(y, h) := Res(\text{numerator } (T_1^p(x, y) - x), \text{ numerator } (I(x, y) - h)), \tag{28}$$

where Res denote the resultant of corresponding expressions. The values of h have to be such that $F(y, h)$ vanishes identically. We need to collect the factors of the above resultant that only depend on h. Denote by $D_p(a, h)$ the product of these factors. We introduce the functions $d_p(a, h)$ as those factors of $D_p(a, h)$ that remain after removing from this polynomial all the factors that already appear in some $D_k(a, h)$ where k is either 1 or a proper divisor of p. We call the conditions $d_p(a, h) = 0$ the resultant p-periodicity conditions associated to the invariant I (RPC from now on), see [7]. The main fact is that the energy levels filled with periodic points must satisfy the RPC what gives us the necessary condition for periodic point because the resultant (28) can contain some spurious factors. We will prove in our examples that the RPC we obtain actually give continua of p-periodic points.

Theorem 5 *The RPC of the map T associated to the invariant I for $p \leq 10$ are given by $d_p(a, h) = 0$, where:*

$$\begin{aligned}
d_2(a, h) &= a \\
d_3(a, h) &= 1 \\
d_4(a, h) &= 1 + h \\
d_5(a, h) &= a - h - 1 \\
d_6(a, h) &= a - h^2 - 3h - 2 \\
d_7(a, h) &= a^2 - ah - a - h^3 - 3h^2 - 3h - 1 \\
d_8(a, h) &= 2a^2 - ah^2 - 5ah - 4a + h^2 + 2h + 1 \\
d_9(a, h) &= -3 + 4a - 3a^2 + a^3 - 12h + 9ah - 3a^2h \\
&\quad - 19h^2 + 6ah^2 - 15h^3 + ah^3 - 6h^4 - h^5 \\
d_{10}(a, h) &= 1 + 5a - 5a^2 + a^3 + 5h + 15ah - 8a^2h + 10h^2 \\
&\quad + 16ah^2 - 3a^2h^2 + 10h^3 + 7ah^3 + 5h^4 + ah^4 + h^5
\end{aligned}$$

Proof. For $p = 2$ we obtain

$$numerator(T_1^2(x,y) - x) = -x^3 - x^2y + ay^2,$$

and

$$Res(numerator(T_1^2(x,y) - x), numerator(I(x,y) - h), x) = ay^3(a^2 + 4ay + 4ahy + 4ay^2 + 3ahy^2 - h^2y^2 - h^3y^2 + 2ay^3 + 2hy^3 + 2h^2y^3 - y^4 + ay^4 - hy^4).$$

So there is no factor of resultant without dependence on the variable y that can be equal to zero since $a > 0$ so we can ensure there are no energy levels formed by continua of period-two points. We come to the same conclusion for $p = 3$, so we continue with $p = 4$ where we have

$$numerator(T_1^4(x,y) - x) = -x^7 + ax^4y - ax^5y - 4x^6y - x^7y + 2a^2x^2y^2 + 2ax^3y^2 - 2ax^4y^2 - 6x^5y^2 - 4x^6y^2 + a^3y^3 + 2a^2xy^3 + ax^2y^3 + a^2x^2y^3 - ax^3y^3 - 4x^4y^3 - 6x^5y^3 + a^2xy^4 - x^3y^4 - 4x^4y^4 - x^3y^5,$$

and

$$Res(numerator(T_1^4(x,y) - x), numerator(I(x,y) - h), x) = a^3(1+h)^2y^8(1+y)$$
$$(a - 2a^2 + a^3 + 2ah - 2a^2h + ah^2 + 8ay - 4a^2y + 20ahy - 4a^2hy + 16ah^2y + 4ah^3y + 8ay^2 - 4a^2y^2 - hy^2 + 22ahy^2 - 5a^2hy^2 - 4h^2y^2 + 19ah^2y^2 - 6h^3y^2 + 5ah^3y^2 - 4h^4y^2 - h^5y^2 + 2y^3 - 4ay^3 + 2a^2y^3 + 8hy^3 - 6ahy^3 + 12h^2y^3 - 2ah^2y^3 + 8h^3y^3 + 2h^4y^3 - y^4 - ay^4 + a^2y^4 - 3hy^4 - ahy^4 - 3h^2y^4 - h^3y^4).$$

The only factor independent of y is $1 + h$ which gives $d_4(a,h) = 1 + h$. In an analogous way we compute $d_5(a,h)$ and $d_6(a,h)$. For $p = 7$ we consider the equation

$$T^4(x,y) = T^{-3}(x,y)$$

so we obtain

$$Res(numerator(T_1^4(x,y) - T_1^{-3}(x,y)), numerator(I(x,y) - h), x) = a^4(-1 - a + a^2 - 3h - ah - 3h^2 - h^3)^2y^{11}(1+y)^3(a + 4ay + 4ay^2 - hy^2 - h^2y^2 + 2y^3 + 2hy^3),$$

and $d_7(a,h) = -1 - a + a^2 - 3h - ah - 3h^2 - h^3$. The computation of $d_8(a,h), d_9(a,h)$ and $d_{10}(a,h)$ is analogous to the previous computation. □

Let us now determine the feasibility region $\mathcal{R}$ of a map T, that is those pairs $(a,h) \in \mathbb{R}^2$ that satisfy the condition

$$\{I(x,y) = h\} \cap \mathbb{R}^2 = \{x^2 + ay - hxy + x^2y + xy^2 = 0\} \cap \mathbb{R}^2 \neq 0.$$

From the property of the invariant I in Lemma 3 we obtain that the equilibrium point $(\bar{y}^2, \bar{y})$ is the absolute minimum of the invariant (25). Let us denote the value of I in the absolute minimum with

$$h_c(\bar{y}) = I(\bar{y}^2, \bar{y}) = \bar{y}(2\bar{y} + 3)$$

and therefore the region

$$\mathcal{R} = \{(a,h), a > 0 \text{ and } h \geq h_c(\bar{y}) \text{ and } a = \bar{y}^3(\bar{y} + 1)\}$$

is a feasibility region for the map T.

5.1 Analysis of the 7-periodic RPC

In this section we will determine the number of the level curves associated to the 7-periodic RPC. We will use the following Lemma from [7],

Lemma 4 *Let*

$$G_a(h) = g_n(a)h^n + g_{n-1}(a)h^{n-1} + ... + g_1(a)h + g_0(a),$$

be a family of real polynomials depending also polynomially on a real parameter a. Set $I_a = (\phi(a), +\infty)$ where $\phi(a)$ is a continuous function. Suppose that there exists an open interval $\Lambda \subset \mathbb{R}$ such that

i) There exists $a_0 \in \Lambda$ such that $G_{a_0}(h)$ has exactly $r \geq 0$ simple roots in I_{a_0}.

ii) For all $a \in \Lambda$, $G_a(\phi(a)) \cdot g_n(a) \neq 0$.

iii) For all $a \in \Lambda$, $\Delta_h(G_a) \neq 0$, where $\Delta_h(G_a)$ is disriminant of the polynomial $G_a(h)$.

Then for all $a \in \Lambda$, $G_a(h)$ has exactly $r \geq 0$ simple roots in I_a.

The disriminant $\Delta_h(G_a)$ of the polynomial $G_a(h)$ is given as

$$\Delta_h(G_a) = (-1)^{\frac{n(n-1)}{2}} \frac{1}{a_n} Res(G_a(h), G_a'(h), h).$$

Let us now for the sake of the convenience rewrite $d_7(a,h)$ as one-parametric family of polynomials in h depending on the parameter $\bar{y}$ where $a = \bar{y}^3(1+\bar{y})$:

$$G_{\bar{y}}(h) := d_7(a,h) = g_3(\bar{y})h^3 + g_2(\bar{y})h^2 + g_1(\bar{y})h + g_0(\bar{y}),$$

where $g_3(\bar{y}) = -1$, $g_2(\bar{y}) = -3$, $g_1(\bar{y}) = -3 - \bar{y}^3(1+\bar{y})$ and $g_0(\bar{y}) = y^3(y+1)\left(y^4 + y^3 - 1\right) - 1$. Since $(a,h) \in \mathcal{R}$ if and only if $h \in [h_c(\bar{y}), +\infty) = [\bar{y}(2\bar{y}+3), +\infty)$ and $a = \bar{y}^3(1+\bar{y})$, we have to study the number of real roots of $G_{\bar{y}}(h) = 0$ in the feasibility region. Let us note that

$$\Delta_h(G_{\bar{y}}(h)) = \bar{y}^9(\bar{y}+1)^3\left(27\bar{y}^4 + 27\bar{y}^3 + 4\right) > 0,$$

so hypotheses (iii) of the Lemma 4 is satisfied.

Further, according to Lemma 4, since $g_3(\bar{y}) \neq 0$, the number of real simple roots in $G_{\bar{y}}(h)$ is constant on any open interval where $G_a(h_c(\bar{y})) \neq 0$. We have

$$G_{\bar{y}}(\bar{y}(2\bar{y}+3)) = (\bar{y}+1)^5\left(\bar{y}^3 - 3\bar{y}^2 - 4\bar{y} - 1\right),$$

and it vanishes in $\bar{y}_0 \approx 4.04892$, which is the only positive root of $\bar{y}^3 - 3\bar{y}^2 - 4\bar{y} - 1 = 0$. Hence, the map T has a constant number of real roots in $I_{\bar{y}} = [h_c(\bar{y}), +\infty) = [\bar{y}(2\bar{y}+3), +\infty)$ for $\bar{y}$ in each of the intervals $(0, \bar{y}_0)$ and $(\bar{y}_0, \infty)$. So we have to determine the number of roots of $G_{\bar{y}}$ in $I_{\bar{y}} = [\bar{y}(2\bar{y}+3), +\infty)$ for the intervals $(0, \bar{y}_0)$ and $(\bar{y}_0, +\infty)$. We can reduce the problem to study one concrete value of $\bar{y}$ in each of the intervals mentioned above. Let us consider the value $\bar{y} = 2 \in (0, \bar{y}_0)$. We can compute

$$G_2(h) = -h^3 - 3h^2 - 27h + 551,$$

and it is easy to see that $G_2(h)$ has no simple roots in $I_{\bar{y}}$. By Lemma 4 we have that $G_{\bar{y}}(h)$ has no simple roots in $I_{\bar{y}} = [\bar{y}(2\bar{y}+3), +\infty)$ for $\bar{y} \in (0, \bar{y}_0)$, i.e. for $a \in (0, \bar{y}_0^3(\bar{y}_0+1))$. Similarly, one can see that $G_{\bar{y}}(h)$ has one simple root in $I_{\bar{y}} = [\bar{y}(2\bar{y}+3), +\infty)$ for $\bar{y} \in (\bar{y}_0, +\infty)$, i.e. $a \in (\bar{y}_0^3(\bar{y}_0+1), +\infty)$.

From the previous discussion we obtain the following theorem:

Theorem 6 *Consider the map T given by (3) with positive parameter a and value $a_0 = \bar{y}_0^3(\bar{y}_0+1) \approx 335.13213$. The set of real 7-periodic points is empty set for $a \in (0, a_0)$ and it is given by smooth non-empty level sets $I_a(x,y) = h$ for the values of h satisfying $d_7(a,h) = 0$ for $a > a_0$, with d_7 given in Theorem 5 and it is formed by one closed curve diffeomorphic to $\mathbb{S}^1$.*

References

[1] A. M. Amleh, E. Camouzis, and G. Ladas, On the Dynamics of a Rational Difference Equation, Part I, *Int. J. Difference Equ.* 3(2008), 1–35.

[2] D. del-Castillo-Negrete, J. M. Greene, E. J. Morrison, Area preserving nontwist maps: periodic orbits and transition to chaos, *Physica D*, 91(1996), 1–23.

[3] A. Cima, A. Gasull, V. Mañosa, Studying discrete dynamical systems through differential equations, *J. Differential Equations* 244 (2008), 630-648.

[4] A. Cima, A. Gasull, V. Mañosa, Non-autonomus 2-periodic Gumovski-Mira difference equations, Internat. *J. Bifur. Chaos Appl. Sci. Engrg.* 22, (2012), 1250264, 14 pp.

[5] E. Denette, M. R. S. Kulenović and E. Pilav, Birkhoff normal forms, KAM theory and time reversal symmetry for certain rational map, *Mathematics, MDPI*, 2016; 4(1):20.

[6] E. Drymmonis, E. Camouzis, G. Ladas, G. and W. Tikjha, Patterns of boundedness of the rational system $x_{n+1} = \frac{\alpha_1}{A_1+B_1x_n+C_1y_n}$ and $y_{n+1} = \frac{\alpha_2+\beta_2x_n+\gamma_2y_n}{A_2+B_2x_n+C_2y_n}$. *J. Difference Equ. Appl.* 18 (2012), 89–110.

[7] A. Gasull, M. Liorens, V. Mañosa, Continua of periodic points for planar integrable rational maps, *Int. J. Difference Equ.*, 11(2016), 37–63.

[8] M. Gidea, J. D. Meiss, I. Ugarcovici, H. Weiss, Applications of KAM Theory to Population Dynamics, *J. Biological Dynamics* 5:1,44-63 (2011).

[9] J. K. Hale and H. Kocak, *Dynamics and Bifurcation*, Springer-Verlag, New York, (1991).

[10] V. L. Kocic, G. Ladas, G. Tzanetopoulos, and E. Thomas, On the stability of Lyness' equation, *Dynam. Contin. Discrete Impuls. Systems*, 1(1995), 245–254.

[11] M. R. S. Kulenović, Invariants and related Liapunov functions for difference equations, *Appl. Math. Lett.* 13(2000), 1-8.

[12] M. R. S. Kulenović and O. Merino, *Discrete Dynamical Systems and Difference Equations with Mathematica,* Chapman and Hall/CRC, Boca Raton, London, 2002.

[13] M. R. S. Kulenović and Z. Nurkanović, Stability of Lyness' Equation with Period-Two Coefficient via KAM Theory, *J. Concr. Appl. Math.*, 6(2008), 229-245.

[14] G. Ladas, G. Tzanetopoulos, and A. Tovbis, On May's host parasitoid model, *J. Difference Equ. Appl.* 2 (1996), 195–204.

[15] R. S. MacKay, *Renormalization in Area-Preserving Maps*, World Scientific, River Edge, NJ, 1993.

[16] R. Moeckel, Generic bifurcations of the twist coefficient, *Ergodic Theory Dyn. Syst.* 10(1) (1990), pp. 185–195.

[17] C. Siegel and J. Moser, *Lectures on Celestial Mechanics*, Springer-Varlag, New York, 1971.

[18] M. Tabor, *Chaos and integrability in nonlinear dynamics. An introduction.* A Wiley-Interscience Publication. John Wiley & Sons, Inc., New York, 1989.

[19] Y. H. Wan, Computation of the stability condition for the Hopf bifurcation of diffeomorphisms on $\mathcal{R}^2$, *SIAM J. Appl. Math.* 34(1) (1978), pp. 167–175.

Durrmeyer type $(p,\ q)$-Baskakov operators for functions of one and two variables

Qing-Bo Cai[a] and Guorong Zhou[b,*]

[a]School of Mathematics and Computer Science, Quanzhou Normal University, Quanzhou 362000, China

[b]School of Applied Mathematics, Xiamen University of Technology, Xiamen 361024, China

E-mail: qbcai@126.com, goonchow@xmut.edu.cn.

Abstract. In this paper, we construct a generalization of Durrmeyer type Baskakov operators based on the concept of $(p,\ q)$-integers and bivariate tensor product form. For the univariate case, we obtain the estimates of moments and central moments of these operators, establish a local approximation theorem, obtain the estimates on the rate of convergence and weighted approximation of those operators. For the bivariate case, we give the rate of convergence by using the weighted modulus of continuity, give some graphs and numerical examples to illustrate the convergent properties of these operators to certain functions. We also compare these operators $D_{n,p,q}$ with another forms.

2000 Mathematics Subject Classification: 41A10, 41A25, 41A36.
Key words and phrases: $(p,\ q)$-integers, Baskakov operators, modulus of continuity, rate of convergence, bivariate tensor product.

1 Introduction

In recent years, $(p,\ q)$-integers have been introduced to linear positive operators to construct new approximation processes. A sequence of $(p,\ q)$-analogue of Bernstein operators was first introduced by Mursaleen [1, 2]. Besides, $(p,\ q)$-analogues of Szász-Mirakyan operators [3] , $(p,\ q)$-Baskakov Kantorovich operators [4, 5], $(p,\ q)$-Baskakov-Beta operators [6] and Kantorovich-type Bernstein-Stancu-Schurer operators [7] were also considered. For further developments, one can also refer to [8, 9, 28]. These operators are double parameters corresponding to p and q versus single parameter q-type operators [11, 12, 13]. The aim of these generalizations is to provide appropriate and powerful tools to these application areas such as numerical analysis, CAGD and solutions of differential equations (see, e. g., [14]).

*Corresponding author.

In 2010, Aral and Gupta [15], Gupta [16] introduced certain Durrmeyer type q-Baskakov operators and got some important approximation properties, motivated by them, in 2012, Cai and Zeng [17] introduced a new modification of Durrmeyer type one. Recently, Acar et al. [18] introduced a generalization of Durrmeyer type (p, q)-Baskakov operators which having Baskakov and Szász basis functions defined by

$$B_n^{p,q}(f;x)=[n]_{p,q}\sum_{k=0}^{\infty}b_{n,k}(p,q;x)\int_0^{\infty}p^{\frac{k(k-1)}{2}}\frac{([n]_{p,q}t)^k\,E_{p,q}\left(-q[n]_{p,q}t\right)}{[k]_{p,q}!}f\left(\frac{p^{k+n-1}}{q^{k-1}}t\right)d_{p,q}t, \quad (1)$$

where

$$b_{n,k}(p,q;x)=\left[\begin{array}{c} n+k-1 \\ k \end{array}\right]_{p,q}p^{k+\frac{n(n-1)}{2}}q^{\frac{k(k-1)}{2}}\frac{x^k}{(1+x)_{p,q}^{n+k}}. \quad (2)$$

From [5], we know $\sum_{k=0}^{\infty}b_{n,k}(p,q;x)=1$. In 2016, Mishra and Pandey [19] introduced the Stancu type base on operators (1).

Inspired by these results, in this paper, we introduce a generalization of Durrmeyer type (p, q)-Baskakov operators $D_{n,p,q}(f;x)$ as

$$D_{n,p,q}(f;x)=[n-1]_{p,q}\sum_{k=0}^{\infty}\widetilde{b_{n,k}}(p,q;\mu(x))\int_0^{\infty}\widetilde{b_{n,k}}(p,q;pu)f(p^ku)d_{p,q}u, \quad (3)$$

where $\mu(x)=\frac{p^{n-2}\left(p^2q[n-2]_{p,q}x-1\right)}{[n]_{p,q}}$, $x\in\left[\frac{1}{p^2q[n-2]_{p,q}},\infty\right)$, $0<q<p\leq 1$ and

$$\widetilde{b_{n,k}}(p,q;x)=\left[\begin{array}{c} n+k-1 \\ k \end{array}\right]_{p,q}p^{\frac{n(n-1)+(k+1)(k+2)}{4}}q^{\frac{k^2-1}{2}}\frac{x^k}{(1+x)_{p,q}^{n+k}}. \quad (4)$$

The paper is organized as follows: In section 2, we give some basic definitions regarding (p, q)-integers and (p, q)-calculus. In section 3, we estimate the moments and central moments of these operators (3). In section 4, we establish a local approximation theorem, obtain the estimates on the rate of convergence and weighted approximation. In section 5, we give some graphs and numerical examples to illustrate the convergent properties for one variable functions. In section 6-7, we propose the bivariate case, give the rate of convergence by using the weighted modulus of continuity and give some graphs and numerical analysis for two variables functions. In the last section, we compare the operators $D_{n,p,q}$ with $\widetilde{D_{n,p,q}}$, and show the former operators give better approximation to f than the latter ones by graphs.

2 Some notations

We mention some definitions based on (p, q)-integers, details can be found in [20, 21, 22, 23, 24]. For any fixed real number $0<q<p\leq 1$ and each nonnegative integer k, we denote (p, q)-integers by $[k]_{p,q}$, where

$$[k]_{p,q}=\frac{p^k-q^k}{p-q}.$$

Also $(p,\,q)$-factorial and $(p,\,q)$-binomial coefficients are defined as follows:

$$[k]_{p,q}! = \begin{cases} [k]_{p,q}[k-1]_{p,q}...[1]_{p,q}, & k = 1,2,..., \\ 1, & k = 0, \end{cases}$$

$$\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} = \frac{[n]_{p,q}!}{[k]_{p,q}![n-k]_{p,q}!}, \quad (n \geq k \geq 0).$$

Let n be a non-negative integer, the $(p,\,q)$-Gamma function is defined as

$$\Gamma_{p,q}(n+1) = \frac{(p-q)_{p,q}^n}{(p-q)^n} = [n]_{p,q}!,$$

where $(p-q)_{p,q}^n = (p-q)(p^2-q^2)...(p^n-q^n)$.

For $m, n \in \mathbb{N}$, the $(p,\,q)$-Beta function of second kind is given by

$$B_{p,q}(m,n) = \int_0^\infty \frac{t^{m-1}}{(1+pt)_{p,q}^{m+n}} d_{p,q}t,$$

where the $(p,\,q)$-power basis is given by

$$(1+pt)_{p,q}^{m+n} = (1+pt)(p+pqt)(p^2+pq^2t)...(p^{m+n-1}+pq^{m+n-1}t).$$

The relationship by the $(p,\,q)$-Beta and Gamma functions is shown as follows

$$B_{p,q}(m,n) = \frac{q\Gamma_{p,q}(m)\Gamma_{p,q}(n)}{\left(p^{m+1}q^{m-1}\right)^{m/2}\Gamma_{p,q}(m+n)},$$

if $p = 1, q \to 1^-$, it reduces to the classic type $B(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$.

The improper $(p,\,q)$-integral of $f(x)$ on $[0,\infty)$ is defined to be

$$\int_0^\infty f(x)d_{p,q}x = \sum_{j=-\infty}^{\infty} \int_{\frac{q^{j+1}}{p^{j+1}}}^{\frac{q^j}{p^j}} f(x)d_{p,q}x = (p-q)\sum_{j=-\infty}^{\infty} \frac{q^j}{p^{j+1}} f\left(\frac{q^j}{p^{j+1}}\right).$$

When $p = 1$, all the definitions of $(p,\,q)$-calculus above are reduced to q-calculus.

3 Auxiliary results

Lemma 3.1. *For $x \in [0,\infty)$ and sufficiently large n, the following equalities hold*

$$D_{n,p,q}(1;x) = 1, \tag{5}$$

$$D_{n,p,q}(t;x) = x, \tag{6}$$

$$D_{n,p,q}(t^2;x) = \frac{[n-2]_{p,q}[n+1]_{p,q}}{q^2[n-3]_{p,q}[n]_{p,q}}x^2 + \frac{\left(p^2+q^2\right)}{p^3q^3[n-3]_{p,q}}x - \frac{2p^{n-2}}{q^3[n-3]_{p,q}[n]_{p,q}}x + \frac{p^{n-4}}{q^4[n-2]_{p,q}[n-3]_{p,q}[n]_{p,q}} - \frac{1}{p^3q^4[n-2]_{p,q}[n-3]_{p,q}}, \tag{7}$$

$$\begin{aligned}
& D_{n,p,q}(t^3;x) \\
= \quad & \frac{[n+1]_{p,q}[n+2]_{p,q}[n-2]_{p,q}^2}{[n-3]_{p,q}[n-4]_{p,q}[n]_{p,q}^2}x^3 + \frac{\left([5]_{p,q}+[2]_{p,q}^2 pq\right)p^{n-2}[2]_{p,q}[n+1]_{p,q}[n+2]_{p,q}}{p^4q^7[n]_{p,q}^2[n-3]_{p,q}[n-4]_{p,q}}x^2 \\
& + \frac{\left([5]_{p,q}q^2+[2]_{p,q}^2pq^3-3p^2\right)[n+1]_{p,q}[n+2]_{p,q}[n-2]_{p,q}}{p^4q^7[n]_{p,q}^2[n-3]_{p,q}[n-4]_{p,q}}x^2 + \frac{q^5+3p^3q^2-p^5}{p^7q^7[n-3]_{p,q}[n-4]_{p,q}}x \\
& + \frac{3p^{2n-4}[2]_{p,q}}{q^8[n]_{p,q}^2[n-3]_{p,q}[n-4]_{p,q}}x - \frac{p^{n-6}\left(2p^4+2q^4+4pq^3+p^3q\right)}{q^8[n]_{p,q}[n-3]_{p,q}[n-4]_{p,q}}x \\
& - \frac{[2]_{p,q}^2}{p^7q^7[n-2]_{p,q}[n-3]_{p,q}[n-4]_{p,q}} + \frac{p^{n-8}\left([5]_{p,q}+[2]_{p,q}pq^2\right)}{q^9[n]_{p,q}[n-2]_{p,q}[n-3]_{p,q}[n-4]_{p,q}} \\
& - \frac{p^{2n+3}[2]_{p,q}}{p^9q^9[n]_{p,q}^2[n-2]_{p,q}[n-3]_{p,q}[n-4]_{p,q}}, \qquad (8)
\end{aligned}$$

$$D_{n,p,q}(t^4;x) = \frac{[n+1]_{p,q}[n+2]_{p,q}[n+3]_{p,q}[n-2]_{p,q}^3}{q^{12}[n-3]_{p,q}[n-4]_{p,q}[n-5]_{p,q}[n]_{p,q}^3}x^4 + O\left(\frac{1}{[n]_{p,q}}\right)\phi(x), \qquad (9)$$

where $\phi(x)$ is depend on x.

Proof. Since

$$\begin{aligned}
\int_0^\infty \frac{u^k}{(1+pu)_{p,q}^{n+k}}d_{p,q}u &= B_{p,q}(k+1,n-1) = \frac{q\Gamma_{p,q}(k+1)\Gamma_{p,q}(n-1)}{\left(p^{k+2}q^k\right)^{\frac{k+1}{2}}\Gamma_{p,q}(n+k)} \\
&= \frac{q[k]_{p,q}![n-2]_{p,q}!}{p^{\frac{(k+1)(k+2)}{2}}q^{\frac{k(k+1)}{2}}[n+k-1]_{p,q}!},
\end{aligned}$$

we have

$$\begin{aligned}
D_{n,p,q}(1;x) &= [n-1]_{p,q}\sum_{k=0}^\infty \widetilde{b_{n,k}}(p,q;\mu(x))\int_0^\infty \widetilde{b_{n,k}}(p,q;pu)d_{p,q}u \\
&= [n-1]_{p,q}\sum_{k=0}^\infty \widetilde{b_{n,k}}(p,q;\mu(x))\frac{[n+k-1]_{p,q}!}{[k]_{p,q}![n-1]_{p,q}!}p^{\frac{n(n-1)+(k+1)(k+2)}{4}}q^{\frac{k^2-1}{2}} \\
&\quad \times \frac{p^kq[k]_{p,q}![n-2]_{p,q}!}{p^{\frac{(k+1)(k+2)}{2}}q^{\frac{k(k+1)}{2}}[n+k-1]_{p,q}!} \\
&= \sum_{k=0}^\infty b_{n,k}(p,q;\mu(x)) = 1.
\end{aligned}$$

Similarly, we get

$$\int_0^\infty \frac{u^{k+1}}{(1+pu)_{p,q}^{n+k}}d_{p,q}u = \frac{q[k+1]_{p,q}![n-3]_{p,q}!}{p^{\frac{(k+2)(k+3)}{2}}q^{\frac{(k+1)(k+2)}{2}}[n+k-1]_{p,q}!},$$

thus,

$$D_{n,p,q}(t;x) = [n-1]_{p,q}\sum_{k=0}^\infty \widetilde{b_{n,k}}(p,q;\mu(x))\int_0^\infty \widetilde{b_{n,k}}(p,q;pu)p^kud_{p,q}u$$

$$
\begin{aligned}
&= [n-1]_{p,q}\sum_{k=0}^{\infty}\widetilde{b_{n,k}}(p,q;\mu(x))\frac{[n+k-1]_{p,q}!}{[k]_{p,q}![n-1]_{p,q}!}p^{\frac{n(n-1)+(k+1)(k+2)}{4}}q^{\frac{k^2-1}{2}}\\
&\quad\times\frac{p^{2k}q[k+1]_{p,q}![n-3]_{p,q}!}{p^{\frac{(k+2)(k+3)}{2}}q^{\frac{(k+1)(k+2)}{2}}[n+k-1]_{p,q}!}\\
&= \sum_{k=0}^{\infty}\widetilde{b_{n,k}}(p,q;\mu(x))\frac{p^{2k}q[k+1]_{p,q}p^{\frac{n(n-1)+(k+1)(k+2)}{4}}q^{\frac{k^2-1}{2}}}{p^{\frac{(k+2)(k+3)}{2}}q^{\frac{(k+1)(k+2)}{2}}[n-2]_{p,q}}.
\end{aligned}
$$

Since $[k+1]_{p,q}=q^k+p[k]_{p,q}$, by simple computations, we have

$$
\begin{aligned}
D_{n,p,q}(t;x) &= \frac{[n]_{p,q}\mu(x)}{p^nq^2[n-2]_{p,q}}\sum_{k=0}^{\infty}\begin{bmatrix} n+k\\ k\end{bmatrix}_{p,q}\frac{p^{k+\frac{n(n+1)}{2}}q^{\frac{k(k-1)}{2}}(\mu(x))^k}{(1+\mu(x))_{p,q}^{n+k+1}}\\
&\quad+\frac{1}{p^2q[n-2]_{p,q}}\sum_{k=0}^{\infty}b_{n,k}(p,q;\mu(x))\\
&= \frac{[n]_{p,q}\mu(x)}{p^nq^2[n-2]_{p,q}}+\frac{1}{p^2q[n-2]_{p,q}}=x.
\end{aligned}
$$

Next,

$$
\int_0^{\infty}\frac{u^{k+2}}{(1+pu)_{p,q}^{n+k}}d_{p,q}u=\frac{q[k+2]_{p,q}![n-4]_{p,q}!}{p^{\frac{(k+3)(k+4)}{2}}q^{\frac{(k+2)(k+3)}{2}}[n+k-1]_{p,q}!},
$$

we get

$$
\begin{aligned}
&D_{n,p,q}(t^2;x)\\
&= [n-1]_{p,q}\sum_{k=0}^{\infty}\widetilde{b_{n,k}}(p,q;\mu(x))\int_0^{\infty}\widetilde{b_{n,k}}(p,q;pu)p^{2k}u^2d_{p,q}u\\
&= [n-1]_{p,q}\sum_{k=0}^{\infty}\widetilde{b_{n,k}}(p,q;\mu(x))\begin{bmatrix} n+k-1\\ k\end{bmatrix}_{p,q}p^{\frac{n(n-1)+(k+1)(k+2)}{4}}q^{\frac{k^2-1}{2}}\\
&\quad\times\int_0^{\infty}\frac{p^{3k}u^{k+2}}{(1+pu)_{p,q}^{n+k}}d_{p,q}u\\
&= [n-1]_{p,q}\sum_{k=0}^{\infty}\widetilde{b_{n,k}}(p,q;\mu(x))\begin{bmatrix} n+k-1\\ k\end{bmatrix}_{p,q}p^{\frac{n(n-1)+(k+1)(k+2)}{4}}q^{\frac{k^2-1}{2}}\\
&\quad\times\frac{p^{3k}q[k+2]_{p,q}![n-4]_{p,q}!}{p^{\frac{(k+3)(k+4)}{2}}q^{\frac{(k+2)(k+3)}{2}}[n+k-1]_{p,q}!}\\
&= \sum_{k=0}^{\infty}\begin{bmatrix} n+k-1\\ k\end{bmatrix}_{p,q}\frac{(\mu(x))^k}{(1+\mu(x))_{p,q}^{n+k}}\frac{p^{\frac{n(n-1)+(k+1)(k+2)}{2}}p^{3k}q^{k^2}}{p^{\frac{(k+3)(k+4)}{2}}q^{\frac{(k+2)(k+3)}{2}}}\frac{[k+1]_{p,q}[k+2]_{p,q}}{[n-2]_{p,q}[n-3]_{p,q}} \qquad (10)
\end{aligned}
$$

Using $[k+1]_{p,q}=q^k+p[k]_{p,q}$ and some computations, we obtain

$$
[k+1]_{p,q}[k+2]_{p,q}=[2]_{p,q}q^{2k}+p[2]_{p,q}^2q^{k-1}[k]_{p,q}+p^4[k]_{p,q}[k-1]_{p,q}. \qquad (11)
$$

Since

$$\begin{aligned}
&\sum_{k=0}^{\infty}\left[\begin{array}{c} n+k-1 \\ k \end{array}\right]_{p,q} \frac{(\mu(x))^k}{(1+\mu(x))_{p,q}^{n+k}} \frac{p^{\frac{n(n-1)+(k+1)(k+2)}{2}} p^{3k} q^{k^2}}{p^{\frac{(k+3)(k+4)}{2}} q^{\frac{(k+2)(k+3)}{2}}} \frac{p^4[k]_{p,q}[k-1]_{p,q}}{[n-2]_{p,q}[n-3]_{p,q}} \\
= &\ \frac{[n]_{p,q}[n+1]_{p,q}x^2}{p^{2n}q^6[n-2]_{p,q}[n-3]_{p,q}} \sum_{k=0}^{\infty}\left[\begin{array}{c} n+k+1 \\ k \end{array}\right]_{p,q} \frac{(\mu(x))^k}{(1+\mu(x))_{p,q}^{n+k+2}} p^{k+\frac{(n+1)(n+2)}{2}} q^{\frac{k(k-1)}{2}} \\
= &\ \frac{[n]_{p,q}[n+1]_{p,q}\,(\mu(x))^2}{p^{2n}q^6[n-2]_{p,q}[n-3]_{p,q}}, \qquad (12)
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{k=0}^{\infty}\left[\begin{array}{c} n+k-1 \\ k \end{array}\right]_{p,q} \frac{(\mu(x))^k}{(1+\mu(x))_{p,q}^{n+k}} \frac{p^{\frac{n(n-1)+(k+1)(k+2)}{2}} p^{3k} q^{k^2}}{p^{\frac{(k+3)(k+4)}{2}} q^{\frac{(k+2)(k+3)}{2}}} \frac{q^{k-1}[k]_{p,q}}{[n-2]_{p,q}[n-3]_{p,q}} \\
= &\ \frac{[n]_{p,q}\mu(x)}{p^{n+4}q^5[n-2]_{p,q}[n-3]_{p,q}} \sum_{k=0}^{\infty}\left[\begin{array}{c} n+k \\ k \end{array}\right]_{p,q} \frac{(\mu(x))^k}{(1+\mu(x))_{p,q}^{n+k+1}} p^{k+\frac{n(n+1)}{2}} q^{\frac{k(k-1)}{2}} \\
= &\ \frac{[n]_{p,q}\mu(x)}{p^{n+4}q^5[n-2]_{p,q}[n-3]_{p,q}}, \qquad (13)
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{k=0}^{\infty}\left[\begin{array}{c} n+k-1 \\ k \end{array}\right]_{p,q} \frac{(\mu(x))^k}{(1+\mu(x))_{p,q}^{n+k}} \frac{p^{\frac{n(n-1)+(k+1)(k+2)}{2}} p^{3k} q^{k^2}}{p^{\frac{(k+3)(k+4)}{2}} q^{\frac{(k+2)(k+3)}{2}}} \frac{[2]_{p,q}q^{2k}}{[n-2]_{p,q}[n-3]_{p,q}} \\
= &\ \frac{[2]_{p,q}}{p^5q^3[n-2]_{p,q}[n-3]_{p,q}} \sum_{k=0}^{\infty} b_{n,k}(p,q;\mu(x)) \\
= &\ \frac{[2]_{p,q}}{p^5q^3[n-2]_{p,q}[n-3]_{p,q}}, \qquad (14)
\end{aligned}$$

combining (10), (11), (12), (13) and (14), we have

$$\begin{aligned}
&D_{n,p,q}(t^2;x) \\
= &\ \frac{[n]_{p,q}[n+1]_{p,q}\,(\mu(x))^2}{p^{2n}q^6[n-2]_{p,q}[n-3]_{p,q}} + \frac{[2]_{p,q}^2[n]_{p,q}\mu(x)}{p^{n+3}q^5[n-2]_{p,q}[n-3]_{p,q}} + \frac{[2]_{p,q}}{p^5q^3[n-2]_{p,q}[n-3]_{p,q}} \\
= &\ \frac{[n-2]_{p,q}[n+1]_{p,q}}{q^2[n-3]_{p,q}[n]_{p,q}}x^2 + \frac{\left(p^2+q^2\right)}{p^3q^3[n-3]_{p,q}}x - \frac{2p^{n-2}}{q^3[n-3]_{p,q}[n]_{p,q}}x \\
&+\frac{p^{n-4}}{q^4[n-2]_{p,q}[n-3]_{p,q}[n]_{p,q}} - \frac{1}{p^3q^4[n-2]_{p,q}[n-3]_{p,q}}.
\end{aligned}$$

Using the same methods, we have

$$\begin{aligned}
&D_{n,p,q}(t^3;x) \\
= &\ [n-1]_{p,q}\sum_{k=0}^{\infty}\widetilde{b_{n,k}}(p,q;\mu(x))\int_0^{\infty}\widetilde{b_{n,k}}(p,q;pu)p^{3k}u^3 d_{p,q}u
\end{aligned}$$

$$= \sum_{k=0}^{\infty} \left[\begin{array}{c} n+k-1 \\ k \end{array} \right]_{p,q} \frac{(\mu(x))^k}{(1+\mu(x))_{p,q}^{n+k}} p^{\frac{n(n-1)+2k-18}{2}} q^{\frac{k^2-7k-12}{2}} \frac{[k+1]_{p,q}[k+2]_{p,q}[k+3]_{p,q}}{[n-2]_{p,q}[n-3]_{p,q}[n-4]_{p,q}},$$

since

$$\begin{aligned} & [k+1]_{p,q}[k+2]_{p,q}[k+3]_{p,q} \\ = \; & p^9[k]_{p,q}[k-1]_{p,q}[k-2]_{p,q} + p^4q^{k-2}\left([5]_{p,q}+[2]_{p,q}^2pq\right)[k]_{p,q}[k-1]_{p,q} \\ & +pq^{2k-2}[2]_{p,q}\left([5]_{p,q}+[2]_{p,q}^2pq\right)[k]_{p,q}+q^{3k}[2]_{p,q}[3]_{p,q}, \end{aligned}$$

by some computations, we have

$$\begin{aligned} & D_{n,p,q}(t^3;x) \\ = \; & \frac{[n]_{p,q}[n+1]_{p,q}[n+2]_{p,q}(\mu(x))^3}{p^{3n}q^{12}[n-2]_{p,q}[n-3]_{p,q}[n-4]_{p,q}} + \frac{\left([5]_{p,q}+[2]_{p,q}^2pq\right)[n]_{p,q}[n+1]_{p,q}(\mu(x))^2}{p^{2n+4}q^{11}[n-2]_{p,q}[n-3]_{p,q}[n-4]_{p,q}} \\ & +\frac{[2]_{p,q}\left([5]_{p,q}+[2]_{p,q}^2pq\right)[n]_{p,q}\mu(x)}{p^{n+7}q^9[n-2]_{p,q}[n-3]_{p,q}[n-4]_{p,q}} + \frac{[2]_{p,q}[3]_{p,q}}{p^9q^6[n-2]_{p,q}[n-3]_{p,q}[n-4]_{p,q}} \\ = \; & \frac{[n+1]_{p,q}[n+2]_{p,q}[n-2]_{p,q}^2}{[n-3]_{p,q}[n-4]_{p,q}[n]_{p,q}^2}x^3 + \frac{\left([5]_{p,q}+[2]_{p,q}^2pq\right)p^{n-2}[2]_{p,q}[n+1]_{p,q}[n+2]_{p,q}}{p^4q^7[n]_{p,q}^2[n-3]_{p,q}[n-4]_{p,q}}x^2 \\ & +\frac{\left([5]_{p,q}q^2+[2]_{p,q}^2pq^3-3p^2\right)[n+1]_{p,q}[n+2]_{p,q}[n-2]_{p,q}}{p^4q^7[n]_{p,q}^2[n-3]_{p,q}[n-4]_{p,q}}x^2 + \frac{q^5+3p^3q^2-p^5}{p^7q^7[n-3]_{p,q}[n-4]_{p,q}}x \\ & +\frac{3p^{2n-4}[2]_{p,q}}{q^8[n]_{p,q}^2[n-3]_{p,q}[n-4]_{p,q}}x - \frac{p^{n-6}\left(2p^4+2q^4+4pq^3+p^3q\right)}{q^8[n]_{p,q}[n-3]_{p,q}[n-4]_{p,q}}x \\ & -\frac{[2]_{p,q}^2}{p^7q^7[n-2]_{p,q}[n-3]_{p,q}[n-4]_{p,q}} + \frac{p^{n-8}\left([5]_{p,q}+[2]_{p,q}pq^2\right)}{q^9[n]_{p,q}[n-2]_{p,q}[n-3]_{p,q}[n-4]_{p,q}} \\ & -\frac{p^{2n+3}[2]_{p,q}}{p^9q^9[n]_{p,q}^2[n-2]_{p,q}[n-3]_{p,q}[n-4]_{p,q}}. \end{aligned}$$

Finally,

$$\begin{aligned} & D_{n,p,q}(t^4;x) \\ = \; & [n-1]_{p,q}\sum_{k=0}^{\infty}\widetilde{b_{n,k}}(p,q;\mu(x))\int_0^{\infty}\widetilde{b_{n,k}}(p,q;pu)p^{4k}u^4d_{p,q}u \\ = \; & \sum_{k=0}^{\infty} \left[\begin{array}{c} n+k-1 \\ k \end{array} \right]_{p,q} p^{\frac{n(n-1)+2k-28}{2}} q^{\frac{k^2-9k-20}{2}} \frac{[k+1]_{p,q}[k+2]_{p,q}[k+3]_{p,q}[k+4]_{p,q}}{[n-2]_{p,q}[n-3]_{p,q}[n-4]_{p,q}[n-5]_{p,q}} \\ & \times\frac{(\mu(x))^k}{(1+\mu(x))_{p,q}^{n+k}}, \end{aligned}$$

since

$$\begin{aligned} & [k+1]_{p,q}[k+2]_{p,q}[k+3]_{p,q}[k+4]_{p,q} \\ = \; & p^{16}[k]_{p,q}[k-1]_{p,q}[k-2]_{p,q}[k-3]_{p,q} + \left([7]_{p,q}+pq[5]_{p,q}+[2]_{p,q}^2p^2q^2\right)p^9q^{k-3}[k]_{p,q} \end{aligned}$$

$$\times[k-1]_{p,q}[k-2]_{p,q}+p^4q^{2k-4}\left([5]_{p,q}+[2]_{p,q}^2pq\right)\left([6]_{p,q}+p^2q^2[2]_{p,q}\right)[k]_{p,q}[k-1]_{p,q}$$
$$+pq^{3k-3}[2]_{p,q}\left([5]_{p,q}^2+[2]_{p,q}^2[5]_{p,q}pq+p^3q^3[3]_{p,q}\right)[k]_{p,q}+q^{4k}[2]_{p,q}[3]_{p,q}[4]_{p,q},$$

we have

$$\begin{aligned}
& D_{n,p,q}(t^4;x)\\
= & \frac{[n]_{p,q}[n+1]_{p,q}[n+2]_{p,q}[n+3]_{p,q}}{p^{4n}q^{20}[n-2]_{p,q}[n-3]_{p,q}[n-4]_{p,q}[n-5]_{p,q}}\left(\mu(x)\right)^4\\
& +\frac{\left([7]_{p,q}+pq[5]_{p,q}+[2]_{p,q}^2p^2q^2\right)[n]_{p,q}[n+1]_{p,q}[n+2]_{p,q}}{p^{3n+5}q^{19}[n-2]_{p,q}[n-3]_{p,q}[n-4]_{p,q}[n-5]_{p,q}}\left(\mu(x)^3\right)\\
& +\frac{\left([5]_{p,q}+[2]_{p,q}^2pq\right)\left([6]_{p,q}+p^2q^2[2]_{p,q}\right)}{p^{2n+9}q^{17}[n-2]_{p,q}[n-3]_{p,q}[n-4]_{p,q}[n-5]_{p,q}}\left(\mu(x)^2\right)\\
& +\frac{[2]_{p,q}\left([5]_{p,q}^2+[2]_{p,q}^2[5]_{p,q}pq+p^3q^3[3]_{p,q}\right)[n]_{p,q}}{p^{n+12}q^{14}[n-2]_{p,q}[n-3]_{p,q}[n-4]_{p,q}[n-5]_{p,q}}\mu(x)\\
& +\frac{[2]_{p,q}[3]_{p,q}[4]_{p,q}}{p^{14}q^{10}[n-2]_{p,q}[n-3]_{p,q}[n-4]_{p,q}[n-5]_{p,q}}\\
= & \frac{[n+1]_{p,q}[n+2]_{p,q}[n+3]_{p,q}[n-2]_{p,q}^3}{q^{12}[n-3]_{p,q}[n-4]_{p,q}[n-5]_{p,q}[n]_{p,q}^3}x^4+O\left(\frac{1}{[n]_{p,q}}\right)\phi(x).
\end{aligned}$$

Lemma 3.1 is proved. □

Lemma 3.2. *For sufficiently large n, we have*

$$D_{n,p,q}(t-x;x)=0, \tag{15}$$

$$\begin{aligned}
& D_{n,p,q}((t-x)^2;x)\\
= & \frac{p^nx^2}{q[n]_{p,q}}+\frac{p^{n-3}x^2}{q[n-3]_{p,q}}+\frac{p^{2n-3}x^2}{q^2[n-3]_{p,q}[n]_{p,q}}+\frac{\left(p^2+q^2\right)x}{p^3q^3[n-3]_{p,q}}-\frac{2p^{n-2}x}{q^3[n-3]_{p,q}[n]_{p,q}}\\
& +\frac{p^{n-4}}{q^4[n-2]_{p,q}[n-3]_{p,q}[n]_{p,q}}-\frac{1}{p^3q^4[n-2]_{p,q}[n-3]_{p,q}}=B_{n,p,q}(x) \qquad (16)\\
= & O\left(\frac{1}{[n]_{p,q}}\right)\left(x^2+x+1\right), \qquad (17)
\end{aligned}$$

$$D_{n,p,q}((t-x)^4;x)=O\left(\frac{1}{[n]_{p,q}}\right)\left(x^4+x^3+x^2+x+1\right). \tag{18}$$

Proof. (15) is obtained by (5) and (6). Since

$$\begin{aligned}
\frac{[n-2]_{p,q}[n+1]_{p,q}}{q^2[n-3]_{p,q}[n]_{p,q}} &= \frac{\left(p^{n-3}+q[n-3]_{p,q}\right)\left(p^n+q[n]_{p,q}\right)}{q^2[n-3]_{p,q}[n]_{p,q}}\\
&= 1+\frac{p^n}{q[n]_{p,q}}+\frac{p^{n-3}}{q[n-3]_{p,q}}+\frac{p^{2n-3}}{q^2[n-3]_{p,q}[n]_{p,q}},
\end{aligned}$$

using lemma 3.1, we have

$$D_{n,p,q}((t-x)^2;x)$$

$$\begin{aligned}
&= D_{n,p,q}(t^2;x) - 2xM_{n,p,q}(t;x) + x^2 \\
&= D_{n,p,q}(t^2;x) - x^2 \\
&= \left[\frac{[n-2]_{p,q}[n+1]_{p,q}}{q^2[n-3]_{p,q}[n]_{p,q}} - 1\right]x^2 + \frac{\left(p^2+q^2\right)}{p^3q^3[n-3]_{p,q}}x - \frac{2p^{n-2}}{q^3[n-3]_{p,q}[n]_{p,q}}x \\
&\quad + \frac{p^{n-4}}{q^4[n-2]_{p,q}[n-3]_{p,q}[n]_{p,q}} - \frac{1}{p^3q^4[n-2]_{p,q}[n-3]_{p,q}} \\
&= \frac{p^nx^2}{q[n]_{p,q}} + \frac{p^{n-3}x^2}{q[n-3]_{p,q}} + \frac{p^{2n-3}x^2}{q^2[n-3]_{p,q}[n]_{p,q}} + \frac{\left(p^2+q^2\right)x}{p^3q^3[n-3]_{p,q}} - \frac{2p^{n-2}x}{q^3[n-3]_{p,q}[n]_{p,q}} \\
&\quad + \frac{p^{n-4}}{q^4[n-2]_{p,q}[n-3]_{p,q}[n]_{p,q}} - \frac{1}{p^3q^4[n-2]_{p,q}[n-3]_{p,q}}.
\end{aligned}$$

Similarly, by some computations, we can obtain (18). □

Lemma 3.3. *(See theorem 2.1 of [25]). For $0 < q_n < p_n \leq 1$, set $q_n := 1-\alpha_n$, $p_n := 1-\beta_n$ such that $0 \leq \beta_n < \alpha_n < 1$, $\alpha_n \to 0$, $\beta_n \to 0$ as $n \to \infty$. The following statements are true*

(A) *If* $\lim\limits_{n\to\infty} e^{n(\beta_n-\alpha_n)} = 1$ *and* $e^{n\beta_n}/n \to 0$, *then* $[n]_{p_n,q_n} \to \infty$.

(B) *If* $\overline{\lim\limits_{n\to\infty}} e^{n(\beta_n-\alpha_n)} < 1$ *and* $e^{n\beta_n}(\alpha_n-\beta_n) \to 0$, *then* $[n]_{p_n,q_n} \to \infty$.

(C) *If* $\underline{\lim}_{n\to\infty} e^{n(\beta_n-\alpha_n)} < 1$, $\overline{\lim\limits_{n\to\infty}} e^{n(\beta_n-\alpha_n)} = 1$ *and* $\max\{e^{n\beta_n}/n, e^{n\beta_n}(\alpha_n-\beta_n)\} \to 0$, *then* $[n]_{p_n,q_n} \to \infty$.

4 Approximation properties

In this section, we establish a local approximation theorem. We give the following definitions at first, the space of all real valued continuous bounded functions f defined on the interval $[0,\infty)$ is denoted by $C_B[0,\infty)$. The norm on $C_B[0,\infty)$ is defined by $||f|| = \sup\{|f(x)| : x \in [0,\infty)\}$. The Peetre's K-functional is given by

$$K_2(f;\delta) = \inf_{g\in W^2}\{||f-g|| + \delta||g''||\}, \tag{19}$$

where $\delta > 0$, $W^2 = \{g \in C_B[0,\infty) : g', g'' \in C_B[0,\infty)\}$. For $f \in C_B[0,\infty)$, the second order modulus of smoothness is defined as

$$\omega_2(f;\sqrt{\delta}) = \sup_{0<h\leq\delta}\sup_{x\in[0,\infty)}|f(x+2h) - 2f(x+h) + f(x)|. \tag{20}$$

By [27], there exists a constant $C > 0$, such that

$$K_2(f;\delta) \leq C\omega_2(f;\sqrt{\delta}). \tag{21}$$

In order to obtain the convergence of operators defined in (3), in the sequel, let $p = \{p_n\}$ and $q = \{q_n\}$ be sequences satisfying $p_n^n \to 1 (n \to \infty)$ and the conditions of lemma 3.3 (A), (B) or (C).

Theorem 4.1. *For $f \in C_B[0,\infty)$ and $n \geq 6$, we have*

$$|D_{n,p,q}(f;x) - f(x)| \leq C\omega_2\left(f; \sqrt{B_{n,p,q}(x)/2}\right), \tag{22}$$

where C is a positive constant, $B_{n,p,q}(x)$ is defined in (16).

Proof. Let $g \in W^2$, by Taylor's expansion, we have

$$g(t) = g(x) + g'(x)(t-x) + \int_x^t (t-u)g''(u)du, \tag{23}$$

applying $D_{n,p,q}$ to (23), using (15), we get

$$D_{n,p,q}(g;x) - g(x) = D_{n,p,q}\left(\int_x^t (t-u)g''(u)du; x\right).$$

Thus, we have,

$$\begin{aligned} |D_{n,p,q}(g;x) - g(x)| &= \left| D_{n,p,q}\left(\int_x^t (t-u)g''(u)du; x\right)\right| \\ &\leq D_{n,p,q}\left(\left|\int_x^t |t-u||g''(u)|du\right|; x\right) \\ &\leq D_{n,p,q}\left((t-x)^2; x\right) ||g''|| \\ &= B_{n,p,q}(x)||g''||. \end{aligned} \tag{24}$$

On the other hand, by (3) and (5), we have

$$D_{n,p,q}(f;x) = [n-1]_{p,q} \sum_{k=0}^{\infty} \widetilde{b_{n,k}}(p,q;x) \int_0^{\infty} \widetilde{b_{n,k}}(p,q;pu)|f(p^k u)|d_{p,q}u \leq ||f||. \tag{25}$$

Now (24) and (25) imply

$$\begin{aligned} |D_{n,p,q}(f;x) - f(x)| &\leq |D_{n,p,q}(f-g;x) - (f-g)(x)| + |D_{n,p,q}(g;x) - g(x)| \\ &\leq 2||f-g|| + B_{n,p,q}(x)||g''||, \end{aligned}$$

from (19), taking infimum on the right hand side over all $g \in W^2$, we obtain

$$|D_{n,p,q}(f;x) - f(x)| \leq 2K_2\left(f; B_{n,p,q}(x)/2\right).$$

Finally, using (21), we get

$$|D_{n,p,q}(f;x) - f(x)| \leq C\omega_2\left(f; \sqrt{B_{n,p,q}(x)/2}\right).$$

Theorem 4.1 is proved. □

Let $B_{x^2}[0,\infty)$ be the set of all functions f defined on $[0,\infty)$ satisfying the condition $|f(x)| \le M_f(1+x^2)$, where M_f is the constant depending only on f. We denote the subspace of all continuous functions belonging to $B_{x^2}[0,\infty)$ by $C_{x^2}[0,\infty)$. Let $C^*_{x^2}[0,\infty)$ be the subspace of all functions $f \in C_{x^2}[0,\infty)$, for which $\lim_{x\to\infty} \frac{f(x)}{1+x^2}$ is finite. The norm on $C^*_{x^2}[0,\infty)$ is

$$||f||_{x^2} = \sup_{x\in[0,\infty)} \frac{|f(x)|}{1+x^2}.$$

We denote the usual modulus of continuity of f on the closed interval $[0,a]$ $(a>0)$ by

$$\omega_a(f;\delta) = \sup_{|t-x|\le\delta} \sup_{x,t\in[0,a]} |f(t)-f(x)|.$$

Obviously, for a function $f \in C_{x^2}[0,\infty)$, the modulus of continuity $\omega_a(f,\delta)$ tends to zero.

Theorem 4.2. *Let $f \in C_{x^2}[0,\infty)$, $\omega_{a+1}(f;\delta)$ be the modulus of continuity on the finite interval $[0,a+1] \subset [0,\infty)$, where $a>0$. Then for $n \ge 6$, we have*

$$||D_{n,p,q}(f)-f||_{C_{x^2}[0,a]} \le 4M_f(1+a^2)B_{n,p,q}(a) + 2\omega_{a+1}\left(f;\sqrt{B_{n,p,q}(a)}\right),$$

where $B_{n,p,q}(a)$ is defined in (16).

Proof. For $x \in [0,a]$ and $t > a+1$, we have

$$|f(t)-f(x)| \le M_f\left(2+x^2+t^2\right) \le M_f\left(2+3x^2+2(t-x)^2\right).$$

Since $t-x \ge t-a > 1$, then $(t-x)^2 > 1$. Thus $2+3x^2+2(t-x)^2 \le (2+3x^2)(t-x)^2 + 2(t-x)^2 = (4+3x^2)(t-x)^2 \le (4+3a^2)(t-x)^2 \le 4(1+a^2)(t-x)^2$. Thus, we obtain

$$|f(t)-f(x)| \le 4M_f(1+a^2)(t-x)^2. \tag{26}$$

For $x \in [0,a]$ and $t \le a+1$, we have

$$|f(t)-f(x)| \le \omega_{a+1}(f;|t-x|) \le \left(1+\frac{|t-x|}{\delta}\right)\omega_{a+1}(f;\delta),\ (\delta>0) \tag{27}$$

From (26) and (27), we get

$$|f(t)-f(x)| \le 4M_f(1+a^2)(t-x)^2 + \left(1+\frac{|t-x|}{\delta}\right)\omega_{a+1}(f;\delta). \tag{28}$$

For $x \in [0,a]$ and $t \ge 0$, by Schwarz's inequality and lemma 3.2, we have

$$\begin{aligned}
& |D_{n,p,q}(f;x)-f(x)| \\
\le\ & D_{n,p,q}(|f(t)-f(x)|;x) \\
\le\ & 4M_f(1+a^2)D_{n,p,q}\left((t-x)^2;x\right) + \omega_{a+1}(f;\delta)\left(1+\frac{1}{\delta}\sqrt{D_{n,p,q}\left((t-x)^2;x\right)}\right) \\
\le\ & 4M_f(1+a^2)B_{n,p,q}(x) + \omega_{a+1}(f;\delta)\left(1+\frac{\sqrt{B_{n,p,q}(x)}}{\delta}\right).
\end{aligned}$$

By taking $\delta = \sqrt{B_{n,p,q}(x)}$, we get the assertion of theorem 4.2. □

Now we discuss the weighted approximation theorem.

Theorem 4.3. *For $f \in C^*_{x^2}[0,\infty)$ and $n \geq 6$, we have*

$$\lim_{n\to\infty} ||D_{n,p_n,q_n}(f) - f||_{x^2} = 0. \tag{29}$$

Proof. By using the Korovkin theorem, we see that it is sufficient to verify the following three conditions

$$\lim_{n\to\infty} ||D_{n,p_n,q_n}(t^i;x) - x^i||_{x^2} = 0, \quad i = 0,1,2. \tag{30}$$

Since $D_{n,p_n,q_n}(1;x) = 1$ and $D_{n,p_n,q_n}(t;x) = x$, equality (30) holds true for $i = 0$ and $i = 1$. Finally, for $i = 2$, from lemma 3.2, we have

$$\begin{aligned}
& ||D_{n,p_n,q_n}(t^2;x) - x^2||_{x^2} \\
= & \sup_{x\in[0,\infty)} \frac{|D_{n,p_n,q_n}(t^2;x) - x^2|}{1+x^2} \\
\leq & \left(\frac{p_n^n}{q_n[n]_{p_n,q_n}} + \frac{p_n^{n-3}}{q_n[n-3]_{p_n,q_n}} + \frac{p_n^{2n-3}}{q_n^2[n-3]_{p_n,q_n}[n]_{p_n,q_n}} \right) \sup_{x\in[0,\infty)} \frac{x^2}{1+x^2} \\
& + \left(\frac{p_n^2+q_n^2}{p_n^3q_n^3[n-3]_{p_n,q_n}} + \frac{2p_n^{n-2}}{q_n^3[n-3]_{p_n,q_n}[n]_{p_n,q_n}} \right) \sup_{x\in[0,\infty)} \frac{x}{1+x^2} \\
& + \left(\frac{p_n^{n-4}}{q_n^4[n-2]_{p_n,q_n}[n-3]_{p_n,q_n}[n]_{p_n,q_n}} + \frac{1}{p_n^3q_n^4[n-2]_{p_n,q_n}[n-3]_{p_n,q_n}} \right) \sup_{x\in[0,\infty)} \frac{1}{1+x^2} \\
\leq & \frac{p_n^n}{q_n[n]_{p_n,q_n}} + \frac{p_n^2+q_n^2+p_n^nq_n^2}{p_n^3q_n^3[n-3]_{p_n,q_n}} + \frac{p_n^{2n-3}q_n + 2p_n^{n-2}}{q_n^3[n-3]_{p_n,q_n}[n]_{p_n,q_n}} + \frac{1}{p_n^3q_n^4[n-2]_{p_n,q_n}[n-3]_{p_n,q_n}} \\
& + \frac{p_n^{n-4}}{q_n^4[n-2]_{p_n,q_n}[n-3]_{p_n,q_n}[n]_{p_n,q_n}}.
\end{aligned}$$

We can obtain $\lim_{n\to\infty} ||D_{n,p_n,q_n}(t^2;x) - x^2|| = 0$ by using lemma 3.3 and $\lim_{n\to\infty} p_n^n = 1$, theorem 4.3 is proved. □

Table 1: The errors of the approximation of $D_{n,p_n,q_n}(t^2;x)$ with $p_n = 0.999999$ and different values of q_n and n .

q_n	$\lVert f(x) - D_{n,p_n,q_n}(f;x)\rVert_\infty$			
	$n=10$	$n=20$	$n=30$	$n=50$
0.95	0.756459	0.471385	0.396404	0.348978
0.99	0.545694	0.264869	0.185654	0.126539
0.999	0.502874	0.224749	0.146152	0.087113
0.9999	0.498679	0.220856	0.142352	0.083372
0.99999	0.498261	0.220468	0.141973	0.083000

Table 2: The errors of the approximation of $D_{n,p_n,q_n}(t^2;x)$ with $q_n = 0.99$ and different values of p_n and n .

$p_n = 1 - 1/10^m$	$\|f(x) - D_{n,p_n,q_n}(f;x)\|_\infty$			
	$n = 10$	$n = 20$	$n = 30$	$n = 50$
$m = 3$	0.545703746	0.264908767	0.185736472	0.126723749
$m = 4$	0.545694253	0.264872541	0.185660670	0.126555374
$m = 5$	0.545693781	0.264869729	0.185654271	0.126540622
$m = 6$	0.545693738	0.264869456	0.185653644	0.126539167
$m = 7$	0.545693733	0.264869429	0.185653581	0.126539022

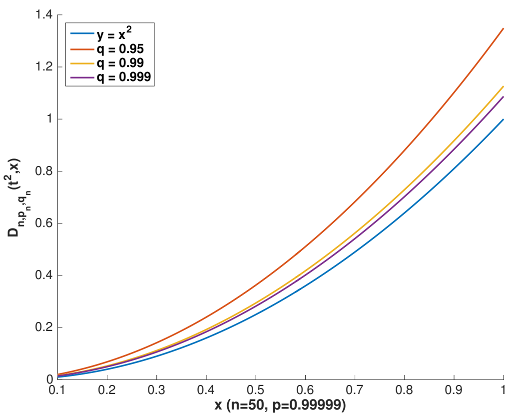


Figure 1: The figures of $D_{n,p_n,q_n}(t^2;x)$ for $n = 50$, $p_n = 0.99999$ and different values of q_n.

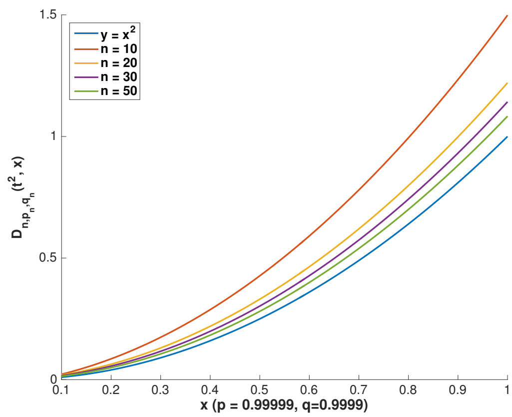


Figure 2: The figures of $D_{n,p_n,q_n}(t^2;x)$ for $p_n = 0.99999$, $q_n = 0.9999$ and different values of n.

5 Graphical and numerical analysis for one variable functions

In this section, we give several graphs and numerical examples to show the convergence of $D_{n,p_n,q_n}(f;x)$ to $f(x)$ with different values of parameters which satisfy the conclusions of lemma 3.3.

Let $f(x) = x^2$, the graphs of $D_{n,p_n,q_n}(f;x)$ with $n = 50$, $p_n = 0.99999$ and different values of q_n is shown in Figure 1. The graphs of $D_{n,p_n,q_n}(f;x)$ with $p_n = 0.99999$, $q_n = 0.9999$ and different values of n is shown in Figure 2. The graphs of $D_{n,p_n,q_n}(f;x)$ with $n = 50$, $q_n = 0.95$ and different values of p_n is shown in Figure 3. Moreover, we give the errors of the approximation of $D_{n,p_n,q_n}(f;x)$ to $f(x)$ with different parameters in Table 1 and Table 2.

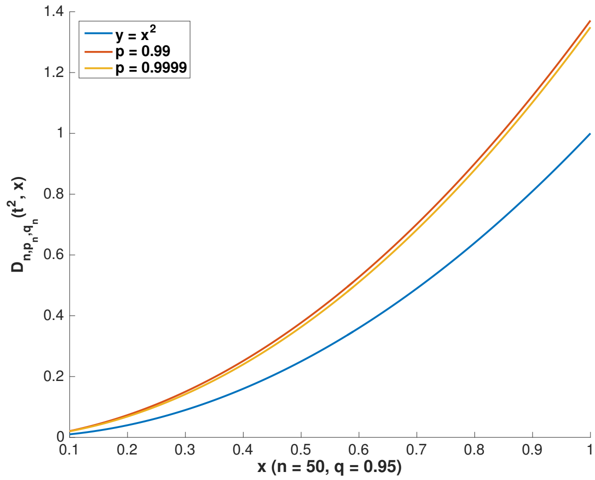


Figure 3: The figures of $D_{n,p_n,q_n}(t^2;x)$ for $n=50$, $q_n=0.95$ and different values of p_n.

6 Construction of bivariate operators and approximation properties

We introduce the bivariate tensor product $(p,\,q)$-analogue of Durrmeyer type Baskakov operators as follows

$$\begin{aligned} & D^{n_1,n_2}_{p_{n_1},q_{n_1},p_{n_2},q_{n_2}}(f;x,y) \\ = \; & [n_1-1]_{p_{n_1},q_{n_1}}[n_2-1]_{p_{n_2},q_{n_2}}\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\widetilde{b_{n_1,k_1}}(p_{n_1},q_{n_1};\mu(x))\widetilde{b_{n_2,k_2}}(p_{n_2},q_{n_2};\nu(y))\int_0^{\infty}\int_0^{\infty} \\ & \widetilde{b_{n_1,k_1}}(p_{n_1},q_{n_1};p_{n_1}u)\widetilde{b_{n_2,k_2}}(p_{n_2},q_{n_2};p_{n_2}v)f\left(p_{n_1}^{k_1}u,p_{n_2}^{k_2}v\right)d_{p_{n_1},q_{n_1}}u\,d_{p_{n_2},q_{n_2}}v, \end{aligned} \tag{31}$$

where

$$\begin{aligned} \mu(x) &= \frac{p_{n_1}^{n_1-2}q_{n_1}\left(p_{n_1}^2q_{n_1}[n_1-2]_{p_{n_1},q_{n_1}}x-1\right)}{[n_1]_{p_{n_1},q_{n_1}}},\ \left(x\geq\frac{1}{p_{n_1}^2q_{n_1}[n_1-2]_{p_{n_1},q_{n_1}}}\right), \\ \nu(y) &= \frac{p_{n_2}^{n_2-2}q_{n_2}\left(p_{n_2}^2q_{n_2}[n_2-2]_{p_{n_2},q_{n_2}}y-1\right)}{[n_2]_{p_{n_2},q_{n_2}}},\ \left(y\geq\frac{1}{p_{n_2}^2q_{n_2}[n_2-2]_{p_{n_2},q_{n_2}}}\right), \end{aligned}$$

$0<q_{n_1},q_{n_2}<p_{n_1},p_{n_2}\leq 1$ and $\widetilde{b_{n,k}}(p,q;x)$ is defined in (4).

Lemma 6.1. *Let $e_{i,j}(x,y)=x^iy^j, i,j\in\mathbb{N}$, $(x,y)\in([0,\infty)\times[0,\infty))$ be the two dimensional test functions and $n_1,n_2\geq 6$, using lemma 3.1, we easily obtain the following equalities*

$$D^{n_1,n_2}_{p_{n_1},q_{n_1},p_{n_2},q_{n_2}}(e_{0,0};x,y)=1, \tag{32}$$
$$D^{n_1,n_2}_{p_{n_1},q_{n_1},p_{n_2},q_{n_2}}(e_{1,0};x,y)=x, \tag{33}$$

$$
\begin{aligned}
&D_{p_{n_1},q_{n_1},p_{n_2},q_{n_2}}^{n_1,n_2}(e_{0,1};x,y) = y, && (34)\\
&D_{p_{n_1},q_{n_1},p_{n_2},q_{n_2}}^{n_1,n_2}(e_{1,1};x,y) = xy, && (35)\\
&D_{p_{n_1},q_{n_1},p_{n_2},q_{n_2}}^{n_1,n_2}(e_{2,0};x,y) = \frac{[n_1-2]_{p_{n_1},q_{n_1}}[n_1+1]_{p_{n_1},q_{n_1}}}{q_{n_1}^2[n_1-3]_{p_{n_1},q_{n_1}}[n_1]_{p_{n_1},q_{n_1}}}x^2 + \frac{\left(p_{n_1}^2+q_{n_1}^2\right)}{p_{n_1}^3 q_{n_1}^3[n_1-3]_{p_{n_1},q_{n_1}}}x\\
&-\frac{2p_{n_1}^{n_1-2}}{q_{n_1}^3[n_1-3]_{p_{n_1},q_{n_1}}[n_1]_{p_{n_1},q_{n_1}}}x + \frac{p_{n_1}^{n_1-4}}{q_{n_1}^4[n_1-2]_{p_{n_1},q_{n_1}}[n_1-3]_{p_{n_1},q_{n_1}}[n_1]_{p_{n_1},q_{n_1}}}\\
&-\frac{1}{p_{n_1}^3 q_{n_1}^4[n_1-2]_{p_{n_1},q_{n_1}}[n_1-3]_{p_{n_1},q_{n_1}}}, && (36)\\
&D_{p_{n_1},q_{n_1},p_{n_2},q_{n_2}}^{n_1,n_2}(e_{0,2};x,y) = \frac{[n_2-2]_{p_{n_2},q_{n_2}}[n_2+1]_{p_{n_2},q_{n_2}}}{q_{n_2}^2[n_2-3]_{p_{n_2},q_{n_2}}[n_2]_{p_{n_2},q_{n_2}}}y^2 + \frac{\left(p_{n_2}^2+q_{n_2}^2\right)}{p_{n_2}^3 q_{n_2}^3[n_2-3]_{p_{n_2},q_{n_2}}}y\\
&-\frac{2p_{n_2}^{n_2-2}}{q_{n_2}^3[n_2-3]_{p_{n_2},q_{n_2}}[n_2]_{p_{n_2},q_{n_2}}}y + \frac{p_{n_2}^{n_2-4}}{q_{n_2}^4[n_2-2]_{p_{n_2},q_{n_2}}[n_2-3]_{p_{n_2},q_{n_2}}[n_2]_{p_{n_2},q_{n_2}}}\\
&-\frac{1}{p_{n_2}^3 q_{n_2}^4[n_2-2]_{p_{n_2},q_{n_2}}[n_2-3]_{p_{n_2},q_{n_2}}}. && (37)
\end{aligned}
$$

Lemma 6.2. *For sufficiently large n_1 and n_2, using lemma 6.1 and lemma 3.2, we have the following statements*

$$
\begin{aligned}
D_{p_{n_1},q_{n_1},p_{n_2},q_{n_2}}^{n_1,n_2}(t-x;x,y) &= 0,\\
D_{p_{n_1},q_{n_1},p_{n_2},q_{n_2}}^{n_1,n_2}(s-y;x,y) &= 0,\\
D_{p_{n_1},q_{n_1},p_{n_2},q_{n_2}}^{n_1,n_2}((t-x)^2;x,y) &= O\left(\frac{1}{[n_1]_{p_{n_1},q_{n_1}}}\right)\left(x^2+x+1\right) = O\left(\frac{1}{[n_1]_{p_{n_1},q_{n_1}}}\right)(x+1)^2,\\
D_{p_{n_1},q_{n_1},p_{n_2},q_{n_2}}^{n_1,n_2}((s-y)^2;x,y) &= O\left(\frac{1}{[n_2]_{p_{n_2},q_{n_2}}}\right)\left(y^2+y+1\right) = O\left(\frac{1}{[n_2]_{p_{n_2},q_{n_2}}}\right)(y+1)^2,\\
D_{p_{n_1},q_{n_1},p_{n_2},q_{n_2}}^{n_1,n_2}((t-x)^4;x,y) &= O\left(\frac{1}{[n_1]_{p_{n_1},q_{n_1}}}\right)\left(x^4+x^3+x^2+x+1\right)\\
&= O\left(\frac{1}{[n_1]_{p_{n_1},q_{n_1}}}\right)(x+1)^4,\\
D_{p_{n_1},q_{n_1},p_{n_2},q_{n_2}}^{n_1,n_2}((s-y)^4;x,y) &= O\left(\frac{1}{[n_2]_{p_{n_2},q_{n_2}}}\right)\left(y^4+y^3+y^2+y+1\right)\\
&= O\left(\frac{1}{[n_2]_{p_{n_2},q_{n_2}}}\right)(y+1)^4.
\end{aligned}
$$

Let B_ρ be the space of all functions f defined on $[0,\infty)\times[0,\infty)$ satisfying the condition $|f(x)| \leq M_f\rho(x,y)$, where M_f is a positive constant depending only on f and $\rho(x,y) = 1+x^2+y^2$ is a weighted function. We denote the subspace of all continuous functions belong to B_ρ by C_ρ. Let C_ρ^* be the subspace of all functions $f \in C_\rho$, for which $\lim_{\sqrt{x^2+y^2}\to\infty} \frac{f(x,y)}{\rho(x,y)}$ is finite. The norm on C_ρ^* is $||f||_\rho = \sup_{x,y\in[0,\infty)} \frac{|f(x,y)|}{\rho(x,y)}$. For the infinite interval $[0,\infty)$,

$f \in C_\rho^*$ and $\delta_1, \delta_2 > 0$, İspir and Atakut [28] introduced the weighted modulus of continuity as

$$\Omega_\rho(f;\delta_1,\delta_2) = \sup_{x,y\in[0,\infty)} \sup_{0\leq|k_1|\leq\delta_1, 0\leq|k_2|\leq\delta_2} \frac{|f(x+k_1,y+k_2)-f(x,y)|}{\rho(x,y)\rho(k_1,k_2)},$$

which satisfy the following inequality

$$\Omega_\rho(f;d_1\delta_1,d_2\delta_2) \leq 4(1+d_1)(1+d_2)(1+\delta_1^2)(1+\delta_2^2)\Omega_\rho(f;\delta_1,\delta_2), \quad d_1,d_2>0. \tag{38}$$

From the definition of Ω_ρ, we have

$$\begin{aligned} & |f(t,s)-f(x,y)| \\ \leq\ & \rho(x,y)\rho(|t-x|,|s-y|)\Omega_\rho(f;|t-x|,|s-y|) \\ \leq\ & \left(1+x^2+y^2\right)\left(1+(t-x)^2\right)\left(1+(s-y)^2\right)\Omega_\rho(f;|t-x|,|s-y|) \end{aligned} \tag{39}$$

Now, we establish the degree approximation of operators $D^{n_1,n_2}_{p_{n_1},q_{n_1},p_{n_2},q_{n_2}}$ in the weighted space C_ρ^* by the weighted modulus of continuity Ω_ρ.

Theorem 6.3. *For $f \in C_\rho^*$, then for sufficiently large n_1, n_2, we have the following inequality*

$$\sup_{x,y\in[0,\infty)} \frac{D^{n_1,n_2}_{p_{n_1},q_{n_1},p_{n_2},q_{n_2}}(f;x,y)-f(x,y)}{\left(\rho(x,y)\right)^3} \leq C\Omega_\rho\left(f;\frac{1}{\sqrt{[n_1]_{p_{n_1},q_{n_1}}}},\frac{1}{\sqrt{[n_2]_{p_{n_2},q_{n_2}}}}\right),$$

where C is a positive constant.

Proof. From (38) and (39), for $\delta_{n_1}, \delta_{n_2} > 0$, we get

$$\begin{aligned} & |f(t,s)-f(x,y)| \\ =\ & 4\left(1+x^2+y^2\right)\left(1+(t-x)^2\right)\left(1+(s-y)^2\right)\left(1+\frac{|t-x|}{\delta_{n_1}}\right)\left(1+\frac{|s-y|}{\delta_{n_2}}\right)\left(1+\delta_{n_1}^2\right) \\ & \times\left(1+\delta_{n_2}^2\right)\Omega_\rho(f;\delta_{n_1},\delta_{n_2}) \\ =\ & 4\left(1+x^2+y^2\right)\left(1+\delta_{n_1}^2\right)\left(1+\delta_{n_2}^2\right)\left(1+\frac{|t-x|}{\delta_{n_1}}+(t-x)^2+\frac{|t-x|}{\delta_{n_1}}(t-x)^2\right) \\ & \times\left(1+\frac{|s-y|}{\delta_{n_2}}+(s-y)^2+\frac{|s-y|}{\delta_{n_2}}(s-y)^2\right)\Omega_\rho(f;\delta_{n_1},\delta_{n_2}), \end{aligned}$$

applying the operators $D^{n_1,n_2}_{p_{n_1},q_{n_1},p_{n_2},q_{n_2}}$ on the above inequality, we have

$$\begin{aligned} & |D^{n_1,n_2}_{p_{n_1},q_{n_1},p_{n_2},q_{n_2}}(f;x,y)-f(x,y)| \\ \leq\ & D^{n_1,n_2}_{p_{n_1},q_{n_1},p_{n_2},q_{n_2}}(|f(t,s)-f(x,y)|;x,y) \\ \leq\ & 4(1+x^2+y^2)\left(1+\delta_{n_1}^2\right)\left(1+\delta_{n_2}^2\right)D^{n_1,n_2}_{p_{n_1},q_{n_1},p_{n_2},q_{n_2}}\left(1+\frac{|t-x|}{\delta_{n_1}}+(t-x)^2\right. \\ & \left.+\frac{|t-x|}{\delta_{n_1}}(t-x)^2;x,y\right)D^{n_1,n_2}_{p_{n_1},q_{n_1},p_{n_2},q_{n_2}}\left(1+\frac{|s-y|}{\delta_{n_2}}+(s-y)^2+\frac{|s-y|}{\delta_{n_2}}(s-y)^2;x,y\right) \end{aligned}$$

$$
\begin{aligned}
&\times\Omega_\rho(f;\delta_{n_1},\delta_{n_2})\\
= \;& 4\left(1+x^2+y^2\right)\left(1+\delta_{n_1}^2\right)\left(1+\delta_{n_2}^2\right)\left(1+\frac{D^{n_1,n_2}_{p_{n_1},q_{n_1},p_{n_2},q_{n_2}}(|t-x|;x,y)}{\delta_{n_1}}\right.\\
&\left.+D^{n_1,n_2}_{p_{n_1},q_{n_1},p_{n_2},q_{n_2}}\left((t-x)^2;x,y\right)+\frac{D^{n_1,n_2}_{p_{n_1},q_{n_1},p_{n_2},q_{n_2}}\left(|t-x|(t-x)^2;x,y\right)}{\delta_{n_1}}\right)\\
&+\left(1+\frac{D^{n_1,n_2}_{p_{n_1},q_{n_1},p_{n_2},q_{n_2}}(|s-y|;x,y)}{\delta_{n_2}}+D^{n_1,n_2}_{p_{n_1},q_{n_1},p_{n_2},q_{n_2}}\left((s-y)^2;x,y\right)\right.\\
&\left.+\frac{D^{n_1,n_2}_{p_{n_1},q_{n_1},p_{n_2},q_{n_2}}\left(|s-y|(s-y)^2;x,y\right)}{\delta_{n_2}}\right)\Omega_\rho(f;\delta_{n_1},\delta_{n_2}).
\end{aligned}
$$

Using Cauchy-Schwarz inequality, we get

$$
\begin{aligned}
&|D^{n_1,n_2}_{p_{n_1},q_{n_1},p_{n_2},q_{n_2}}(f;x,y)-f(x,y)|\\
\leq\;& 4(1+x^2+y^2)\left(1+\frac{\sqrt{D^{n_1,n_2}_{p_{n_1},q_{n_1},p_{n_2},q_{n_2}}\left((t-x)^2;x,y\right)}}{\delta_{n_1}}+D^{n_1,n_2}_{p_{n_1},q_{n_1},p_{n_2},q_{n_2}}\left((t-x)^2;x,y\right)\right.\\
&\left.+\frac{\sqrt{D^{n_1,n_2}_{p_{n_1},q_{n_1},p_{n_2},q_{n_2}}\left((t-x)^2;x,y\right)}\sqrt{D^{n_1,n_2}_{p_{n_1},q_{n_1},p_{n_2},q_{n_2}}\left((t-x)^4;x,y\right)}}{\delta_{n_1}}\right)\\
&\times\left(1+\delta_{n_1}^2\right)\left(1+\delta_{n_2}^2\right)\left(1+\frac{\sqrt{D^{n_1,n_2}_{p_{n_1},q_{n_1},p_{n_2},q_{n_2}}\left((s-y)^2;x,y\right)}}{\delta_{n_2}}\right.\\
&+D^{n_1,n_2}_{p_{n_1},q_{n_1},p_{n_2},q_{n_2}}\left((s-y)^2;x,y\right)+\sqrt{D^{n_1,n_2}_{p_{n_1},q_{n_1},p_{n_2},q_{n_2}}\left((s-y)^2;x,y\right)}\\
&\left.\times\frac{\sqrt{D^{n_1,n_2}_{p_{n_1},q_{n_1},p_{n_2},q_{n_2}}\left((s-y)^4;x,y\right)}}{\delta_{n_2}}\right)\Omega_\rho(f;\delta_{n_1},\delta_{n_2}).
\end{aligned}
$$

Using lemma 6.2, we have

$$
\begin{aligned}
&|D^{n_1,n_2}_{p_{n_1},q_{n_1},p_{n_2},q_{n_2}}(f;x,y)-f(x,y)|\\
\leq\;& 4(1+x^2+y^2)(1+\delta_{n_1}^2)(1+\delta_{n_2}^2)\left(1+\frac{1}{\delta_{n_1}}\sqrt{O\left(\frac{1}{[n_1]_{p_{n_1},q_{n_1}}}\right)(x+1)^2}\right.\\
&\left.+O\left(\frac{1}{[n_1]_{p_{n_1},q_{n_1}}}\right)(x+1)^2+\frac{1}{\delta_{n_1}}\sqrt{O\left(\frac{1}{[n_1]_{p_{n_1},q_{n_1}}}\right)(x+1)^2}\sqrt{O\left(\frac{1}{[n_1]_{p_{n_1},q_{n_1}}}\right)(x+1)^4}\right)\\
&\times\left(1+\frac{1}{\delta_{n_2}}\sqrt{O\left(\frac{1}{[n_2]_{p_{n_2},q_{n_2}}}\right)(y+1)^2}+O\left(\frac{1}{[n_2]_{p_{n_2},q_{n_2}}}\right)(y+1)^2\right.\\
&\left.+\frac{1}{\delta_{n_2}}\sqrt{O\left(\frac{1}{[n_2]_{p_{n_2},q_{n_2}}}\right)(y+1)^2}\sqrt{O\left(\frac{1}{[n_2]_{p_{n_2},q_{n_2}}}\right)(y+1)^4}\right)\Omega_\rho(f;\delta_{n_1},\delta_{n_2}).
\end{aligned}
$$

Then, we have

$$\begin{aligned}
&|D^{n_1,n_2}_{p_{n_1},q_{n_1},p_{n_2},q_{n_2}}(f;x,y)-f(x,y)| \\
\leq\ & 4(1+x^2+y^2)(1+\delta_{n_1}^2)(1+\delta_{n_2}^2)\left(1+\frac{1}{\delta_{n_1}}\sqrt{\frac{C_1}{[n_1]_{p_{n_1},q_{n_1}}}(x+1)+\frac{C_1}{[n_1]_{p_{n_1},q_{n_1}}}(x+1)^2}\right. \\
&\left.+\frac{1}{\delta_{n_1}}\sqrt{\frac{C_1^2}{[n_1]^2_{p_{n_1},q_{n_1}}}(x+1)^3}\right)\left(1+\delta_{n_2}\sqrt{\frac{C_2}{[n_2]_{p_{n_2},q_{n_2}}}(y+1)+\frac{C_2}{[n_2]_{p_{n_2},q_{n_2}}}(y+1)^2}\right. \\
&\left.+\frac{1}{\delta_{n_2}}\sqrt{\frac{C_2^2}{[n_2]^2_{p_{n_2},q_{n_2}}}(y+1)^3}\right)\Omega_\rho(f;\delta_{n_1},\delta_{n_2}).
\end{aligned}$$

Let $\delta_{n_1}=\frac{1}{\sqrt{[n_1]_{p_{n_1},q_{n_1}}}}$ and $\delta_{n_2}=\frac{1}{\sqrt{[n_2]_{p_{n_2},q_{n_2}}}}$, we have

$$\begin{aligned}
&|D^{n_1,n_2}_{p_{n_1},q_{n_1},p_{n_2},q_{n_2}}(f;x,y)-f(x,y)| \\
\leq\ & 4(1+x^2+y^2)\left(1+\frac{1}{[n_1]_{p_{n_1},q_{n_1}}}\right)\left(1+\frac{1}{[n_2]_{p_{n_2},q_{n_2}}}\right)C(1+x^2+y^2)^2\Omega_\rho(f;\delta_{n_1},\delta_{n_2}),
\end{aligned}$$

where C is a positive constant. Theorem 6.3 is proved. □

7 Graphical and numerical analysis for two variables functions

In this section, we give several graphs and numerical examples to show the convergence of $D^{n_1,n_2}_{p_{n_1},q_{n_1},p_{n_2},q_{n_2}}(f;x,y)$ to $f(x,y)$ with different values of parameters which satisfy the conclusions of lemma 3.3.

Let $f(x,y)=x^2y^2$, the graphs of $D^{n_1,n_2}_{p_{n_1},q_{n_1},p_{n_2},q_{n_2}}(f;x,y)$ with $n_1=n_2=30$, $p_{n_1}=p_{n_2}=0.9999$, $q_{n_1}=q_{n_2}=0.999$ and $f(x,y)=x^2y^2$ are shown in Figure 4. The graphs of $D^{n_1,n_2}_{p_{n_1},q_{n_1},p_{n_2},q_{n_2}}(f;x,y)$ with $n_1=n_2=50$, $p_{n_1}=p_{n_2}=0.99999$, $q_{n_1}=q_{n_2}=0.9999$ and $f(x,y)=x^2y^2$ are shown in Figure 5. Moreover, we give the errors of the approximation of $D^{n_1,n_2}_{p_{n_1},q_{n_1},p_{n_2},q_{n_2}}(f;x,y)$ to $f(x,y)$ with different parameters in Table 3 and Table 4.

Table 3: The errors of the approximation of $D^{n_1,n_2}_{p_{n_1},q_{n_1},p_{n_2},q_{n_2}}(f;x,y)$ with $p_{n_1}=p_{n_2}=0.99999$ and different values of $q_{n_1}=q_{n_2}=q$ and $n_1=n_2=n$.

q	$\|f(x)-D^{n_1,n_2}_{p_{n_1},q_{n_1},p_{n_2},q_{n_2}}(f;x,y)\|_\infty$			
	$n=10$	$n=20$	$n=30$	$n=50$
0.95	2.085144	1.164978	0.949957	0.819780
0.99	1.389169	0.599895	0.405776	0.269094
0.999	1.258631	0.500011	0.313666	0.181816
0.9999	1.246041	0.490490	0.3049678	0.173697

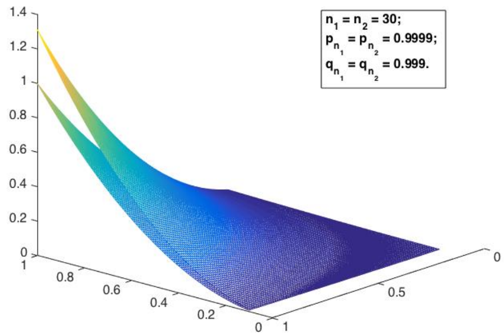


Figure 4: The figures of (the upper one) $D^{n_1,n_2}_{p_{n_1},q_{n_1},p_{n_2},q_{n_2}}(f;x,y)$ for $n_1 = n_2 = 30$, $p_{n_1} = p_{n_2} = 0.9999$, $q_{n_1} = q_{n_2} = 0.999$, and (the below one) $f(x,y) = x^2y^2$.

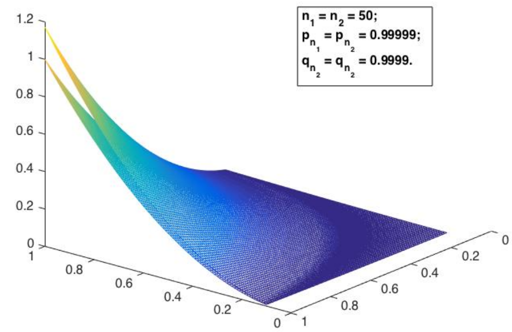


Figure 5: The figures of (the upper one) $D^{n_1,n_2}_{p_{n_1},q_{n_1},p_{n_2},q_{n_2}}(f;x,y)$ for $n_1 = n_2 = 50$, $p_{n_1} = p_{n_2} = 0.99999$, $q_{n_1} = q_{n_2} = 0.9999$, and (the below one) $f(x,y) = x^2y^2$.

Table 4: The errors of the approximation of $D^{n_1,n_2}_{p_{n_1},q_{n_1},p_{n_2},q_{n_2}}(f;x,y)$ with $q_{n_1} = q_{n_2} = 0.999$ and different values of $p_{n_1} = p_{n_2} = p$ and $n_1 = n_2 = n$.

$p = 1 - 1/10^m$	$\|f(x) - D^{n_1,n_2}_{p_{n_1},q_{n_1},p_{n_2},q_{n_2}}(f;x,y)\|_\infty$			
	$n = 10$	$n = 20$	$n = 30$	$n = 50$
$m = 4$	1.25863727	0.50001245	0.31366763	0.18181861
$m = 5$	1.25863086	0.50001052	0.31366566	0.18181572
$m = 6$	1.25863023	0.50001034	0.31366548	0.18181546
$m = 7$	1.25863016	0.50001033	0.31366546	0.18181544

8 Further discussion

If we consider the following modified forms $\widetilde{D_{n,p,q}}$,

$$\widetilde{D_{n,p,q}}(f;x) = [n-1]_{p,q} \sum_{k=0}^{\infty} \widetilde{b_{n,k}}(p,q;x) \int_0^{\infty} \widetilde{b_{n,k}}(p,q;pu) f(p^k u) d_{p,q}u, \tag{40}$$

where $x \in [0,\infty)$, $\widetilde{b_{n,k}}(p,q;x)$ is defined in (4). Here we omit the bivariate forms of operators (40). By similar computations in section 3, we know these operators (40) reproduce only constant functions, but not linear functions. We also provide two graphs to show that the operators $D_{n,p,q}$ give a better approximation to f than $\widetilde{D_{n,p,q}}$ and so is the bivariate case (See Figure 6 and Figure 7), hence it is more appropriate to consider the operators $D_{n,p,q}$ and the bivariate ones defined in (31).

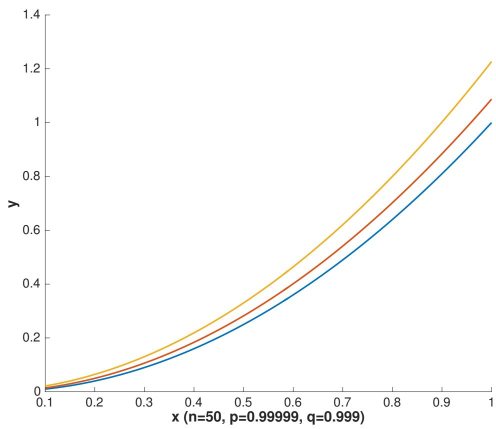


Figure 6: The figures of $D_{n,p_n,q_n}(f;x)$ (the red one), $\widetilde{D}_{n,p_n,q_n}(f;x)$ (the yellow one)for $n = 50$, $p_n = 0.99999$, $q_n = 0.999$, and $f(x) = x^2$ (the blue one).

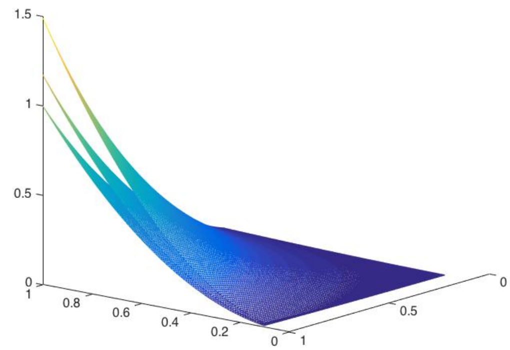


Figure 7: The figures of (the middle one) $D^{n_1,n_2}_{p_{n_1},q_{n_1},p_{n_2},q_{n_2}}(f;x,y)$ and $\widetilde{D}^{n_1,n_2}_{p_{n_1},q_{n_1},p_{n_2},q_{n_2}}(f;x,y)$ (the upper one) for $n_1 = n_2 = 50$, $p_{n_1} = p_{n_2} = 0.99999$, $q_{n_1} = q_{n_2} = 0.9999$, and $f(x,y) = x^2y^2$ (the below one).

Acknowledgement

This work is supported by the National Natural Science Foundation of China (Grant Nos. 11601266 and 11626201), the Natural Science Foundation of Fujian Province of China (Grant No. 2016J05017) and the Program for New Century Excellent Talents in Fujian Province University. We also thank Fujian Provincial Key Laboratory of Data Intensive Computing and Key Laboratory of Intelligent Computing and Information Processing of Fujian Province University.

References

[1] M. Mursaleen, K. J. Ansari, A. Khan, On (p, q)-analogue of Bernstein operators. Appl. Math. Comput., **266** (2015), 874-882.

[2] M. Mursaleen, K. J. Ansari, A. Khan, Erratum to "On (p, q)-analogue of Bernstein operators [Appl. Math. Comput. 266 (2015) 874-882]", Appl. Math. Comput., **278** (2016), 70-71.

[3] T. Acar, (p, q)-Generalization of Szász-Mirakyan operators, Math. Methods Appl. Sci., **39(10)** (2016), 2685-2695.

[4] T. Acar, A. Aral, S. A. Mohiuddine, On Kantorovich modification of (p, q)-Baskakov operators, J. Ineq. App., (2016), Doi: 10.1186/s13660-016-1045-9.

[5] V. Gupta, (p, q)-Baskakov-Kantorovich operators, Appl. Math. Inf. Sci., **10(4)** (2016), 1551-1556.

[6] N. Malik, V. Gupta, Approximation by (p, q)-Baskakov-Beta operators, Appl. Math. Comput., **293** (2017), 49-56.

[7] Q. -B. Cai, G. Zhou, On (p, q)-analogue of Kantorovich type Bernstein-Stancu-Schurer operators, Appl. Math. Comput., **276** (2016), 12-20.

[8] T. Acar, On pointwise convergence of q-Bernstein operators and their q-derivatives, Nurmer. Funct. Anal. Optim., **36(3)** (2015), 287-304.

[9] T. Acar, P. Agrawal, A. Kumar, On a modification of (p, q)-Szász-Mirakyan operators, Complex Anal. Oper. Theory, (2016), Doi: 10.1007/s11785-016-0613-9.

[10] H. Ilarslan, T. Acar, Approximation by bivariate (p, q)-Baskakov-Kantorovich operators, Georgian Math. J., (2016), Doi: 10.1515/gmj-2016-0057.

[11] G. M. Phillips, Bernstein polynomials based on the q-integers, Ann. Number. Math., **4** (1997), 511-518.

[12] V. Gupta, T. Kim, On the rate of approximation by q modified Beta operators, J. Math. Anal. Appl., **377** (2011), 471-480.

[13] V. Gupta, A. Aral, Convergence of the q analogue of Szász-Beta operators, Appl. Math. Comput., **216** (2010), 374-380.

[14] K. Khan, D. K. Lobiyal, Bézier curves based on Lupas (p, q)-analogue of Bernstein functions in CAGD, J. Comput. Appl. Math., **317** (2017), 458-477.

[15] A. Aral, V. Gupta, On the Durrmeyer type modification of the q-Baskakov type operators, Nonlinear Anal., **72** (2010), 1171-1180.

[16] V. Gupta, On certain Durrmeyer type q Baskakov operators, Ann. Univ. Ferrara, **56** (2010), 295-303.

[17] Q. -B. Cai, X. -M. Zeng, Convergence of modification of the Durrmeyer type q-Baskakov operators, Georgian Math. J., **19** (2012), 49-61.

[18] T. Acar, A. Aral, M. Mursaleen, Approximation by Baskakov-Durrmeyer operators based on (p, q)-integers, arXiv: submit/1450876 [math. CA].

[19] V. N. Mishra, S. Pandey, On (p, q)-Baskakov-Durrmeyer-Stancu operators, (2016), arXiv: 1602. 06719.

[20] M. N. Hounkonnou, J. Désiré, B. Kyemba, $R(p, q)$-calculus: differentiation and integration, SUT Journal of Mathematics, **49** (2013), 145-167.

[21] R. Jagannathan, K. S. Rao, Two-parameter quantum algebras, twin-basic numbers, and associated generalized hypergeometric series, Proceedings of the International Conference on Number Theory and Mathematical Physics, (2005) 20-21.

[22] J. Katriel, M. Kibler, Normal ordering for deformed boson operators and operator-valued deformed Stirling numbers, J. Phys. A: Math. Gen. (1992) 24, 2683-2691, printed in the UK.

[23] P. N. Sadjang, On the fundamental theorem of (p, q)-calculus and some (p, q)-Taylor formulas, (2015) arXiv: 1309.3934v1.

[24] V. Sahai, S. Yadav, Representations of two parameter quantum algebras and p, q-special functions, J. Math. Anal. Appl., **335**(2007), 268-279.

[25] Q. -B. Cai, X. -W. Xu, A basic problem of (p, q)-Bernstein operators, J. Ineq. Appl., **140** (2017), Doi: 10. 1186/s13660-017-1413-0.

[26] G. A. Anastassiou, S. G. Gal, Approximation theory: moduli of continuity and global smoothness preservation, Birkhauser, Boston, 2000.

[27] R. A. DeVore, G. G. Lorentz, Constructive Approximation, Springer, Berlin, 1993.

[28] N. İspir, C. Atakut, Approximation by modified Szász-Mirakjan operators on weighted spaces, Proc. Indian Acad. Sci. Math. Sci., **112(4)** (2002), 571-578.

A subclass of analytic functions defined by a fractional integral operator

Alb Lupaş Alina
Department of Mathematics and Computer Science
University of Oradea
str. Universitatii nr. 1, 410087 Oradea, Romania
dalb@uoradea.ro, alblupas@gmail.com

Abstract

Making use the fractional integral associated with the convolution product of Sălăgean operator and Ruscheweyh derivative, we introduce a new class of analytic functions $\mathcal{D}(\mu,\lambda,\alpha,\beta)$ defined on the open unit disc, and investigate its various characteristics. Further we obtain distortion bounds, extreme points and radii of close-to-convexity, starlikeness and convexity for functions belonging to the class $\mathcal{D}(\mu,\lambda,\alpha,\beta)$.

Keywords: Analytic functions, univalent functions, radii of starlikeness and convexity, neighborhood property, Salagean operator, Ruscheweyh operator.
2000 Mathematical Subject Classification: 30C45, 30A20, 34A40.

1 Introduction

Denote by U the unit disc of the complex plane, $U=\{z\in\mathbb{C}:\ |z|<1\}$ and $\mathcal{H}(U)$ the space of holomorphic functions in U.

Let $\mathcal{A}(p,t)=\{f\in\mathcal{H}(U):\ f(z)=z^p+\sum_{j=p+t}^{\infty}a_jz^j,\ \ z\in U\}$, with $\mathcal{A}(1,t)=\mathcal{A}_t$ and $\mathcal{H}[a,t]=\{f\in\mathcal{H}(U):\ f(z)=a+a_tz^t+a_{t+1}z^{t+1}+\ldots,\ z\in U\}$, where $p,t\in\mathbb{N}$, $a\in\mathbb{C}$.

Definition 1.1 *(Sălăgean [4]) For $f\in\mathcal{A}_t$, and $n\in\mathbb{N}$, the operator S^n is defined by $S^n:\mathcal{A}_t\to\mathcal{A}_t$,*

$$S^0f(z)=f(z),\ \ S^1f(z)=zf'(z),...,\ \ S^{n+1}f(z)=z\left(S^nf(z)\right)',\ \ z\in U.$$

Remark 1.1 *If $f\in\mathcal{A}_t$, $f(z)=z+\sum_{j=t+1}^{\infty}a_jz^j$, then $S^nf(z)=z+\sum_{j=t+1}^{\infty}j^na_jz^j$, $z\in U$.*

Definition 1.2 *(Ruscheweyh [3]) For $f\in\mathcal{A}_t$ and $n\in\mathbb{N}$, the operator R^n is defined by $R^n:\mathcal{A}_t\to\mathcal{A}_t$,*

$$R^0f(z)=f(z),\ \ R^1f(z)=zf'(z),...,\ \ (n+1)R^{n+1}f(z)=z\left(R^nf(z)\right)'+nR^nf(z),\ \ z\in U.$$

Remark 1.2 *If $f\in\mathcal{A}_t$, $f(z)=z+\sum_{j=t+1}^{\infty}a_jz^j$, then $R^nf(z)=z+\sum_{j=t+1}^{\infty}\frac{\Gamma(n+j)}{\Gamma(n+1)\Gamma(j)}a_jz^j$ for $z\in U$.*

Definition 1.3 *Let $n,m\in\mathbb{N}$. Denote by $SR_{\lambda}^{m,n}:\mathcal{A}_t\to\mathcal{A}_t$ the operator given by the Hadamard product of the Sălăgean operator S^m and the Ruscheweyh derivative R^n, $SR^{m,n}f(z)=(S^m*R^n)f(z)$, for any $z\in U$ and each nonnegative integers m,n.*

Remark 1.3 *If $f\in\mathcal{A}_t$ and $f(z)=z+\sum_{j=t+1}^{\infty}a_jz^j$, then $SR^{m,n}f(z)=z+\sum_{j=t+1}^{\infty}j^m\frac{\Gamma(n+j)}{\Gamma(n+1)\Gamma(j)}a_j^2z^j$, $z\in U$.*

Definition 1.4 *([2]) The fractional integral of order λ ($\lambda>0$) is defined for a function f by $D_z^{-\lambda}f(z)=\frac{1}{\Gamma(\lambda)}\int_0^z\frac{f(t)}{(z-t)^{1-\lambda}}dt$, where f is an analytic function in a simply-connected region of the z-plane containing the origin, and the multiplicity of $(z-t)^{\lambda-1}$ is removed by requiring $\log(z-t)$ to be real, when $(z-t)>0$.*

From Definition 1.3 and Definition 1.4, we get the fractional integral associated with the convolution product of Sălăgean operator and Ruscheweyh derivative, which has the following form

$$D_z^{-\lambda}SR^{m,n}f(z)=\frac{1}{\Gamma(\lambda+2)}z^{\lambda+1}+\sum_{j=t+1}^{\infty}\frac{j^{m+1}\Gamma(n+j)}{\Gamma(n+1)\Gamma(j+\lambda+1)}a_j^2z^{j+\lambda},$$

for a function $f(z)=z+\sum_{j=t+1}^{\infty}a_jz^j\in\mathcal{A}_t$.

Following the work from [1] we can define the class $\mathcal{D}(\mu,\lambda,\alpha,\beta)$ as follows.

Definition 1.5 *For $\mu \geq 0$, $\lambda \in \mathbb{N}$, $\alpha \in \mathbb{C}-\{0\}$ and $0<\beta \leq 1$, let $\mathcal{D}(\mu,\lambda,\alpha,\beta)$ be the subclass of $\mathcal{A}_t$ consisting of functions that satisfying the inequality*

$$\left|\frac{\lambda\left(1-\mu\right)\frac{D_z^{-\lambda}SR^{m,n}f(z)}{z}+\mu\left(D_z^{-\lambda}SR^{m,n}f\left(z\right)\right)'}{\lambda\left(1-\mu\right)\frac{D_z^{-\lambda}SR^{m,n}f(z)}{z}+\mu\left(D_z^{-\lambda}SR^{m,n}f\left(z\right)\right)'-\alpha}\right|<\beta \tag{1.1}$$

2 Coefficient bounds

In this section we obtain coefficient bounds and extreme points for functions is $\mathcal{D}(\mu,\lambda,\alpha,\beta)$.

Theorem 2.1 *Let the function $f \in \mathcal{A}_t$. Then $f \in \mathcal{D}(\mu,\lambda,\alpha,\beta)$ if and only if*

$$\sum_{j=t+1}^{\infty}\frac{\left(\beta+1\right)\left(\lambda+\mu j\right)j^{m+1}\Gamma\left(n+j\right)}{\Gamma\left(n+1\right)\Gamma\left(j+\lambda+1\right)}a_j^2<\beta\left|\alpha\right|-\frac{\left(\beta+1\right)\left(\lambda+\mu\right)}{\Gamma\left(\lambda+2\right)}. \tag{2.1}$$

The result is sharp for the function $F(z)$ defined by $F(z)=z+\sqrt{\frac{\left(\beta|\alpha|-\frac{(\beta+1)(\lambda+\mu)}{\Gamma(\lambda+2)}\right)\Gamma(n+1)\Gamma(j+\lambda+1)}{(\beta+1)(\lambda+\mu j)j^{m+1}\Gamma(n+j)}}z^j$, $j \geq t+1$.

Proof. Suppose f satisfies (2.1). Then for $|z|<1$, we have
$\left|\lambda\left(1-\mu\right)\frac{D_z^{-\lambda}SR^{m,n}f(z)}{z}+\mu\left(D_z^{-\lambda}SR^{m,n}f\left(z\right)\right)'\right|-\beta\left|\lambda\left(1-\mu\right)\frac{D_z^{-\lambda}SR^{m,n}f(z)}{z}+\mu\left(D_z^{-\lambda}SR^{m,n}f\left(z\right)\right)'-\alpha\right|=$
$\left|\frac{\lambda+\mu}{\Gamma(\lambda+2)}z^\lambda+\sum_{j=t+1}^{\infty}\frac{(\lambda+\mu j)j^{m+1}\Gamma(n+j)}{\Gamma(n+1)\Gamma(j+\lambda+1)}a_j^2z^{j+\lambda-1}\right|-\beta\left|\frac{\lambda+\mu}{\Gamma(\lambda+2)}z^\lambda+\sum_{j=t+1}^{\infty}\frac{(\lambda+\mu j)j^{m+1}\Gamma(n+j)}{\Gamma(n+1)\Gamma(j+\lambda+1)}a_j^2z^{j+\lambda-1}-\alpha\right|\leq$
$\left|\frac{\lambda+\mu}{\Gamma(\lambda+2)}z^\lambda\right|+\left|\sum_{j=t+1}^{\infty}\frac{(\lambda+\mu j)j^{m+1}\Gamma(n+j)}{\Gamma(n+1)\Gamma(j+\lambda+1)}a_j^2z^{j+\lambda-1}\right|-\beta\left|\alpha\right|+\beta\left|\frac{\lambda+\mu}{\Gamma(\lambda+2)}z^\lambda\right|+\beta\left|\sum_{j=t+1}^{\infty}\frac{(\lambda+\mu j)j^{m+1}\Gamma(n+j)}{\Gamma(n+1)\Gamma(j+\lambda+1)}a_j^2z^{j+\lambda-1}\right|<$
$\frac{(\beta+1)(\lambda+\mu)}{\Gamma(\lambda+2)}-\beta\left|\alpha\right|+\sum_{j=t+1}^{\infty}\frac{(\beta+1)(\lambda+\mu j)j^{m+1}\Gamma(n+j)}{\Gamma(n+1)\Gamma(j+\lambda+1)}a_j^2<0.$
Hence, by using the maximum modulus Theorem and (1.1), $f \in \mathcal{D}(\mu,\lambda,\alpha,\beta)$. Conversely, assume that
$\left|\frac{\lambda(1-\mu)\frac{D_z^{-\lambda}SR^{m,n}f(z)}{z}+\mu\left(D_z^{-\lambda}SR^{m,n}f(z)\right)'}{\lambda(1-\mu)\frac{D_z^{-\lambda}SR^{m,n}f(z)}{z}+\mu\left(D_z^{-\lambda}SR^{m,n}f(z)\right)'-\alpha}\right|=\left|\frac{\frac{\lambda+\mu}{\Gamma(\lambda+2)}z^\lambda+\sum_{j=t+1}^{\infty}\frac{(\lambda+\mu j)j^{m+1}\Gamma(n+j)}{\Gamma(n+1)\Gamma(j+\lambda+1)}a_j^2z^{j+\lambda-1}}{\frac{\lambda+\mu}{\Gamma(\lambda+2)}z^\lambda+\sum_{j=t+1}^{\infty}\frac{(\lambda+\mu j)j^{m+1}\Gamma(n+j)}{\Gamma(n+1)\Gamma(j+\lambda+1)}a_j^2z^{j+\lambda-1}-\alpha}\right|<\beta,\ z\in U.$

Since $Re(z) \leq |z|$ for all $z \in U$, we have $Re\left\{\frac{\frac{\lambda+\mu}{\Gamma(\lambda+2)}z^\lambda+\sum_{j=t+1}^{\infty}\frac{(\lambda+\mu j)j^{m+1}\Gamma(n+j)}{\Gamma(n+1)\Gamma(j+\lambda+1)}a_j^2z^{j+\lambda-1}}{\frac{\lambda+\mu}{\Gamma(\lambda+2)}z^\lambda+\sum_{j=t+1}^{\infty}\frac{(\lambda+\mu j)j^{m+1}\Gamma(n+j)}{\Gamma(n+1)\Gamma(j+\lambda+1)}a_j^2z^{j+\lambda-1}-\alpha}\right\}<\beta$. By choosing choose values of z on the real axis so that $\lambda\left(1-\mu\right)\frac{D_z^{-\lambda}SR^{m,n}f(z)}{z}+\mu\left(D_z^{-\lambda}SR^{m,n}f\left(z\right)\right)'$ is real and letting $z\to 1$ through real values, we obtain the desired inequality (2.1). ■

Corollary 2.2 *If $f \in \mathcal{A}_t$ be in $\mathcal{D}(\mu,\lambda,\alpha,\beta)$, then*

$$a_j\leq\sqrt{\frac{\left(\beta\left|\alpha\right|-\frac{(\beta+1)(\lambda+\mu)}{\Gamma(\lambda+2)}\right)\Gamma\left(n+1\right)\Gamma\left(j+\lambda+1\right)}{\left(\beta+1\right)\left(\lambda+\mu j\right)j^{m+1}\Gamma\left(n+j\right)}},\quad j\geq t+1, \tag{2.2}$$

with equality only for functions of the form $F(z)$.

Theorem 2.3 *Let $f_1(z)=z$ and $f_j(z)=z-\sqrt{\frac{\left(\beta|\alpha|-\frac{(\beta+1)(\lambda+\mu)}{\Gamma(\lambda+2)}\right)\Gamma(n+1)\Gamma(j+\lambda+1)}{(\beta+1)(\lambda+\mu j)j^{m+1}\Gamma(n+j)}}z^j$, $j\geq t+1$, for $\mu\geq 0$, $\lambda\in\mathbb{N}$, $\alpha\in\mathbb{C}-\{0\}$ and $0<\beta\leq 1$. Then $f(z)$ is in the class $\mathcal{D}(\mu,\lambda,\alpha,\beta)$ if and only if it can be expressed in the form*

$$f(z)=\sum_{j=1}^{\infty}\omega_jf_j(z), \tag{2.3}$$

where $\omega_j\geq 0$ and $\sum_{j=1}^{\infty}\omega_j=1$.

Proof. Suppose $f(z)$ can be written as in (2.3). Then $f(z)=z-\sum_{j=t+1}^{\infty}\omega_j\sqrt{\frac{\left(\beta|\alpha|-\frac{(\beta+1)(\lambda+\mu)}{\Gamma(\lambda+2)}\right)\Gamma(n+1)\Gamma(j+\lambda+1)}{(\beta+1)(\lambda+\mu j)j^{m+1}\Gamma(n+j)}}z^j$.
Now, $\sum_{j=t+1}^{\infty}\sqrt{\frac{(\beta+1)(\lambda+\mu j)j^{m+1}\Gamma(n+j)}{\left(\beta|\alpha|-\frac{(\beta+1)(\lambda+\mu)}{\Gamma(\lambda+2)}\right)\Gamma(n+1)\Gamma(j+\lambda+1)}}\omega_j\sqrt{\frac{\left(\beta|\alpha|-\frac{(\beta+1)(\lambda+\mu)}{\Gamma(\lambda+2)}\right)\Gamma(n+1)\Gamma(j+\lambda+1)}{(\beta+1)(\lambda+\mu j)j^{m+1}\Gamma(n+j)}}=\sum_{j=t+1}^{\infty}\omega_j=1-\omega_1\leq 1$. Thus $f\in\mathcal{D}(\mu,\lambda,\alpha,\beta)$.

Conversely, let $f\in\mathcal{D}(\mu,\lambda,\alpha,\beta)$. Then by using (2.2), setting $\omega_j=\sqrt{\frac{\left(\beta|\alpha|-\frac{(\beta+1)(\lambda+\mu)}{\Gamma(\lambda+2)}\right)\Gamma(n+1)\Gamma(j+\lambda+1)}{(\beta+1)(\lambda+\mu j)j^{m+1}\Gamma(n+j)}}a_j$, $j\geq t+1$ and $\omega_1=1-\sum_{j=2}^{\infty}\omega_j$, we have $f(z)=\sum_{j=1}^{\infty}\omega_jf_j(z)$. And this completes the proof of Theorem 2.3. ■

3 Distortion bounds

In this section we obtain distortion bounds for the class $\mathcal{D}(\mu,\lambda,\alpha,\beta)$.

Theorem 3.1 *If $f\in\mathcal{D}(\mu,\lambda,\alpha,\beta)$, then*

$$r-\sqrt{\frac{\left(\beta\left|\alpha\right|-\frac{(\beta+1)(\lambda+\mu)}{\Gamma(\lambda+2)}\right)\Gamma\left(n+1\right)\Gamma\left(t+\lambda+2\right)}{\left(\beta+1\right)\left(\lambda+\mu t+\mu\right)\left(t+1\right)^{m+1}\Gamma\left(n+t+1\right)}}r^{t+1}\leq|f(z)| \tag{3.1}$$

$$\leq r+\sqrt{\frac{\left(\beta\left|\alpha\right|-\frac{(\beta+1)(\lambda+\mu)}{\Gamma(\lambda+2)}\right)\Gamma\left(n+1\right)\Gamma\left(t+\lambda+2\right)}{\left(\beta+1\right)\left(\lambda+\mu t+\mu\right)\left(t+1\right)^{m+1}\Gamma\left(n+t+1\right)}}r^{t+1}$$

holds if the sequence $\{\sigma_j(\mu,\lambda,\alpha,\beta)\}_{j=t+1}^{\infty}$ is non-decreasing, and

$$1-(t+1)\sqrt{\frac{\left(\beta\left|\alpha\right|-\frac{(\beta+1)(\lambda+\mu)}{\Gamma(\lambda+2)}\right)\Gamma\left(n+1\right)\Gamma\left(t+\lambda+2\right)}{\left(\beta+1\right)\left(\lambda+\mu t+\mu\right)\left(t+1\right)^{m+1}\Gamma\left(n+t+1\right)}}r^{t}\leq|f'(z)| \tag{3.2}$$

$$\leq 1+(t+1)\sqrt{\frac{\left(\beta\left|\alpha\right|-\frac{(\beta+1)(\lambda+\mu)}{\Gamma(\lambda+2)}\right)\Gamma\left(n+1\right)\Gamma\left(t+\lambda+2\right)}{\left(\beta+1\right)\left(\lambda+\mu t+\mu\right)\left(t+1\right)^{m+1}\Gamma\left(n+t+1\right)}}r^{t}$$

holds if the sequence $\{\frac{\sigma_j(\mu,\lambda,\beta)}{j}\}_{j=t+1}^{\infty}$ is non- decreasing, where $\sigma_j(\mu,\lambda,\beta)=\sqrt{\frac{(\beta+1)(\lambda+\mu j)j^{m+1}\Gamma(n+j)}{\Gamma(n+1)\Gamma(j+\lambda+1)}}$.

The bounds in (3.1) and (3.2) are sharp, for $f(z)$ given by $f(z)=z+\sqrt{\frac{\left(\beta|\alpha|-\frac{(\beta+1)(\lambda+\mu)}{\Gamma(\lambda+2)}\right)\Gamma(n+1)\Gamma(t+\lambda+2)}{(\beta+1)(\lambda+\mu t+\mu)(t+1)^{m+1}\Gamma(n+t+1)}}z^{t+1}$, $z=\pm r$.

Proof. In view of Theorem 2.1, we have $\sum_{j=t+1}^{\infty}a_j\leq\sqrt{\frac{\left(\beta|\alpha|-\frac{(\beta+1)(\lambda+\mu)}{\Gamma(\lambda+2)}\right)\Gamma(n+1)\Gamma(t+\lambda+2)}{(\beta+1)(\lambda+\mu t+\mu)(t+1)^{m+1}\Gamma(n+t+1)}}$. We obtain $|z|-|z|^{t+1}\sum_{j=t+1}^{\infty}a_j\leq|f(z)|\leq|z|+|z|^{t+1}\sum_{j=t+1}^{\infty}a_j$. Thus

$$r-\sqrt{\frac{\left(\beta\left|\alpha\right|-\frac{(\beta+1)(\lambda+\mu)}{\Gamma(\lambda+2)}\right)\Gamma\left(n+1\right)\Gamma\left(t+\lambda+2\right)}{\left(\beta+1\right)\left(\lambda+\mu t+\mu\right)\left(t+1\right)^{m+1}\Gamma\left(n+t+1\right)}}r^{t+1}\leq|f(z)| \tag{3.3}$$

$$\leq r+\sqrt{\frac{\left(\beta\left|\alpha\right|-\frac{(\beta+1)(\lambda+\mu)}{\Gamma(\lambda+2)}\right)\Gamma\left(n+1\right)\Gamma\left(t+\lambda+2\right)}{\left(\beta+1\right)\left(\lambda+\mu t+\mu\right)\left(t+1\right)^{m+1}\Gamma\left(n+t+1\right)}}r^{t+1}.$$

Hence (3.1) follows from (3.3). Further, $\sum_{j=t+1}^{\infty}ja_j\leq\sqrt{\frac{\left(\beta|\alpha|-\frac{(\beta+1)(\lambda+\mu)}{\Gamma(\lambda+2)}\right)\Gamma(n+1)\Gamma(t+\lambda+2)}{(\beta+1)(\lambda+\mu t+\mu)(t+1)^{m+1}\Gamma(n+t+1)}}$. Hence (3.2) follows from $1-r^t\sum_{j=t+1}^{\infty}ja_j\leq|f'(z)|\leq1+r^t\sum_{j=t+1}^{\infty}ja_j$. ■

4 Radius of starlikeness and convexity

The radii of close-to-convexity, starlikeness and convexity for the class $\mathcal{D}(\mu,\lambda,\alpha,\beta)$ are given in this section.

Theorem 4.1 *Let the function $f\in\mathcal{A}_t$ belong to the class $\mathcal{D}(\mu,\lambda,\alpha,\beta)$, Then $f(z)$ is close -to-convex of order δ, $0\leq\delta<1$ in the disc $|z|<r$, where $r:=\inf_{j\geq t+1}\left[\sqrt{\frac{(1-\delta)^2(\beta+1)(\lambda+\mu j)j^{m-1}\Gamma(n+j)}{\left(\beta|\alpha|-\frac{(\beta+1)(\lambda+\mu)}{\Gamma(\lambda+2)}\right)\Gamma(n+1)\Gamma(j+\lambda+1)}}\right]^{\frac{1}{t}}$. The result is sharp, with extremal function $f(z)$ given by (2.2).*

Proof. For given $f\in\mathcal{A}_t$ we must show that

$$|f'(z)-1|<1-\delta. \tag{4.1}$$

By a simple calculation we have $|f'(z)-1| \leq \sum_{j=t+1}^{\infty} j a_j |z|^t$. The last expression is less than $1-\delta$ if $\sum_{j=t+1}^{\infty} \frac{j}{1-\delta} a_j |z|^t < 1$. Using the fact that $f \in \mathcal{D}(\mu,\lambda,\alpha,\beta)$ if and only if $\sum_{j=t+1}^{\infty} \frac{(\beta+1)(\lambda+\mu j)j^{m+1}\Gamma(n+j)}{\left(\beta|\alpha|-\frac{(\beta+1)(\lambda+\mu)}{\Gamma(\lambda+2)}\right)\Gamma(n+1)\Gamma(j+\lambda+1)} a_j^2 < 1$, (4.1) holds true if $\frac{j}{1-\delta}|z|^t \leq \sum_{j=t+1}^{\infty} \sqrt{\frac{(\beta+1)(\lambda+\mu j)j^{m+1}\Gamma(n+j)}{\left(\beta|\alpha|-\frac{(\beta+1)(\lambda+\mu)}{\Gamma(\lambda+2)}\right)\Gamma(n+1)\Gamma(j+\lambda+1)}}$.

Or, equivalently, $|z|^t \leq \sum_{j=t+1}^{\infty} \sqrt{\frac{(1-\delta)^2(\beta+1)(\lambda+\mu j)j^{m-1}\Gamma(n+j)}{\left(\beta|\alpha|-\frac{(\beta+1)(\lambda+\mu)}{\Gamma(\lambda+2)}\right)\Gamma(n+1)\Gamma(j+\lambda+1)}}$, which completes the proof. ■

Theorem 4.2 *Let $f \in \mathcal{D}(\mu,\lambda,\alpha,\beta)$. Then*

1. f is starlike of order δ, $0 \leq \delta < 1$, in the disc $|z| < r_1$ where,

$r_1 = \inf_{j \geq t+1} \left\{ \sqrt{\frac{(1-\delta)^2(\beta+1)(\lambda+\mu j)j^{m+1}\Gamma(n+j)}{\left(\beta|\alpha|-\frac{(\beta+1)(\lambda+\mu)}{\Gamma(\lambda+2)}\right)\Gamma(n+1)(j+\delta-2)^2\Gamma(j+\lambda+1)}} \right\}^{\frac{1}{t}}$.

2. f is convex of order δ, $0 \leq \delta < 1$, in the disc $|z| < r_2$ where,

$r_2 = \inf_{j \geq t+1} \left\{ \sqrt{\frac{(1-\delta)^2(\beta+1)(\lambda+\mu j)j^{m-1}\Gamma(n+j)}{\left(\beta|\alpha|-\frac{(\beta+1)(\lambda+\mu)}{\Gamma(\lambda+2)}\right)\Gamma(n+1)(j-1)^2\Gamma(j+\lambda+1)}} \right\}^{\frac{1}{t}}$.

Each of these results is sharp for the extremal function $f(z)$ given by (2.3).

Proof. 1. For $0 \leq \delta < 1$ we need to show that

$$\left|\frac{zf'(z)}{f(z)} - 1\right| < 1-\delta. \tag{4.2}$$

We have $\left|\frac{zf'(z)}{f(z)} - 1\right| \leq \left|\frac{\sum_{j=t+1}^{\infty}(j-1)a_j|z|^t}{1+\sum_{j=t+1}^{\infty} a_j|z|^t}\right|$. The last expresion is less than $1-\delta$ if $\sum_{j=t+1}^{\infty} \frac{(j+\delta-2)}{1-\delta} a_j |z|^t < 1$. Using the fact that $f \in \mathcal{D}(\mu,\lambda,\alpha,\beta)$ if and only if $\sum_{j=t+1}^{\infty} \frac{(\beta+1)(\lambda+\mu j)j^{m+1}\Gamma(n+j)}{\left(\beta|\alpha|-\frac{(\beta+1)(\lambda+\mu)}{\Gamma(\lambda+2)}\right)\Gamma(n+1)\Gamma(j+\lambda+1)} a_j^2 < 1$, (4.2) holds true if $\frac{j+\delta-2}{1-\delta}|z|^t < \sqrt{\frac{(\beta+1)(\lambda+\mu j)j^{m+1}\Gamma(n+j)}{\left(\beta|\alpha|-\frac{(\beta+1)(\lambda+\mu)}{\Gamma(\lambda+2)}\right)\Gamma(n+1)\Gamma(j+\lambda+1)}}$.

Or, equivalently, $|z|^t < \sqrt{\frac{(1-\delta)^2(\beta+1)(\lambda+\mu j)j^{m+1}\Gamma(n+j)}{\left(\beta|\alpha|-\frac{(\beta+1)(\lambda+\mu)}{\Gamma(\lambda+2)}\right)\Gamma(n+1)(j+\delta-2)^2\Gamma(j+\lambda+1)}}$, which yields the starlikeness of the family.

2. Using the fact that f is convex if and only zf' is starlike, we can prove (2) with a similar way of the proof of (1). The function f is convex if and only if

$$|zf''(z)| < 1-\delta. \tag{4.3}$$

We have $|zf''(z)| \leq \left|\sum_{j=t+1}^{\infty} j(j-1)a_j |z|^{t-1}\right| < 1-\delta$, i.e. $\sum_{j=t+1}^{\infty} \frac{j(j-1)}{1-\delta} a_j |z|^{t-1} < 1$. Using the fact that $f \in \mathcal{D}(\mu,\lambda,\alpha,\beta)$ if and only if $\sum_{j=t+1}^{\infty} \frac{(\beta+1)(\lambda+\mu j)j^{m+1}\Gamma(n+j)}{\left(\beta|\alpha|-\frac{(\beta+1)(\lambda+\mu)}{\Gamma(\lambda+2)}\right)\Gamma(n+1)\Gamma(j+\lambda+1)} a_j^2 < 1$, (4.3) holds true if $\frac{j(j-1)}{1-\delta}|z|^{t-1} < \sqrt{\frac{(\beta+1)(\lambda+\mu j)j^{m+1}\Gamma(n+j)}{\left(\beta|\alpha|-\frac{(\beta+1)(\lambda+\mu)}{\Gamma(\lambda+2)}\right)\Gamma(n+1)\Gamma(j+\lambda+1)}}$, or, equivalently, $|z|^{t-1} < \sqrt{\frac{(1-\delta)^2(\beta+1)(\lambda+\mu j)j^{m-1}\Gamma(n+j)}{\left(\beta|\alpha|-\frac{(\beta+1)(\lambda+\mu)}{\Gamma(\lambda+2)}\right)\Gamma(n+1)(j-1)^2\Gamma(j+\lambda+1)}}$, which yields the convexity of the family. ■

References

[1] A. Alb Lupas, *Aspects of univalent holomorphic functions involving Sălăgean operator and Ruscheweyh derivative,* J. of Concrete Applicable Math., 13 (2015), No.'s 1-2, 51-59.

[2] N.E.Cho, A.M.K. Aouf, *Some applications of fractional calculus operators to a certain subclass of analytic functions with negative coefficients*, Tr. J. of Mathematics, Vol. 20, 1996, 553-562.

[3] St. Ruscheweyh, *New criteria for univalent functions*, Proc. Amet. Math. Soc., 49(1975), 109-115.

[4] G. St. Sălăgean, *Subclasses of univalent functions*, Lecture Notes in Math., Springer Verlag, Berlin, 1013 (1983), 362-372.

Properties on a subclass of analytic functions defined by a fractional integral operator

Alb Lupaş Alina
Department of Mathematics and Computer Science
University of Oradea
str. Universitatii nr. 1, 410087 Oradea, Romania
dalb@uoradea.ro, alblupas@gmail.com

Abstract

In this paper we have introduced and studied the subclass $\mathcal{L}(\lambda, d, \alpha, \beta)$ using the fractional integral associated with the convolution product of Sălăgean operator and Ruscheweyh derivative. The main object is to investigate several properties such as coefficient estimates, distortion theorems, closure theorems, neighborhoods and the radii of starlikeness, convexity and close-to-convexity of functions belonging to the class $\mathcal{L}(\lambda, d, \alpha, \beta)$.

Keywords: Analytic functions, univalent functions, radii of starlikeness and convexity, neighborhood property, Salagean operator, Ruscheweyh operator.
2000 Mathematical Subject Classification: 30C45, 30A20, 34A40.

1 Introduction

Denote by U the unit disc of the complex plane, $U = \{z \in \mathbb{C} : \ |z| < 1\}$ and $\mathcal{H}(U)$ the space of holomorphic functions in U.

Let $\mathcal{A}(p,t) = \{f \in \mathcal{H}(U) : \ f(z) = z^p + \sum_{j=p+t}^{\infty} a_j z^j, \ \ z \in U\}$, with $\mathcal{A}(1,1) = \mathcal{A}$ and $\mathcal{H}[a,t] = \{f \in \mathcal{H}(U) : f(z) = a + a_t z^t + a_{t+1} z^{t+1} + \dots, \ z \in U\}$, where $p, t \in \mathbb{N}$, $a \in \mathbb{C}$.

Definition 1.1 *(Sălăgean [4]) For $f \in \mathcal{A}$, and $n \in \mathbb{N}$, the operator S^n is defined by $S^n : \mathcal{A} \to \mathcal{A}$,*

$$S^0 f(z) = f(z), \ S^1 f(z) = z f'(z), \ \dots, \ S^{n+1} f(z) = z\left(S^n f(z)\right)', \quad z \in U.$$

Remark 1.1 *If $f \in \mathcal{A}$, $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$, then $S^n f(z) = z + \sum_{j=2}^{\infty} j^n a_j z^j$, $z \in U$.*

Definition 1.2 *(Ruscheweyh [3]) For $f \in \mathcal{A}$ and $n \in \mathbb{N}$, the operator R^n is defined by $R^n : \mathcal{A} \to \mathcal{A}$,*

$$R^0 f(z) = f(z), \ R^1 f(z) = z f'(z), \ \dots, \ (n+1) R^{n+1} f(z) = z\left(R^n f(z)\right)' + n R^n f(z), \quad z \in U.$$

Remark 1.2 *If $f \in \mathcal{A}$, $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$, then $R^n f(z) = z + \sum_{j=2}^{\infty} \frac{\Gamma(n+j)}{\Gamma(n+1)\Gamma(j)} a_j z^j$ for $z \in U$.*

Definition 1.3 *Let $m, n \in \mathbb{N}$. Denote by $SR_{\lambda}^{m,n} : \mathcal{A} \to \mathcal{A}$ the operator given by the Hadamard product of the Sălăgean operator S^m and the Ruscheweyh derivative R^n, $SR^{m,n} f(z) = \left(S^m * R^n\right) f(z)$,for any $z \in U$ and each nonnegative integers m, n.*

Remark 1.3 *If $f \in \mathcal{A}$ and $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$, then $SR^{m,n} f(z) = z + \sum_{j=2}^{\infty} j^m \frac{\Gamma(n+j)}{\Gamma(n+1)\Gamma(j)} a_j^2 z^j$, $z \in U$.*

Definition 1.4 *([2]) The fractional integral of order λ ($\lambda > 0$) is defined for a function f by $D_z^{-\lambda} f(z) = \frac{1}{\Gamma(\lambda)} \int_0^z \frac{f(t)}{(z-t)^{1-\lambda}} dt$, where f is an analytic function in a simply-connected region of the z-plane containing the origin, and the multiplicity of $(z-t)^{\lambda-1}$ is removed by requiring $\log(z-t)$ to be real, when $(z-t) > 0$.*

From Definition 1.3 and Definition 1.4, we get the fractional integral associated with the convolution product of Sălăgean operator and Ruscheweyh derivative, which has the following form $D_z^{-\lambda} SR^{m,n} f(z) = \frac{1}{\Gamma(\lambda+2)} z^{\lambda+1} + \sum_{j=t+1}^{\infty} \frac{j^{m+1}\Gamma(n+j)}{\Gamma(n+1)\Gamma(j+\lambda+1)} a_j^2 z^{j+\lambda}$, for a function $f(z) = z + \sum_{j=2}^{\infty} a_j z^j \in \mathcal{A}$.

We follow the works from [1].

Definition 1.5 *Let the function $f \in \mathcal{A}$. Then $f(z)$ is said to be in the class $\mathcal{L}(\lambda, d, \alpha, \beta)$ if it satisfies the following criterion:*

$$\left| \frac{1}{d} \left(\frac{z(D_z^{-\lambda} SR^{m,n} f(z))' + \alpha z^2 (D_z^{-\lambda} SR^{m,n} f(z))''}{(1-\alpha) D_z^{-\lambda} SR^{m,n} f(z) + \alpha z (D_z^{-\lambda} SR^{m,n} f(z))'} - 1 \right) \right| < \beta, \tag{1.1}$$

where $\lambda > 0$, $\in \mathbb{C} - \{0\}$, $0 \le \alpha \le 1$, $0 < \beta \le 1$, $m, n \in \mathbb{N}$, $z \in U$.

In this paper we shall first deduce a necessary and sufficient condition for a function $f(z)$ to be in the class $\mathcal{L}(\lambda, d, \alpha, \beta)$. Then obtain the distortion and growth theorems, closure theorems, neighborhood and radii of univalent starlikeness, convexity and close-to-convexity of order δ, $0 \le \delta < 1$, for these functions.

2 Coefficient Inequality

Theorem 2.1 *Let the function $f \in \mathcal{A}$. Then $f(z)$ is said to be in the class $\mathcal{L}(\lambda, d, \alpha, \beta)$ if and only if*

$$\sum_{j=2}^{\infty} \frac{j^{m+1}\Gamma(n+j)}{\Gamma(j+\lambda+1)} \left\{ \alpha j^2 + [\alpha(2\lambda - 2 + \beta|d|) + 1] j + [\alpha(\lambda-1)+1](\lambda - 1 + \beta|d|) \right\} a_j^2 \le$$

$$(\alpha\lambda + 1)(\beta|d| - \lambda) \frac{\Gamma(n+1)}{\Gamma(\lambda+2)}, \tag{2.1}$$

where $\lambda > 0$, $\in \mathbb{C} - \{0\}$, $0 \le \alpha \le 1$, $0 < \beta \le 1$, $m, n \in \mathbb{N}$, $z \in U$.

Proof. Let $f(z) \in \mathcal{L}(\lambda, d, \alpha, \beta)$. Assume that inequality (2.1) holds true. Then we find that
$\left| \frac{z(D_z^{-\lambda} SR^{m,n} f(z))' + \alpha z^2 (D_z^{-\lambda} SR^{m,n} f(z))''}{(1-\alpha) D_z^{-\lambda} SR^{m,n} f(z) + \alpha z (D_z^{-\lambda} SR^{m,n} f(z))'} - 1 \right| =$
$\left| \frac{\frac{\lambda(\alpha\lambda+1)}{\Gamma(\lambda+2)} z^{\lambda+1} + \sum_{j=2}^{\infty} \frac{j^{m+1}\Gamma(n+j)}{\Gamma(n+1)\Gamma(j+\lambda+1)} \left\{ \alpha j^2 + [2\alpha(\lambda-1)+1] j + (\lambda-1)[\alpha(\lambda-1)+1] \right\} a_j^2 z^{j+\lambda}}{\frac{\alpha\lambda+1}{\Gamma(\lambda+2)} z^{\lambda+1} + \sum_{j=2}^{\infty} \frac{j^{m+1}\Gamma(n+j)}{\Gamma(n+1)\Gamma(j+\lambda+1)} [\alpha j + \alpha(\lambda-1)+1] a_j^2 z^{j+\lambda}} \right| \le$
$\frac{\frac{\lambda(\alpha\lambda+1)}{\Gamma(\lambda+2)} + \sum_{j=2}^{\infty} \frac{j^{m+1}\Gamma(n+j)}{\Gamma(n+1)\Gamma(j+\lambda+1)} \left\{ \alpha j^2 + [2\alpha(\lambda-1)+1] j + (\lambda-1)[\alpha(\lambda-1)+1] \right\} a_j^2 \left| z^{j-1} \right|}{\frac{\alpha\lambda+1}{\Gamma(\lambda+2)} - \sum_{j=2}^{\infty} \frac{j^{m+1}\Gamma(n+j)}{\Gamma(n+1)\Gamma(j+\lambda+1)} [\alpha j + \alpha(\lambda-1)+1] a_j^2 |z^{j-1}|} \le \beta|d|.$
Choosing values of z on real axis and letting $z \to 1^-$, we have
$\sum_{j=2}^{\infty} \frac{j^{m+1}\Gamma(n+j)}{\Gamma(j+\lambda+1)} \left\{ \alpha j^2 + [\alpha(2\lambda-2+\beta|d|)+1] j + [\alpha(\lambda-1)+1](\lambda-1+\beta|d|) \right\} a_j^2 \le$
$(\alpha\lambda+1)(\beta|d|-\lambda) \frac{\Gamma(n+1)}{\Gamma(\lambda+2)}.$
Conversely, assume that $f(z) \in \mathcal{L}(\lambda, d, \alpha, \beta)$, then we get the following inequality
$Re \left\{ \frac{z(D_z^{-\lambda} SR^{m,n} f(z))' + \alpha z^2 (D_z^{-\lambda} SR^{m,n} f(z))''}{(1-\alpha) D_z^{-\lambda} SR^{m,n} f(z) + \alpha z (D_z^{-\lambda} SR^{m,n} f(z))'} - 1 \right\} > -\beta|d|$
$Re \left\{ \frac{\frac{(\lambda+1)(\alpha\lambda+1)}{\Gamma(\lambda+2)} z^{\lambda+1} + \sum_{j=2}^{\infty} \frac{j^{m+1}\Gamma(n+j)}{\Gamma(n+1)\Gamma(j+\lambda+1)} \left[\alpha j^2 + (2a\lambda - \alpha + 1) j + \lambda(\alpha\lambda - \alpha + 1) \right] a_j^2 z^{j+\lambda}}{\frac{\alpha\lambda+1}{\Gamma(\lambda+2)} z^{\lambda+1} + \sum_{j=2}^{\infty} \frac{j^{m+1}\Gamma(n+j)}{\Gamma(n+1)\Gamma(j+\lambda+1)} [\alpha j + \alpha(\lambda-1)+1] a_j^2 z^{j+\lambda}} - 1 + \beta|d| \right\} > 0$
$Re \frac{\frac{(\alpha\lambda+1)(\beta|d|-\lambda)}{\Gamma(\lambda+2)} z^{\lambda+1} + \sum_{j=2}^{\infty} \frac{j^{m+1}\Gamma(n+j)}{\Gamma(n+1)\Gamma(j+\lambda+1)} \left\{ \alpha j^2 + [\alpha(2\lambda-2+\beta|d|)+1] j + [\alpha(\lambda-1)+1](\lambda-1+\beta|d|) \right\} a_j^2 z^{j+\lambda}}{\frac{\alpha\lambda+1}{\Gamma(\lambda+2)} z^{\lambda+1} + \sum_{j=2}^{\infty} \frac{j^{m+1}\Gamma(n+j)}{\Gamma(n+1)\Gamma(j+\lambda+1)} [\alpha j + \alpha(\lambda-1)+1] a_j^2 z^{j+\lambda}} > 0.$
Since $Re(-e^{i\theta}) \ge -|e^{i\theta}| = -1$, the above inequality reduces to
$\frac{\frac{(\alpha\lambda+1)(\beta|d|-\lambda)}{\Gamma(\lambda+2)} r^{\lambda+1} - \sum_{j=2}^{\infty} \frac{j^{m+1}\Gamma(n+j)}{\Gamma(n+1)\Gamma(j+\lambda+1)} \left\{ \alpha j^2 + [\alpha(2\lambda-2+\beta|d|)+1] j + [\alpha(\lambda-1)+1](\lambda-1+\beta|d|) \right\} a_j^2 r^{j+\lambda j}}{\frac{\alpha\lambda+1}{\Gamma(\lambda+2)} r^{\lambda+1} - \sum_{j=2}^{\infty} \frac{j^{m+1}\Gamma(n+j)}{\Gamma(n+1)\Gamma(j+\lambda+1)} [\alpha j + \alpha(\lambda-1)+1] a_j^2 r^{j+\lambda}} > 0.$

Letting $r \to 1^-$ and by the mean value theorem we have desired inequality (2.1). This completes the proof of Theorem 2.1 ■

Corollary 2.2 *Let the function $f \in \mathcal{A}$ be in the class $\mathcal{L}(\lambda, d, \alpha, \beta)$.*
Then $a_j \le \sqrt{\frac{(\alpha\lambda+1)(\beta|d|-\lambda)\frac{\Gamma(n+1)}{\Gamma(\lambda+2)}}{\frac{j^{m+1}\Gamma(n+j)}{\Gamma(j+\lambda+1)} \{\alpha j^2 + [\alpha(2\lambda-2+\beta|d|)+1] j + [\alpha(\lambda-1)+1](\lambda-1+\beta|d|)\}}}$, $j \ge 2$.

3 Distortion Theorems

Theorem 3.1 *Let the function $f \in \mathcal{A}$ be in the class $\mathcal{L}(\lambda, d, \alpha, \beta)$. Then for $|z| = r < 1$, we have*
$r - \sqrt{\frac{(\alpha\lambda+1)(\lambda+2)(\beta|d|-\lambda)}{2^{m+1}(n+1)\{(\lambda+1)[\alpha(\lambda+1+\beta|d|)+1]+\beta|d|\}}} r^2 \le |f(z)| \le r + \sqrt{\frac{(\alpha\lambda+1)(\lambda+2)(\beta|d|-\lambda)}{2^{m+1}(n+1)\{(\lambda+1)[\alpha(\lambda+1+\beta|d|)+1]+\beta|d|\}}} r^2.$
The result is sharp for the function $f(z)$ given by $f(z) = z + \sqrt{\frac{(\alpha\lambda+1)(\lambda+2)(\beta|d|-\lambda)}{2^{m+1}(n+1)\{(\lambda+1)[\alpha(\lambda+1+\beta|d|)+1]+\beta|d|\}}} z^2.$

Proof. Given that $f(z) \in \mathcal{L}(\lambda, d, \alpha, \beta)$, from the equation (2.1) and since $2^{m+1}(n+1)\{(\lambda+1)[\alpha(\lambda+1+\beta|d|)+1]+\beta|d|\}$ is non decreasing and positive for $j \geq 2$, then we have $\sqrt{2^{m+1}(n+1)\{(\lambda+1)[\alpha(\lambda+1+\beta|d|)+1]+\beta|d|\}}\sum_{j=2}^{\infty} a_j \leq$
$\sum_{j=2}^{\infty}\sqrt{\frac{j^{m+1}\Gamma(n+j)}{\Gamma(j+\lambda+1)}\{\alpha j^2+[\alpha(2\lambda-2+\beta|d|)+1]j+[\alpha(\lambda-1)+1](\lambda-1+\beta|d|)\}}a_j \leq$
$\sqrt{(\alpha\lambda+1)(\beta|d|-\lambda)\frac{\Gamma(n+1)}{\Gamma(\lambda+2)}}$, which is equivalent to, $\sum_{j=2}^{\infty} a_j \leq \sqrt{\frac{(\alpha\lambda+1)(\lambda+2)(\beta|d|-\lambda)}{2^{m+1}(n+1)\{(\lambda+1)[\alpha(\lambda+1+\beta|d|)+1]+\beta|d|\}}}$.
We obtain for $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$,
$|f(z)| \leq |z| + \sum_{j=2}^{\infty} a_j |z|^j \leq r + \sum_{j=2}^{\infty} a_j r^j \leq r + r^2\sum_{j=2}^{\infty} a_j \leq r + \sqrt{\frac{(\alpha\lambda+1)(\lambda+2)(\beta|d|-\lambda)}{2^{m+1}(n+1)\{(\lambda+1)[\alpha(\lambda+1+\beta|d|)+1]+\beta|d|\}}}r^2$.
Similarly, $|f(z)| \geq r - \sqrt{\frac{(\alpha\lambda+1)(\lambda+2)(\beta|d|-\lambda)}{2^{m+1}(n+1)\{(\lambda+1)[\alpha(\lambda+1+\beta|d|)+1]+\beta|d|\}}}r^2$. This completes the proof of Theorem 3.1. ■

Theorem 3.2 *Let the function $f \in \mathcal{A}$ be in the class $\mathcal{L}(\lambda, d, \alpha, \beta)$. Then for $|z| = r < 1$, we have*
$-\sqrt{\frac{(\alpha\lambda+1)(\lambda+2)(\beta|d|-\lambda)}{2^{m-1}(n+1)\{(\lambda+1)[\alpha(\lambda+1+\beta|d|)+1]+\beta|d|\}}}r \leq |f'(z)| \leq \sqrt{\frac{(\alpha\lambda+1)(\lambda+2)(\beta|d|-\lambda)}{2^{m-1}(n+1)\{(\lambda+1)[\alpha(\lambda+1+\beta|d|)+1]+\beta|d|\}}}r$.
The result is sharp for the function $f(z)$ given by $f(z) = z + \sqrt{\frac{(\alpha\lambda+1)(\lambda+2)(\beta|d|-\lambda)}{2^{m+1}(n+1)\{(\lambda+1)[\alpha(\lambda+1+\beta|d|)+1]+\beta|d|\}}}z^2$.

Proof. We have $f'(z) = 1 + \sum_{j=2}^{\infty} j a_j z^{j-1}$ and
$|f'(z)| \leq 1 - \sum_{j=2}^{\infty} j a_j |z|^{j-1} \leq 1 + \sum_{j=2}^{\infty} j a_j r^{j-1} \leq 1 + \sqrt{\frac{(\alpha\lambda+1)(\lambda+2)(\beta|d|-\lambda)}{2^{m-1}(n+1)\{(\lambda+1)[\alpha(\lambda+1+\beta|d|)+1]+\beta|d|\}}}r$.
Similarly, $|f'(z)| \geq 1 - \sqrt{\frac{(\alpha\lambda+1)(\lambda+2)(\beta|d|-\lambda)}{2^{m-1}(n+1)\{(\lambda+1)[\alpha(\lambda+1+\beta|d|)+1]+\beta|d|\}}}r$. This completes the proof of Theorem 3.2. ■

4 Closure Theorems

Theorem 4.1 *Let the functions f_k, $k = 1, 2, ..., l$, defined by $f_k(z) = z + \sum_{j=2}^{\infty} a_{j,k} z^j$, $a_{j,k} \geq 0$, be in the class $\mathcal{L}(\lambda, d, \alpha, \beta)$. Then the function $h(z)$ defined by $h(z) = \sum_{k=1}^{l} \mu_k f_k(z)$, $\mu_k \geq 0$, is also in the class $\mathcal{L}(\lambda, d, \alpha, \beta)$, where $\sum_{k=1}^{l} \mu_k = 1$.*

Proof. We can write $h(z) = \sum_{k=1}^{l} \mu_k z + \sum_{k=1}^{l}\sum_{j=2}^{\infty} \mu_k a_{j,k} z^j = z + \sum_{j=2}^{\infty}\sum_{k=1}^{l} \mu_k a_{j,k} z^j$. Furthermore, since the functions $f_k(z)$, $k = 1, 2, ..., l$, are in the class $\mathcal{L}(\lambda, d, \alpha, \beta)$, then from Corollary 2.2 we have
$\sum_{j=2}^{\infty}\sqrt{\frac{j^{m+1}\Gamma(n+j)}{\Gamma(j+\lambda+1)}\{\alpha j^2+[\alpha(2\lambda-2+\beta|d|)+1]j+[\alpha(\lambda-1)+1](\lambda-1+\beta|d|)\}}a_j \leq$
$\sqrt{(\alpha\lambda+1)(\beta|d|-\lambda)\frac{\Gamma(n+1)}{\Gamma(\lambda+2)}}$.
Thus it is enough to prove that
$\sum_{j=2}^{\infty}\sqrt{\frac{j^{m+1}\Gamma(n+j)}{\Gamma(j+\lambda+1)}\{\alpha j^2+[\alpha(2\lambda-2+\beta|d|)+1]j+[\alpha(\lambda-1)+1](\lambda-1+\beta|d|)\}}\left(\sum_{k=1}^{m}\mu_k a_{j,k}\right) =$
$\sum_{k=1}^{m}\mu_k\sum_{j=2}^{\infty}\sqrt{\frac{j^{m+1}\Gamma(n+j)}{\Gamma(j+\lambda+1)}\{\alpha j^2+[\alpha(2\lambda-2+\beta|d|)+1]j+[\alpha(\lambda-1)+1](\lambda-1+\beta|d|)\}}a_{j,k} \leq$
$\sum_{k=1}^{m}\mu_k\sqrt{(\alpha\lambda+1)(\beta|d|-\lambda)\frac{\Gamma(n+1)}{\Gamma(\lambda+2)}} = \sqrt{(\alpha\lambda+1)(\beta|d|-\lambda)\frac{\Gamma(n+1)}{\Gamma(\lambda+2)}}$. Hence the proof is complete. ■

Corollary 4.2 *Let the functions f_k, $k = 1, 2$, defined by $f_k(z) = z + \sum_{j=2}^{\infty} a_{j,k} z^j$, $a_{j,k} \geq 0$ be in the class $\mathcal{L}(\lambda, d, \alpha, \beta)$. Then the function $h(z)$ defined by $h(z) = (1-\zeta)f_1(z) + \zeta f_2(z)$, $0 \leq \zeta \leq 1$, is also in the class $\mathcal{L}(\lambda, d, \alpha, \beta)$.*

Theorem 4.3 *Let $f_1(z) = z$, and* $f_j(z) = z + \sqrt{\frac{(\alpha\lambda+1)(\beta|d|-\lambda)\frac{\Gamma(n+1)}{\Gamma(\lambda+2)}}{\frac{j^{m+1}\Gamma(n+j)}{\Gamma(j+\lambda+1)}\{\alpha j^2+[\alpha(2\lambda-2+\beta|d|)+1]j+[\alpha(\lambda-1)+1](\lambda-1+\beta|d|)\}}}z^j$, $j \geq 2$.
Then the function $f(z)$ is in the class $\mathcal{L}(\lambda, d, \alpha, \beta)$ if and only if it can be expressed in the form $f(z) = \mu_1 f_1(z) + \sum_{j=2}^{\infty} \mu_j f_j(z)$, where $\mu_1 \geq 0$, $\mu_j \geq 0$, $j \geq 2$ and $\mu_1 + \sum_{j=2}^{\infty} \mu_j = 1$.

Proof. Assume that $f(z)$ can be expressed in the form
$f(z) = \mu_1 f_1(z) + \sum_{j=2}^{\infty} \mu_j f_j(z) = z + \sum_{j=2}^{\infty}\sqrt{\frac{(\alpha\lambda+1)(\beta|d|-\lambda)\frac{\Gamma(n+1)}{\Gamma(\lambda+2)}}{\frac{j^{m+1}\Gamma(n+j)}{\Gamma(j+\lambda+1)}\{\alpha j^2+[\alpha(2\lambda-2+\beta|d|)+1]j+[\alpha(\lambda-1)+1](\lambda-1+\beta|d|)\}}}\mu_j z^j$. Thus
$\sum_{j=2}^{\infty}\sqrt{\frac{\frac{j^{m+1}\Gamma(n+j)}{\Gamma(j+\lambda+1)}\{\alpha j^2+[\alpha(2\lambda-2+\beta|d|)+1]j+[\alpha(\lambda-1)+1](\lambda-1+\beta|d|)\}}{(\alpha\lambda+1)(\beta|d|-\lambda)\frac{\Gamma(n+1)}{\Gamma(\lambda+2)}}}\cdot$
$\sqrt{\frac{(\alpha\lambda+1)(\beta|d|-\lambda)\frac{\Gamma(n+1)}{\Gamma(\lambda+2)}}{\frac{j^{m+1}\Gamma(n+j)}{\Gamma(j+\lambda+1)}\{\alpha j^2+[\alpha(2\lambda-2+\beta|d|)+1]j+[\alpha(\lambda-1)+1](\lambda-1+\beta|d|)\}}}\mu_j = \sum_{j=2}^{\infty} \mu_j = 1 - \mu_1 \leq 1$. Hence $f(z) \in \mathcal{L}(\lambda, d, \alpha, \beta)$.

Conversely, assume that $f(z)\in\mathcal{L}(\lambda,d,\alpha,\beta)$. Setting $\mu_j=\sqrt{\frac{\frac{j^{m+1}\Gamma(n+j)}{\Gamma(j+\lambda+1)}\{\alpha j^2+[\alpha(2\lambda-2+\beta|d|)+1]j+[\alpha(\lambda-1)+1](\lambda-1+\beta|d|)\}}{(\alpha\lambda+1)(\beta|d|-\lambda)\frac{\Gamma(n+1)}{\Gamma(\lambda+2)}}}a_j$, since $\mu_1=1-\sum_{j=2}^{\infty}\mu_j$. Thus $f(z)=\mu_1f_1(z)+\sum_{j=2}^{\infty}\mu_jf_j(z)$. Hence the proof is complete. ■

Corollary 4.4 *The extreme points of the class $\mathcal{L}(d,\alpha,\beta)$ are the functions $f_1(z)=z$, and $f_j(z)=z+\sqrt{\frac{(\alpha\lambda+1)(\beta|d|-\lambda)\frac{\Gamma(n+1)}{\Gamma(\lambda+2)}}{\frac{j^{m+1}\Gamma(n+j)}{\Gamma(j+\lambda+1)}\{\alpha j^2+[\alpha(2\lambda-2+\beta|d|)+1]j+[\alpha(\lambda-1)+1](\lambda-1+\beta|d|)\}}}z^j$, $j\geq 2$.*

5 Inclusion and Neighborhood Results

We define the δ- neighborhood of a function $f(z)\in\mathcal{A}$ by $N_\delta(f)=\{g\in\mathcal{A}:g(z)=z+\sum_{j=2}^{\infty}b_jz^j$ and $\sum_{j=2}^{\infty}j|a_j-b_j|\leq\delta\}$.

In particular, for $e(z)=z$, $N_\delta(e)=\{g\in\mathcal{A}:g(z)=z+\sum_{j=2}^{\infty}b_jz^j$ and $\sum_{j=2}^{\infty}j|b_j|\leq\delta\}$.

Furthermore, a function $f\in\mathcal{A}$ is said to be in the class $\mathcal{L}^{\xi}(\lambda,d,\alpha,\beta)$ if there exists a function $h(z)\in\mathcal{L}(\lambda,d,\alpha,\beta)$ such that $\left|\frac{f(z)}{h(z)}-1\right|<1-\xi,\ z\in U,\ 0\leq\xi<1$.

Theorem 5.1 *If $\delta=\sqrt{\frac{(\alpha\lambda+1)(\lambda+2)(\beta|d|-\lambda)}{2^{m-1}(n+1)\{(\lambda+1)[\alpha(\lambda+1+\beta|d|)+1]+\beta|d|\}}}$, then $\mathcal{L}(\lambda,d,\alpha,\beta)\subset N_\delta(e)$.*

Proof. Let $f\in\mathcal{L}(\lambda,d,\alpha,\beta)$. Then in view of assertion of Corollary 2.2 and since $\frac{j^{m+1}\Gamma(n+j)}{\Gamma(j+\lambda+1)}\left\{\alpha j^2+\left[\alpha\left(2\lambda-2+\beta\left|d\right|\right)+1\right]j+\left[\alpha\left(\lambda-1\right)+1\right]\left(\lambda-1+\beta\left|d\right|\right)\right\}\geq$ $\frac{2^{m-1}\Gamma(n+2)}{\Gamma(\lambda+3)}\left\{(\lambda+1)\left[\alpha\left(\lambda+1+\beta\left|d\right|\right)+1\right]+\beta\left|d\right|\right\}$ for $j\geq 2$, we get
$\sqrt{\frac{2^{m-1}\Gamma(n+2)}{\Gamma(\lambda+3)}\left\{(\lambda+1)\left[\alpha\left(\lambda+1+\beta\left|d\right|\right)+1\right]+\beta\left|d\right|\right\}}\sum_{j=2}^{\infty}a_j\leq$
$\sum_{j=2}^{\infty}\sqrt{\frac{j^{m+1}\Gamma(n+j)}{\Gamma(j+\lambda+1)}\left\{\alpha j^2+\left[\alpha\left(2\lambda-2+\beta\left|d\right|\right)+1\right]j+\left[\alpha\left(\lambda-1\right)+1\right]\left(\lambda-1+\beta\left|d\right|\right)\right\}}a_j\leq$
$\sqrt{(\alpha\lambda+1)\left(\beta\left|d\right|-\lambda\right)\frac{\Gamma(n+1)}{\Gamma(\lambda+2)}}$, which implise $\sum_{j=2}^{\infty}a_j\leq\sqrt{\frac{(\alpha\lambda+1)(\lambda+2)(\beta|d|-\lambda)}{2^{m+1}(n+1)\{(\lambda+1)[\alpha(\lambda+1+\beta|d|)+1]+\beta|d|\}}}$.

Applying assertion of Corollary 2.2, we obtain $\sum_{j=2}^{\infty}ja_j\leq\sqrt{\frac{(\alpha\lambda+1)(\lambda+2)(\beta|d|-\lambda)}{2^{m-1}(n+1)\{(\lambda+1)[\alpha(\lambda+1+\beta|d|)+1]+\beta|d|\}}}=\delta$, so we have $f\in N_\delta(e)$. This completes the proof of the Theorem 5.1. ■

Theorem 5.2 *If $h\in\mathcal{L}(\lambda,d,\alpha,\beta)$ and*

$$\xi=1+\frac{\delta}{2}\sqrt{\frac{(\alpha\lambda+1)\left(\lambda+2\right)\left(\beta\left|d\right|-\lambda\right)}{2^{m+1}\left(n+1\right)\left\{(\lambda+1)\left[\alpha\left(\lambda+1+\beta\left|d\right|\right)+1\right]+\beta\left|d\right|\right\}}},\tag{5.1}$$

then $N_\delta(h)\subset\mathcal{L}^{\xi}(d,\alpha,\beta)$.

Proof. Suppose that $f\in N_\delta(h)$, we then find that $\sum_{j=2}^{\infty}j|a_j-b_j|\leq\delta$, which readily implies the following coefficient inequality $\sum_{j=2}^{\infty}|a_j-b_j|\leq\frac{\delta}{2}$.

Next, since $h\in\mathcal{L}(d,\alpha,\beta)$, we have $\sum_{j=2}^{\infty}b_j\leq\sqrt{\frac{(\alpha\lambda+1)(\lambda+2)(\beta|d|-\lambda)}{2^{m+1}(n+1)\{(\lambda+1)[\alpha(\lambda+1+\beta|d|)+1]+\beta|d|\}}}$ and we get $\left|\frac{f(z)}{h(z)}-1\right|\leq\frac{\sum_{j=2}^{\infty}|a_j-b_j|}{1-\sum_{j=2}^{\infty}b_j}\leq\frac{\delta}{2\left(1-\sqrt{\frac{(\alpha\lambda+1)(\lambda+2)(\beta|d|-\lambda)}{2^{m+1}(n+1)\{(\lambda+1)[\alpha(\lambda+1+\beta|d|)+1]+\beta|d|\}}}\right)}=1-\xi$, provided that ξ is given by (5.1), thus $f\in\mathcal{L}^{\xi}(\lambda,d,\alpha,\beta)$, where ξ is given by (5.1). ■

6 Radii of Starlikeness, Convexity and Close-to-Convexity

Theorem 6.1 *Let the function $f\in\mathcal{A}$ be in the class $\mathcal{L}(\lambda,d,\alpha,\beta)$. Then f is univalent starlike of order δ, $0\leq\delta<1$, in $|z|<r_1$, where $r_1=\inf_j\left\{\frac{(1-\delta)^2\frac{j^{m+1}\Gamma(n+j)}{\Gamma(j+\lambda+1)}\left\{\alpha j^2+[\alpha(2\lambda-2+\beta|d|)+1]j+[\alpha(\lambda-1)+1](\lambda-1+\beta|d|)\right\}}{(\alpha\lambda+1)(\beta|d|-\lambda)\frac{\Gamma(n+1)}{\Gamma(\lambda+2)}(j-\delta)^2}\right\}^{\frac{1}{2(j-1)}}$.*

The result is sharp for the function $f(z)$ given by

$$f_j(z)=z+\sqrt{\frac{(\alpha\lambda+1)\left(\beta\left|d\right|-\lambda\right)\frac{\Gamma(n+1)}{\Gamma(\lambda+2)}}{\frac{j^{m+1}\Gamma(n+j)}{\Gamma(j+\lambda+1)}\left\{\alpha j^2+\left[\alpha\left(2\lambda-2+\beta\left|d\right|\right)+1\right]j+\left[\alpha\left(\lambda-1\right)+1\right]\left(\lambda-1+\beta\left|d\right|\right)\right\}}}z^j,\quad j\geq 2.\tag{6.1}$$

Proof. It suffices to show that $\left|\frac{zf'(z)}{f(z)}-1\right| \leq 1-\delta$, $|z|<r_1$. Since $\left|\frac{zf'(z)}{f(z)}-1\right| = \left|\frac{\sum_{j=2}^{\infty}(j-1)a_jz^{j-1}}{1+\sum_{j=2}^{\infty}a_jz^{k-1}}|\right| \leq \frac{\sum_{j=2}^{\infty}(j-1)a_j|z|^{j-1}}{1-\sum_{j=2}^{\infty}a_j|z|^{j-1}}$. To prove the theorem, we must show that $\frac{\sum_{j=2}^{\infty}(j-1)a_j|z|^{j-1}}{1-\sum_{j=2}^{\infty}a_j|z|^{j-1}} \leq 1-\delta$.

It is equivalent to $\sum_{j=2}^{\infty}(j-\delta)a_j|z|^{j-1} \leq 1-\delta$, using Theorem 2.1, we obtain

$|z| \leq \left\{\frac{(1-\delta)^2\frac{j^{m+1}\Gamma(n+j)}{\Gamma(j+\lambda+1)}\left\{\alpha j^2+[\alpha(2\lambda-2+\beta|d|)+1]j+[\alpha(\lambda-1)+1](\lambda-1+\beta|d|)\right\}}{(\alpha\lambda+1)(\beta|d|-\lambda)\frac{\Gamma(n+1)}{\Gamma(\lambda+2)}(j-\delta)^2}\right\}^{\frac{1}{2(j-1)}}$. Hence the proof is complete. ∎

Theorem 6.2 *Let the function $f \in \mathcal{A}$ be in the class $\mathcal{L}(\lambda, d, \alpha, \beta)$. Then f is univalent convex of order δ, $0 \leq \delta \leq 1$, in $|z| < r_2$, where* $r_2 = \inf_j \left\{\frac{(1-\delta)^2\frac{j^{m-1}\Gamma(n+j)}{\Gamma(j+\lambda+1)}\left\{\alpha j^2+[\alpha(2\lambda-2+\beta|d|)+1]j+[\alpha(\lambda-1)+1](\lambda-1+\beta|d|)\right\}}{(\alpha\lambda+1)(\beta|d|-\lambda)\frac{\Gamma(n+1)}{\Gamma(\lambda+2)}(j-\delta)^2}\right\}^{\frac{1}{2(j-1)}}$.

The result is sharp for the function $f(z)$ given by (6.1).

Proof. It suffices to show that $\left|\frac{zf''(z)}{f'(z))}\right| \leq 1-\delta$, $|z|<r_2$. Since $\left|\frac{zf''(z)}{f'(z)}\right| = \left|\frac{\sum_{j=2}^{\infty}j(j-1)a_jz^{j-1}}{1+\sum_{j=2}^{\infty}ja_jz^{j-1}}\right| \leq \frac{\sum_{j=2}^{\infty}j(j-1)a_j|z|^{j-1}}{1-\sum_{j=2}^{\infty}ja_j|z|^{j-1}}$. To prove the theorem, we must show that $\frac{\sum_{j=2}^{\infty}j(j-1)a_j|z|^{j-1}}{1-\sum_{j=2}^{\infty}ja_j|z|^{j-1}} \leq 1-\delta$, i.e. $\sum_{j=2}^{\infty}j(j-\delta)a_j|z|^{j-1} \leq 1-\delta$, using Theorem 2.1, we obtain $|z|^{j-1} \leq \frac{(1-\delta)}{j(j-\delta)}\sqrt{\frac{\frac{j^{m+1}\Gamma(n+j)}{\Gamma(j+\lambda+1)}\{\alpha j^2+[\alpha(2\lambda-2+\beta|d|)+1]j+[\alpha(\lambda-1)+1](\lambda-1+\beta|d|)\}}{(\alpha\lambda+1)(\beta|d|-\lambda)\frac{\Gamma(n+1)}{\Gamma(\lambda+2)}}}$, or

$|z| \leq \left\{\frac{(1-\delta)^2\frac{j^{m-1}\Gamma(n+j)}{\Gamma(j+\lambda+1)}\left\{\alpha j^2+[\alpha(2\lambda-2+\beta|d|)+1]j+[\alpha(\lambda-1)+1](\lambda-1+\beta|d|)\right\}}{(\alpha\lambda+1)(\beta|d|-\lambda)\frac{\Gamma(n+1)}{\Gamma(\lambda+2)}(j-\delta)^2}\right\}^{\frac{1}{2(j-1)}}$. Hence the proof is complete. ∎

Theorem 6.3 *Let the function $f \in \mathcal{A}$ be in the class $\mathcal{L}(\lambda, d, \alpha, \beta)$. Then f is univalent close-to-convex of order δ, $0 \leq \delta < 1$, in $|z| < r_3$, where* $r_3 = \inf_j \left\{\frac{(1-\delta)^2\frac{j^{m-1}\Gamma(n+j)}{\Gamma(j+\lambda+1)}\left\{\alpha j^2+[\alpha(2\lambda-2+\beta|d|)+1]j+[\alpha(\lambda-1)+1](\lambda-1+\beta|d|)\right\}}{(\alpha\lambda+1)(\beta|d|-\lambda)\frac{\Gamma(n+1)}{\Gamma(\lambda+2)}}\right\}^{\frac{1}{2(j-1)}}$.

The result is sharp for the function $f(z)$ given by (6.1).

Proof. It suffices to show that $|f'(z)-1| \leq 1-\delta$, $|z| < r_3$. Then $|f'(z)-1| = \left|\sum_{j=2}^{\infty}ja_jz^{j-1}\right| \leq \sum_{j=2}^{\infty}ja_j|z|^{j-1}$. Thus $|f'(z)-1| \leq 1-\delta$ if $\sum_{j=2}^{\infty}\frac{ja_j}{1-\delta}|z|^{j-1} \leq 1$. Using Theorem 2.1, the above inequality holds true if $|z|^{j-1} \leq \frac{(1-\delta)}{j}\sqrt{\frac{\frac{j^{m+1}\Gamma(n+j)}{\Gamma(j+\lambda+1)}\{\alpha j^2+[\alpha(2\lambda-2+\beta|d|)+1]j+[\alpha(\lambda-1)+1](\lambda-1+\beta|d|)\}}{(\alpha\lambda+1)(\beta|d|-\lambda)\frac{\Gamma(n+1)}{\Gamma(\lambda+2)}}}$ or

$|z| \leq \left\{\frac{(1-\delta)^2\frac{j^{m-1}\Gamma(n+j)}{\Gamma(j+\lambda+1)}\left\{\alpha j^2+[\alpha(2\lambda-2+\beta|d|)+1]j+[\alpha(\lambda-1)+1](\lambda-1+\beta|d|)\right\}}{(\alpha\lambda+1)(\beta|d|-\lambda)\frac{\Gamma(n+1)}{\Gamma(\lambda+2)}}\right\}^{\frac{1}{2(j-1)}}$. Hence the proof is complete. ∎

References

[1] A. Alb Lupaş, *Properties on a subclass of univalent functions defined by using Sălăgean operator and Ruscheweyh derivative*, J. of Comput. Anal. Appl., 21 (2016), No.7, 1213-1217.

[2] N.E.Cho, A.M.K. Aouf, *Some applications of fractional calculus operators to a certain subclass of analytic functions with negative coefficients*, Tr. J. of Mathematics, Vol. 20, 1996, 553-562.

[3] St. Ruscheweyh, *New criteria for univalent functions*, Proc. Amet. Math. Soc., 49(1975), 109-115.

[4] G. St. Sălăgean, *Subclasses of univalent functions*, Lecture Notes in Math., Springer Verlag, Berlin, 1013 (1983), 362-372.

Normal criteria of meromorphic functions concerning holomorphic functions*

Da-Wei Meng[1], San-Yang Liu[1], and Hong-Yan Xu[1,2]

[1] School of Mathematics and statistics, Xidian University, Xi'an 710071, Shaanxi, P. R. China

[2] Department of Informatics and Engineering, Jingdezhen Ceramic Institute, Jingdezhen, Jiangxi, 333403, China

E-mail: Goths511@163.com(Da-Wei Meng); liusanyang@126.com(San-Yang Liu); xhyhhh@126.com

Abstract

In this paper, we mainly investigate the problem of normal families of meromorphic functions concerning shared functions, and obtain some normality criteria of meromorphic functions sharing a holomorphic function. Our results generalize or extend the previous theorems given by Ding J. J., Ding L. W. and Yuan W. J..

1 Introduction and main results

Let $\mathscr{F}$ be a family of meromorphic functions defined in a domain D. In the sense of Montel, $\mathscr{F}$ is said to be normal in D, if for any sequence $\{f_n\} \subset \mathscr{F}$ there exists a

---

2010 Mathematics Subject Classification. Primary 30D35, 30D45.

*This work is supported by Natural Science Foundation of China (Grant No.11271227, No.11201360, No.61373174 and No.11561033), and the Fundamental Research Funds for the Central Universities of China (Grant No.800272125771).

Key words: meromorphic functions; normal family; sharing a holomorphic function.

subsequence $\{f_{n_j}\}$ which converges spherically locally uniformly in D, to a meromorphic function or ∞. For simplicity, we take $\to$ to stand for convergence, $\rightrightarrows$ for convergence spherically locally uniformly, and $\mathcal{M}(D)$ (resp. $\mathcal{A}(D)$) for the set of meromorphic (resp. holomorphic) functions on D. Let F and G two non-constant meromorphic functions defined in D. Then we say that f and g share a IM if $F-a$ and $G-a$ assume the same zeros ignoring multiplicity. The zeros of $F-a$ mean the poles of F when $a=\infty$.

In 1959, Hayman [9] proposed a conjecture: if $F\in\mathcal{M}(\mathbb{C})$ is transcendental, then F^nF' assumes every finite non-zero complex number infinitely often for any positive integer n. The conjecture is showed to be true by many authors, such as Hayman [10], Mues [17], Clunie [6], Bergweiler and Eremenko [2], Chen and Fang [4]. Accordingly, Hayman [10] conjectured that if $\mathscr{F}$ is the family of $\mathcal{M}(D)$ such that each $f\in\mathscr{F}$ satisfies $f^nf'\neq a$ for a positive integer n and a non-zero complex number a, then $\mathscr{F}$ is normal. This conjecture has been confirmed by some authors, such as Yang and Zhang [26] , Gu [8], Pang [20], Oshkin [18] and Pang [20]. In 2008, from the point of shared values, Zhang [29] concluded that if $\mathscr{F}$ is the family of $\mathcal{M}(D)$ such that each pair (f,g) of $\mathscr{F}$ satisfies that f^nf' and g^ng' share a finite non-zero complex number a IM for $n\geq 2$, then $\mathscr{F}$ is normal. Recently, Jiang and Gao [12] generalized Zhang's result based on the ideas of shared functions. For other generations, we can refer to [3, 15, 24].

For the case of $F^nF^{(k)}$, Zhang and Li [31] proved that if $F\in\mathcal{M}(\mathbb{C})$ is transcendental, then $F^nL[F]-a$ has infinitely many zeros for $n\geq 2$ and $a\neq 0,\infty$, where $L[F]=a_kF^{(k)}+a_{k-1}F^{(k-1)}+\cdots+a_0F$ in which a_i $(i=0,1,2,\cdots,k)$ are small functions of F. Pang and Zalcman [22] further obtained the corresponding normality criterion as follows: If $\mathscr{F}$ is the family of $\mathcal{A}(D)$ such that zeros of each $f\in\mathscr{F}$ have multiplicities at least k and such that each $f\in\mathscr{F}$ satisfies $f^nf^{(k)}\neq a$ for a non-zero complex number a, then $\mathscr{F}$ is normal. Recently, Meng and Hu [16] extended Pang's result, by replacing $f^nf^{(k)}\neq a$ into the condition that f^nf^k and g^ng^k share a IM. Similarly, we also have analogues related to some conditions of $f\left(f^{(k)}\right)^l$ for a positive integer l (refer to [1, 11, 13, 30]).

In 2013, considering the general case of $F^n(F^{(k)})^l$ from the view of shared values, Ding, Ding and Yuan [7] proved a normality criterion as follows: Let a be a non-zero value, if $\mathscr{F}$ is the family of $\mathcal{M}(D)$ such that each pair (f,g) of $\mathscr{F}$ satisfies that $f^n(f^{(k)})^l$ and $g^n(g^{(k)})^l$ share a non-zero value a, where each $f\in\mathcal{F}$ has only zeros of multiplicity at least $max(k,2)$, then $\mathscr{F}$ is normal. Naturally we ask: whether there exists normality theorem when a is a function?

Take four integers $k\geq 1$, $m\geq 0$, $n\geq 1$ and $l\geq 2$. Let a $(\not\equiv 0)$ be a holomorphic function in a domain D such that multiplicities of zeros of a are at most m and divisible by $n+l$. In this paper, we prove the following normality criterion:

Theorem 1.1. *Let $\mathscr{F}$ be the family of $\mathcal{M}(D)$ such that multiplicities of zeros of each $f \in \mathscr{F}$ are at least $k+m+1$ and such that multiplicities of poles of f are at least $m+1$ whenever f have zeros and poles. If each pair (f,g) of $\mathscr{F}$ satisfies that $f^n(f^{(k)})^l$ and $g^n(g^{(k)})^l$ share a IM, then $\mathscr{F}$ is normal in D.*

In special, when $k=1$, we may modify Theorem 1.1 as follows:

Theorem 1.2. *Suppose $a = a(z)$ as in Theorem 1.1, if $\mathscr{F}$ is the family of $\mathcal{M}(D)$ such that each $f \in \mathscr{F}$ satisfies that $f^n(f')^l \neq a$, then $\mathscr{F}$ is normal in D.*

Similar to the proof of Theorem 1.2, we conclude the following result:

Theorem 1.3. *Suppose $a = a(z)$ as in Theorem 1.1, if $\mathscr{F}$ is the family of $\mathcal{M}(D)$ such that each $f \in \mathscr{F}$ satisfies that $f^n(f'(z))^l = a$ implies $|f(z)| > A$ for a positive number A, then $\mathscr{F}$ is normal in D.*

As a matter of fact, Theorem 1.3 is inspired by the ideas of papers [11, 13] initially.

2 Preliminary lemmas

First of all, we introduce the following *Zalcman's lemma* [28]:

Lemma 2.1. *Take a positive integer k. Let $\mathscr{F}$ be a family of meromorphic functions in the unit disc $\triangle$ with the property that zeros of each $f \in \mathscr{F}$ are of multiplicity at least k. If $\mathscr{F}$ is not normal at a point $z_0 \in \Delta$, then for $0 \leq \alpha < k$, there exist a sequence $\{z_n\} \subset \Delta$ of complex numbers with $z_n \to z_0$; a sequence $\{f_n\}$ of $\mathscr{F}$; and a sequence $\{\rho_n\}$ of positive numbers with $\rho_n \to 0$ such that $g_n(\xi) = \rho_n^{-\alpha} f_n(z_n + \rho_n \xi)$ locally uniformly (with respect to the spherical metric) to a nonconstant meromorphic function $g(\xi)$ on $\mathbb{C}$. Moreover, the zeros of $g(\xi)$ are of multiplicity at least k, and the function $g(\xi)$ may be taken to satisfy the normalization $g^\sharp(\xi) \leq g^\sharp(0) = 1$ for any $\xi \in \mathbb{C}$. In particular, $g(\xi)$ has at most order 2.*

This Lemma is Pang's generalization [19, 21, 25] to the Main Lemma in [27] (where α is taken to be 0), with improvements due to Schwick [23], Chen and Gu [5].

Next, by applying the results from [1, 14, 31, 30] we can deduce the following lemma:

Lemma 2.2. *Let f be a transcendental meromorphic function in the complex plane. Let n, l, k be three positive integers and $a = a(z) \not\equiv 0$ be a polynomial. Then for $l \geq 2$, $f^n(f^{(k)})^l - a$ has infinitely many zeros.*

Finally, we investigate the zeros of $f^n(f^{(k)})^l - a$ if f is rational, and thus give Lemma 2.3 and 2.4:

Lemma 2.3. *Let $p \geq 0$, $n, k \geq 1$ and $l \geq 2$ be four integers, and let a be a non-zero polynomial of degree p. If f is a non-constant rational function which has only zeros of multiplicity at least $k+p+1$ and has only poles of multiplicity at least $p+1$, then $f^n(f^{(k)})^l - a$ has at least two distinct zeros.*

Proof. Firstly, we assume that f is a non-constant polynomial. It follows that $f^{(k)} \not\equiv 0$ from f has only zeros of multiplicity at least $k+p+1$. Hence we have

$$\deg\left(f^n(f^{(k)})^l\right) \geq n(k+p+1) + l(p+1) > p = \deg(a).$$

Therefore, it follows that $f^n(f^{(k)})^l - a$ is also a non-constant polynomial, and hence $f^n(f^{(k)})^l - a$ has at least one zero.

Further, we claim that $f^n(f^{(k)})^l - a$ has at least two distinct zeros if f is a non-constant polynomial. To the contrary, suppose that $f^n(f^{(k)})^l - a$ has only one zero z_0, which means

$$f^n(z)(f^{(k)})^l(z) - a(z) = A'(z - z_0)^d,$$

where A' is a non-zero constant and d is a positive integer. Since f is a non-constant polynomial which has only zeros of multiplicity at least $k+p+1$, we find $f^{(k)} \not\equiv 0$, and hence

$$d = \deg(f^n(f^{(k)})^l - a) > \deg(f^n) \geq n(k+p+1) \geq p+2.$$

By computing we find

$$\left\{f^n(f^{(k)})^l\right\}^{(p+1)}(z) = A'd(d-1)...(d-p)(z-z_0)^{d-p-1},$$

hence $\left\{f^n(f^{(k))^l}\right\}^{(p+1)}$ has a unique zero z_0. Take a zero ξ_0 of f, then it is a zero of f^n with multiplicity at least $n(k+p+1)$. It follows that ξ_0 is a zero of $\left\{f^n(f^{(k)})^l\right\}^{(p)}$ and $\left\{f^n(f^{(k)})^l\right\}^{(p+1)}$, which further implies that $\xi_0 = z_0$. Therefore, we obtain $\left\{f^n(f^{(k)})^l\right\}^{(p)}(z_0) = 0$. On the other hand, we get $\left\{f^n(f^{(k)})^l\right\}^{(p)}(z) = b + A'd(d-1)...(d-p+1)(z-z_0)^{d-p}$, in which b is a non-zero constant such that $b = a^{(p)}(z)$. This yields that $\left\{f^n(f^{(k)})^l\right\}^{(p)}(z_0) = b \neq 0$, which is contradictory to $\left\{f^n(f^{(k)})^l\right\}^{(p)}(z_0) = 0$. The claim is proved.

Secondly, we assume that f has poles, and then set

$$f(z) = \frac{A(z-\alpha_1)^{m_1}(z-\alpha_2)^{m_2}\cdots(z-\alpha_s)^{m_s}}{(z-\beta_1)^{n_1}(z-\beta_2)^{n_2}\cdots(z-\beta_t)^{n_t}}, \tag{2.1}$$

where A is a non-zero constant, α_i distinct zeroes of f with $s \geq 0$, and β_j distinct poles of f with $t \geq 1$. For simplicity, we put

$$m_1 + m_2 + \cdots + m_s = M \geq (k+p+1)s, \tag{2.2}$$

4

$$n_1 + n_2 + \cdots + n_t = N \geq (p+1)t. \tag{2.3}$$

From (2.1), we obtain

$$f^{(k)}(z) = \frac{(z-\alpha_1)^{m_1-k}(z-\alpha_2)^{m_2-k}\cdots(z-\alpha_s)^{m_s-k}g(z)}{(z-\beta_1)^{n_1+k}(z-\beta_2)^{n_2+k}\cdots(z-\beta_t)^{n_t+k}}, \tag{2.4}$$

where g is a polynomial of degree $\leq kl(s+t-1)$. From (2.1) and (2.4), we get

$$f^n(z)(f^{(k)})^l(z) = \frac{A^n(z-\alpha_1)^{M_1}(z-\alpha_2)^{M_2}\cdots(z-\alpha_s)^{M_s}g^l(z)}{(z-\beta_1)^{N_1}(z-\beta_2)^{N_2}\cdots(z-\beta_t)^{N_t}}, \tag{2.5}$$

in which

$$M_i = (n+l)m_i - kl, \ i = 1, 2, \cdots, s,$$

$$N_j = (n+l)n_j + kl, \ j = 1, 2, \cdots, t.$$

Differentiating (2.5) yields

$$\left\{f^n(f^{(k)})^l\right\}^{(p+1)}(z) = \frac{(z-\alpha_1)^{M_1-p-1}(z-\alpha_2)^{M_2-p-1}\cdots(z-\alpha_s)^{M_s-p-1}g_0(z)}{(z-\beta_1)^{N_1+p+1}\cdots(z-\beta_t)^{N_t+p+1}}, \tag{2.6}$$

where $g_0(z)$ is a polynomial of degree $\leq (p+kl+1)(s+t-1)$.

We claim that $f^n(f^{(k)})^l - a$ has at least one zero if f is a non-polynomial rational function. In order to prove this claim, suppose the contrary holds, thus we set

$$f^n(z)(f^{(k)})^l(z) = a(z) + \frac{C}{(z-\beta_1)^{N_1}(z-\beta_2)^{N_2}\cdots(z-\beta_t)^{N_t}}, \tag{2.7}$$

where C is a non-zero constant. Subsequently, (2.7) yields

$$\left\{f^n(f^{(k)})^l\right\}^{(p+1)}(z) = \frac{g_1(z)}{(z-\beta_1)^{N_1+p+1}\cdots(z-\beta_t)^{N_t+p+1}}, \tag{2.8}$$

where $g_1(z)$ is a polynomial of degree $\leq (p+1)(t-1)$. Comparing (2.6) with (2.8), we get

$$(p+1)(t-1) \geq \deg(g_1) \geq (n+l)M - kls - (p+1)s,$$

and hence

$$M < \frac{p+kl+1}{n+l}s + \frac{p+1}{n+l}t. \tag{2.9}$$

On the other hand, from (2.5) and (2.7) we have

$$(n+l)N + klt + p = (n+l)M - kls + \deg(g^l).$$

Since $\deg(g^l) \leq kl(s+t-1)$, we find

$$(n+l)N \leq (n+l)M - kls + kl(s+t-1) - klt - p,$$

implying that $(n+l)N < (n+l)M$, and thus we have

$$N < M. \tag{2.10}$$

By (2.9), (2.10) and noting that $M \geq (k+p+1)s$, $N \geq (p+1)t$, we deduce that

$$M < \frac{p+kl+1}{n+l}s + \frac{p+1}{n+l}t \leq \left\{\frac{p+kl+1}{(n+l)(k+p+1)} + \frac{1}{n+l}\right\}M. \tag{2.11}$$

Noting that $l \geq 2$ we immediately obtain

$$\frac{p+kl+1}{(n+l)(k+p+1)} + \frac{1}{n+l} = \frac{(1+l)k+2p+2}{(n+l)(k+p+1)} \leq 1.$$

Hence it follows from (2.11) that $M < M$, which is a contradiction. The claim is proved.

Now we suppose that $f^n(f^{(k)})^l - a$ has only one zero z_0, where f is a non-polynomial rational function, then we find

$$f^n(z)(f^{(k)})^l(z) = a(z) + \frac{C'(z-z_0)^d}{(z-\beta_1)^{N_1}(z-\beta_2)^{N_2}\cdots(z-\beta_t)^{N_t}}, \tag{2.12}$$

where C' is a non-zero constant and d is a positive integer. We distinguish two cases to deduce contradictions.

Case 1. $p \geq d$. Since $p \geq d$, the expression (2.5) together with (2.12) imply that

$$(n+l)N + klt + p = (n+l)M - kls + \deg(g^l).$$

Therefore, we can also conclude (2.10), that is, $N < M$. Differentiating (2.12), we obtain

$$\left\{f^n(f^{(k)})^l\right\}^{(p+1)}(z) = \frac{g_2(z)}{(z-\beta_1)^{N_1+p+1}\cdots(z-\beta_t)^{N_t+p+1}},$$

where $g_2(z)$ is a polynomial of degree at most $(p+1)t-(p+1)+d$, and hence

$$(p+1)t-(p+1)+d \geq \deg(g_2) \geq (n+l)M - kls - (p+1)s.$$

Then we have

$$\frac{p+1-d}{n+l} \leq \frac{p+kl+1}{n+l}s + \frac{p+1}{n+1}t - M \leq \left\{\frac{p+kl+1}{(n+l)(k+p+1)} + \frac{1}{n+l} - 1\right\}M \tag{2.13}$$

since $M \geq (k+p+1)s, N \geq (p+1)t, M > N$. It follows that

$$\frac{p+kl+1}{(n+l)(k+p+1)} + \frac{1}{n+l} < 1$$

since $l \geq 2$. Therefore, from (2.13) we conclude that $\frac{p+1-d}{n+l} < 0$, a contradiction with the assumption $p \geq d$.

Case 2. $d > p$. The expression (2.12) yields

$$\left\{f^n(f^{(k)})^l\right\}^{(p+1)}(z) = \frac{(z-z_0)^{d-p-1}g_3(z)}{(z-\beta_1)^{N_1+p+1}\cdots(z-\beta_t)^{N_t+p+1}}, \tag{2.14}$$

where $g_3(z)$ is a polynomial with $\deg(g_3) \le (p+1)t$. We claim that $z_0 \ne \alpha_i$ for each i. Otherwise, if $z_0 = \alpha_i$ for some i, then (2.12) yields

$$a^{(p)}(z_0) = \left\{f^n(f^{(k)})^l\right\}^{(p)}(z_0) = \left\{f^n(f^{(k)})^l\right\}^{(p)}(\alpha_i) = 0$$

because each α_i is a zero of $f^n(f^{(k)})^l$ of multiplicity $> n(k+p+1) \ge p+2$. This is impossible since $\deg(a) = p$. Hence $(z-z_0)^{d-p-1}$ is a factor of the polynomial g_0 in (2.6). By (2.6) and (2.14), we conclude that

$$(p+1)t \ge \deg(g_3) \ge (n+l)M - kls - (p+1)s,$$

which is equivalent to

$$M \le \frac{p+kl+1}{n+l}s + \frac{p+1}{n+l}t. \tag{2.15}$$

If $d \ne (n+l)N + klt + p$, then (2.5) together with (2.12) implies

$$(n+l)N + klt + p \le (n+l)M - kls + \deg(g^l),$$

so we get $N < M$ from $\deg(g^l) \le kl(s+t-1)$. Therefore, by using the facts $M \ge (k+p+1)s, N \ge (p+1)t$, (2.15) implies a contradiction

$$M < \left\{\frac{p+kl+1}{(n+l)(k+p+1)} + \frac{1}{n+l}\right\} M \le M.$$

Hence $d = (n+l)N + klt + p$.

Now we must have $N \ge M$, otherwise, when $N < M$, we can deduce the contradiction $M < M$ from (2.15). Comparing (2.6) with (2.14), we find

$$(p+kl+1)(s+t-1) \ge \deg(g_0) \ge d - p - 1$$

since $(z-z_0)^{l-p-1} | g_0$, and hence

$$(n+l)N + klt + p = d \le (p+kl+1)s + (p+kl+1)t - kl,$$

which further yields

$$N < \frac{p+k+1}{n+1}s + \frac{p+1}{n+1}t.$$

Since $M \ge (k+p+1)s$ and $N \ge (p+1)t$, it follows from that

$$N < \frac{p+kl+1}{(n+l)(k+p+1)}M + \frac{1}{n+l}N.$$

Hence $N \geq M$ yields

$$N < \left\{ \frac{p+kl+1}{(n+l)(k+p+1)} + \frac{1}{n+l} \right\} N. \tag{2.16}$$

Since $l \geq 2$, we obtain consequently

$$\frac{p+kl+1}{(n+l)(k+p+1)} + \frac{1}{n+l} \leq 1.$$

Hence (2.16) yields $N < N$. This is a contradiction. Proof of Lemma 2.3 is completed. $\square$

Lemma 2.4. *Let $p \geq 0$,$n \geq 1$ and $l \geq 2$ be three integers such that p is divisible by $n+l$, and let a be a non-zero polynomial of degree p. If f is a non-constant rational function, then $f^n(f')^l - a$ has at least one zero.*

Proof. If f is a non-constant polynomial, then $f' \not\equiv 0$. We consequently conclude that

$$\deg\left(f^n(f')^l\right) = (n+l)\deg(f) - l \neq p$$

since p is divisible by $n+l$. It follows that $f^n(f')^l - a$ is also a non-constant polynomial, so that $f^n(f')^l - a$ has at least one zero.

If f has poles, we can express f by (2.1) again, and then, by differentiating (2.1), we deduce that

$$f'(z) = \frac{(z-\alpha_1)^{m_1-1}(z-\alpha_2)^{m_2-1}\cdots(z-\alpha_s)^{m_s-1}h(z)}{(z-\beta_1)^{n_1+1}(z-\beta_2)^{n_2+1}\cdots(z-\beta_t)^{n_t+1}}, \tag{2.17}$$

where $h(z)$ is a polynomial of form

$$h(z) = (M-N)z^{s+t-1} + \cdots.$$

From (2.1) and (2.17), we obtain

$$f^n(f')^l = \frac{P}{Q},$$

in which

$$P(z) = A^n(z-\alpha_1)^{(n+l)m_1-l}(z-\alpha_2)^{(n+l)m_2-l}\cdots(z-\alpha_s)^{(n+l)m_s-l}h^l(z),$$

$$Q(z) = (z-\beta_1)^{(n+l)n_1+l}(z-\beta_2)^{(n+l)n_2+l}\cdots(z-\beta_t)^{(n+l)n_t+l}.$$

We suppose, to the contrary, that $f^n f' - a$ has no zero. When $M \neq N$, we have

$$f^n f' = a + \frac{B}{Q} = \frac{P}{Q},$$

where B is a non-zero constant. Therefore, we obtain

$$\deg(P) = \deg(Qa + B) = \deg(Q) + p.$$

8

Note $deg(h^l) \geq l(s+t-1)$ implies that

$$(n+l)M - ls + l(s+t-1) \geq (n+l)N + lt + p,$$

or equivalently

$$(n+l)M - N \geq (p+l),$$

which further yields $M > N$ and $deg(h) = s+t-1$. It follows that

$$(n+l)M - ls + l(s+t-1) = (n+l)N + lt + p.$$

Thus we immediately obtain

$$M - N = \frac{p+l}{n+l},$$

which is impossible since $M-N$ is an integer. Therefore, $f^n f' - a$ has at least one zero. □

3 Proof of Theorem 1.1

Without loss of generality, we may assume that $D = \{z \in \mathbb{C} \mid |z| < 1\}$. For any point z_0 in D, either $a(z_0) = 0$ or $a(z_0) \neq 0$ holds. For simplicity, we assume $z_0 = 0$ and distinguish two cases.

Case 1. $a(0) \neq 0$. To the contrary, we suppose that $\mathscr{F}$ is not normal at $z_0 = 0$. Then, by Lemma 2.1, there exist a sequence $\{z_j\}$ of complex numbers with $z_j \to 0$ $(j \to \infty)$; a sequence $\{f_j\}$ of $\mathscr{F}$; and a sequence $\{\rho_j\}$ of positive numbers with $\rho_j \to 0$ $(j \to \infty)$ such that

$$g_j(\xi) = \rho_j^{-\frac{lk}{n+l}} f_j(z_j + \rho_j \xi)$$

converges uniformly to a non-constant meromorphic function $g(\xi)$ in $\mathbb{C}$ with respect to the spherical metric. Moreover, $g(\xi)$ is of order at most 2. By Hurwitz's theorem, the zeros of $g(\xi)$ have at least multiplicity $k+m+1$.

On every compact subset of $\mathbb{C}$ which contains no poles of g, we have uniformly

$$\begin{aligned} & f_j^n(z_j+\rho_j\xi)(f_j^{(k)}(z_j+\rho_j\xi))^l - a(z_j+\rho_j\xi) \\ = \ & g_j^n(\xi)(g_j^{(k)}(\xi))^l - a(z_j+\rho_j\xi) \rightrightarrows g^n(\xi)(g^{(k)}(\xi))^l - a(0). \end{aligned} \tag{3.1}$$

If $g^n(g^{(k)})^l \equiv a(0)$, then g has no zeros and poles. Then there exist constants c_i such that $(c_1, c_2) \neq (0,0)$, and

$$g(\xi) = e^{c_0 + c_1\xi + c_2\xi^2}$$

since g is a non-constant meromorphic function of order at most 2. Obviously, this is contrary to the case $g^n(g^{(k)})^l \equiv a(0)$. Hence we have $g^n(g^{(k)})^l \not\equiv a(0)$.

By Lemma 2.2 and Lemma 2.3, the function $g^n(g^{(k)})^l - a(0)$ has two distinct zeros ξ_0 and ξ_0^*. We choose a positive number δ small enough such that $D_1 \cap D_2 = \emptyset$ and such that $g^n(g^{(k)})^l - a(0)$ has no other zeros in $D_1 \cup D_2$ except for ξ_0 and ξ_0^*, where

$$D_1 = \{\xi \in \mathbb{C} \mid |\xi - \xi_0| < \delta\},\ D_2 = \{\xi \in \mathbb{C} \mid |\xi - \xi_0^*| < \delta\}.$$

By (3.1) and Hurwitz's theorem, there exist points $\xi_j \in D_1$, $\xi_j^* \in D_2$ such that

$$f_j^n(z_j + \rho_j\xi_j)(f_j^{(k)}(z_j + \rho_j\xi_j))^l - a(z_j + \rho_j\xi_j) = 0,$$

and

$$f_j^n(z_j + \rho_j\xi_j^*)(f_j^{(k)}(z_j + \rho_j\xi_j^*))^l - a(z_j + \rho_j\xi_j^*) = 0$$

for sufficiently large j.

By the assumption in Theorem 1.1, $f_1^n(f_1^{(k)})^l$ and $f_j^n(f_j^{(k)})^l$ share a IM for each j. It follows

$$f_1^n(z_j + \rho_j\xi_j)(f_1^{(k)}(z_j + \rho_j\xi_j))^l - a(z_j + \rho_j\xi_j) = 0,$$

and

$$f_1^n(z_j + \rho_j\xi_j^*)(f_1^{(k)}(z_j + \rho_j\xi_j^*))^l - a(z_j + \rho_j\xi_j^*) = 0.$$

By letting $j \to \infty$, and noting $z_j + \rho_j\xi_j \to 0$, $z_j + \rho_j\xi_j^* \to 0$, we obtain

$$f_1^n(0)(f_1^{(k)}(0))^l - a(0) = 0.$$

Since the zeros of $f_1^n(\xi)(f_1^{(k)}(\xi))^l - a(\xi)$ has no accumulation points, in fact we have

$$z_j + \rho_j\xi_j = 0,\ z_j + \rho_j\xi_j^* = 0,$$

or equivalently

$$\xi_j = -\frac{z_j}{\rho_j}, \qquad \xi_j^* = -\frac{z_j}{\rho_j}.$$

This contradicts with the facts that $\xi_j \in D_1$, $\xi_j^* \in D_2$, $D_1 \cap D_2 = \emptyset$. Thus $\mathscr{F}$ is normal at $z_0 = 0$.

Case 2. $a(0) = 0$. We assume that $z_0 = 0$ is a zero of a of multiplicity p. Then we have $p \le m$ by the assumption. Write $a(z) = z^p b(z)$, in which $b(0) = b_p \neq 0$. Since multiplicities of all zeros of a are divisible by $n + l$, then $d = \frac{p}{n+l}$ is just a positive integer. Thus we obtain a new family of $\mathcal{M}(D)$ as follows

$$\mathscr{H} = \left\{ \frac{f(z)}{z^d} \;\middle|\; f \in \mathscr{F} \right\}.$$

We claim that $\mathscr{H}$ is normal at 0.

Otherwise, if $\mathscr{H}$ is not normal at 0, then by lemma 2.1 there exist a sequence $\{z_j\}$ of complex numbers with $z_j \to 0$ $(j \to \infty)$; a sequence $\{h_j\}$ of $\mathscr{H}$; and a sequence $\{\rho_j\}$ of positive numbers with $\rho_j \to 0$ $(j \to \infty)$ such that

$$g_j(\xi) = \rho_j^{-\frac{lk}{n+l}} h_j(z_j + \rho_j \xi) \tag{3.2}$$

converges uniformly to a non-constant meromorphic function $g(\xi)$ in $\mathbb{C}$ with respect to the spherical metric, where $g^\sharp(\xi) \leq 1$, $\mathrm{ord}(g) \leq 2$, and h_j has the following form

$$h_j(z) = \frac{f_j(z)}{z^d}.$$

We will deduce contradiction by distinguishing two cases.

Subcase 2.1. There exists a subsequence of $\frac{z_j}{\rho_j}$, for simplicity we still denote it as $\frac{z_j}{\rho_j}$, such that $\frac{z_j}{\rho_j} \to c$ as $j \to \infty$, where c is a finite number. Thus we have

$$F_j(\xi) = \frac{f_j(\rho_j\xi)}{\rho_j^{\frac{lk}{n+l}+d}} = \frac{(\rho_j\xi)^d h_j(z_j + \rho_j(\xi - \frac{z_j}{\rho_j}))}{(\rho_j)^d (\rho_j)^{\frac{lk}{n+l}}} \rightrightarrows \xi^d g(\xi - c) = h(\xi),$$

and

$$F_j^n(\xi)(F_j^{(k)}(\xi))^l - \frac{a(\rho_j\xi)}{\rho_j^p} = \frac{f_j^n(\rho_j\xi)(f_j^{(k)}(\rho_j\xi))^l - a(\rho_j\xi)}{\rho_j^p} \rightrightarrows h^n(\xi)(h^{(k)}(\xi))^l - b_p\xi^p. \tag{3.3}$$

Noting $p \leq m$, it follows from Lemma 2.2 and Lemma 2.3 that $h^n(\xi)(h^{(k)}(\xi))^l - b_p\xi^p$ has at least two distinct zeros. Similar to the proof of Case1, we can obtain a contradiction.

Subcase 2.2. There exists a subsequence of $\frac{z_j}{\rho_j}$, for simplicity we still denote it as $\frac{z_j}{\rho_j}$, such that $\frac{z_j}{\rho_j} \to \infty$ as $j \to \infty$. Then we deduce that

$$\begin{aligned} f_j^{(k)}(z_j + \rho_j\xi) &= \Big\{(z_j + \rho_j\xi)^d h_j(z_j + \rho_j\xi)\Big\}^{(k)} \\ &= (z_j + \rho_j\xi)^d h_j^{(k)}(z_j + \rho_j\xi) + \sum_{i=1}^{k} a_i (z_j + \rho_j\xi)^{d-i} h_j^{(k-i)}(z_j + \rho_j\xi) \\ &= (z_j + \rho_j\xi)^d \rho_j^{-\frac{nk}{n+l}} g_j^{(k)}(\xi) + \sum_{i=1}^{k} a_i (z_j + \rho_j\xi)^{d-i} \rho_j^{-\frac{nk}{n+l}+i} g_j^{(k-i)}(\xi), \end{aligned}$$

in which $a_i (i = 1, 2, ..., k)$ are all constants. Thus the expansion of $(f_j^{(k)}(z_j + \rho_j\xi))^l$ can be stated as

$$(g_j^{(k)}(\xi))^l (z_j + \rho_j\xi)^{ld} (\rho_j\xi)^{-\frac{nlk}{n+l}} + \sum_{l_0<l} \prod_{i=0}^{k} (a_i g_j^{(k-i)}(\xi))^{l_i} (z_j + \rho_j\xi)^{ld} (\rho_j\xi)^{-\frac{nlk}{n+l}} \left(\frac{\rho_j}{z_j + \rho_j\xi}\right)^{\sum_{i=1}^{k} i l_i},$$

where $l_i (i = 0, 1, ..., k)$ are arbitrary non-negative integers satisfying $\sum_{i=0}^{k} l_i = l$.

11

Since $\frac{z_j}{\rho_j} \to \infty$, $b(z_j + \rho_j\xi) \to b_p$ as $j \to \infty$, it follows that

$$\begin{aligned}
& b_p \frac{f_j^n(z_j+\rho_j\xi)(f_j^{(k)}(z_j+\rho_j\xi))^l}{a(z_j+\rho_j\xi)} - b_p \\
= & \frac{b_p(z_j+\rho_j\xi)^{(n+l)d} g_j^n(\xi)(g_j^{(k)}(\xi))^l}{b(z_j+\rho_j\xi)(z_j+\rho_j\xi)^p} \\
& + \sum_{l_0<l} \frac{b_p(z_j+\rho_j\xi)^{(n+l)d} g_j^n(\xi)\prod_{i=0}^{k}(a_i g_j^{(k-i)}(\xi))^{l_i}}{b(z_j+\rho_j\xi)(z_j+\rho_j\xi)^p}\left(\frac{\rho_j}{z_j+\rho_j\xi}\right)^{\sum_{i=1}^{k} i l_i} - b_p \\
\rightrightarrows & \; g^n(\xi)(g^{(k)}(\xi))^l - b_p
\end{aligned} \tag{3.4}$$

on every compact subset of $\mathbb{C}$ which contains no poles of g.

Since all zeros of $f_j \in \mathscr{F}$ have at least multiplicity $k+m+1$, then multiplicities of zeros of g are at least $k+1$. Then from Lemma 2.2 and Lemma 2.3, the function $g^n(\xi)(g^{(k)}(\xi))^l - b_p$ has at least two distinct zeros. With similar discussion to the proof of Case1, we can get a contradiction.

Hence the claim is proved, that is, $\mathscr{H}$ is normal at $z_0 = 0$. Therefore, for any sequence $\{f_t\} \subset \mathscr{F}$ there exist $\Delta_r = \{z : |z| < r\}$ and a subsequence $\{h_{t_k}\}$ of $\{h_t(z) = f_t(z)/z^d\} \subset \mathscr{H}$ such that $h_{t_k} \rightrightarrows I$ or ∞ in Δ_r, where I is a meromorphic function. Next we distinguish two cases.

Case A. Assume $f_{t_k}(0) \neq 0$ when k is sufficiently large. Then $I(0) = \infty$, and hence for arbitrary $R > 0$, there exists a positive number δ with $0 < \delta < r$ such that $|I(z)| > R$ when $z \in \Delta_\delta$. Hence when k is sufficiently large, we have $|h_{t_k}(z)| > \frac{R}{2}$, which means that $\frac{1}{f_{t_k}}$ is holomorphic in Δ_δ. In fact, when $|z| = \frac{\delta}{2}$,

$$\left|\frac{1}{f_{t_k}(z)}\right| = \left|\frac{1}{h_{t_k}(z)z^d}\right| \leq M = \frac{2^{d+1}}{R\delta^d}.$$

By applying maximum principle, we have

$$\left|\frac{1}{f_{t_k}(z)}\right| \leq M$$

for $z \in \Delta_{\delta/2}$. It follows from Motel's normal criterion that there exists a convergent subsequence of $\{f_{t_k}\}$, that is, $\mathscr{F}$ is normal at 0.

Case B. There exists a subsequence of f_{t_k}, for simplicity we still denote it as f_{t_k}, such that $f_{t_k}(0) = 0$. Then we get $I(0) = 0$ since $h_{t_k}(z) = \frac{f_{t_k}(z)}{z^d} \rightrightarrows I(z)$, and hence there exists a positive number ρ with $0 < \rho < r$ such that $I(z)$ is holomorphic in Δ_ρ and has a unique zero $z = 0$ in Δ_ρ. Therefore, we have $f_{t_k}(z) \rightrightarrows z^d I(z)$ in Δ_ρ since h_{t_k} converges spherically locally uniformly to a holomorphic function I in Δ_ρ. Thus $\mathscr{F}$ is normal at 0.

Similarly, we can prove that $\mathscr{F}$ is normal at arbitrary $z_0 \in D$, hence $\mathscr{F}$ is normal in D.

12

4 Proof of Theorem 1.2

Similar to the proof of Theorem 1.1, we assume that $D = \{z \in \mathbb{C} \mid |z| < 1\}$ and $z_0 = 0$, and then distinguish two cases by either $a(0) = 0$ or $a(0) \neq 0$.

Case 1. $a(0) \neq 0$. To the contrary, we suppose that $\mathscr{F}$ is not normal at 0. By using the notations in the proof of Theorem 1.1, we also obtain

$$\begin{aligned} & f_j^n(z_j+\rho_j\xi)(f_j'(z_j+\rho_j\xi))^l - a(z_j+\rho_j\xi) \\ = \ & g_j^n(\xi)(g_j'(\xi))^l - a(z_j+\rho_j\xi) \rightrightarrows g^n(\xi)(g'(\xi))^l - a(0), \end{aligned} \tag{4.1}$$

where $g^n(g^{(k)})^l \not\equiv a(0)$.

By Lemma 2.2 and Lemma 2.4, the function $g^n(g')^l - a(0)$ has a zero η_0. By (4.1) and Hurwitz's theorem, there exist points $\eta_j \to \eta_0$ $(j \to \infty)$ such that for sufficiently large j, $z_j + \rho_j\eta_j \in D$ and

$$f_j^n(z_j+\rho_j\eta_j)(f_j'(z_j+\rho_j\eta_j))^l - a(z_j+\rho_j\eta_j) = 0,$$

which contradicts the assumption that $f^n(f')^l \neq a$.

Case 2. $a(0) = 0$. By using the notations in the proof of Theorem 1.1, we also get the formulas (3.1)–(3.4). Therefore, with the similar method in Case 1, we can prove that $\mathscr{F}$ is normal at $z_0 = 0$, and hence $\mathscr{F}$ is normal in D.

5 Proof of Theorem 1.3

We also take the assumptions in the proof of Theorem 1.1, distinguishing two cases as follows:

Case 1. $a(0) \neq 0$. Similar to the proof of proof of Theorem 1.2, we get that $g^n(g')^l - a(0)$ has a zero η_0. By Hurwitz's theorem, there exist points $\eta_j \to \eta_0$ $(j \to \infty)$ such that for sufficiently large j, $z_j + \rho_j\eta_j \in D$ and $f_j^n(z_j+\rho_j\eta_j)(f_j'(z_j+\rho_j\eta_j))^l = a(z_j+\rho_j\eta_j)$. Consequently, we have $|g_j(\eta_j)| = |\rho_j^{-\frac{lk}{n+l}} f_j(z_j+\rho_j\eta_j)| \geq |\rho_j^{-\frac{lk}{n+l}}|A$, which implies that $g(\eta_0) = \infty$. This contradicts the assumption that $g^n(g'(\eta_0))^l = a(0)$. Hence $\mathscr{F}$ is normal at $z_0 = 0$.

Case 2. $a(0) = 0$. We also obtain the formulas (3.1)–(3.4), thus we can prove that $\mathscr{F}$ is normal by using the similar method of Case 1.

6 Acknowledgements

We would like to repress their hearty thanks to the referee for his/her valuable comments and suggestions made to this paper. This research is supported by Natural Science Foundation of China (Grant No.11271227, No.11201360) and the Fundamental Research Funds for the Central Universities of China (Grant No.800272125771).

References

[1] Alotaibi, A., On the zeros of $af\left(f^{(k)}\right)^n - 1$ for $n \geq 2$, Comput. Methods Funct. Theory 4(1) (2004), 227-235.

[2] Bergweiler, W. and Eremenko, A., On the singularities of the inverse to a meromorphic function of finite order, Rev, Mat. Iberoamericana 11 (1995), 355-373.

[3] Chang, J. M. and Fang, M. L., Normality and shared functions of holomorphic functions and their derivatives, Michigan Math. J. 53(2005), 625-645.

[4] Chen, H. H. and Fang, M. L., On the value distribution of $f^n f'$, Sci. China Ser. A 38 (1995), 789-798.

[5] Chen, H. H. and Gu, Y. X., An improvement of Marty's criterion and its applications, Sci. China Ser. A 36 (1993), 674-681.

[6] Clunie, J., On a result of Hayman, J. London Math. Soc. 42 (1967), 389-392.

[7] Ding, J. J., Ding, L. W. and Yuan, W. J., Normal families of meromorphic functions concerning shared values, Complex Var. Elliptic Equ. 58(1) (2013), 113-121.

[8] Gu, Y. X., Sur les familles normales de fonctions méromorphes, Sci. Sinica 21 (1978), 431-445.

[9] Hayman, W., Picard value of meromorphic functions and their derivatives, Annals of Mathematics 70 (1959), 9-42.

[10] Hayman, W., Research problems in function theory, Athlone Press (University of London), London, 1967.

[11] Hu, P. C. and Meng, D. W., Normal criteria of meromorphic fucntions with multiple zeros, J. Math. Anal. Appl. 357 (2009), 323-329.

[12] Jiang, Y. B. and Gao, Z. S., Normal families of meromorphic functions sharing values and functions, J. Inequal. Appl. 72 (2011).

[13] Jiang, Y. B. and Gao, Z. S., Normal families of meromorphic functions sharing a holomorphic function and the converse of the Bloch principle, Acta Math. Sci. 4 (2012), 1503-1512.

[14] Li, G. W. and Gao, L. Y., On the distribution of $f^m(f^{(k)})^n - \varphi$, J. Jinan Univ. 29 (2008), 424-426.

[15] Li, Y. T. and Gu, Y. X., On normal families of meromorphic functions, J. Math. Anal. Appl. 354 (2009), 421-425.

[16] Meng, D. W. and Hu, P. C., Normality criteria of meromorphic functions sharing a holomorphic function, Bull. Malays. Math. Sci. Soc. 38 (2015), 1331-1347.

[17] Mues, E., Über ein problem von Hayman, Math. Z. 164 (1979), 239-259.

[18] Oshkin, I. B., On a test for the normality of families of holomorphic functions, Uspehi Mat. Nauk 37(2) (1982), 221-222; Russian Math. Surveys 37(2) (1982), 237-238.

[19] Pang, X. C., Normality conditions for differential polynomials (in Chinese), Kexue Tongbao 33 (1988), 1690-1693.

[20] Pang, X. C., Bloch principle and normality criterion, Sci. Sinica Ser. A 11 (1988), 1153-1159; Sci. China Ser. A 32 (1989), 782-791.

[21] Pang, X. C., On normal criterion of meromorphic functions, Sci. China Ser. A 33 (1990), 521-527.

[22] Pang, X. C. and L. Zalcmam, On theorems of Hayman and clunie, NewZealand J. Math. 28 (1999), 71-75.

[23] Schwick, W., Normal criteria for families of meromorphic function, J. Anal. Math. 52 (1989), 241-289.

[24] Xu, Y. and Chang, J. M., Normality criteria and multiple values II, Annales Polonici Mathematici 102.1(2011), 91-99.

[25] Xue, G. F. and Pang, X. C., A criterion for normality of a family of meromorphic functions (Chinese), J. East China Norm. Univ. Natur. Sci. Ed. 2 (1988), 15-22.

[26] Yang, L. and Zhang, G. H., Recherches sur la normalité des familes de fonctions analytiques à des valeurs multiples, I. Un nouveau critère et quelques applications, Scientia Sinica, Series A 14 (1965), 1258-1271; II. Géneralisations, ibid., 15 (1966), 433-453.

[27] Zalcman, L., A heuristic principle in complex function theory, Amer. Math. Monthly 82 (1975), 813-817.

[28] Zalcman, L., Normal families: New perspectives, Bull. Amer. Math. Soc. 35 (1998) 215-230.

[29] Zhang, Q. C., Some normality criteria of meromorphic functions, Complex Var. Elliptic Equ. 53(8) (2008), 791-795.

[30] Zhang, Z. F. and Song, G. D., On the zeros of $f\left(f^{(k)}\right)^n - a(z)$, Chinese Ann. Math. Ser. A 19(2) (1998), 275-282.

[31] Zhang, Z. L. and Li, W., Picard exceptional values for two class differential polynomials, Acta Math. Sinica 34 (1994), 828-835.

Mixed Weakly Monotone Mappings and its Application to System of Integral Equations via Fixed Point Theorems

Deepak Singh

Department of Applied Sciences

NITTTR, (Under Ministry of HRD, Govt. of India) Bhopal,462002, India.

E-mail: dk.singh1002@gmail.com

Om Prakash Chauhan

Department of Applied Mathematics, Jabalpur Engineering College, Jabalpur, (M.P.), India.

E-mail: chauhaan.op@gmail.com

Afrah A N Abdou[1]

Department of Mathematics, King Abdulaziz University, Jeddah 21589, Saudi Arabia.

E-mail: aabdou@kau.edu.sa

Garima Singh

University Institute of Technology, Barkatullah University, Bhopal, (M.P.), India.

E-mail: drgarimasingh@ymail.com

Abstract: In this note, the notion of a mixed weakly monotone pair of mappings is invoked to prove some coupled common fixed point theorems in partially ordered b-metric spaces. Results proved in this note are authenticated by some innovative examples. Demonstrative surfaces leading to better understanding of calculation done. Moreover as an application our results are utilized to establish existence of solutions of system of integral equations which is also substantiated by an example.

Key Words. Cauchy sequence, Partially ordered b-metric Space, mixed weakly monotone mappings, coupled common fixed point, integral equation.

2010 AMS Subject classification. 47H10, 54H25.

1 Introduction and Preliminaries

In 1989, Bakhtin [2] introduced the notion of b-metric spaces and studied the concept of b-metric spaces as a generalization of metric spaces. Also he proved the Banach contraction principle in b-metric spaces. After that the study of fixed point theorems in b-metric spaces is followed by some other mathematicians (see [1], [5], [13]).

In 2011, Ran and Rarings [12] introduced the existence of fixed point in partially ordered metric spaces and studied

[1]Corresponding Author

some applications to matrix equations.

Guo and Lakshmikantham [8] introduced the concept of coupled fixed point. later on Bhaskar and Lakshmikantham [3] introduced the notions of a mixed monotone mappings and then established some coupled fixed point theorems for mixed monotone mappings. They also discussed the existence and uniqueness of the solution for periodic boundary value problems.

In 2009, Lakshmikantham and Ciric [9] defined g- monotone property and proved coupled coincidence and coupled common fixed points theorems for nonlinear mappings satisfying certain contractive conditions in partially ordered metric spaces. Some remarkable contributions on this line can be seen in [4], [10], [11].

In 2012, Gordji et al. [7], proved some coupled fixed point theorems for a contractive-type mappings with the mixed weakly monotone property in partially ordered metric spaces. In this article, utilizing the notion of a mixed weakly monotone pair of mappings we prove coupled common fixed points theorems for mappings on partially ordered b-metric spaces.

First we recall some basic definitions, notions, lemmas, and examples which will be needed in the sequel.

Definition 1.1. *[5] Let X be a (nonempty) set and $s \geq 1$ be a given real number. A function $d : X \times X \to [0, \infty)$ is called a b-metric on X if, for all $x, y, z \in X$, the following conditions hold:*

(i) $d(x, y) = 0$ *if and only if* $x = y$*;*

(ii) $d(x, y) = d(y, x)$*;*

(iii) $d(x, z) \leq s[d(x, y) + d(y, z)]$.

In this case, the pair (X, d) is called a b-metric space. If $(X, \precsim)$ is still a partially ordered set, then $(X, \precsim, d)$ is called a partially ordered b-metric space.

Definition 1.2. *[3] An element $(x, y) \in X \times X$ is called coupled fixed point of a mapping $F : X \times X \to X$ if* $x = F(x, y)$ *and* $y = F(y, x)$.

Definition 1.3. *[3] Let $(X, d, \precsim)$ be a partially ordered set and $f : X \times X \to X$ be mapping. We say that f has the mixed monotone property on X if, for all $x, y \in X$,*

$$x_1, x_2 \in X, x_1 \precsim x_2 \Rightarrow f(x_1, y) \precsim f(x_2, y)$$
$$and \quad y_1, y_2 \in X, y_1 \precsim y_2 \Rightarrow f(x, y_1) \succsim f(x, y_2).$$

Definition 1.4. *[7] let $(X, \precsim)$ be a partially ordered set and $f, g : X \times X \to X$ be mappings. We say that a pair (f, g) has the mixed weakly monotone property on X, if for all $x, y \in X$,we have*

$$x \precsim kf(x, y), f(y, x) \precsim y \;\; implies \;\; f(x, y) \precsim g(f(x, y), f(y, x)), g(f(y, x), f(x, y)) \precsim f(y, x)$$
$$and \quad x \precsim g(x, y), g(y, x) \precsim y \;\; implies \;\; g(x, y) \precsim f(g(x, y), g(y, x)), f(g(y, x), g(x, y)) \precsim g(y, x).$$

For mixed weakly monotone property related examples one is suggested to refer [7]

Remark 1.1. *[7] Let $(X, \precsim)$ be a partially ordered set, $f : X \times X \to X$ be a map with the mixed monotone property on X. Then for all $n \in \mathbb{N}$, the pair (f^n, f^n) has the mixed weakly monotone property on X.*

Lemma 1.1. *[7] Let (X, d) be a metric space. Then $X \times X$ is a metric space with the metric D_d given by $D_d((x, y), (u, v)) = d(x, u) + d(y, v)$, for all $x, y, u, v \in X$.*

2 Main result

In this section, some fixed point theorems for contraction conditions described by rational expressions are proved.

Lemma 2.1. *Let (X, d) be a b-metric space. Then $X \times X$ is a b-metric space with the b-metric D given by $D((x, y), (u, v)) = d(x, u) + d(y, v)$, for all $x, y, u, v, w, t \in X$.*

Proof. For all $x, y, u, v, w, t \in X$, we have $D((x, y), (u, v)) \in [0, \infty)$ and
$D((x, y), (u, v)) = 0$ if and only if $d(x, u) + d(y, v) = 0$
if and only if $x = u, y = v$, that is $(x, y) = (u, v)$ and

$$\begin{aligned}
D((x,y),(u,v)) &= d(x,u) + d(y,v) \\
&= d(u,x) + d(v,y) = D((u,v),(x,y)). \\
Also, \quad D((x,y),(u,v)) &= d(x,u) + d(y,v) \\
&\leq s[d(x,w) + d(w,u)] + s[d(y,t) + d(t,v)] \\
&\leq s[d(x,w) + d(y,t)] + s[d(w,u) + d(t,v)] \\
&\leq s[D((x,y),(w,t)) + D((w,t),(u,v))].
\end{aligned}$$

Hence, D is a b-metric on $X \times X$. □

Let $(X, d, \precsim)$ be a partially ordered complete metric space. We consider the product space $X \times X$ with the following partial order, for all $(x, y), (u, v) \in X \times X$

$$(x, y) \precsim (u, v) \Leftrightarrow x \precsim u, y \succsim v.$$

Also let $(X \times X, D)$ be a b-metric space with the following metric

$$D((x, y), (u, v)) = d(x, u) + d(y, v), \quad for\ all\ (x, y), (u, v) \in X \times X.$$

Theorem 2.1. *Let $(X, d, \precsim)$ be a partially ordered complete b-metric space. Let $f, g : X \times X \to X$ be the mappings such that the pair (f, g) has the mixed weakly monotone property on X. Suppose there exists $\alpha, \beta, \gamma, \delta \in [0, 1)$ with $\alpha + \beta + 2s(\gamma + \delta) < \frac{1}{2}$ such that*

$$\begin{aligned} d(f(x,y), g(u,v)) \precsim \alpha &\frac{[1 + D((x,y),(f(x,y), f(y,x)))].D((u,v),(g(u,v), g(v,u)))}{[1 + D((x,y),(u,v))]} + \beta D((x,y),(u,v)) \\ &+ \gamma[D((x,y),(f(x,y), f(y,x))) + D((u,v),(g(u,v), g(v,u)))] \\ &+ \delta[D((u,v),(f(x,y), f(y,x))) + D((x,y),(g(u,v), g(v,u)))] \end{aligned} \tag{2.1}$$

for all $x, y, u, v \in X$ with $x \precsim u$ and $y \succsim v$ and D is defined as in Lemma 2.1. Let $x_0, y_0 \in X$ be such that $x_0 \precsim f(x_0, y_0), y_0 \succsim f(y_0, x_0)$ or $x_0 \precsim g(x_0, y_0), y_0 \succsim g(y_0, x_0)$. If f or g is continuous, then f and g have a coupled common fixed point in X.

Proof. We construct two Cauchy sequence in X. Let $x_0, y_0 \in X$, be such that $x_0 \precsim f(x_0, y_0), y_0 \succsim f(y_0, x_0)$.
Put $x_1 = f(x_0, y_0), y_1 = f(y_0, x_0), x_2 = g(x_1, y_1), y_2 = g(y_1, x_1)$
Continuing this, way $\quad x_{2n+1} = f(x_{2n}, y_{2n}), y_{2n+1} = f(y_{2n}, x_{2n})$,

$$x_{2n+2} = g(x_{2n+1}, y_{2n+1}), y_{2n+2} = g(y_{2n+1}, x_{2n+1}) \;\; for \;\; all \; n \in N.$$

From the choice of x_0, y_0 and the (f, g) has mixed weakly monotone property, we have

$$\begin{aligned} x_1 =& f(x_0, y_0) \precsim g(f(x_0, y_0), f(y_0, x_0) = g(x_1, y_1) = x_2 \Rightarrow x_1 \precsim x_2, \\ x_2 =& g(x_1, y_1) \precsim f(g(x_1, y_1), g(y_1, x_1) = f(x_2, y_2) = x_3 \Rightarrow x_2 \precsim x_3. \\ Similarly, \quad y_1 =& f(y_0, x_0) \succsim g(f(y_0, x_0), f(x_0, y_0) = g(y_1, x_1) = y_2 \Rightarrow y_1 \succsim y_2, \\ y_2 =& g(y_1, x_1) \succsim f(g(y_1, x_1), g(x_1, y_1) = f(y_2, x_2) = y_3 \Rightarrow y_2 \succsim y_3. \end{aligned}$$

Therefore, we acquire

$$x_0 \precsim x_1 \precsim x_2 \precsim \ldots x_n \precsim x_{n+1} \precsim \cdots$$

$$y_0 \succsim y_1 \succsim y_2 \succsim \cdots y_n \succsim y_{n+1} \succsim \cdots$$

the sequences $\{x_n\}$ and $\{y_n\}$ are monotone increasing and decreasing respectively. Applying (2.1), we obtain

$$\begin{aligned} &d(x_{2n+1}, x_{2n+2}) = d(f(x_{2n}, y_{2n}), g(x_{2n+1}, y_{2n+1})) \\ &\precsim \alpha \tfrac{[1+D((x_{2n},y_{2n}),(f(x_{2n},y_{2n}),f(y_{2n},x_{2n})))]D((x_{2n+1},y_{2n+1}),(g(x_{2n+1},y_{2n+1}),g(y_{2n+1},x_{2n+1})))}{[1+D((x_{2n},y_{2n}),(x_{2n+1},y_{2n+1}))]} \\ &\quad + \beta D((x_{2n}, y_{2n}), (x_{2n+1}, y_{2n+1})) \\ &\quad + \gamma[D((x_{2n}, y_{2n}), (f(x_{2n}, y_{2n}), f(y_{2n}, x_{2n}))) + D((x_{2n+1}, y_{2n+1}), (g(x_{2n+1}, y_{2n+1}), g(y_{2n+1}, x_{2n+1})))] \\ &\quad + \delta[D((x_{2n+1}, y_{2n+1}), (f(x_{2n}, y_{2n}), f(y_{2n}, x_{2n}))) + D((x_{2n}, y_{2n}), (g(x_{2n+1}, y_{2n+1}), g(y_{2n+1}, x_{2n+1})))] \end{aligned}$$

$$\begin{aligned}
d(x_{2n+1},x_{2n+2}) \precsim \alpha & \frac{[1+D((x_{2n},y_{2n}),(x_{2n+1},y_{2n+1}))]D((x_{2n+1},y_{2n+1}),(x_{2n+2},y_{2n+2}))}{[1+D((x_{2n},y_{2n}),(x_{2n+1},y_{2n+1}))]} \\
& + \beta D((x_{2n},y_{2n}),(x_{2n+1},y_{2n+1}))+ \\
& \gamma[D((x_{2n},y_{2n}),(x_{2n+1},y_{2n+1}))+D((x_{2n+1},y_{2n+1}),(x_{2n+2},y_{2n+2}))]+ \\
& \delta[D((x_{2n+1},y_{2n+1}),(x_{2n+1},y_{2n+1}))+D((x_{2n},y_{2n}),(x_{2n+2},y_{2n+2}))]
\end{aligned}$$

$$\begin{aligned}
d(x_{2n+1},x_{2n+2}) \precsim & (\alpha+\gamma)D((x_{2n+1},y_{2n+1}),(x_{2n+2},y_{2n+2}))+ \\
& (\beta+\gamma)D((x_{2n},y_{2n}),(x_{2n+1},y_{2n+1}))+\delta D((x_{2n},y_{2n}),(x_{2n+2},y_{2n+2})) \\
\precsim & (\alpha+\gamma)D((x_{2n+1},y_{2n+1}),(x_{2n+2},y_{2n+2}))+ \\
& (\beta+\gamma)D((x_{2n},y_{2n}),(x_{2n+1},y_{2n+1}))+ \\
& s\delta[D((x_{2n},y_{2n}),(x_{2n+1},y_{2n+1}))+D((x_{2n+1},y_{2n+1}),(x_{2n+2},y_{2n+2}))] \\
\precsim & (\alpha+\gamma)[d(x_{2n+1},x_{2n+2})+d(y_{2n+1},y_{2n+2})]+ \\
& (\beta+\gamma)[d(x_{2n},x_{2n+1})+d(y_{2n},y_{2n+1})]+ \\
& s\delta[d(x_{2n},x_{2n+1})+d(x_{2n+1},x_{2n+2})+d(y_{2n},y_{2n+1})+d(y_{2n+1},y_{2n+2})]
\end{aligned}$$

$$\begin{aligned}
d(x_{2n+1},x_{2n+2}) \precsim & (\alpha+\gamma+s\delta)[d(x_{2n+1},x_{2n+2})+d(y_{2n+1},y_{2n+2})]+ \\
& (\beta+\gamma+s\delta)[d(x_{2n},x_{2n+1})+d(y_{2n},y_{2n+1})] \quad for \;\; all \;\; n\in N.
\end{aligned} \tag{2.2}$$

Similarly, we have

$$\begin{aligned}
d(y_{2n+1},y_{2n+2}) \precsim & (\alpha+\gamma+s\delta)[d(y_{2n+1},y_{2n+2})+d(x_{2n+1},x_{2n+2})]+ \\
& (\beta+\gamma+s\delta)[d(y_{2n},y_{2n+1})+d(x_{2n},x_{2n+1})] \quad for \;\; all \;\; n\in N.
\end{aligned} \tag{2.3}$$

Thus it follows from (2.2) and (2.3) that

$$\begin{aligned}
d(x_{2n+1},x_{2n+2})+d(y_{2n+1},y_{2n+2}) \precsim & 2(\alpha+\gamma+s\delta)[d(x_{2n+1},x_{2n+2})+d(y_{2n+1},y_{2n+2})]+ \\
& 2(\beta+\gamma+s\delta)[d(x_{2n},x_{2n+1})+d(y_{2n},y_{2n+1})]
\end{aligned}$$

or

$$d(x_{2n+1},x_{2n+2})+d(y_{2n+1},y_{2n+2}) \precsim \frac{2(\beta+\gamma+s\delta)}{1-2(\alpha+\gamma+s\delta)}[d(x_{2n},x_{2n+1})+d(y_{2n},y_{2n+1})] \quad for \;\; all \;\; n\in N. \tag{2.4}$$

Moreover, if we apply (2.1), then we have

$$\begin{aligned}
& d(x_{2n+2},x_{2n+3}) = d(g(x_{2n+1},y_{2n+1}),f(x_{2n+2},y_{2n+2})) \\
& \precsim \alpha\frac{[1+D((x_{2n+1},y_{2n+1}),(g(x_{2n+1},y_{2n+1}),g(y_{2n+1},x_{2n+1})))]D((x_{2n+2},y_{2n+2}),(f(x_{2n+2},y_{2n+2}),f(y_{2n+2},x_{2n+2})))}{[1+D((x_{2n+1},y_{2n+1}),(x_{2n+2},y_{2n+2}))]} \\
& +\beta D((x_{2n+1},y_{2n+1}),(x_{2n+2},y_{2n+2}))+\gamma[D((x_{2n+1},y_{2n+1}),(g(x_{2n+1},y_{2n+1}),g(y_{2n+1},x_{2n+1}))) \\
& +D((x_{2n+2},y_{2n+2}),(f(x_{2n+2},y_{2n+2}),f(y_{2n+2},x_{2n+2})))]+\delta[D((x_{2n+2},y_{2n+2}),(g(x_{2n+1},y_{2n+1}),g(y_{2n+1},x_{2n+1})))
\end{aligned}$$

$$+D((x_{2n+1},y_{2n+1}),(f(x_{2n+2},y_{2n+2}),f(y_{2n+2},x_{2n+2})))]$$

$$\begin{aligned}
d(x_{2n+2},x_{2n+3}) \precsim \alpha & \frac{[1+D((x_{2n+1},y_{2n+1}),(x_{2n+2},y_{2n+2}))]D((x_{2n+2},y_{2n+2}),(x_{2n+3},y_{2n+3}))}{[1+D((x_{2n+1},y_{2n+1}),(x_{2n+2},y_{2n+2}))]} \\
& + \beta D((x_{2n+1},y_{2n+1}),(x_{2n+2},y_{2n+2}))+ \\
& \gamma[D((x_{2n+1},y_{2n+1}),(x_{2n+2},y_{2n+2}))+D((x_{2n+2},y_{2n+2}),(x_{2n+3},y_{2n+3}))]+ \\
& \delta[D((x_{2n+2},y_{2n+2}),(x_{2n+2},y_{2n+2}))+D((x_{2n+1},y_{2n+1}),(x_{2n+3},y_{2n+3}))]
\end{aligned}$$

$$\begin{aligned}
d(x_{2n+2},x_{2n+3}) \precsim & (\alpha+\gamma)D((x_{2n+2},y_{2n+2}),(x_{2n+3},y_{2n+3}))+ \\
& (\beta+\gamma)D((x_{2n+1},y_{2n+1}),(x_{2n+2},y_{2n+2}))+\delta D((x_{2n+1},y_{2n+1}),(x_{2n+3},y_{2n+3})) \\
\precsim & (\alpha+\gamma)D((x_{2n+2},y_{2n+2}),(x_{2n+3},y_{2n+3}))+ \\
& (\beta+\gamma)D((x_{2n+1},y_{2n+1}),(x_{2n+2},y_{2n+2}))+ \\
& s\delta[D((x_{2n+1},y_{2n+1}),(x_{2n+2},y_{2n+2}))+D((x_{2n+2},y_{2n+2}),(x_{2n+3},y_{2n+3}))] \\
& (\alpha+\gamma+s\delta)[D(x_{2n+2},y_{2n+2}),d(x_{2n+3},y_{2n+3})]+ \\
& (\beta+\gamma+s\delta)[D(x_{2n+1},y_{2n+1}),d(x_{2n+2},y_{2n+2})]
\end{aligned}$$

$$\begin{aligned}
d(x_{2n+2},x_{2n+3}) \precsim & (\alpha+\gamma+s\delta)[d(x_{2n+2},x_{2n+3})+d(y_{2n+2},y_{2n+3})]+ \\
& (\beta+\gamma+s\delta)[d(x_{2n+1},x_{2n+2})+d(y_{2n+1},y_{2n+2})], \quad for \ \ all \ \ n\in N.
\end{aligned} \tag{2.5}$$

Similarly, we have

$$\begin{aligned}
d(y_{2n+2},y_{2n+3}) \precsim & (\alpha+\gamma+s\delta)[d(y_{2n+2},y_{2n+3})+d(x_{2n+2},x_{2n+3})]+ \\
& (\beta+\gamma+s\delta)[d(y_{2n+1},y_{2n+2})+d(x_{2n+1},x_{2n+2})], \quad for \ \ all \ \ n\in N.
\end{aligned} \tag{2.6}$$

Thus it follows from (2.5) and (2.6) that

$$\begin{aligned}
d(x_{2n+2},x_{2n+3})+d(y_{2n+2},y_{2n+3}) \precsim & 2(\alpha+\gamma+s\delta)[d(x_{2n+2},x_{2n+3})+d(y_{2n+2},y_{2n+3})]+ \\
& 2(\beta+\gamma+s\delta)[d(x_{2n+1},x_{2n+2})+d(y_{2n+1},y_{2n+2})]
\end{aligned}$$

or

$$d(x_{2n+2},x_{2n+3})+d(y_{2n+2},y_{2n+3}) \precsim \frac{2(\beta+\gamma+s\delta)}{1-2(\alpha+\gamma+s\delta)}[d(x_{2n+1},x_{2n+2})+d(y_{2n+1},y_{2n+2})], \quad for \ \ all \ \ n\in N. \tag{2.7}$$

Moreover, it follows from (2.4) and (2.7) that

$$d(x_{2n+2},x_{2n+3})+d(y_{2n+2},y_{2n+3}) \precsim \Big[\frac{2(\beta+\gamma+s\delta)}{1-2(\alpha+\gamma+s\delta)}\Big]^2[d(x_{2n},x_{2n+1})+d(y_{2n},y_{2n+1})], \quad for \ \ all \ \ n\in N. \tag{2.8}$$

Let $\lambda = \frac{2(\beta+\gamma+s\delta)}{1-2(\alpha+\gamma+s\delta)}$. Then $0 \le \lambda < 1$ and

$$\begin{aligned}
d(x_{2n+1}, x_{2n+2}) + d(y_{2n+1}, y_{2n+2}) &\precsim \lambda[d(x_{2n}, x_{2n+1}) + d(y_{2n}, y_{2n+1})]+ \\
&\precsim \lambda^3[d(x_{2n-2}, x_{2n-1}) + d(y_{2n-2}, y_{2n-1})] \\
&\precsim \lambda^5[d(x_{2n-4}, x_{2n-3}) + d(y_{2n-4}, y_{2n-3})] \\
&\ \vdots \\
&\precsim \lambda^{2n+1}[d(x_0, x_1) + d(y_0, y_1)] \\
and\ \ d(x_{2n+2}, x_{2n+3}) + d(y_{2n+2}, y_{2n+3}) &\precsim \lambda[d(x_{2n+1}, x_{2n+2}) + d(y_{2n+1}, y_{2n+2})]+ \\
&\precsim \lambda.\lambda^{2n+1}[d(x_0, x_1) + d(y_0, y_1)] \\
&\precsim \lambda^{2n+2}[d(x_0, x_1) + d(y_0, y_1)] \ \ for\ \ all\ \ n \in N.
\end{aligned}$$

Now, for all $m, n \ge 1$ with $n \le m$, we have

$$\begin{aligned}
d(x_{2n+1}, x_{2m+1}) + d(y_{2n+1}, y_{2m+1}) &\precsim s[d(x_{2n+1}, x_{2n+2}) + d(x_{2n+2}, x_{2m+1})] \\
&\quad + s[d(y_{2n+1}, y_{2n+2}) + d(y_{2n+2}, x_{2m+1})] \\
&\precsim s[(x_{2n+1}, x_{2n+2}) + d(y_{2n+1}, y_{2n+2}] + s[x_{2n+2}, x_{2m+1}) + d(y_{2n+2}, y_{2m+1}] \\
&\precsim s[(x_{2n+1}, x_{2n+2}) + d(y_{2n+1}, y_{2n+2}] + s^2[x_{2n+2}, x_{2n+3}) + d(y_{2n+2}, y_{2n+3}] \\
&\quad + s^2[d(x_{2n+3}, x_{2m+1}) + d(y_{2n+3}, y_{2m+1})] \\
&\precsim s[(x_{2n+1}, x_{2n+2}) + d(y_{2n+1}, y_{2n+2}] + s^2[x_{2n+2}, x_{2n+3}) + d(y_{2n+2}, y_{2n+3}] \\
&\quad + \ldots + s^{2(m-n)}[d(x_{2m}, x_{2m+1}) + d(y_{2m}, y_{2m+1})] \\
&\precsim s\lambda^{2n+1}[d(x_0, x_1) + d(y_0, y_1)] + s^2\lambda^{2n+2}[d(x_0, x_1) + d(y_0, y_1)] \\
&\quad + \ldots + s^{2(m-n)}\lambda^{2m}[d(x_0, x_1) + d(y_0, y_1)] \\
&\precsim s\lambda^{2n+1}[1 + s\lambda + (s\lambda)^2 + \ldots + (s\lambda)^{2(m-n)-1}][d(x_0, x_1) + d(y_0, y_1)] \\
&\precsim \frac{s\lambda^{2n+1}}{1-s\lambda}[d(x_0, x_1) + d(y_0, y_1)].
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
d(x_{2n}, x_{2m+1}) + d(y_{2n}, y_{2m+1}) &\precsim s\lambda^{2n}[1 + s\lambda + (s\lambda)^2 + \ldots + (s\lambda)^{2(m-n)}][d(x_0, x_1) + d(y_0, y_1)] \\
&\precsim \frac{s\lambda^{2n}}{1-s\lambda}[d(x_0, x_1) + d(y_0, y_1)].
\end{aligned}$$

$$\begin{aligned}
d(x_{2n}, x_{2m}) + d(y_{2n}, y_{2m}) &\precsim s\lambda^{2n}[1 + s\lambda + (s\lambda)^2 + \ldots + (s\lambda)^{2(m-n)-1}][d(x_0, x_1) + d(y_0, y_1)] \\
&\precsim \frac{s\lambda^{2n}}{1-s\lambda}[d(x_0, x_1) + d(y_0, y_1)]
\end{aligned}$$

and

$$\begin{aligned}d(x_{2n+1},x_{2m})+d(y_{2n+1},y_{2m}) &\precsim s\lambda^{2n+1}[1+s\lambda+(s\lambda)^2+\ldots+(s\lambda)^{2(m-n-1)}][d(x_0,x_1)+d(y_0,y_1)]\\ &\precsim \frac{s\lambda^{2n+1}}{1-s\lambda}[d(x_0,x_1)+d(y_0,y_1)].\end{aligned}$$

Hence for all $m,n\geq 1$ with $n\leq m$, it follows that

$$d(x_n,x_m)+d(y_n,y_m)\precsim \frac{s\lambda^n}{1-s\lambda}[d(x_0,x_1)+d(y_0,y_1)].$$

Since $0\leq\lambda<1$, we can conclude that

$d(x_n,x_m)+d(y_n,y_m)\to 0$ as $n\to\infty$, which implies that $d(x_n,x_m)\to 0$ and $d(y_n,y_m)\to 0$ as $m,n\to\infty$.

Therefore the sequences $\{x_n\}$ and $\{y_n\}$ are Cauchy sequence in X.

Since (X,d) be a partially ordered complete b-metric space, then there exist $x,y\in X$ such that $x_n\to x$ and $y_n\to y$ as $n\to\infty$.

Suppose that f is a continuous then we have

$$\begin{aligned} x &= \lim_{n\to\infty} x_{2n+1} = \lim_{n\to\infty} f(x_{2n},y_{2n}) = f(x,y)\\ and\ \ y &= \lim_{n\to\infty} y_{2n+1} = \lim_{n\to\infty} f(y_{2n},x_{2n}) = f(y,x).\end{aligned}$$

this implies (x,y) is coupled fixed point of f.

Taking $u=x$ and $v=y$ in (2.1), we have

$$\begin{aligned}
&d(f(x,y),g(x,y))+d(f(y,x),g(y,x))\\
&\qquad\precsim \alpha\frac{[1+D((x,y),(f(x,y),f(y,x)))]D((x,y),(g(x,y),g(y,x)))}{[1+D((x,y),(x,y))]}+\beta D((x,y),(x,y))\\
&\qquad+\gamma[D((x,y),(f(x,y),f(y,x)))+D((x,y),(g(x,y),g(y,x)))]\\
&\qquad k+\delta[D((x,y),(f(x,y),f(y,x)))+D((x,y),(g(x,y),g(y,x)))]\\
&\qquad+\alpha\frac{[1+D((y,x),(f(y,x),f(x,y)))]D((y,x),(g(y,x),g(x,y)))}{[1+D((y,x),(y,x))]}+\beta D((y,x),(y,x))\\
&\qquad+\gamma[D((y,x),(f(y,x),f(x,y)))+D((y,x),(g(y,x),g(x,y)))]\\
&\qquad+\delta[D((y,x),(f(y,x),f(x,y)))+D((y,x),(g(y,x),g(x,y)))]\\
&\qquad\precsim \alpha[1+D((x,y),(x,y))]D((x,y),(g(x,y),g(y,x)))\\
&\qquad+\gamma[D((x,y),(x,y)))+D((x,y),(g(x,y),g(y,x)))]\\
&\qquad+\delta[D((x,y),(x,y)))+D((x,y),(g(x,y),g(y,x)))]\\
&\qquad+\alpha[1+D((y,x),(y,x))]D((y,x),(g(y,x),g(x,y)))\\
&\qquad+\gamma[D((y,x),(y,x))+D((y,x),(g(y,x),g(x,y)))]\\
&\qquad+\delta[D((y,x),(y,x))+D((y,x),(g(y,x),g(x,y)))].
\end{aligned}$$

Hence, we have

$$\begin{aligned} d(x,g(x,y))+d(y,,g(y,x)) &\precsim (\alpha+\gamma+\delta)[D((x,y),(g(x,y),g(y,x)))+ \\ &D((y,x),(g(y,x),g(x,y)))] \\ &\precsim 2(\alpha+\gamma+\delta)[d(x,g(x,y))+d(y,g(y,x))] \end{aligned}$$

Since $2(\alpha+\gamma+\delta)<1$, we get $d(x,g(x,y))=0,\ \ d(y,g(y,x))=0\ \ \Rightarrow x=g(x,y),\ \ y=g(y,x))$.
This implies (x,y) is a coupled fixed point of g. Hence (x,y) is a coupled common fixed point of f and g when f is continuous.
Similarly, we can prove that (x,y) is a coupled common fixed point of f and g when g is continuous. □

Next result is proved, relaxing continuity.

Theorem 2.2. *Let $(X,d,\precsim)$ be a partially ordered complete b-metric space. Assume that X has the following property:*

1 if $\{x_n\}$ is a increasing sequence with $x_n\to x$, then $x_n\precsim x$ for all $n\in N$;

2 if $\{y_n\}$ is a decreasing sequence with $y_n\to y$, then $y_n\succsim y$ for all $n\in N$.
Let $f,g:X\times X\to X$ be the mappings such that the pair (f,g) has the mixed weakly monotone property on X. Also, Suppose there exists $\alpha,\beta,\gamma,\delta\in[0,1)$ with $\alpha+\beta+2s(\gamma+\delta)<\frac{1}{2}$ such that

$$\begin{aligned} d(f(x,y),g(u,v)) \precsim \alpha\frac{[1+D((x,y),(f(x,y),f(y,x)))].D((u,v),(g(u,v),g(v,u)))}{[1+D((x,y),(u,v))]}+\beta D((x,y),(u,v)) \\ +\gamma[D((x,y),(f(x,y),f(y,x)))+D((u,v),(g(u,v),g(v,u)))] \\ +\delta[D((u,v),(f(x,y),f(y,x)))+D((x,y),(g(u,v),g(v,u)))] \end{aligned}$$

for all $x,y,u,v\in X$ with $x\precsim u$ and $y\succsim v$ and D is defined as in Lemma 2.1. Let $x_0,y_0\in X$ be such that $x_0\precsim f(x_0,y_0),y_0\succsim f(y_0,x_0)$ or $x_0\precsim g(x_0,y_0),y_0\succsim g(y_0,x_0)$, then f and g have a coupled common fixed point in X.

Proof. Following the proof of Theorem 2.1, we only have to show that

$$f(x,y)=g(x,y)=x, f(y,x)=g(y,x)=y.$$

It is clear that

$$\begin{aligned}
&D((x,y),(f(x,y),f(y,x))) \precsim s[D((x,y),(x_{2n+2},y_{2n+2}))+D((x_{2n+2},y_{2n+2}),(f(x,y),f(y,x)))]\\
&\quad = sD((x,y),(x_{2n+2},y_{2n+2}))+sD((g(x_{2n+1},y_{2n+1}),g(y_{2n+1},x_{2n+1})),(f(x,y),f(y,x)))\\
&\quad = sD((x,y),(x_{2n+2},y_{2n+2}))+sd(f(x,y),g(x_{2n+1},y_{2n+1}))+sd(f(y,x),g(y_{2n+1},x_{2n+1}))\\
&\quad \precsim sD((x,y),(x_{2n+2},y_{2n+2}))+\\
&\quad s\alpha\Big\{\frac{[1+D((x,y),(f(x,y),f(y,x)))]D((x_{2n+1},y_{2n+1}),(g(x_{2n+1},y_{2n+1}),g(y_{2n+1},x_{2n+1})))}{1+D((x,y),(x_{2n+1},y_{2n+1}))}+\\
&\quad \beta D((x,y),(x_{2n+1},y_{2n+1}))+\\
&\quad \gamma[D((x,y),(f(x,y),f(y,x)))+D((x_{2n+1},y_{2n+1}),(g(x_{2n+1},y_{2n+1}),g(y_{2n+1},x_{2n+1})))]+\\
&\quad \delta[D((x_{2n+1},y_{2n+1}),(f(x,y),f(y,x)))+D((x,y),(g(x_{2n+1},y_{2n+1}),g(y_{2n+1},x_{2n+1})))]\Big\}\\
&\quad +s\alpha\Big\{\frac{[1+D((x,y),(f(x,y),f(y,x)))]D((y_{2n+1},x_{2n+1}),(g(y_{2n+1},x_{2n+1}),g(x_{2n+1},y_{2n+1})))}{1+D((y,x),(y_{2n+1},x_{2n+1}))}+\\
&\quad \beta D((y,x),(y_{2n+1},x_{2n+1}))+\\
&\quad \gamma[D((y,x),(f(y,x),f(x,y)))+D((y_{2n+1},x_{2n+1}),(g(y_{2n+1},x_{2n+1}),g(x_{2n+1},y_{2n+1})))]+\\
&\quad \delta[D((y_{2n+1},x_{2n+1}),(f(y,x),f(x,y)))+D((y,x),(g(y_{2n+1},x_{2n+1}),g(x_{2n+1},y_{2n+1})))]\Big\}.
\end{aligned} \tag{2.9}$$

Letting $n\to\infty$ in (2.9), we obtain

$$\begin{aligned}
d(x,f(x,y)+d(y,f(y,x))) &\precsim s\gamma D((x,y),(f(x,y),f(y,x)))+\\
&s\delta D((y_{2n+1},x_{2n+1}),(f(x,y),f(y,x)))+\\
&s\gamma D((y,x),(f(y,x),f(x,y)))+\\
&s\delta D((y_{2n+1},x_{2n+1}),(f(y,x),f(x,y)))\\
&\precsim 2s(\gamma+\delta)[d(x,f(x,y)+d(y,f(y,x)))]
\end{aligned}$$

and, since $2s(\gamma+\delta)<\frac{1}{2}$, we have $d(x,f(x,y))+d(y,f(y,x))=0$ and so $f(x,y)=x,\quad f(y,x)=y$. Similarly we can show that $g(x,y)=x\ and\ g(y,x)=y$. Therefore (x,y) is a coupled common fixed point of f and g. □

Following example establishes validity of Theorem 2.1.

Examplex 2.1. *Consider* $(\mathbb{R},d,\leq)$, *where* $\leq$ *represents the b-metric with usual order relation and metric* $d(x,y)=(|x-y|)^2=(x-y)^2$ *on* $\mathbb{R}$, *where* $s=2$.

Let $f,g:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ *be two functions defined by*

$$f(x,y)=\frac{10x-5y+115}{120},\quad g(x,y)=\frac{16x-8y+184}{192}.$$

Then the pair (f, g) has the mixed weakly monotone property.

In order to verify the condition (2.1)*, first we notice that*

$$0 \leq \alpha \frac{[1 + D((x, y), (f(x, y), f(y, x)))].D((u, v), (g(u, v), g(v, u)))}{[1 + D((x, y), (u, v))]},$$

$$0 \leq \gamma[D((x, y), (f(x, y), f(y, x))) + D((u, v), (g(u, v), g(v, u)))],$$

$$0 \leq \delta[D((u, v), (f(x, y), f(y, x))) + D((x, y), (g(u, v), g(v, u)))] \ \ for \ \ all \ \ x, y \in R.$$

Thus it is sufficient to show that $d(f(x, y), g(u, v)) \leq \beta D((x, y), (y, v))$.

$$\begin{aligned} Now, \ \ d(f(x, y), g(u, v)) =& (|f(x, y) - g(u, v)|)^2 \\ =& \left(\left|\frac{10x - 5y + 115}{120} - \frac{16u - 8v + 184}{192}\right|\right)^2 \\ \leq& \left(\frac{1}{12}|x - u| + \frac{1}{24}|y - v|\right)^2 \leq \left(\frac{1}{12}(|x - u| + |y - v|)\right)^2 \\ \leq& \left[\frac{1}{3}(|x - u|^2 + |y - v|^2)\right], \quad \forall x, y \in R \\ \leq& \beta D((x, y), (y, v)). \end{aligned}$$

For $\beta = \frac{1}{3}$ *and choosing* $\alpha, \gamma, \delta \geq 0$ *such that* $\alpha + \beta + 2s(\gamma + \delta) \leq \frac{1}{2}$*, Thus condition* (2.1) *is satisfied. Following Figure 1 and Figure 2 show that* $(1, 1)$ *is the coupled fixed point of mapping* f.

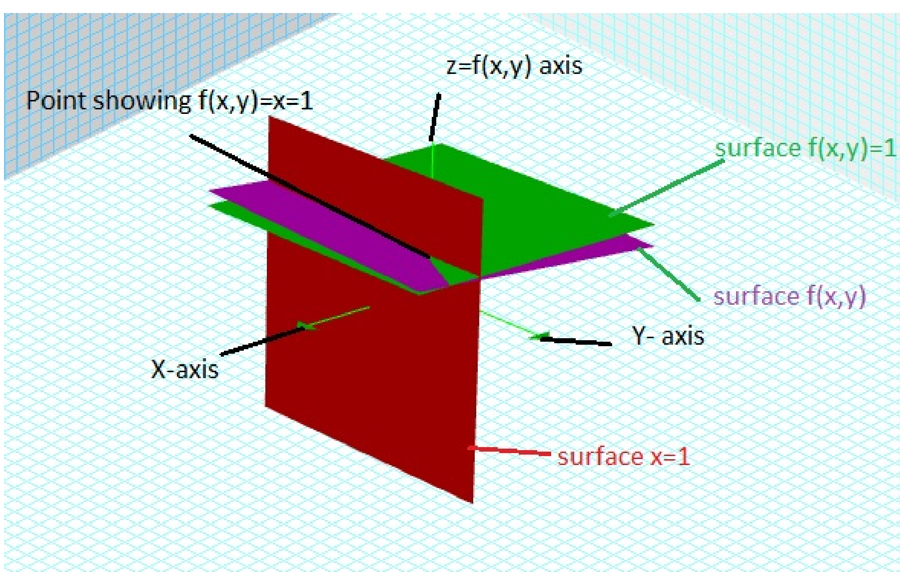


Figure 1: [Figure showing x=f(x,y)]

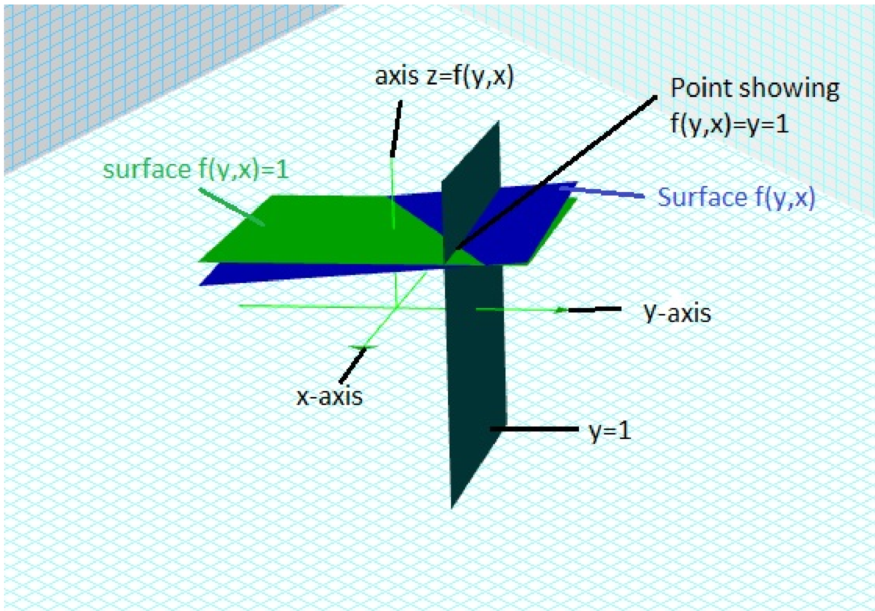


Figure 2: [Figure showing y=f(y,x)]

Next two Figure 3 and Figure 4 are demonstrating that $(1, 1)$ *is coupled fixed point of mapping* g *also.*

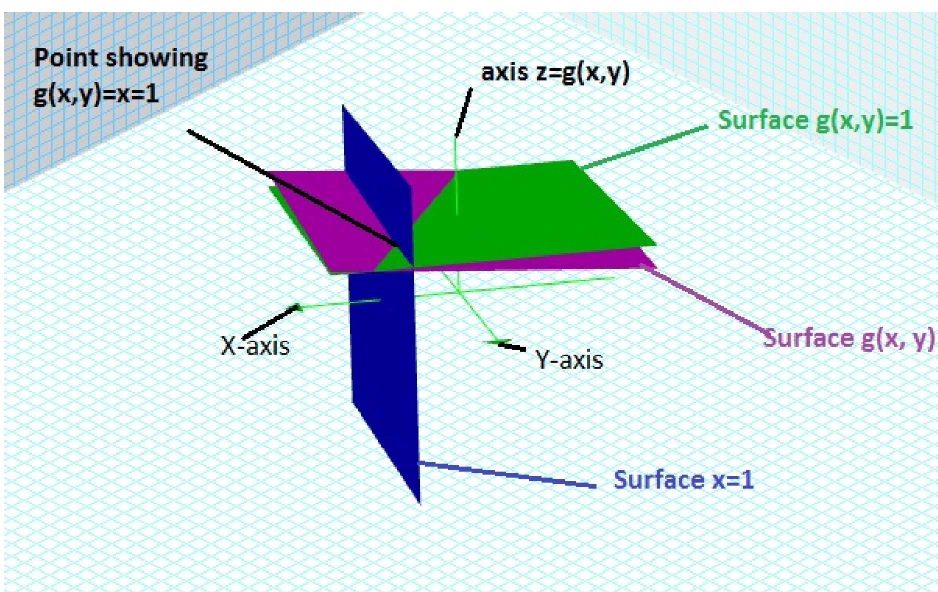


Figure 3: [Figure showing x=g(x,y)]

Thus we conclude that $(1, 1)$ *is a coupled common fixed point of* f *and* g.

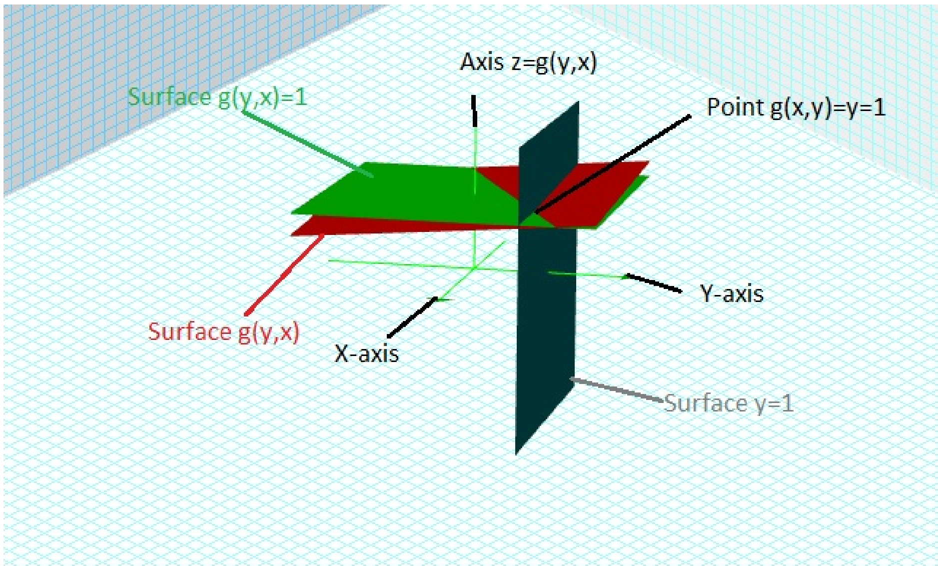


Figure 4: [Figure showing y=g(y,x)]

Theorem 2.3. *In Theorems 2.1 and 2.2, if X is a total ordered set with ordering $\precsim$, then a coupled common fixed point of f and g is unique and $x = y$.*

Proof. If $(x^*, y^*) \in X \times X$ is another coupled common fixed point of f and g, then, invoking (2.1), we have

$$
\begin{aligned}
d(x,x^*) + d(y,y^*) &= d(f(x,y), g(x^*,y^*)) + d(f(y,x), g(y^*,x^*)) \\
&\precsim \alpha \frac{[1 + D((x,y),(f(x,y),f(y,x)))].D((x^*,y^*),(g(x^*,y^*),g(y^*,x^*)))}{[1 + D((x,y),(x^*,y^*))]} + \\
&\beta D((x,y),(x^*,y^*)) + \\
&\gamma[D((x,y),(f(x,y),f(y,x))) + D((x^*,y^*),(g(x^*,y^*),g(y^*,x^*)))] + \\
&\delta[D((x^*,y^*),(f(x,y),f(y,x))) + D((x,y),(g(x^*,y^*),g(y^*,x^*)))] + \\
&\alpha \frac{[1 + D((y,x),(f(y,x),f(x,y)))].D((y^*,x^*),(g(y^*,x^*),g(x^*,y^*)))}{[1 + D((y,x),(y^*,x^*))]} + \\
&\beta D((y,x),(y^*,x^*)) + \\
&\gamma[D((y,x),(f(y,x),f(x,y))) + D((y^*,x^*),(g(y^*,x^*),g(x^*,y^*)))] + \\
&\delta[D((y^*,x^*),(f(y,x),f(x,y))) + D((y,x),(g(y^*,x^*),g(x^*,y^*)))] \\
&= 2\beta(d(x,x^*) + d(y,y^*)) + \\
&2\delta(d(x^*,f(x,y)) + d(y^*,f(y,x)) + d(x,g(x^*,y^*)) + d(y,g(y^*,x^*))) \\
&= (2\beta + 4\delta)(d(x,x^*) + d(y,y^*))
\end{aligned}
$$

$$
and \;\; hence \;\; d(x,x^*) + d(y,y^*) = (2\beta + 4\delta)(d(x,x^*) + d(y,y^*))
$$

Since, $(2\beta+4\delta) \leq \frac{1}{2}$, we have $d(x, x^*) + d(y, y^*) = 0$ which implies that $x = x^*\ and\ y = y^*$. On the other hand, we have

$$\begin{aligned}
d(x, y) &= d(f(x, y), g(y, x)) \\
&\preceq \alpha \frac{[1 + D((x, y), (f(x, y), f(y, x)))].D((y, x), (g(y, x), g(x, y)))}{[1 + D((x, y), (y, x))]} + \beta D((x, y), (y, x)) \\
&+ \gamma[D((x, y), (f(x, y), f(y, x))) + D((y, x), (g(y, x), g(x, y)))] \\
&+ \delta[D((y, x), (f(x, y), f(y, x))) + D((x, y), (g(y, x), g(x, y)))] \\
&\preceq (\beta + 2\delta)(d(x, y) + d(y, x)) \\
&\preceq (2\beta + 4\delta)d(x, y).
\end{aligned}$$

Since $(2\beta + 4\delta) \leq \frac{1}{2}$, we have $d(x, y) = 0$ and $x = y$. This complete the proof. □

Let $f : X \times X \to X$ be a mapping. Now we denote $f^{n+1}(x, y) = f(f^n(x, y), f^n(y, x))$, for all $x, y \in X$ and $n \in N$ k8

3 Application to metric space

Taking $s = 1$, $f = g$ and $\alpha = \gamma = \delta = 0$ in Theorem 2.1, we get the following:

Corollary 3.1. *Let $(X, d, \precsim)$ be a partially ordered complete metric space. Let $f : X \times X \to X$ be the mapping such that f has the mixed monotone property on X. Suppose there exists $\beta \in [0, 1)$ with $\beta < \frac{1}{2}$ such that*

$$d(f(x, y), f(u, v)) \precsim \beta D((x, y), (u, v))$$

for all $x, y, u, v \in X$ with $x \precsim u$ and $y \succsim v$ and D is defined as in Lemma 2.1. Let $x_0, y_0 \in X$ be such that $x_0 \precsim f(x_0, y_0), y_0 \succsim f(y_0, x_0)$. If f is continuous, then f has a coupled fixed point in X.

3.1 Application to system of integral equations

Consider the following system of integral equations:

$$\begin{aligned}
u(t) &= p(t) + \int_0^T \lambda(t, s)[f_1(s, u(s)) + f_2(s, v(s))]ds \\
v(t) &= p(t) + \int_0^T \lambda(t, s)[f_1(s, v(s)) + f_2(s, u(s))]ds.
\end{aligned} \tag{3.1}$$

We consider the space $X = C([0, T], \mathbb{R})$ of continuous functions defined on $[0, T]$. Obviously, the space with the metric given by

$$d(u, v) = \max_{t \in [0,T]} |u(t) - v(t)|, \quad u, v \in C([0, T], \mathbb{R})$$

is a complete metric space. Consider on $X = C([0,T],\mathbb{R})$ the natural partial order relation, that is,

$$u, v \in C([0,T],\mathbb{R}), \quad u \le v \iff u(t) \le v(t), \quad t \in [0,T].$$

Theorem 3.1. *Consider the problem (3.1) and assume that the following conditions are satisfied:*

(i) $f_1,\ f_2 : [0,T] \times \mathbb{R} \to \mathbb{R}$ *are continuous;*

(ii) $p : [0,T] \to \mathbb{R}$ *is continuous;*

(iii) $\lambda : [0,T] \times \mathbb{R} \to [0,\infty)$ *is continuous;*

(iv) there exists $c > 0$ *and* $\beta \in [0,1)$ *with* $\beta < \frac{1}{2}$ *such that for all* $u, v \in \mathbb{R},\ v \ge u$,

$$0 \le f_1(s,v) - f_1(s,u) \le c\ \beta(v-u)$$

$$0 \le f_2(s,u) - f_2(s,v) \le c\ \beta(v-u);$$

(v) assume that $c\ \max_{t\in[0,1]} \int_0^1 \lambda(t,s)ds \le 1$;

(vi) there exists $x_0, y_0 \in X$ *such that*

$$x_0(t) \ge p(t) + \int_0^T \lambda(t,s)[f_1(s,x_0(s)) + f_2(s,y_0(s))]ds$$
$$y_0(t) \le p(t) + \int_0^T \lambda(t,s)[f_1(s,y_0(s)) + f_2(s,x_0(s))]ds.$$

Then the system of integral equation (3.1) has a unique solution in X^2 *with* $(X = C([0,T],\mathbb{R}))$.

Proof. Consider the mapping $F : X \times X \to X$ defined by

$$F(u,v)(t) = p(t) + \int_0^T \lambda(t,s)[f_1(s,u(s)) + f_2(s,v(s))]ds, \tag{3.2}$$

for all $u, v \in X$ and $t \in [0,T]$. Now, we shall show that all the conditions of Corollary 3.1 are satisfied. From the condition (iv) of the Theorem 3.1, it is easy to prove that F has mixed monotone property.

Now, for $x, y, u, v \in X$ with $x \ge u,\ y \le v$ we have

$$\begin{aligned}
d(F(x,y),F(u,v)) &= \max_{t\in[0,T]} |F(x,y)(t) - F(u,v)(t)| \\
&= \max_{t\in[0,T]} \Big| \int_0^T \lambda(t,s)\Big[f_1(s,x(s)) + f_2(s,y(s))\Big]ds - \int_0^T \lambda(t,s)\Big[f_1(s,u(s)) + f_2(s,v(s))\Big]ds\Big| \\
&\le \max_{t\in[0,T]} \Big[\int_0^T |f_1(s,x(s)) - f_1(s,u(s))|.|\lambda(t,s)|ds + \int_0^T |f_1(s,y(s)) - f_1(s,v(s))|.|\lambda(t,s)|ds\Big] \\
&= \max_{t\in[0,T]} c\ \beta\Big[\int_0^T |x(s) - u(s)|.|\lambda(t,s)|ds + \int_0^T |y(s) - v(s)|.|\lambda(t,s)|ds\Big] \\
&\le \Big[\max_{t\in[0,T]} |x(t) - u(t)| + \max_{t\in[0,T]} |y(t) - v(t)|\Big] c\ \beta \int_0^T |\lambda(t,s)|ds \\
&\le \beta(d(x,u) + d(y,v)) \\
&= \beta D((x,y),(u,v)).
\end{aligned}$$

Which implies $d(F(x,y),F(u,v)) \leq \beta D((x,y),(u,v))$.

Which is just the contractive condition given in Corollary 3.1. Therefore, from Corollary 3.1, we deduce that, F has a coupled fixed point (x,y) in X, that is the system of integral equations has a solution. □

The following example shows that the superiority of Theorem3.1

Examplex 3.1. *Consider the following integral equation in* $X = C([0,1],R)$

$$F(u,v)(t) = \frac{t^2+8}{5} + \int_0^1 \frac{s^2}{35(t+4)}\Big[u(s) + \frac{1}{v(s)+1}\Big]ds. \tag{3.3}$$

It is easy to verify that the aforesaid equation is the special case of equation 3.2, in which

$$p(t) = \frac{t^2+8}{5}, \quad \lambda(t,s) = \frac{s^2}{35(t+4)}, \quad f_1(s,t) = t, \quad f_2(s,t) = \frac{1}{t+1}.$$

Indeed, the function p, λ, f_1 *and* f_2 *are continuous. Hence the assumption (i)-(iii) are fulfilled. Further, for all* $u, v \in R, v \geq u$ *there exist* $C = 16 > 0$ *and* $\beta = \frac{1}{4} \in [0,1)$ *with* $\beta < \frac{1}{2}$ *such that*

$$0 \leq f_1(s,v) - f_1(s,u) \leq c\beta(v-u),$$

$$0 \leq f_2(s,u) - f_2(s,v) \leq c\beta(v-u).$$

Thus the condition (iv) of Theorem 3.1 is satisfied. For condition (v), we have

$$c \max_{t\in[0,1]} \int_0^T \lambda(t,s)ds = 16 \max_{t\in[0,1]} \int_0^1 \frac{s^2}{35(t+4)}ds = \max_{t\in[0,1]} \frac{16}{105(t+4)} \leq 1$$

shows the validity of condition (v).

Consider $x_0(t) = 1$ *and* $y_0(t) = 1$, *then we get*

$$\begin{aligned} p(t) + \int_0^1 \lambda(t,s)[f_1(s,x_0(s)) + f_2(s,y_0(s))]ds &= \frac{t^2+8}{5} + \int_0^1 \frac{s^2}{35(t+4)}[f_1(s,1) + f_2(s,1)]ds \\ &= \frac{t^2+8}{5} + \int_0^1 \frac{s^2}{35(t+4)}\Big[1 + \frac{1}{2}\Big]ds \\ &= \frac{t^2+8}{5} + \frac{3}{70(t+4)}\Big(\frac{s^3}{3}\Big)_0^1 \\ &= \frac{t^2+8}{5} + \frac{3}{70(t+4)} \geq 1 \end{aligned}$$

that is $x_0 \geq f(x_0,y_0)$.

Similarly, one can show that $y_0 \leq f(y_0,x_0)$. *It follows that all the conditions are satisfied. Thus the integral equation* (3.3) *has a solution in* X^2 *with* $X = C([0,1],R)$.

References

[1] H. Aydi, Monica-F Bota, E. Karapinar, S. Mitrovic, A fixed point theorem for set-valued quasi contractions in b-metric spaces, *Fixed Point Theory and Applications*,2012:88(12012).

[2] I. A. Bakhtin, The contraction principle in quasi metric spaces, *Functional Analysis*,30,26-37(1989).

[3] T. G. Bhaskar, V. Lakshmikantham, Fixed point theorems in partially ordered metric spaces and applications, *Nonlinear Anal.*,65,1379-1393(2006).

[4] B. S. Choudhury, A. Kundu, A coupled coincidence point result in partially ordered metric spaces for compatible mappings, *Nonlinear Anal.*,73,2524-2531(2010).

[5] S. Czerwik, Contraction mappings in b-metric spaces, *Acta Mathematica et Informatica Universitatis Ostraviensis*,1,5-11(1993).

[6] N. V. Dung, On coupled common fixed points for mixed weakly monotone maps in partially ordered S-metric spaces, *Fixed Point Theory and Applications*,2013:48(2013).

[7] M. E. Gordji, E. Akbartabar, Y. J. Cho, M. Ramezani, Coupled common fixed point theorems for mixed weakly monotone mappings in partially ordered metric spaces, *Fixed Point Theory Appl.* ,2012:95(2012).

[8] D. Guo, V. Lakshmikantham, Coupled fixed points of nonlinear operators with applications, *Nonlinear Anal.*,11,623-632(1987).

[9] V. Lakshmikantham, L. Ciric, Coupled fixed points theorems for nonlinear contractions in partially ordered metric spaces, *Nonlinear Anal.*,70,4341-4349(2009).

[10] N. V. Luong, N. X. Thuan, Coupled fixed points in partially ordered metric spaces and application, *Nonlinear Anal.*,74,983-992(2011).

[11] H. K. Nashine, B. Samet, C. Vetro,Coupled coincidence points for compatible mappings satisfying mixed monotone property, *J. Nonlinear Sci. Appl.*,5(2):104-114(2012).

[12] A. C. M. Ran, M. C. B. Reurings, A fixed point theorem in partially ordered sets and some application to matrix equations, *Proc. Am. Math. Soc.*,132:1435-1443(2004).

[13] D. Singh, O.P. Chauhan, N. Singh, V. Joshi, Common fixed point theorems in complex valued b- metric spaces, *J. Math. Comput. Sci.*, 5,(3),412-429(2013).

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 27, NO.3, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC

FUNCTIONAL INEQUALITIES IN FUZZY NORMED SPACES AND ITS STABILITY

GILJUN HAN AND CHANG IL KIM*

Abstract. In this paper, we investigate the functional inequality

$$N(f(x+2y)-3f(x+y)+3f(x)-f(x-y)-6f(y),t)$$
$$\geq N(f(2x+y)+f(2x-y)-2f(x+y)-2f(x-y)-12f(x),kt)$$

for some fixed nonzero real number k and prove the generalized Hyers-Ulam stability for it in fuzzy Banach spaces by fixed point methods.

1. Introduction

In 1940, Ulam proposed the following stability problem (cf. [25]):

"Let G_1 be a group and G_2 a metric group with the metric d. Given a constant $\delta > 0$, does there exist a constant $c > 0$ such that if a mapping $f : G_1 \longrightarrow G_2$ satisfies $d(f(xy), f(x)f(y)) < c$ for all $x, y \in G_1$, then there exists a unique homomorphism $h : G_1 \longrightarrow G_2$ with $d(f(x), h(x)) < \delta$ for all $x \in G_1$?"

In the next year, Hyers [12] gave a partial solution of Ulam's problem for the case of approximate additive mappings. Subsequently, his result was generalized by Aoki ([1]) for additive mappings and by Rassias [24] for linear mappings to consider the stability problem with unbounded Cauchy differences. During the last decades, the stability problem of functional equations have been extensively investigated by a number of mathematicians (see [4], [5], [6], [9], and [19]).

In 2001, Rassias [23] introduced the following cubic functional equation

$$f(x+2y)-3f(x+y)+3f(x)-f(x-y)-6f(y)=0 \tag{1.1}$$

and every solution of the cubic functional equation is called *a cubic mapping* and in ([14]), the following cubic functional equation was investigated

$$f(2x+y)+f(2x-y)=2f(x+y)+2f(x-y)+12f(x). \tag{1.2}$$

Katsaras [15] defined a fuzzy norm on a vector space to construct a fuzzy vector topological structure on the space. Later, some mathematicians have defined fuzzy norms on a vector space in different points of view. In particular, Bag and Samanta [2], following Cheng and Mordeson [3], gave an idea of fuzzy norm in such a manner that the corresponding fuzzy metric is of Kramosil and Michalek type [16].

In [10], Glányi showed that if a mapping $f : X \longrightarrow Y$ satisfies the following functional inequality

$$\|2f(x)+2f(y)-f(xy^{-1})\| \leq \|f(xy)\|, \tag{1.3}$$

2010 *Mathematics Subject Classification.* 39B62, 39B72, 54A40, 47H10.

Key words and phrases. Hyers-Ulam stability, fuzzy normed space, fixed point theorem.

* Corresponding author.

The second author was supported by the research fund of Dankook University in 2018.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 27, NO.3, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC

then f satisfies the Jordan-Von Neumann functional equation

$$2f(x) + 2f(y) - f(xy^{-1}) = f(xy).$$

Glányi [11] and Fechner [8] proved the Hyers-Ulam stability of (1.3). Park, Cho, and Han [22] proved the Hyers-Ulam stability of the following functional inequality:

$$\|f(x) + f(y) + f(z)\| \leq \|f(x + y + z)\|. \tag{1.4}$$

Further, Park [21] proved the generalized Hyers-Ulam stability of the Cauchy additive functional inequality (1.4) in fuzzy Banach spaces using the fixed point method if f is an odd mapping.

In this paper, we investigate the following functional inequality related by (1.1) and (1.2)

$$\begin{aligned} &N(f(x + 2y) - 3f(x + y) + 3f(x) - f(x - y) - 6f(y), t) \\ &\geq N(f(2x + y) + f(2x - y) - 2f(x + y) - 2f(x - y) - 12f(x), kt) \end{aligned} \tag{1.5}$$

for some fixed nonzero real number k and prove the generalized Hyers-Ulam stability for (1.5) in fuzzy Banach spaces by fixed point methods.

2. PRELIMINARIES

In this paper, we use the definition of fuzzy normed spaces given in [2], [17], and [18].

Definition 2.1. Let X be a real vector space. A function $N : X \times \mathbb{R} \longrightarrow [0, 1]$ is called *a fuzzy norm on X* if for any $x, y \in X$ and any $s, t \in \mathbb{R}$,
(N1) $N(x, t) = 0$ for $t \leq 0$;
(N2) $x = 0$ if and only if $N(x, t) = 1$ for all $t > 0$;
(N3) $N(cx, t) = N(x, \frac{t}{|c|})$ if $c \neq 0$;
(N4) $N(x + y, s + t) \geq \min\{N(x, s), N(y, t)\}$;
(N5) $N(x, \cdot)$ is a nondecreasing function of $\mathbb{R}$ and $\lim_{t\to\infty} N(x, t) = 1$;
(N6) for any $x \neq 0$, $N(x, \cdot)$ is continuous on $\mathbb{R}$.
In this case, the pair (X, N) is called *a fuzzy normed space.*

Let (X, N) be a fuzzy normed space and $\{x_n\}$ a sequence in X. Then (i) $\{x_n\}$ is said to be *Cauchy in* (X, N) if for any $\epsilon > 0$, there exists an $m \in \mathbb{N}$ such that $N(x_{n+p} - x_n, t) > 1 - \epsilon$ for all $n \geq m$, all positive integer p, and any $t > 0$ and (ii) $\{x_n\}$ is said to be *convergent in* (X, N) if there exists an $x \in X$ such that $\lim_{n\to\infty} N(x_n - x, t) = 1$ for all $t > 0$. In this case, x is called *the limit of the sequence* $\{x_n\}$ *in* X and one denotes it by $N - \lim_{n\to\infty} x_n = x$.

Example 2.2. For example, it is well known that for any normed space $(X, ||\cdot||)$ and any nonnegative real number ϵ, the mapping $N_X : X \times \mathbb{R} \longrightarrow [0, 1]$, defined by

$$N_X(x, t) = \begin{cases} 0, & \text{if } t \leq 0 \\ \frac{t}{t+\epsilon||x||}, & \text{if } t > 0 \end{cases},$$

is a fuzzy norm on X([17], [18], and [19]).

It is well known that every convergent sequence in a fuzzy normed space is Cauchy. A fuzzy normed space is said to be *complete* if each Cauchy sequence in it is convergent and a complete fuzzy normed space is called *a fuzzy Banach space.*

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 27, NO.3, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC

In 1996, Isac and Rassias [13] were the first to provide applications of stability theory of functional equations for the proof of new fixed point theorems with applications.

Theorem 2.3. [7] *Let (X, d) be a complete generalized metric space and let $J : X \longrightarrow X$ be a strictly contractive mapping with some Lipschitz constant L with $0 < L < 1$. Then for each given element $x \in X$, either $d(J^n x, J^{n+1}x) = \infty$ for all nonnegative integer n or there exists a positive integer n_0 such that*
(1) *$d(J^n x, J^{n+1}x) < \infty$ for all $n \geq n_0$;*
(2) *the sequence $\{J^n x\}$ converges to a fixed point y^* of J ;*
(3) *y^* is the unique fixed point of J in the set $Y = \{y \in X \mid d(J^{n_0}x, y) < \infty\}$;*
(4) *$d(y, y^*) \leq \dfrac{1}{1-L}\, d(y, Jy)$ for all $y \in Y$.*

Throughout this paper, we assume that X is a linear space, (Y, N) is a fuzzy Banach space, and (Z, N') is a fuzzy normed space.

3. Solutions of (1.5)

In this section, we investigate the solution and prove the generalized Hyers-Ulam stability of the functional inequality (1.5) in fuzzy Banach spaces. For any mapping $f : X \longrightarrow Y$, let

$$A_f(x, y) = f(2x + y) + f(2x - y) - 2f(x + y) - 2f(x - y) - 12f(x)$$

and

$$B_f(x, y) = f(x + 2y) - 3f(x + y) + 3f(x) - f(x - y) - 6f(y).$$

By (N5), we can easily shown the following lemma.

Lemma 3.1. *Let $\alpha_i : [0, \infty) \longrightarrow [0, \infty)(i = 1, 2, \cdots, n)$ be a mapping and r a positive real numbers with $r > 1$ and $y, z, z_1, z_2, \cdot, z_n \in Y$. Suppose that*

$$N(y, t) \geq \min\{N(z, r^n t), N(z_1, \alpha_1(t)), N(z_2, \alpha_2(t)), \cdots, N(z_n, \alpha_n(t))\}$$

for all $t > 0$ and all $n \in \mathbb{N}$. Then

$$N(y, t) \geq \min\{N(z_1, \alpha_1(t)), N(z_2, \alpha_2(t)), \cdots, N(z_n, \alpha_n(t))\}$$

for all $t > 0$.

By Lemma 3.1, we have the following corollary.

Corollary 3.2. *Let r be a real number with $r > 1$ and $y \in Y$. Suppose that*

$$N(y, t) \geq N(y, rt)$$

for all $t > 0$. Then $y = 0$.

Using Lemma 3.1 and Corollary 3.2, we will prove the following theorem :

Theorem 3.3. *Let $f : X \longrightarrow Y$ be a mapping. Suppose that k is a real number with $k > 4$. Then f is cubic if and only if f is a solution of (1.5).*

Proof. Letting $x = 0$ and $y = 0$ in (1.5), we have

$$N(f(0), t) \geq N\Big(f(0), \frac{3}{7}kt\Big)$$

for all $t > 0$ and sicne $\frac{3}{7}k > 1$, by Corollary 3.2, we get $f(0) = 0$.

Suppose that f is a solution of (1.5). Letting $x=0$ in (1.5), we have

$$N(f(2y)-9f(y)-f(-y),t)\geq N(f(y)+f(-y),kt) \tag{3.1}$$

for all $y\in X$ and all $t>0$ and letting $y=-x$ in (1.5), we have

$$N(f(2x)-3f(x)+5f(-x),t)\geq N(f(3x)-2f(2x)-11f(x),kt) \tag{3.2}$$

for all $x\in X$ and all $t>0$. Letting $y=x$ in (1.5), we have

$$N(f(3x)-3f(2x)-3f(x),t)\geq N(f(3x)-2f(2x)-11f(x),kt) \tag{3.3}$$

for all $x\in X$ and all $t>0$. By (3.1) and (3.2), we get

$$\begin{aligned}&N(6f(x)+6f(-x),t)\\ &\geq \min\Big\{N\Big(f(2x)-9f(x)-f(-x),\frac{t}{2}\Big),N\Big(f(2x)-3f(x)+5f(-x),\frac{t}{2}\Big)\Big\}\\ &\geq \min\Big\{N\Big(f(x)+f(-x),\frac{kt}{2}\Big),N\Big(f(3x)-2f(2x)-11f(x),\frac{kt}{2}\Big)\Big\}\end{aligned}$$

for all $x\in X$ and all $t>0$ and so we obtain

$$\begin{aligned}&N(f(x)+f(-x),t)\\ &\geq \min\{N(f(x)+f(-x),3kt),N(f(3x)-2f(2x)-11f(x),3kt)\}\end{aligned} \tag{3.4}$$

for all $x\in X$ and all $t>0$. For any $x\in X$, let

$$G(x)=f(3x)-2f(2x)-11f(x),\ \ H(x)=f(x)+f(-x)$$

for all $x\in X$. By (3.1), (3.4), and (N5), we have

$$\begin{aligned}N(f(2x)-8f(x),t)&\geq \min\Big\{N\Big(f(2x)-9f(x)-f(-x),\frac{t}{2}\Big),N\Big(H(x),\frac{t}{2}\Big)\Big\}\\ &\geq \min\Big\{N\Big(H(x),\frac{kt}{2}\Big),N\Big(H(x),\frac{3kt}{2}\Big),N\Big(G(x),\frac{3kt}{2}\Big)\Big\}\\ &\geq \min\Big\{N\Big(H(x),\frac{kt}{2}\Big),N\Big(G(x),\frac{3kt}{2}\Big)\Big\}\end{aligned} \tag{3.5}$$

for all $x\in X$ and all $t>0$. Further, by (3.3), (3.5), and (N5), we have

$$\begin{aligned}N(G(x),t)&\geq \min\Big\{N\Big(f(3x)-3f(2x)-3f(x),\frac{t}{2}\Big),N\Big(f(2x)-8f(x),\frac{t}{2}\Big)\Big\}\\ &\geq \min\Big\{N\Big(G(x),\frac{kt}{2}\Big),N\Big(H(x),\frac{kt}{4}\Big),N\Big(G(x),\frac{3kt}{4}\Big)\Big\}\\ &\geq \min\Big\{N\Big(G(x),\frac{kt}{2}\Big),N\Big(H(x),\frac{kt}{4}\Big)\Big\}\end{aligned} \tag{3.6}$$

for all $x\in X$ and all $t>0$ and since $k>4$, by (3.6) and (N5), we have

$$\begin{aligned}N(G(x),t)&\geq \min\Big\{N\Big(G(x),\frac{kt}{2}\Big),N\Big(H(x),\frac{kt}{4}\Big)\Big\}\\ &\geq \min\Big\{N\Big(G(x),\frac{k^2t}{2^2}\Big),N\Big(H(x),\frac{k^2t}{2^3}\Big),N\Big(H(x),\frac{kt}{4}\Big)\Big\}\\ &\geq \min\Big\{N\Big(G(x),\frac{k^2t}{2^2}\Big),N\Big(H(x),\frac{kt}{4}\Big)\Big\}\end{aligned}$$

for all $x\in X$ and all $t>0$. Hence by induction, we get

$$N(G(x),t)\geq \min\Big\{N\Big(G(x),\frac{k^nt}{2^n}\Big),N\Big(H(x),\frac{kt}{4}\Big)\Big\} \tag{3.7}$$

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 27, NO.3, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC

for all $x \in X$, all $t > 0$ and all $n \in \mathbb{N}$. By Lemma 3.1 and (3.7), we obtain

$$N(G(x), t) \geq N\Big(H(x), \frac{kt}{4}\Big)$$

for all $x \in X$ and all $t > 0$. By (3.4) and (N5), we have

$$\begin{aligned} N(G(x), t) &\geq N\Big(H(x), \frac{kt}{4}\Big) \\ &\geq \min\Big\{N\Big(H(x), \frac{(3k)^2 t}{12}\Big), N\Big(G(x), \frac{(3k)^2 t}{12}\Big)\Big\} \\ &\geq \min\Big\{N\Big(H(x), \frac{(3k)^3 t}{12}\Big), N\Big(G(x), \frac{(3k)^3 t}{12}\Big), N\Big(G(x), \frac{(3k)^2 t}{12}\Big)\Big\} \\ &\geq \min\Big\{N\Big(H(x), \frac{(3k)^3 t}{12}\Big), N\Big(G(x), \frac{(3k)^2 t}{12}\Big)\Big\} \end{aligned} \tag{3.8}$$

for all $x \in X$ and all $t > 0$. By induction and (3.8), we get

$$N(G(x), t) \geq N\Big(G(x), \frac{(3k)^2 t}{12}\Big) \tag{3.9}$$

for all $x \in X$ and all $t > 0$. By (3.9) and Corollary 3.2, we get

$$G(x) = f(3x) - 2f(2x) - 11f(x) = 0 \tag{3.10}$$

for all $x \in X$. By (3.4) and (3.10), we get

$$N(H(x), t) \geq N(H(x), 3kt) \tag{3.11}$$

for all $x \in X$ and by Corollary 3.2, we have

$$H(x) = f(x) + f(-x) = 0 \tag{3.12}$$

for all $x \in X$. Hence f is an odd mapping. Further, by (3.5), (3.10), and (3.12), we get

$$f(2x) = 8f(x) \tag{3.13}$$

for all $x \in X$. Now, letting $x = 2y$ in (1.5), by (3.13), we have

$$\begin{aligned} &N(8f(x+y) - 3f(2x+y) + 24f(x) - f(2x-y) - 6f(y), t) \\ &\geq N(f(4x+y) + f(4x-y) - 2f(2x+y) - 2f(2x-y) - 96f(x), kt) \end{aligned} \tag{3.14}$$

for all $x, y \in X$ and all $t > 0$ and letting $y = -y$ in (3.14), by (3.12), we have

$$\begin{aligned} &N(8f(x-y) - 3f(2x-y) + 24f(x) - f(2x+y) + 6f(y), t) \\ &\geq N(f(4x+y) + f(4x-y) - 2f(2x+y) - 2f(2x-y) - 96f(x), kt) \end{aligned} \tag{3.15}$$

for all $x, y \in X$ and all $t > 0$. By (3.14) and (3.15), we have

$$N(4A_f(x, y), t) \geq N\Big(A_f(2x, y), \frac{kt}{2}\Big)$$

for all $x, y \in X$ and all $t > 0$ and so we have

$$N(A_f(x, y), t) \geq N(A_f(2x, y), 2kt) \tag{3.16}$$

for all $x, y \in X$ and all $t > 0$. Letting $y = 2y$ in (1.5), we have

$$\begin{aligned} &N(f(x+4y) - 3f(x+2y) + 3f(x) - f(x-2y) - 48f(y), t) \\ &\geq N(8f(x+y) + 8f(x-y) - 2f(x+2y) - 2f(x-2y) - 12f(x), kt) \end{aligned} \tag{3.17}$$

for all $x, y \in X$ and all $t > 0$ and interchang x and y in (3.17), by (3.12) and (1.5), we get

$$\begin{aligned}
&N(f(4x+y) - 3f(2x+y) + 3f(y) + f(2x-y) - 48f(x), t)\\
&\geq N(8f(x+y) - 8f(x-y) - 2f(2x+y) + 2f(2x-y) - 12f(y), kt)\\
&= N(-2A_f(x,y) - 4B_f(y,-x), kt)\\
&\geq \min\Big\{N\Big(A_f(x,y), \frac{kt}{6}\Big), N\Big(B_f(y,-x), \frac{kt}{6}\Big)\Big\} \qquad (3.18)\\
&\geq \min\Big\{N\Big(A_f(x,y), \frac{kt}{6}\Big), N\Big(A_f(y,-x), \frac{k^2t}{6}\Big)\Big\}\\
&\geq \min\Big\{N\Big(A_f(x,y), \frac{kt}{6}\Big), N\Big(A_f(y,x), \frac{k^2t}{6}\Big)\Big\}
\end{aligned}$$

for all $x, y \in X$ and all $t > 0$. Letting $y = -y$ in (3.18), we get

$$\begin{aligned}
&N(f(4x-y) - 3f(2x-y) - 3f(y) + f(2x+y) - 48f(x), t)\\
&\geq \min\Big\{N\Big(A_f(x,-y), \frac{kt}{6}\Big), N\Big(A_f(-y,-x), \frac{k^2t}{6}\Big)\Big\} \qquad (3.19)\\
&\geq \min\Big\{N\Big(A_f(x,y), \frac{kt}{6}\Big), N\Big(A_f(y,x), \frac{k^2t}{6}\Big)\Big\}
\end{aligned}$$

for all $x, y \in X$. By (3.16), (3.18), (3.19), and (N5), we have

$$N(A_f(x,y),t) \geq N(A_f(2x,y), 2kt) \geq \min\Big\{N\Big(A_f(x,y), \frac{k^2t}{6}\Big), N\Big(A_f(y,x), \frac{k^3t}{6}\Big)\Big\}$$

for all $x, y \in X$ and all $t > 0$. By Lemma 3.1 and induction, we get

$$A_f(x,y) = f(2x+y) + f(2x-y) - 2f(x+y) - 2f(x-y) - 12f(x) = 0$$

for all $x, y \in X$. Thus f is a cubic mapping. □

By Theorem 3.3, we have the following corollaries :

Corollary 3.4. *Let $f : X \longrightarrow Y$ be a mapping. Suppose that a, b are real numbers with $|a| > 4|b| > 0$. Then f is cubic if and only if f satisfies the following inequlaity*

$$N(aB_f(x,y),t) \geq N(bA_f(x,y),t) \qquad (3.20)$$

for all $x, y \in X$ and all $t > 0$.

Corollary 3.5. *Let $f : X \longrightarrow Y$ be a mapping. Suppose that a is a real number with $|a| > 8$. Then f is cubic if and only if f satisfies the following inequlaity*

$$N(aB_f(x,y) + A_f(x,y),t) \geq N(A_f(x,y),t) \qquad (3.21)$$

for all $x, y \in X$ and all $t > 0$.

Proof. By (3.21) and (N5), we have

$$\begin{aligned}
N(B_f(x,y),t) &= N(aB_f(x,y), |a|t)\\
&\geq \min\Big\{N\Big(aB_f(x,y) + A_f(x,y), \frac{|a|}{2}t\Big), N\Big(A_f(x,y), \frac{|a|}{2}t\Big)\Big\}\\
&= N\Big(A_f(x,y), \frac{|a|}{2}t\Big)
\end{aligned}$$

for all $x, y \in X$ and all $t > 0$. Hence by Theorem 3.3, we have the result. □

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 27, NO.3, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC

Using the fuzzy norm $N_X : X\times\mathbb{R} \longrightarrow [0,1]$ in Exmaple 2.2, we have the following corollary :

Corollary 3.6. *Let $(X, ||\cdot||)$ be a normed space and $f : X \longrightarrow Y$ a mapping. Suppose that a is a real number with $|a| > 8$. Then f is cubic if and only if f satisfies the following inequlaity*

$$\|aB_f(x,y) + A_f(x,y)\| \le \|A_f(x,y)\| \tag{3.22}$$

for all $x, y \in X$.

4. The generalized Hyers-Ulam stability for (1.5)

Now, we will prove the generalized Hyers-Ulam stability for (1.5) in fuzzy normed spaces.

Theorem 4.1. *Assume that $\phi : X^3 \longrightarrow [0,\infty)$ is a function such that*

$$N'(\phi(2x,2y),t) \ge N'(8L\phi(x,y),t) \tag{4.1}$$

for all $x, y \in X$, $t > 0$ and some real number L with $0 < L < 1$. Let $f : X \longrightarrow Y$ be a mapping such that $f(0) = 0$ and

$$N(B_f(x,y),t) \ge \min\{N(A_f(x,y),kt), N'(\phi(x,y),t)\} \tag{4.2}$$

for all $x, y \in X$ and $t > 0$. Then there exists a unique cubic mapping $C : X \longrightarrow Y$ such that

$$N(f(x) - C(x), \frac{1}{8(1-L)}t) \ge \Psi(x,t) \tag{4.3}$$

for all $x \in X$, $t > 0$, and some real number k with $k > 4$,
where $\Psi(x,t) = \min\Big\{N'\Big(\phi(x.x), \frac{3kt}{4}\Big), N'\Big(\phi(x.-x), \frac{3t}{2}\Big), N'\Big(\phi(0,x), \frac{t}{2}\Big)\Big\}$.

Proof. Letting $x = 0$ in (4.2), by (N2), we have

$$N(f(2y) - 9f(y) - f(-y),t) \ge \min\{N(H(y),kt), N'(\phi(0,y),t)\} \tag{4.4}$$

for all $y \in X$ and $t > 0$ and letting $y = -x$ in (4.2), by (N2), we have

$$N(f(2x) + 5f(-x) - 3f(x),t) \ge \min\{N(G(x),kt), N'(\phi(x.-x),t)\} \tag{4.5}$$

for all $x \in X$ and $t > 0$. Letting $y = x$ in (4.2), we have

$$N(f(3x) - 3f(2x) - 3f(x),t) \ge \min\{N(G(x),kt), N'(\phi(x.x),t)\} \tag{4.6}$$

for all $x \in X$ and all $t > 0$. By (4.4) and (4.5), we get

$$\begin{aligned} &N(H(x),t) \\ &\ge \min\{N(H(x),3kt), N(G(x),3kt), N'(\phi(0,x),3t), N'(\phi(x.-x),3t)\} \end{aligned} \tag{4.7}$$

for all $x \in X$ and all $t > 0$. Similar to the proof of Theorem 3.3, by (4.7), we have

$$N(H(x),t) \ge \min\{N(G(x),3kt), N'(\phi(0,x),3t), N'(\phi(x.-x),3t)\} \tag{4.8}$$

for all $x \in X$ and all $t > 0$. By (4.4), (4.8), and (N5), we get

(4.9)

$$
\begin{aligned}
N(f(2x) - 8f(x), t) &\geq \min\Big\{N\Big(H(x), \frac{t}{2}\Big), N\Big(f(2x) - 9f(x) - f(-x), \frac{t}{2}\Big)\Big\} \\
&\geq \min\Big\{N\Big(H(x), \frac{t}{2}\Big), N\Big(H(x), \frac{kt}{2}\Big), N'\Big(\phi(0, x), \frac{t}{2}\Big)\Big\} \\
&\geq \min\Big\{N\Big(G(x), \frac{3kt}{2}\Big), N'\Big(\phi(x. - x), \frac{3t}{2}\Big), N'\Big(\phi(0, x), \frac{t}{2}\Big)\Big\}
\end{aligned}
$$

for all $x \in X$ and all $t > 0$ and by (4.6), (4.9), and (N5), we get

(4.10)

$$
\begin{aligned}
N(G(x), t) &\geq \min\Big\{N\Big(f(3x) - 3f(2x) + 5f(-x), \frac{t}{2}\Big), N\Big(f(2x) - 8f(x), \frac{t}{2}\Big)\Big\} \\
&\geq \min\Big\{N\Big(G(x), \frac{kt}{2}\Big), N'\Big(\phi(x.x), \frac{t}{2}\Big), N'\Big(\phi(x. - x), \frac{3t}{4}\Big), N'\Big(\phi(0, x), \frac{t}{4}\Big)\Big\}
\end{aligned}
$$

for all $x \in X$ and all $t > 0$. Since $k > 4$, by (4.10) and (N5), we obtain

$$
N(G(x), t) \geq \min\Big\{N'\Big(\phi(x.x), \frac{t}{2}\Big), N'\Big(\phi(x. - x), \frac{3t}{4}\Big), N'\Big(\phi(0, x), \frac{t}{4}\Big)\Big\} \tag{4.11}
$$

for all $x \in X$ and all $t > 0$. By (4.9), (4.11), and (N5), we get

$$
\begin{aligned}
&N(f(2x) - 8f(x), t) \\
&\geq \min\Big\{N'\Big(\phi(x.x), \frac{3kt}{4}\Big), N'\Big(\phi(x. - x), \frac{9kt}{8}\Big), N'\Big(\phi(0, x), \frac{3kt}{8}\Big), \\
&N'\Big(\phi(x. - x), \frac{3t}{2}\Big), N'\Big(\phi(0, x), \frac{t}{2}\Big)\Big\} \\
&\geq \min\Big\{N'\Big(\phi(x.x), \frac{3kt}{4}\Big), N'\Big(\phi(x. - x), \frac{3t}{2}\Big), N'\Big(\phi(0, x), \frac{t}{2}\Big)\Big\}
\end{aligned} \tag{4.12}
$$

for all $x \in X$ and all $t > 0$.

Consider the set $S = \{g \mid g : X \longrightarrow Y\}$ and the generalized metric d on S defined by

$$
d(g, h) = \inf\{c \in [0, \infty) \mid N(g(x) - h(x), ct) \geq \Psi(x, t), \forall x \in X, \forall t > 0\}.
$$

Then (S, d) is a complete metric space(See [20]). Define a mapping $J : S \longrightarrow S$ by $Jg(x) = \frac{1}{8}g(2x)$ for all $x \in X$ and all $g \in S$.

Let $g, h \in S$ and $d(g, h) \leq c$ for some $c \in [0, \infty)$. Then by (4.1), we have

$$
N(Jg(x) - Jh(x), ct) = N(g(2x) - h(2x), 8ct) \geq \Psi(2x, 8t) \geq \Psi(x, \frac{t}{L})
$$

for all $x \in X$ and $t > 0$. Hence $N(Jg(x) - Jh(x), cLt) \geq \Psi(x, t)$ for all $x \in X$ and $t > 0$ and thus $d(Jg, Jh) \leq Ld(g, h)$ for any $g, h \in S$. Moreover, by (4.12), we have $d(Jf, f) \leq \frac{1}{8} < \infty$. By Theorem 2.3, there exists a mapping $C : X \longrightarrow Y$ which is a fixed point of J such that $d(J^n f, A) \to 0$ as $n \to \infty$. That is,

$$
C(x) = N - \lim_{n\to\infty} \frac{f(2^n x)}{2^{3n}} \tag{4.13}
$$

for all $x \in X$. Replacing x, and y by $2^n x$ and $2^n y$ in (4.2), respectively, by (4.1), we have

$$
\begin{aligned}
&N(B_f(2^n x, 2^n), 2^{3n} t) \\
&\geq \min\Big\{N(A_f(2^n x, 2^n y), 2^{3n} t), N'\Big(\phi(x, y), \frac{1}{L^n} t\Big)\Big\}
\end{aligned} \tag{4.14}
$$

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 27, NO.3, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC

for all $x, y \in X$ and all $n \in \mathbb{N}$. Letting $n \to \infty$ in (4.14), C is a solution of (1.5) and so by Theorem 3.3, C is a cubic mapping. Since $d(f, Jf) \leq \frac{1}{8}$, by Theorem 2.3, we have (4.17).

Now, we show the uniqueness of C. Let C_0 be another cubic mapping with (4.17). Then for any positive integer n,

$$C(x) = \frac{C(2^n x)}{2^{3n}}, \ C_0(x) = \frac{C_0(2^n x)}{2^{3n}}$$

for all $x \in X$. Hence by (4.17), (N3) and (N4), we have

$$\begin{aligned} &N(C(x) - C_0(x), t) = N(C(2^n x) - C_0(2^n x), 2^{3n} t) \geq \Psi(2^n x, 2^{3n} 8(1-L)t) \\ &\geq \Psi\Big(x, \frac{8(1-L)t}{L^n}\Big) \end{aligned}$$

for all $x \in X$, $t > 0$, and all $n \in \mathbb{N}$. Hence, letting $n \to \infty$ in the above inequality, we have $C(x) = C_0(x)$ for all $x \in X$. □

By Corollary 3.5 and Theorem 4.1, we can show that the following corollaries:

Corollary 4.2. *Let ϵ and p be real numbers with $\epsilon \geq 0$ and $0 < p < \frac{3}{2}$. Let $f : X \longrightarrow Y$ be a mapping such that*

$$(4.15) \quad N(B_f(x,y), t) \geq \min\Big\{N(A_f(x,y), kt), \ \frac{t}{t + \epsilon(\|x\|^{2p} + \|y\|^{2p} + \|x\|^p \|y\|^p)}\Big\}$$

for all $x, y \in X$, all $t > 0$ and some real number k with $k > 4$. Then there exists a unique cubic mapping $C : X \longrightarrow Y$ such that

$$N(f(x) - C(x), t) \geq \frac{(8 - 2^{2p})t}{(8 - 2^{2p})t + 2\epsilon\|x\|^{2p}}$$

for all $x \in X$ and all $t > 0$.

Corollary 4.3. *Assume that $\phi : X^3 \longrightarrow [0, \infty)$ is a function with (4.1) Let $f : X \longrightarrow Y$ be a mapping such that $f(0) = 0$ and*

$$(4.16) \qquad N(aB_f(x,y) + A_f(x,y), t) \geq \min\{N(A_f(x,y), t), N'(\phi(x,y), t)\}$$

for all $x, y \in X$, all $t > 0$ and some real numbers a with $|a| > 8$. Then there exists a unique cubic mapping $C : X \longrightarrow Y$ such that

$$(4.17) \qquad N(f(x) - C(x), \frac{1}{8(1-L)} t) \geq \Psi(x, t)$$

*for all $x \in X$, $t > 0$, and some fixed real number k with $k > 4$,
where $\Psi(x,t) = \min\Big\{N'\Big(\phi(x.x), \frac{3|a|t}{8}\Big), N'\Big(\phi(x.-x), \frac{3t}{2}\Big), N'\Big(\phi(0,x), \frac{t}{2}\Big)\Big\}$.*

Proof. By (N5) and (4.16), we have

$$\begin{aligned} N(B_f(x,y), t) &\geq \min\Big\{N\Big(aB_f(x,y) + A_f(x,y), \frac{|a|}{2} t\Big), N\Big(A_f(x,y), \frac{|a|}{2} t\Big)\Big\} \\ &\geq \min\Big\{N\Big(A_f(x,y), \frac{|a|}{2} t\Big), N'\Big(\phi(x,y), \frac{|a|}{2} t\Big)\Big\} \\ &\geq \min\Big\{N\Big(A_f(x,y), \frac{|a|}{2} t\Big), N'\Big(\phi(x,y), t\Big)\Big\} \end{aligned}$$

for all $x, y \in X$, all $t > 0$. Hence we have the results. □

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 27, NO.3, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC

Corollary 4.4. *Let ϵ and p be real numbers with $\epsilon \geq 0$ and $0 < p < \frac{3}{2}$. Let $f : X \longrightarrow Y$ be a mapping such that*

$$\begin{aligned} &N(aB_f(x,y) + A_f(x,y), t) \\ &\geq \min\Big\{N(A_f(x,y),t),\ \frac{t}{t + \epsilon(\|x\|^{2p} + \|y\|^{2p} + \|x\|^p\|y\|^p)}\Big\} \end{aligned} \tag{4.18}$$

for all $x, y \in X$, all $t > 0$ and some real number a with $|a| > 8$. Then there exists a unique cubic mapping $C : X \longrightarrow Y$ such that

$$N(f(x) - C(x), t) \geq \frac{(8 - 2^{2p})t}{(8 - 2^{2p})t + 2\epsilon\|x\|^{2p}}$$

for all $x \in X$ and all $t > 0$.

Related with Theorem 4.1, we can also have the following theorem. And the proof is similar to that of Theorem 4.1.

Theorem 4.5. *Assume that $\phi : X^3 \longrightarrow [0,\infty)$ is a function such that*

$$N'\Big(\phi\Big(\frac{x}{2}, \frac{y}{2}\Big), t\Big) \geq N'\Big(\frac{L}{8}\phi(x,y), t\Big) \tag{4.19}$$

for all $x, y \in X$, $t > 0$ and some L with $0 \leq L < 1$. Let $f : X \longrightarrow Y$ be a mapping with $f(0) = 0$ and (4.2). Then there exists a unique cubic mapping $C : X \longrightarrow Y$ such that

$$N\Big(f(x) - C(x), \frac{L}{1-L}t\Big) \geq \Psi_0(x,t) \tag{4.20}$$

for all $x \in X$, $t > 0$, and some fixed real number k with $k > 4$,
where $\Psi_0(x,t) = \min\Big\{N'\Big(\phi(x.x), 6kt\Big), N'\Big(\phi(x.-x), 12t\Big), N'\Big(\phi(0,x), 4t\Big)\Big\}$.

Proof. By (4.12) in Theorem 4.1, we get

$$\begin{aligned} &N\Big(f(x) - 8f\Big(\frac{x}{2}\Big), t\Big) \\ &\geq \min\Big\{N'\Big(\phi(x.x), \frac{6kt}{L}\Big), N'\Big(\phi(x.-x), \frac{12t}{L}\Big), N'\Big(\phi(0,x), \frac{4t}{L}\Big)\Big\} \end{aligned} \tag{4.21}$$

for all $x \in X$ and all $t > 0$.

Consider the set $S = \{g \mid g : X \longrightarrow Y\}$ and the generalized metric d on S defined by

$$d(g,h) = \inf\{c \in [0,\infty) \mid N(g(x) - h(x), ct) \geq \Psi_0(x,t), \forall x \in X, \forall t > 0\}.$$

Then (S,d) is a complete metric space(See [20]). Define a mapping $J : S \longrightarrow S$ by $Jg(x) = 8g\Big(\frac{1}{2}x\Big)$ for all $x \in X$ and all $g \in S$.

Let $g, h \in S$ and $d(g,h) \leq c$ for some $c \in [0,\infty)$. Then by (4.19), we have

$$N(Jg(x) - Jh(x), ct) = N\Big(8g\Big(\frac{1}{2}x\Big) - 8h\Big(\frac{1}{2}x\Big), ct\Big) \geq \Psi_0\Big(\frac{1}{2}x, \frac{t}{8}\Big) \geq \Psi_0\Big(x, \frac{t}{L}\Big)$$

for all $x \in X$ and $t > 0$. Hence $N(Jg(x) - Jh(x), cLt) \geq \Psi_0(x,t)$ for all $x \in X$ and $t > 0$ and thus $d(Jg, Jh) \leq Ld(g,h)$ for any $g, h \in S$. Moreover, by (4.21) we have $d(f, Jf) \leq L < \infty$. The rest of the proof is similar to Theorem 4.1. □

By Corollary 3.6 and Theorem 4.5, we can show that the following corollaries:

Corollary 4.6. *Let ϵ and p be real numbers with $\epsilon \geq 0$ and $p > \frac{3}{2}$. Let $f : X \longrightarrow Y$ be a mapping such that*

$$N(B_f(x,y),t) \geq \min\Big\{N(A_f(x,y),kt),\ \frac{t}{t+\epsilon(\|x\|^{2p}+\|y\|^{2p}+\|x\|^p\|y\|^p)}\Big\} \tag{4.22}$$

for all $x, y \in X$, all $t > 0$ and some real number k with $k > 4$. Then there exists a unique cubic mapping $C : X \longrightarrow Y$ such that

$$N(f(x)-C(x),t) \geq \frac{(2^{2p}-8)t}{(2^{2p}-8)t+2\epsilon\|x\|^{2p}}$$

for all $x \in X$ and all $t > 0$.

Corollary 4.7. *Assume that $\phi : X^3 \longrightarrow [0,\infty)$ is a function with (4.1) Let $f : X \longrightarrow Y$ be a mapping such that $f(0) = 0$ and*

$$N(aB_f(x,y)+A_f(x,y),t) \geq \min\{N(A_f(x,y),t), N'(\phi(x,y),t)\} \tag{4.23}$$

for all $x, y \in X$, all $t > 0$ and some real numbers a, b with $|a| > 8$. Then there exists a unique cubic mapping $C : X \longrightarrow Y$ such that

$$N(f(x)-C(x),\frac{L}{1-L}t) \geq \Psi_0(x,t) \tag{4.24}$$

for all $x \in X$, $t > 0$, and some fixed real number k with $k > 4$,
where $\Psi_0(x,t) = \min\Big\{N'\Big(\phi(x.x),3|a|t\Big), N'\Big(\phi(x.-x),12t\Big), N'\Big(\phi(0,x),4t\Big)\Big\}$.

Corollary 4.8. *Let ϵ and p be real numbers with $\epsilon \geq 0$ and $p > \frac{3}{2}$. Let $f : X \longrightarrow Y$ be a mapping such that*

$$\begin{aligned} &N(aB_f(x,y)+A_f(x,y),t) \\ &\geq \min\Big\{N(A_f(x,y),t),\ \frac{t}{t+\epsilon(\|x\|^{2p}+\|y\|^{2p}+\|x\|^p\|y\|^p)}\Big\} \end{aligned} \tag{4.25}$$

for all $x, y \in X$, all $t > 0$ and some real number a with $|a| > 8$. Then there exists a unique cubic mapping $C : X \longrightarrow Y$ such that

$$N(f(x)-C(x),t) \geq \frac{(2^{2p}-8)t}{(2^{2p}-8)t+2\epsilon\|x\|^{2p}}$$

for all $x \in X$ and all $t > 0$.

References

[1] T. Aoki, *On the stability of the linear transformation in Banach spaces*, J. Math. Soc. Japan **2**(1950), 64-66.

[2] T. Bag and S. K. Samanta, *Finite dimensional fuzzy normed linear spaces*, J. Fuzzy Math. **11**(2003), 687-705.

[3] S. C. Cheng and J. N. Mordeson, *Fuzzy linear operator and fuzzy normed linear spaces, Bull. Calcutta Math. Soc.* **86***(1994), 429-436.*

[4] P.W.Cholewa, Remarkes on the stability of functional equations*, Aequationes Math.* **27***(1984), 76-86.*

[5] K Cieplinski, Applications of fixed point theorems to the hyers-ulam stability of functional equation-A survey*, Ann. Funct. Anal.* **3***(2012), no. 1, 151-164.*

[6] S. Czerwik, On the stability of the quadratic mapping in normed spaces*, Bull. Abh. Math. Sem. Univ. Hamburg* **62***(1992), 59-64.*

[7] J. B. Diaz and B. Margolis, A fixed point theorem of the alternative for contractions on a generalized complete metric space*, Bull. Amer. Math. Soc.* **74** *(1968), 305-309.*

[8] W. Fechner, Stability of a functional inequalty associated with the Jordan-Von Neumann functional equation*, Aequationes Math.* **71***(2006), 149-161.*

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 27, NO.3, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC

[9] P. Găvruta, A generalization of the Hyer-Ulam-Rassias stability of approximately additive mappings, J. Math. Anal. Appl. **184**(1994), 431-436. .

[10] A. Gilányi, *Eine zur Parallelogrammgleichung äquivalente Ungleichung*, Aequationes Mathematicae, **62**(2001), 303-309.

[11] A. Gilányi, *On a problem by K. Nikoden, Mathematical Inequalities and Applications*, **5**(2002), 701-710.

[12] D. H. Hyers, *On the stability of the linear functional equation*, Proc. Nat. Acad. Sci. **27**(1941), 222-224.

[13] G. Isac and Th. M. Rassias, *Stability of ψ-additive mappings, Appications to nonlinear analysis*, Internat. J. Math. and Math. Sci. **19**(1996), 219-228.

[14] K. W. Jun and H. M. Kim, *The generalized Hyers-Ulam-Rassias stability of a cubic functional equation*, J. Math. Anal. Appl. **274**(2002), 867-878.

[15] A. K. Katsaras, *Fuzzy topological vector spaces II*, Fuzzy Sets Syst **12**(1984), 143-154.

[16] I. Kramosil and J. Michálek, *Fuzzy metric and statistical metric spaces*, Kybernetica **11**(1975), 336-344.

[17] A. K. Mirmostafaee and M. S. Moslehian, *Fuzzy versions of Hyers-Ulam-Rassias theorem*, Fuzzy Sets Syst. **159**(2008), 720-729.

[18] A. K. Mirmostafaee, M. Mirzavaziri, and M. S. Moslehian, *Fuzzy stability of the Jensen functional equation*, Fuzzy Sets Syst. **159**(2008), 730-738.

[19] M. Mirzavaziri and M. S. Moslehian, *A fixed point approach to stability of a quadratic equation*, Bulletin of the Brazilian Mathematical Society **37**(2006), no. 3, 361-376

[20] M. S. Moslehian and T. H. Rassias, *Stability of functional equations in non-Archimedean spaces*, Applicable Anal. Discrete Math. **1**(2007), 325-334.

[21] C. Park, *Fuzzy Stability of Additive Functional Inequalities with the Fixed Point Alternative*, J. Inequal. Appl. **2009**(2009), 1-17.

[22] C. Park, Y. S. Cho, and M. H. Han, *Functional inequalities associated with Jordan-von Neumann type additive functional equations*, J. Inequal. Appl. **2007**(2007), 1-13.

[23] J. M. Rassias, *Solution of the Ulam stability problem for cubic mappings*, Glasnik Matematički, **36**(2001), 63-72.

[24] Th. M. Rassias, *On the stability of the linear mapping in Banach sapces*, Proc. Amer. Math. Sco. **72**(1978), 297-300.

[25] S. M. Ulam, Problems in modern mathematics, Science Editions John Wiley and Sons, Inc., New York, 1964.

Department of Mathematics Education, Dankook University, 152, Jukjeon-ro, Suji-gu, Yongin-si, Gyeonggi-do, 448-701, Korea

E-mail address: gilhan@dankook.ac.kr

Department of Mathematics Education, Dankook University, 126, Jukjeon, Yongin, Gyeonggi, South Korea 448-701, KOREA

E-mail address: kci206@hanmail.net

TABLE OF CONTENTS, JOURNAL OF COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 27, NO. 3, 2019

TABLE OF CONTENTS, JOURNAL OF COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 27, NO. 3, 2019

(continued)

Volume 27, Number 4 October 15, 2019
ISSN:1521-1398 PRINT,1572-9206 ONLINE

Journal of

Computational

Analysis and

Applications

EUDOXUS PRESS,LLC

Journal of Computational Analysis and Applications
ISSNno.'s:1521-1398 PRINT,1572-9206 ONLINE

SCOPE OF THE JOURNAL

An international publication of Eudoxus Press, LLC
(fifteen times annually)

Editor in Chief: George Anastassiou
Department of Mathematical Sciences,
University of Memphis, Memphis, TN 38152-3240, U.S.A
ganastss@memphis.edu
http://www.msci.memphis.edu/~ganastss/jocaaa

The main purpose of "J.Computational Analysis and Applications" is to publish high quality research articles from all subareas of Computational Mathematical Analysis and its many potential applications and connections to other areas of Mathematical Sciences. Any paper whose approach and proofs are computational,using methods from Mathematical Analysis in the broadest sense is suitable and welcome for consideration in our journal, except from Applied Numerical Analysis articles. Also plain word articles without formulas and proofs are excluded. The list of possibly connected mathematical areas with this publication includes, but is not restricted to: Applied Analysis, Applied Functional Analysis, Approximation Theory, Asymptotic Analysis, Difference Equations, Differential Equations, Partial Differential Equations, Fourier Analysis, Fractals, Fuzzy Sets, Harmonic Analysis, Inequalities, Integral Equations, Measure Theory, Moment Theory, Neural Networks, Numerical Functional Analysis, Potential Theory, Probability Theory, Real and Complex Analysis, Signal Analysis, Special Functions, Splines, Stochastic Analysis, Stochastic Processes, Summability, Tomography, Wavelets, any combination of the above, e.t.c.

"J.Computational Analysis and Applications" is a peer-reviewed Journal. See the instructions for preparation and submission of articles to JoCAAA. Assistant to the Editor: Dr.Razvan Mezei,mezei_razvan@yahoo.com, Madison,WI,USA.

Journal of Computational Analysis and Applications(JoCAAA) is published by **EUDOXUS PRESS,LLC**,1424 Beaver Trail Drive,Cordova,TN38016,USA,anastassioug@yahoo.com http://www.eudoxuspress.com. **Annual Subscription Prices**:For USA and Canada,Institutional:Print $800, Electronic OPEN ACCESS. Individual:Print $400. For any other part of the world add $160 more(handling and postages) to the above prices for Print. No credit card payments.

JoCAAA is reviewed and abstracted by AMS Mathematical Reviews,MATHSCI,and Zentralblaat MATH.

Editorial Board
Associate Editors of Journal of Computational Analysis and Applications

Dartmouth College
8000 Cummings Hall,
Hanover, NH 03755-8000
603-646-3843 (X 3546 Secr.)
e-mail:george.cybenko@dartmouth.edu
Approximation Theory and Neural Networks

Sever S. Dragomir
School of Computer Science and Mathematics, Victoria University,
PO Box 14428,
Melbourne City,
MC 8001, AUSTRALIA
Tel. +61 3 9688 4437
Fax +61 3 9688 4050
sever.dragomir@vu.edu.au
Inequalities, Functional Analysis, Numerical Analysis, Approximations, Information Theory, Stochastics.

Oktay Duman
TOBB University of Economics and Technology,
Department of Mathematics, TR-06530,
Ankara, Turkey,
oduman@etu.edu.tr
Classical Approximation Theory, Summability Theory, Statistical Convergence and its Applications

Saber N. Elaydi
Department Of Mathematics
Trinity University
715 Stadium Dr.
San Antonio, TX 78212-7200
210-736-8246
e-mail: selaydi@trinity.edu
Ordinary Differential Equations, Difference Equations

J .A. Goldstein
Department of Mathematical Sciences
The University of Memphis
Memphis, TN 38152
901-678-3130
jgoldste@memphis.edu
Partial Differential Equations, Semigroups of Operators

H. H. Gonska
Department of Mathematics
University of Duisburg
Duisburg, D-47048
Germany
011-49-203-379-3542
e-mail: heiner.gonska@uni-due.de
Approximation Theory, Computer Aided Geometric Design

John R. Graef
Department of Mathematics
University of Tennessee at Chattanooga
Chattanooga, TN 37304 USA
John-Graef@utc.edu
Ordinary and functional differential equations, difference equations, impulsive systems, differential inclusions, dynamic equations on time scales, control theory and their applications

Weimin Han
Department of Mathematics
University of Iowa
Iowa City, IA 52242-1419
319-335-0770
e-mail: whan@math.uiowa.edu
Numerical analysis, Finite element method, Numerical PDE, Variational inequalities, Computational mechanics

Tian-Xiao He
Department of Mathematics and Computer Science
P.O. Box 2900, Illinois Wesleyan University
Bloomington, IL 61702-2900, USA
Tel (309)556-3089
Fax (309)556-3864
the@iwu.edu
Approximations, Wavelet, Integration Theory, Numerical Analysis, Analytic Combinatorics

Margareta Heilmann
Faculty of Mathematics and Natural Sciences, University of Wuppertal
Gaußstraße 20
D-42119 Wuppertal, Germany,
heilmann@math.uni-wuppertal.de
Approximation Theory (Positive Linear Operators)

Xing-Biao Hu
Institute of Computational Mathematics
AMSS, Chinese Academy of Sciences
Beijing, 100190, CHINA
hxb@lsec.cc.ac.cn

Computational Mathematics

Jong Kyu Kim
Department of Mathematics
Kyungnam University
Masan Kyungnam,631-701,Korea
Tel 82-(55)-249-2211
Fax 82-(55)-243-8609
jongkyuk@kyungnam.ac.kr
Nonlinear Functional Analysis, Variational Inequalities, Nonlinear Ergodic Theory, ODE, PDE, Functional Equations.

Robert Kozma
Department of Mathematical Sciences
The University of Memphis
Memphis, TN 38152, USA
rkozma@memphis.edu
Neural Networks, Reproducing Kernel Hilbert Spaces,
Neural Percolation Theory

Mustafa Kulenovic
Department of Mathematics
University of Rhode Island
Kingston, RI 02881,USA
kulenm@math.uri.edu
Differential and Difference Equations

Irena Lasiecka
Department of Mathematical Sciences
University of Memphis
Memphis, TN 38152
PDE, Control Theory, Functional Analysis, lasiecka@memphis.edu

Burkhard Lenze
Fachbereich Informatik
Fachhochschule Dortmund
University of Applied Sciences
Postfach 105018
D-44047 Dortmund, Germany
e-mail: lenze@fh-dortmund.de
Real Networks, Fourier Analysis, Approximation Theory

Hrushikesh N. Mhaskar
Department Of Mathematics
California State University
Los Angeles, CA 90032
626-914-7002
e-mail: hmhaska@gmail.com
Orthogonal Polynomials, Approximation Theory, Splines, Wavelets, Neural Networks

Ram N. Mohapatra
Department of Mathematics
University of Central Florida
Orlando, FL 32816-1364
tel.407-823-5080
ram.mohapatra@ucf.edu
Real and Complex Analysis, Approximation Th., Fourier Analysis, Fuzzy Sets and Systems

Gaston M. N'Guerekata
Department of Mathematics
Morgan State University
Baltimore, MD 21251, USA
tel: 1-443-885-4373
Fax 1-443-885-8216
Gaston.N'Guerekata@morgan.edu
nguerekata@aol.com
Nonlinear Evolution Equations, Abstract Harmonic Analysis, Fractional Differential Equations, Almost Periodicity & Almost Automorphy

M.Zuhair Nashed
Department Of Mathematics
University of Central Florida
PO Box 161364
Orlando, FL 32816-1364
e-mail: znashed@mail.ucf.edu
Inverse and Ill-Posed problems, Numerical Functional Analysis, Integral Equations, Optimization, Signal Analysis

Mubenga N. Nkashama
Department OF Mathematics
University of Alabama at Birmingham
Birmingham, AL 35294-1170
205-934-2154
e-mail: nkashama@math.uab.edu
Ordinary Differential Equations, Partial Differential Equations

Vassilis Papanicolaou
Department of Mathematics
National Technical University of Athens
Zografou campus, 157 80
Athens, Greece
tel:: +30(210) 772 1722
Fax +30(210) 772 1775
papanico@math.ntua.gr
Partial Differential Equations, Probability

Instructions to Contributors
Journal of Computational Analysis and Applications

An international publication of Eudoxus Press, LLC, of TN.

Editor in Chief: George Anastassiou
Department of Mathematical Sciences
University of Memphis
Memphis, TN 38152-3240, U.S.A.

1. Manuscripts files in Latex and PDF and in English, should be submitted via email to the Editor-in-Chief:

Prof.George A. Anastassiou
Department of Mathematical Sciences
The University of Memphis
Memphis,TN 38152, USA.
Tel. 901.678.3144
e-mail: ganastss@memphis.edu

Authors may want to recommend an associate editor the most related to the submission to possibly handle it.

Also authors may want to submit a list of six possible referees, to be used in case we cannot find related referees by ourselves.

2. Manuscripts should be typed using any of TEX,LaTEX,AMS-TEX,or AMS-LaTEX and according to EUDOXUS PRESS, LLC. LATEX STYLE FILE. (Click HERE to save a copy of the style file.)They should be carefully prepared in all respects. Submitted articles should be brightly typed (not dot-matrix), double spaced, in ten point type size and in 8(1/2)x11 inch area per page. Manuscripts should have generous margins on all sides and should not exceed 24 pages.

3. Submission is a representation that the manuscript has not been published previously in this or any other similar form and is not currently under consideration for publication elsewhere. A statement transferring from the authors(or their employers,if they hold the copyright) to Eudoxus Press, LLC, will be required before the manuscript can be accepted for publication.The Editor-in-Chief will supply the necessary forms for this transfer.Such a written transfer of copyright,which previously was assumed to be implicit in the act of submitting a manuscript,is necessary under the U.S.Copyright Law in order for the publisher to carry through the dissemination of research results and reviews as widely and effective as possible.

4. The paper starts with the title of the article, author's name(s) (no titles or degrees), author's affiliation(s) and e-mail addresses. The affiliation should comprise the department, institution (usually university or company), city, state (and/or nation) and mail code.

The following items, 5 and 6, should be on page no. 1 of the paper.

5. An abstract is to be provided, preferably no longer than 150 words.

6. A list of 5 key words is to be provided directly below the abstract. Key words should express the precise content of the manuscript, as they are used for indexing purposes.

The main body of the paper should begin on page no. 1, if possible.

7. All sections should be numbered with Arabic numerals (such as: 1. INTRODUCTION) .
Subsections should be identified with section and subsection numbers (such as 6.1. Second-Value Subheading).
If applicable, an independent single-number system (one for each category) should be used to label all theorems, lemmas, propositions, corollaries, definitions, remarks, examples, etc. The label (such as Lemma 7) should be typed with paragraph indentation, followed by a period and the lemma itself.

8. Mathematical notation must be typeset. Equations should be numbered consecutively with Arabic numerals in parentheses placed flush right, and should be thusly referred to in the text [such as Eqs.(2) and (5)]. The running title must be placed at the top of even numbered pages and the first author's name, et al., must be placed at the top of the odd numbed pages.

9. Illustrations (photographs, drawings, diagrams, and charts) are to be numbered in one consecutive series of Arabic numerals. The captions for illustrations should be typed double space. All illustrations, charts, tables, etc., must be embedded in the body of the manuscript in proper, final, print position. In particular, manuscript, source, and PDF file version must be at camera ready stage for publication or they cannot be considered.

Tables are to be numbered (with Roman numerals) and referred to by number in the text. Center the title above the table, and type explanatory footnotes (indicated by superscript lowercase letters) below the table.

10. List references alphabetically at the end of the paper and number them consecutively. Each must be cited in the text by the appropriate Arabic numeral in square brackets on the baseline.
References should include (in the following order):
initials of first and middle name, last name of author(s)
title of article,

name of publication, volume number, inclusive pages, and year of publication.

Authors should follow these examples:

Journal Article

1. H.H.Gonska,Degree of simultaneous approximation of bivariate functions by Gordon operators, (journal name in italics) *J. Approx. Theory*, 62,170-191(1990).

Book

2. G.G.Lorentz, (title of book in italics) *Bernstein Polynomials* (2nd ed.), Chelsea,New York,1986.

Contribution to a Book

3. M.K.Khan, Approximation properties of beta operators,in(title of book in italics) *Progress in Approximation Theory* (P.Nevai and A.Pinkus,eds.), Academic Press, New York,1991,pp.483-495.

11. All acknowledgements (including those for a grant and financial support) should occur in one paragraph that directly precedes the References section.

12. Footnotes should be avoided. When their use is absolutely necessary, footnotes should be numbered consecutively using Arabic numerals and should be typed at the bottom of the page to which they refer. Place a line above the footnote, so that it is set off from the text. Use the appropriate superscript numeral for citation in the text.

13. After each revision is made please again submit via email Latex and PDF files of the revised manuscript, including the final one.

14. Effective 1 Nov. 2009 for current journal page charges, contact the Editor in Chief. Upon acceptance of the paper an invoice will be sent to the contact author. The fee payment will be due one month from the invoice date. The article will proceed to publication only after the fee is paid. The charges are to be sent, by money order or certified check, in US dollars, payable to Eudoxus Press, LLC, to the address shown on the Eudoxus homepage.

No galleys will be sent and the contact author will receive one (1) electronic copy of the journal issue in which the article appears.

15. This journal will consider for publication only papers that contain proofs for their listed results.

Some Fixed Point Results of Caristi Type in $G-$Metric Spaces

Hamed M. Obiedat[1] and Ameer A. Jaber[2]
[1,2]Department of Mathematics
Hashemite University
P.O.Box150459
Zarqa13115-Jordan
email[1]: hobiedat@hu.edu.jo, email[2]: ameerj@hu.edu.jo,

September 4, 2017

Abstract

In this paper, we prove several fixed point results for mappings of Caristi type in the setting of $G-$metric spaces.

1 Introduction

The class of $G-$metric spaces introduced by Z. Mustafa and B. Sims (See [7]) was to provide a new class of generalized metric spaces and to extend the fixed point theory for a variety of mappings. Moreover, many theorems were proved in this new setting with most of them recognizable as counterparts of well-known metric space theorems (See [6], [8], [9]).

Caristi's fixed point theorem provides a generalization of Banach's contraction mapping principle (See [2]). Due to the importance of Caristi's fixed point theorem, it has been improved, generalized, extended and used in many application (See [1], [3], [4], [5]). In this paper, we prove several fixed point results for mappings of Caristi type in the setting of $G-$metric spaces.

[0] *2000 Mathematics Subject Classification.* 47H10, 54E50.
Key words and phrases. Caristi's Fixed Point Theorem; G-Metric Spaces; Lower semi-Continuous Functions.

Definition 1 *([7]) G-metric space is a pair (X, G), where X is a nonempty set, and G is a nonnegative real-valued function defined on $X \times X \times X$ such that for all $x, y, z, a \in X$, we have:*

(G1) $G(x, y, z) = 0$ if $x = y = z$;

(G2) $0 < G(x, x, y)$, for all $x, y \in X$, with $x \neq y$;

(G3) $G(x, x, y) \leq G(x, y, z)$, for all $x, y, z \in X$, with $z \neq y$;

(G4) $G(x, y, z) = G(p\{x, z, y\})$ (symmetry in all three variables);

(G5) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$, (rectangle inequality).

The function G is called a $G-$metric on X .

Definition 2 *([7])A sequence (x_n) in a $G-$metric space X is said to converge if there exists $x \in X$ such that $\lim\limits_{n,m\to\infty} G(x, x_n, x_m) = 0$, and one say that the sequence (x_n) is $G-$convergent to x.*

Proposition 3 *([7])Let X be $G-$metric space. Then the following statements are equivalent.*

1. *(x_n) is $G-$convergent to x.*
2. *$G(x_n, x_n, x) \to 0$, as $n \to \infty$.*
3. *$G(x_n, x, x) \to 0$, as $n \to \infty$.*
4. *$G(x_m, x_n, x) \to 0$, as $m, n \to \infty$.*

In a $G-$metric space X, a sequence (x_n) is said to be $G-$Cauchy if given $\varepsilon > 0$, there is $N_\varepsilon \in \mathbb{N}$ such that $G(x_n, x_m, x_l) < \varepsilon$, for all $n, m, l \geq N_\varepsilon$.

Proposition 4 *([7])In a $G-$metric space X, the following statements are equivalent.*

1. The sequence (x_n) is $G-$Cauchy.
2. For every $\varepsilon > 0$, there exists $N_\varepsilon \in \mathbb{N}$ such that $G(x_n, x_m, x_m) < \varepsilon$, for all $n, m \geq N$.

Definition 5 *([7])Let (X,G) and (X',G') be two $G-$metric spaces, and let $f:(X,G)\to(X',G')$ be a function, then f is said to be $G-$continuous at a point $a\in X$ if and only if, given $\varepsilon>0$, there exists $\delta>0$ such that $x,y\in X$; and $G(a,x,y)<\delta$ implies $G'(f(a),f(x),f(y))<\varepsilon$.*

A function f is $G-$continuous on X if it is $G-$continuous at all $a\in X$.

Proposition 6 *([7])Let (X,G) and (X',G') be two G-metric spaces. Then a function $f:(X,G)\to(X',G')$ is $G-$continuous at a point $x\in X$ if and only if it is $G-$sequentially continuous at x; that is, whenever (x_n) is $G-$convergent to x we have $(f(x_n))$ is $G-$convergent to $f(x)$.*

A $G-$metric space (X,G) is called symmetric $G-$metric space if $G(x,y,y)=G(y,x,x)$ for all $x,y\in X$, and called nonsymmetric if it is not symmetric.

Proposition 7 *([7])Let X be a $G-$metric space, then the function $G(x,y,z)$ is jointly continuous in all three of its variables. A $G-$metric space X is said to be complete if every $G-$Cauchy sequence in X is $G-$convergent in X.*

Definition 8 *With $\mathcal{M}$ we indicate the space of functions ρ, where*

1. $\rho:[0,\infty)\to[0,\infty)$ is strictly increasing, continuous and concave,
2. $\rho(0)=0$.

Lemma 9 *Let (X,G) be a complete $G-$metric space and let $\rho\in\mathcal{M}$. Then $(X,\rho\circ G)$ is a complete $G-$metric space.*

Proof. First let us prove that ρ is subadditive. To do so, let $x,y\in[0,\infty)$ and set $k=\dfrac{\rho(x)}{\rho(x)+\rho(y)}$. Then since ρ is increasing, we have

$$\rho(x+y)=\rho(\frac{k}{k}x+\frac{1-k}{1-k}y)\leq\max\{\rho(\frac{x}{k}),\rho(\frac{y}{1-k})\}.$$

Since ρ is concave and $\rho(0)=0$, we have

$$\rho(x)=\rho(\frac{k}{k}x)=\rho(\frac{k}{k}x+(1-k).0)\geq k\rho(\frac{1}{k}x)+(1-k)\rho(0)=k\rho(\frac{1}{k}x)$$

which implies that $\frac{1}{k}\rho(x) \geq \rho(\frac{1}{k}x)$. Similarly $\frac{1}{1-k}\rho(y) \geq \rho(\frac{1}{1-k}x)$. Therefore,

$$\rho(x+y) \leq \max\{\rho(\frac{x}{k}), \rho(\frac{y}{1-k})\} \leq \max\{\frac{1}{k}\rho(x), \frac{1}{1-k}\rho(y)\} \leq \rho(x) + \rho(y).$$

This completes the proof that ρ is subadditive. Now to prove that $\rho \circ G$ defines $G-$metric on X, we let $x, y, z, a \in X$. Then

G1) Since ρ is strictly increasing and $\rho(0) = 0$ then $\rho \circ G(x, y, z) = 0$ implies $G(x, y, z) = 0$ which means $x = y = z$;

G2) Since $0 < G(x, x, y)$; with $x \neq y$ and ρ is strictly increasing with $\rho(0) = 0$, then $0 < \rho \circ G(x, x, y)$; with $x \neq y$;

G3) Since $G(x, x, y) \leq G(x, y, z)$, with $z \neq y$ and ρ is strictly increasing then $\rho \circ G(x, x, y) \leq \rho \circ G(x, y, z)$, with $z \neq y$,

G4) Since $G(x, y, z) = G(p\{x, z, y\})$ and ρ is strictly increasing(injective) then $\rho \circ G(x, y, z) = \rho \circ G(p\{x, z, y\})$ (symmetry in all three variables); ■

G5) Since $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ and ρ is strictly increasing and subadditive then

$$\rho \circ G(x, y, z) \leq \rho(G(x, a, a) + G(a, y, z)) \leq \rho \circ G(x, a, a) + \rho \circ G(a, y, z),$$

which proves that $\rho \circ G$ defines $G-$metric on X. We still need to prove that $(X, \rho \circ G)$ is complete, so let $\{x_n\}$ be a Cauchy sequence in $(X, \rho \circ G)$. Then $\lim\limits_{n,m\to\infty} \rho \circ G(x_n, x_m, x_m) = 0$. Since ρ is continuous and strictly increasing with $\rho(0) = 0$, we have $\rho(\lim\limits_{n,m\to\infty} G(x_n, x_m, x_m)) = 0$. This implies $\lim\limits_{n,m\to\infty} G(x_n, x_m, x_m) = 0$, which means that $\{x_n\}$ is Cauchy sequence in the complete $G-$metric space (X, G). Therefore, there exists $x^* \in X$ such that $\{x_n\}$ G-converges to $x^* \in X$. Hence $\lim\limits_{n\to\infty} G(x_n, x^*, x^*) = 0$, which implies $\rho(\lim\limits_{n\to\infty} G(x_n, x^*, x^*)) = \rho(0) = 0$. By continuity of ρ, we have $\lim\limits_{n\to\infty} \rho(G(x_n, x^*, x^*)) = 0$, which implies $\{x_n\}$ is $\rho \circ G-$convergent in $(X, \rho \circ G)$. This completes the proof of Lemma 9.

Definition 10 *With $\mathcal{L}(X)$ we indicate the space of functions ϕ, where $\phi : X \to \mathbb{R}^+$ is lower semi-continuous.*

Remark 11 *Let (X, G) be a $G-$metric space and $\phi \in \mathcal{L}(X)$. Define $\preceq$ on X by*

$$x \preceq y \Longleftrightarrow G(x, y, y) \leq \phi(x) - \phi(y) \;\forall x, y \in X,$$

4

then $(X, G, \preceq)$ is partially ordered $G-$metric space. In fact, $\forall x, y, z \in X$ the following conditions are satisfied

i) since $0 = G(x, x, x) \leq \phi(x) - \phi(x) = 0$, we have that $x \preceq x$

ii) if $x \preceq y$ and $y \preceq x$,then $0 \leq G(x, y, y) \leq \phi(x) - \phi(y) = -(\phi(y) - \phi(x)) \leq -G(y, x, x) \leq 0$. This implies that $G(x, y, y) = G(y, x, x) = 0$. Hence, $x = y$.

iii) if $x \preceq y$ and $y \preceq z$,then

$$\begin{aligned} G(x, z, z) &\leq G(x, y, y) + G(y, z, z) \text{ by rectangle inequality} \\ &\leq \phi(x) - \phi(y) + \phi(y) - \phi(z) \\ &= \phi(x) - \phi(z), \end{aligned}$$

which implies $x \preceq z$.

2 Main Results

In this section, we introduce several fixed point results for mappings of Caristi type in the setting of $G-$metric spaces. We use the existence of a maximal element to prove Caristi's fixed point theorem in the setting of $G-$metric spaces.

Theorem 12 *Let $(X, G, \preceq)$ be a partially ordered $G-$metric space with $\preceq$ as defined in Remark 11. Then the following statements are equivalent:*

1 Any selfmapping T on X satisfies $G(x, Tx, Tx) \leq \phi(x) - \phi(Tx)$ has a fixed point.

2 X has a maximal element.

Proof. $1 \Longrightarrow 2$) Suppose that $T : X \to X$ has a fixed point, say x^*, and $x_1 \preceq x_2 \preceq ... \preceq$ be a chain in X. Fix x_j, then $G(x_j, x^*, x^*) = G(x_j, Tx^*, Tx^*) \leq \phi(x_j) - \phi(Tx^*) = \phi(x_j) - \phi(x^*)$ which implies that $x_j \preceq x^*$. Hence X has x^* as the maximal element.

$2 \Longrightarrow 1$) Suppose X has x^* as a maximal element, then $Tx^* \preceq x^*$. Since T satisfies $G(x^*, Tx^*, Tx^*) \leq \phi(x^*) - \phi(Tx^*)$ which implies that $x^* \preceq Tx^*$. Therefore, $Tx^* = x^*$. ∎

Theorem 13 *Let $(X, G, \preceq)$ be a partially ordered complete $G-$metric space, $\phi : X \to \mathbb{R}^+$ be a lower semi-continuous and $T : X \to X$ be selfmapping satisfying the inequality; $G(x, Tx, Tx) \leq \phi(x) - \phi(Tx)$. Then T has a fixed point.*

Proof. Let $\mathcal{C} = \{x_t : t \in \triangle\} \subseteq X$ be any chain in X and let $\{t_n\}$ be any increasing sequence of elements of $\triangle$. We prove first that $\phi(\mathcal{C})$ is a decreasing net. To do so, let c_t and c_s be any pair of elements in $\mathcal{C}$ with $x_t \preceq x_s$ for $t, s \in \triangle$. Then $G(x_t, x_s, x_s) \leq \phi(x_t) - \phi(x_s)$, which implies that $\phi(x_s) \leq \phi(x_t) - G(x_t, x_s, x_s)$. Therefore, $\{\phi(x_t)\}_{t\in\triangle}$ is a decreasing net of positive real numbers. Thus $\inf\{\phi(x_t) : t \in \triangle\}$ exists by completeness property of $\mathbb{R}$. Now choose $\{t_n\}_{n\in\mathbb{N}}$ to be an increasing sequence of $\triangle$ such that $\lim_{n\to\infty} \phi(x_{t_n}) = \inf\{\phi(x_t) : t \in \triangle\}$. Then $\{x_{t_n}\}$ is $G-$Cauchy since for $n, m \in \mathbb{N}$, we have

$$G(x_{t_n}, x_{t_m}, x_{t_m}) \leq \phi(x_{t_n}) - \phi(x_{t_m}). \tag{1}$$

Thus passing to the limit in the inequality (1) implies $G(x_{t_n}, x_{t_m}, x_{t_m}) = 0$ as $n, m \to \infty$. Since $(X, G, \preceq)$ is $G-$complete then there exists $x^* \in X$ such that $\{x_{t_n}\}$ converges to x^*. To prove that x^* is an upper bound of the set $\mathcal{C}$, let $m, n \in \mathbb{N}$ since $\{x_{t_n}\}$ converges to x^* and $\{x_{t_n}\}$ is increasing imply $x_{t_n} \preceq x^*$ $\forall n \geq 1$. Therefore,

$$\begin{aligned} G(x_{t_n}, x^*, x^*) &= \lim_{m\to\infty} G(x_{t_n}, x_{t_m}, x_{t_m}) \\ &\leq \phi(x_{t_n}) - \lim_{m\to\infty} \phi(x_{t_m}) \\ &\leq \phi(x_{t_n}) - \lim_{m\to\infty} \phi(x_{t_m}) \\ &\leq \phi(x_{t_n}) - \phi(x^*). \end{aligned}$$

Then $\phi(x^*) \leq \phi(x_{t_n})$ $\forall n \geq 1$ which implies that $\phi(x^*) \leq \inf\{\phi(x_t) : t \in \triangle\}$. Hence $x_t \preceq x^*$ $\forall t \in \triangle$ since ϕ is decreasing which means that x^* is an upper bound of the chain $\mathcal{C}$. Therefore Zorn's lemma implies that $(X, \preceq)$ has a maximal element. By Theorem 12 any selfmapping $T : X \to X$ satisfies the inequality $G(x, Tx, Tx) \leq \phi(x) - \phi(Tx)$ has a fixed point. ■

Corollary 14 *Let (X, G) be a partially ordered $G-$metric space. Suppose $f : X \to X$ is any function and $T : X \to X$ be $G-$continuous. If there exists a real number $r < 0$ such that for all $x \in X$*

$$G(f(x), Tf(x), Tf(x) \leq G(x, Tx, Tx) + rG(x, f(x), f(x)),$$

then f has a fixed point.

Proof. Define $\phi : X \to \mathbb{R}^+$ by $\phi(x) = -\dfrac{G(x,Tx,Tx)}{r}$. Then the lower semi continuity of ϕ follows from the $G-$continuity of T. Now

$$G(f(x),Tf(x),Tf(x) = -r\phi(f(x)) \le -r\phi(x) + rG(x,f(x),f(x)).$$

Then

$$\phi(f(x)) \le \phi(x) - G(x,f(x),f(x)),$$

which implies

$$G(x,f(x),f(x)) \le \phi(x) - \phi(f(x)).$$

Define $\preceq$ on X by

$$x \preceq y \Longleftrightarrow G(x,y,y) \le \phi(x) - \phi(y)\ \forall x,y \in X.$$

Then by Theorem 13, there exists $x^* \in X$ such that $f(x^*) = x^*$. ■

Corollary 15 *Let (X,G) be a complete $G-$metric space and let $\rho \in \mathcal{M}$. Then $(X,\rho\circ G)$ is a complete $G-$metric space. Then any selfmapping T on X satisfies $\rho\circ G(x,Tx,Tx) \le \phi(x) - \phi(Tx)$ has a fixed point.*

Corollary 16 *Let (X,G) be a complete $G-$metric space and let $\rho \in \mathcal{M}$. Suppose $f : X \to X$ is any function and $T : X \to X$ is $G-$continuous. If for all $x \in X$*

$$G(f(x),Tf(x),Tf(x) \le G(x,Tx,Tx) - \rho\circ G(x,f(x),f(x)).$$

Then f has a fixed point.

Proof. Define $\phi(x) = \rho^{-1}\circ G(x,Tx,Tx)$. Then lower semi continuity of ϕ follows from the $G-$continuity of T and continuity of ρ^{-1}. Now

$$G(f(x),Tf(x),Tf(x) = \rho(\phi(f(x))) \le \rho(\phi(x)) - \rho\circ G(x,f(x),f(x)).$$

Then

$$\rho(\phi(f(x))) \le \rho(\phi(x)) - \rho\circ G(x,f(x),f(x)).$$

By the subaddivity of ρ, the above inequality becomes

$$\begin{aligned}\rho(\phi(f(x)) + G(x,f(x),f(x))) &\le \rho(\phi(f(x))) + \rho\circ G(x,f(x),f(x)) \\ &\le \rho(\phi(x)).\end{aligned}$$

Now since ρ is increasing, we obtain

$$\phi(f(x)) + G(x, f(x), f(x)) \leq \phi(x).$$

Hence

$$G(x, f(x), f(x)) \leq \phi(x) - \phi(f(x)).$$

Define $\preceq$ on X by

$$x \preceq y \Longleftrightarrow G(x, y, y) \leq \phi(x) - \phi(y)\ \forall x, y \in X.$$

Then by Theorem 13 there exists $x^* \in X$ such that $f(x^*) = x^*$. ■

Corollary 17 *Let (X, G) be a complete $G-$metric space and Suppose $f : X \to X$ is any function and $T : X \to X$ is $G-$continuous. If there exist a real number $r < 0$ and $n \in \mathbb{N}$ such that for all $x, y \in X$*

$$G(f(x), Tf(x), T^n f(x) \leq G(x, Tx, T^n x) + rG(x, f(x), f(x))$$

then f has a fixed point.

Proof. Define $\phi : X \to \mathbb{R}^+$ by $\phi(x) = -\dfrac{G(x, Tx, T^n x)}{r}$. Then lower semi continuity of ϕ follows from the $G-$continuity of T. Then

$$\begin{aligned} -r\phi(f(x)) &= G(x, Tf(x), T^n f(x) \\ &\leq -r\phi(x) + rG(x, f(x), f(x)). \end{aligned}$$

Then we obtain

$$G(x, f(x), f(x)) \leq \phi(x) - \phi(f(x)).$$

Define $\preceq$ on X by

$$x \preceq y \Longleftrightarrow G(x, y, y) \leq \phi(x) - \phi(y)\ \forall x, y \in X.$$

Then by Theorem 13 there exists $x^* \in X$ such that $f(x^*) = x^*$.

Corollary 18 *Let (X, G) be a complete $G-$metric space and let $\rho \in \mathcal{M}$. Suppose $f : X \to X$ is any function and $T : X \to X$ is $G-$continuous. If there exist $\rho \in \mathcal{M}$ and $n \in \mathbb{N}$ such that for all $x \in X$*

$$G(f(x), Tf(x), T^n f(x) \leq G(x, Tx, T^n x) - \rho \circ G(x, f(x), f(x)),$$

then f has a fixed point.

■

The following theorem gives a natural generalization of Caristi type mapping in the setting of $G-$metric spaces.

Theorem 19 *Let (X, G) be a complete $G-$metric space. Suppose $T : X \to X$ is $G-$continuous. If there exists $\phi_y \in \mathcal{L}(X)$ for all $y \in X$ such that for all $x \in X$*

$$G(Tx, T^2x, Ty) \leq \phi_y(x) - \phi_y(Tx),$$

then T has a fixed point.

Proof. Fix $x_0 \in X$ and let $x_n = T^n x_0$ $n = 1, 2, 3, ...$ Then

$$\begin{aligned} G(x_n, x_{n+1}, Ty) &= G(Tx_{n-1}, Tx_n, Ty) = G(Tx_{n-1}, T^2x_{n-1}, Ty) \\ &\leq \phi_y(x_{n-1}) - \phi_y(Tx_{n-1}) \\ &= \phi_y(x_{n-1}) - \phi_y(x_n). \end{aligned}$$

Then for each $y \in X$,

$$\begin{aligned} \sum_{j=1}^{n} G(x_n, x_{n+1}, Ty) &= G(x_1, x_2, Ty) + G(x_2, x_3, Ty) + ... + G(x_n, x_{n+1}, Ty) \\ &\leq \phi_y(x_0) - \phi_y(x_1) + \phi_y(x_1) - \phi_y(x_2) + ... - \phi_y(x_{n-1}) + \phi_y(x_{n-1}) - \phi_y(x_n) \\ &\leq \phi_y(x_0) - \phi_y(x_n) \\ &\leq \phi_y(x_0) + C, \end{aligned}$$

where $C > 0$, which implies that

$$\sum_{j=1}^{\infty} G(x_n, x_{n+1}, Ty) \leq \phi_y(x_0) + C < \infty.$$

Then $\sum\limits_{j=1}^{\infty} G(x_n, x_{n+1}, Ty)$ is a convergent series. Hence $\lim\limits_{n\to\infty} G(x_n, x_{n+1}, Ty) = 0$. Therefore,

$$\begin{aligned} G(x_n, x_m, x_m) &\leq G(x_n, Ty, Ty) + G(Ty, x_m, x_m) \\ &\leq G(x_n, x_m, Ty) + G(x_n, x_m, Ty) \\ &\leq 2\sum_{j=n}^{m} G(x_j, x_{j+1}, Ty) \to 0 \text{ as } m, n \to \infty \end{aligned}$$

9

which implies that $\{x_n\}$ is Cauchy in complete $G-$metric space. That is, there exists $x^* \in X$ such that $\{x_n\}$ converges to x^*. Now

$$G(Tx^*, T^2x^*, x^*) = \lim_{n\to\infty} G(Tx_n, T^2x_n, x_n) = \lim_{n\to\infty} G(x_{n+1}, x_{n+2}, x_n) = 0,$$

which implies that x^* is a fixed point of T. ■

Corollary 20 *Let (X, G) be a complete $G-$metric space. Suppose $T : X \to X$ is $G-$continuous and for all $x \in X$*

$$G(Tf(x), T^2f(x), Ty) \leq G(x, Tx, Ty) - G(f(x), f^2(x), Ty).$$

Then f has a fixed point.

Proof. For each $y \in X$, choose $\phi_y(x) = G(x, Tx, Ty)$ then the result follows by applying Theorem 19 ■

References

[1] R. P. Agarwal, M. A. Khamsi, Extension of Caristi's fixed point theorem to vector valued metric space. *Nonlinear Anal.* 74, (2011), 141–145.

[2] J. Caristi, Fixed point theorems for mappings satisfying inwardness conditions, *Trans. Amer. Math. Soc.* 215 (1976), 241-251.

[3] W.A. Kirk, Caristi's fixed-point theorem and metric convexity, *Colloq.Math.*36 (1976), 81-86.

[4] D. Downing, W. A. Kirk, A generalization of Caristi's theorem with applications to nonlinear mapping theory. *Pacific J. Math.* 69, (1977), 339–345.

[5] M.A. Khamsi, Remarks on Caristi's fixed point theorem, *Nonlinear Anal.* TMA70 (2009), 4341-4349.

[6] Mustafa, Z, *A new structure for generalized metric spaces-with applications to fixed point theory. PhD thesis*, the University of Newcastle, Australia (2005).

[7] Z. Mustafa, B. Sims, A new approach to generalized metric spaces. *J. Nonlinear Convex Anal.* 7(2), (2006), 289–297.

[8] Z. Mustafa, H. Obiedat , A fixed point theorem of Reich in G-metric spaces. *CUBO*. 12(1), (2010), 83–93. Publisher Full Text OpenURL.

[9] Z. Mustafa, H. Obiedat, F. Awawdeh, Some fixed point theorem for mapping on complete G-metric spaces. *Fixed Point Theory Appl*. 2008, Article ID 189870 (2008).

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 27, NO.4, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC

Meir-Keeler contraction mappings in M_b-metric Spaces

N. Mlaiki[1], N. Souayah[2], K. Abodayeh[1], T. Abdeljawad[1]
Department of Math & General Sciences, Prince Sultan University[1]
Department of Natural Sciences, Community College, King Saud University[2]
Riyadh 11586, Saudi Arabia
E-mail: nmlaiki@psu.edu.sa
nsouayah@ksu.edu.sa
kamal@psu.edu.sa
tabdeljawad@psu.edu.sa

Abstract

In this paper, we generalize the notion of Meir-Keeler contraction condition in M_b-metric spaces. We prove some fixed point theorems for this class of contractions which enables us to extend and generalize the recent results of Gholmian and Khanehgir [2].

1 Introduction and preliminaries

First of all, we would like to mention that this work is inspired by the work of Gholmian and Khanehgir [2]. In 1922 Banach established one of the most important theorem in fixed point theory known as the "Banach contraction principle". Subsequently, many authors have extended this theorem in many different ways. For example, in 1969, Meir and Keeler [3] generalize the Banach's theorem using the weakly uniformly strict contraction and proved the following theorem:

Theorem 1. *Let (X, d) be a complete metric space and f a mapping of X into itself satisfying the following condition:*

$$\text{given } \epsilon > 0, \text{ there exists } \delta > 0 \text{ such that } \epsilon \leq d(x, y) < \epsilon + \delta \text{ implies } d(f(x), f(y)) < \epsilon.$$

Then f has a unique fixed point ξ. Moreover, For any $x \in X$, $\lim_{n\to\infty} f^n(x) = \xi$.

The Theorem 1 has been extended in many different metric spaces under several contractive definitions, see [2], [5].
On the other hand, several types of generalized metric spaces are proposed and a series of fixed point theorems for various classes of mapping are obtained, see [4], [6], [8], [9], [10], [11], [12].

M-metric spaces was introduced by Asadi see [1], which is an extension of partial metric spaces. So, first we remind the reader of the definition of an M-metric spaces along with some other notations.

1

Notation 1. *[1]*
 1. $m_{x,y} := min\{m(x,x), m(y,y)\}$
 2. $M_{x,y} := max\{m(x,x), m(y,y)\}$

Definition 1. *[1] Let X be a nonempty set, if the function $m : X^2 \to R^+$ satisfies the following conditions: for all $x, y, z \in X$*
 (1) $m(x,x) = m(y,y) = m(x,y)$ *if and only if* $x = y$,
 (2) $m_{x,y} \leq m(x,y)$,
 (3) $m(x,y) = m(y,x)$,
 (4) $(m(x,y) - m_{x,y}) \leq (m(x,z) - m_{x,z}) + (m(z,y) - m_{z,y})$.
Then the pair (X,m) is called an M-metric space.

Recently, Mlaiki et al. [7] developed the concept of M_b-metric spaces which extends the M-metric spaces and some fixed point theorems are established. Motivated by the properties of this original metric space, we introduce the notion of generalized Meir-Keeler contraction mappings in the M_b-metric spaces.

Now, let's recall some definitions and notations of M_b-metric spaces.

Notation 2. *[7]*
 1. $m_{bx,y} := min\{m_b(x,x), m_b(y,y)\}$
 2. $M_{bx,y} := max\{m_b(x,x), m_b(y,y)\}$

Definition 2. *[7] An M_b-metric space on a nonempty set X is a function $m_b : X^2 \to R^+$ that satisfies the following conditions, for all $x, y, z \in X$ we have*
 (1) $m_b(x,x) = m_b(y,y) = m_b(x,y)$ *if and only if* $x = y$,
 (2) $m_{bx,y} \leq m_b(x,y)$,
 (3) $m_b(x,y) = m_b(y,x)$,
 (4) *There exists a real number $s \geq 1$ such that for all $x, y, z \in X$ we have*

$$(m_b(x,y) - m_{bx,y}) \leq s[(m_b(x,z) - m_{bx,z}) + (m_b(z,y) - m_{bz,y})] - m_b(z,z).$$

The number s is called the coefficient of the M_b-metric space (X, m_b).

Now, we give an example of an M_b-metric which is not an M-metric space.

Example 1. *[7] Let $X = [0,\infty)$ and $p > 1$ be constant and $m_b : X^2 \to [0,\infty)$ defined by for all $x, y \in X$ we have*

$$m_b(x,y) = max\{x,y\}^p + |x-y|^P.$$

Note that (X, m_b) is an M_b-metric with coefficient $s = 2^p$. Now, we show that (X, m_b) is not an M-metric space. Take $x = 5$, $y = 1$ and $z = 4$, we get $m_b(x,y) - m_{bx,y} = 5^p + 4^p - 1$ and $(m_b(x,z) - m_{bx,z}) + (m_b(z,y) - m_{bz,y}) = 5^p + 1 - 4^p + 4^p + 3^p - 1 = 5^p + 3^p$. Therefore,

$$m_b(x,y) - m_{bx,y} > (m_b(x,z) - m_{bx,z}) + (m_b(z,y) - m_{bz,y}),$$

as required.

2

Definition 3. *[7] Let (X, m_b) be a M_b-metric space. Then:*
1) A sequence $\{x_n\}$ in X converges to a point x if and only if

$$\lim_{n\to\infty}(m_b(x_n, x) - m_{bx_n,x}) = 0.$$

2) A sequence $\{x_n\}$ in X is said to be m_b-Cauchy sequence if and only if

$$\lim_{n,m\to\infty}(m_b(x_n, x_m) - m_{bx_n,x_m}), \text{ and } \lim_{n\to\infty}(M_{bx_n,x_m} - m_{bx_n,x_m})$$

exist and finite.
3) An M_b-metric space is said to be complete if every m_b-Cauchy sequence $\{x_n\}$ converges to a point x such that

$$\lim_{n\to\infty}(m_b(x_n, x) - m_{bx_n,x}) = 0 \text{ and } \lim_{n\to\infty}(M_{bx_n,x} - m_{bx_n,x}) = 0.$$

Definition 4. *Each m_b-metric generates a topology τ_{m_b} on X whose base is the family of open m_b-balls $\{B_{m_b}(x, \epsilon) \mid x \in X, \epsilon > 0\}$ where $B_{m_b}(x, \epsilon) = \{y \in X \mid m_b(x, y) - m_{bx,y} < \epsilon\}$.*

Definition 5. *Let X be a nonempty set, $T : X \longrightarrow X$ be a mapping and $\alpha : X\times X \longrightarrow [0, \infty)$ be a function. Then, T is said to be α-admissible if for all $x, y \in X$ we have*

$$\alpha(x, y) \geq 1 \Longrightarrow \alpha(Tx, Ty) \geq 1. \tag{1}$$

Definition 6. *A mapping $T : X \to X$ is called triangular α-admissible if it is α-admissible and it satisfies the following condition:*

$$\alpha(x, y) \geq 1 \text{ and } \alpha(y, z) \geq 1, \text{ then } \alpha(x, z) \geq 1 \text{ where } x, y, z \in X.$$

Definition 7. *Let (X, m_b) be an m_b-metric space with coefficient s, an α-admissible mapping $T : X \longrightarrow X$ is said to be generalized Meir-Keeler contraction of type (I) if for every $\epsilon > 0$ there exists $\delta > 0$ such that*

$$\epsilon \leq \beta(m_b(x, y))M(x, y) < \epsilon + \delta \text{ implies } \alpha(x, y)m_b(Tx, Ty) < \epsilon \tag{2}$$

where

$$M(x, y) = \max\{m_b(x, y), m_b(Tx, x), m_b(Ty, y)\}, \text{ for all } x, y \in \mathbb{N} \tag{3}$$

and $\beta : [0, \infty) \longrightarrow (0, \frac{1}{s})$ is a given function.

Definition 8. *Let (X, m_b) be an m_b-metric space with coefficient s. A triangular α-admissible mapping $T : X \longrightarrow X$ is said to be generalized Meir-Keeler contraction of type (II) if for every $\epsilon > 0$ there exists $\delta > 0$ such that*

$$\epsilon \leq \beta(m_b(x, y))N(x, y) < \epsilon + \delta \text{ implies } \alpha(x, y)m_b(Tx, Ty) < \epsilon \tag{4}$$

where

$$N(x, y) = \max\{m_b(x, y), \frac{1}{2}[m_b(Tx, x) + m_b(Ty, y)]\}, \text{ forall } x, y \in \mathbb{N} \tag{5}$$

and $\beta : [0, \infty) \longrightarrow (0, \frac{1}{s})$ is a given function.

Remark 1. *1. Suppose that $T : X \longrightarrow X$ is a generalized Meir-Keeler contraction of type (I). Then*

$$\alpha(x, y)m_b(Tx, Ty) < \beta(m_b(x, y))M(x, y) \tag{6}$$

for all $x, y \in X$ when $M(x, y) > 0$.

2. Note that for all $x, y \in X$, we have $N(x, y) \leq M(x, y)$.

3

2 Main Results

Theorem 2. *Let (X, m_b) be a complete M_b metric space and $T : X \to X$ be a triangular α-admisible mapping. Suppose that the following conditions hold:*

(a) There exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$, $\alpha(Tx_0, x_0) \geq 1$.

(b) If $\{x_n\}$ is a sequence in X that converges to z as $n \to \infty$, and $\alpha(x_n, x_m) \geq 1$ for all $n, m \in \mathbb{N}$, then $\alpha(x_n, z) \geq 1$ for all $n \in \mathbb{N}$.

(c) If for each $\epsilon > 0$ there exists $\delta > 0$ such that

$$2s\epsilon \leq m_b(y, Ty)\frac{1 + m_b(x, Tx)}{1 + M(x, y)} + N(x, y) < s(2\epsilon + \delta),$$

then we have $\alpha(x, y)m_b(Tx, Ty) < \epsilon$.

Then, T has a fixed point in X.

Proof. Note that condition (c) implies that

$$\alpha(x, y)m_b(Tx, Ty) < \frac{1}{2s}m_b(y, Ty)\frac{1 + m_b(x, Tx)}{1 + M(x, y)} + \frac{1}{2s}N(x, y).$$

Let $x_0 \in X$ that satisfies condition (a) and define the sequence $\{x_n\}$ by $x_1 = Tx_0$ and $x_{n+1} = Tx_n$ for all $n \in \mathbb{N}$. If there exists an n such that $x_{n+1} = x_n$, then we are done. Without lost of generality, we may assume that $x_{n+1} \neq x_n$ for all $n \in \mathbb{N}$.
Since T is α-admisible, we have $\alpha(x_0, x_1) = \alpha(x_0, Tx_0) \geq 1$ and thus $\alpha(Tx_0, Tx_1) = \alpha(x_1, x_2) \geq 1$. By repeating the same argument, we get $\alpha(x_n, x_{n+1}) \geq 1$, for all $n \in \mathbb{N}$. Hence,

$$\begin{aligned}
m_b(x_{n+1}, x_{n+2}) &= m_b(Tx_n, Tx_{n+1}) \\
&\leq \alpha(x_n, x_{n+1})m_b(Tx_n, Tx_{n+1}) \\
&< \frac{1}{2s}m_b(x_{n+1}, x_{n+2})\frac{1 + m_b(x_n, x_{n+1})}{1 + M(x_n, x_{n+1})} + \frac{1}{2s}N(x_n, x_{n+1}).
\end{aligned}$$

Note that $M(x_n, x_{n+1}) = \max\{m_b(x_n, x_{n+1}), m_b(x_{n+1}, x_{n+2})\}$. So, if $M(x_n, x_{n+1}) = m_b(x_{n+1}, x_{n+2})$ then we have

$$\begin{aligned}
m_b(x_{n+1}, x_{n+2}) = m_b(Tx_n, Tx_{n+1}) &\leq \alpha(x_n, x_{n+1})m_b(Tx_n, Tx_{n+1}) \\
&< \frac{1}{2s}m_b(x_{n+1}, x_{n+2})\frac{1 + m_b(x_n, x_{n+1})}{1 + m_b(x_{n+1}, x_{n+2})} + \frac{1}{2s}m_b(x_{n+1}, x_{n+2}) \\
&< \frac{1}{2s}m_b(x_{n+1}, x_{n+2}) + \frac{1}{2s}m_b(x_{n+1}, x_{n+2}) \\
&= \frac{1}{s}m_b(x_{n+1}, x_{n+2}) \leq m_b(x_{n+1}, x_{n+2}),
\end{aligned}$$

which leads to a contradiction. Therefore, $M(x_n, x_{n+1}) = m_b(x_n, x_{n+1})$. Also note that $N(x_n, x_{n+1}) \leq M(x_n, x_{n+1})$ and hence

$$\begin{aligned} m_b(x_{n+1}, x_{n+2}) &< \frac{1}{2s} m_b(x_{n+1}, x_{n+2}) \frac{1 + m_b(x_n, x_{n+1})}{1 + m_b(x_n, x_{n+1})} + \frac{1}{2s} m_b(x_n, x_{n+1}) \\ &= \frac{1}{2s} m_b(x_{n+1}, x_{n+2}) + \frac{1}{2s} m_b(x_n, x_{n+1}) \\ &\leq \frac{1}{2s} m_b(x_n, x_{n+1}) + \frac{1}{2s} m_b(x_n, x_{n+1}) \\ &= \frac{1}{s} m_b(x_n, x_{n+1}) \leq m_b(x_n, x_{n+1}). \end{aligned}$$

Therefore, $m_b(x_{n+1}, x_{n+2}) < m_b(x_n, x_{n+1})$ and thus the sequence $\{m_b(x_n, x_{n+1})\}$ is a strictly decreasing positive sequence that converges to some number say $r \geq 0$. By Condition (c) in the hypothesis of the theorem, choose $\epsilon = \frac{r}{s}$. Note that, $\lim_{n\to\infty}[m_b(x_{n+1}, x_{n+2}) + m_b(x_n, x_{n+1})] = 2r$. Hence, there exists $N_0 \in \mathbb{N}$ such that

$$2r < m_b(x_{N_0+1}, x_{N_0+2}) + m_b(x_{N_0}, x_{N_0+1}) < 2r + \delta.$$

Therefore,

$$\begin{aligned} 2s\epsilon &< m_b(x_{N_0+1}, x_{N_0+2}) + m_b(x_{N_0}, x_{N_0+1}) \\ &= m_b(x_{N_0+1}, Tx_{N_0+1}) \left[\frac{1 + m_b(x_{N_0}, Tx_{N_0})}{1 + M(x_{N_0}, x_{N_0+1})}\right] + N(x_{N_0}, x_{N_0+1}) \\ &< 2s\epsilon + \delta < s(2\epsilon + \delta). \end{aligned}$$

Using the fact that $\alpha(x_{N_0}, x_{N_0+1}) \geq 1$ and condition (c), we deduce that

$$m_b(x_{N_0+1}, x_{N_0+2}) \leq \alpha(x_{N_0}, x_{N_0+1}) m_b(Tx_{N_0}, Tx_{N_0+1}) < \epsilon = \frac{r}{s} \leq r.$$

But we know that for all $n \in \mathbb{N}$, $r \leq m_b(x_n, x_{n+1})$ which leads to a contradiction. Thus $r = 0$; that is, $\lim_{n\to\infty} m_b(x_n, x_{n+1}) = 0$. Now let $\epsilon > 0$ and $\delta' = \min\{\delta, \epsilon, 1\}$. Since $\lim_{n\to\infty} m_b(x_n, x_{n+1}) = 0$, there exists $k \in \mathbb{N}$ such that $m_b(x_m, x_{m+1}) < \frac{\delta'}{4}$, for all $m \geq k$. Let $\eta = s(2\epsilon + \frac{\delta'}{2})$ and consider the set

$$B[x_k, \eta] = \{x_i | \, i \geq k, m_b(x_i, x_k) - m_{bx_i, x_k} < \eta\}.$$

We prove that T maps $B[x_k, \eta]$ to itself. Let $x_l \in B[x_k, \eta]$. Then we have $m_b(x_l, x_k) - m_{bx_l, x_k} < \eta$. If $l = k$, then we have $Tx_l = Tx_k = x_{k+1} \in B[x_k, \eta]$. So we may assume that $l > k$. Suppose that $2s\epsilon \leq m_b(x_l, x_k)$, so that

$$2s\epsilon \leq m_b(x_l, x_k) - m_{bx_l, x_k} < \eta.$$

Note $m_b(x_l, x_k) \leq N(x_l, x_k)$. Hence, $2s\epsilon \leq m_b(x_l, x_k)$ and this implies that $\epsilon \leq \frac{1}{2s} m_b(x_l, x_k)$. Thus,

$$\epsilon \leq \frac{1}{2s} m_b(x_k, x_{k+1}) \frac{1 + m_b(x_l, x_{l+1})}{1 + M(x_l, x_k)} + \frac{1}{2s} N(x_l, x_k).$$

5

Therefore,

$$\frac{1}{2s}m_b(x_k,x_{k+1})\frac{1+m_b(x_l,x_{l+1})}{1+M(x_l,x_k)}+\frac{1}{2s}N(x_l,x_k)<\epsilon+\frac{\delta'}{2},$$

and this implies that

$$2s\epsilon\leq m_b(x_k,Tx_k)\frac{1+m_b(x_l,Tx_l)}{1+M(x_l,x_k)}+N(x_l,x_k)<s(2\epsilon+\delta').$$

Thus, by part (c) of the theorem, we have $m_b(Tx_l,Tx_k)\leq\alpha(x_l,x_k)m_b(Tx_l,Tx_k)<\epsilon$. Therefore,

$$\begin{aligned}
m_b(Tx_l,x_k)-m_{bTx_l,x_k} &\leq m_b(Tx_l,x_k)\\
&\leq s\left[(m_b(Tx_l,Tx_k)-m_{bTx_l,Tx_k})+(m_b(Tx_k,x_k)-m_{bTx_k,x_k})\right]\\
&\leq s\left[m_b(Tx_l,Tx_k)+m_b(Tx_k,x_k)\right]\\
&< s[\epsilon+\frac{\delta'}{4}]\\
&< s[2\epsilon+\frac{\delta'}{2}]
\end{aligned}$$

which implies that $x_{l+1}\in B[x_k,\eta]$ as desired. Now assume that $m_b(x_l,x_k)<2s\epsilon$. Then we have

$$\begin{aligned}
m_b(Tx_l,x_k)-m_{bTx_l,x_k} &\leq m_b(Tx_l,x_k)\\
&\leq s\left[(m_b(Tx_l,Tx_k)-m_{bTx_l,Tx_k})+(m_b(Tx_l,x_k)-m_{bTx_l,x_k})\right]\\
&\leq s\left[m_b(Tx_l,Tx_k)+m_b(Tx_k,x_k)\right]\\
&\leq s\alpha(x_l,x_k)m_b(Tx_l,Tx_k)+sm_b(Tx_k,x_k)\\
&< s\left[\frac{1}{2s}m_b(x_k,x_{k+1})\frac{1+m_b(x_l,Tx_{l+1})}{1+M(x_l,x_k)}+\frac{1}{2s}N(x_l,x_k)\right]+sm_b(x_{k+1},x_k)\\
&\leq \frac{1}{2}m_b(x_k,x_{k+1})+\frac{m_b(x_k,x_{k+1})m_b(x_l,x_{l+1})}{2(1+m_b(x_l,x_k))}+\frac{1}{2}N(x_l,x_k)+sm_b(x_{k+1},x_k)\\
&\leq \frac{\delta'}{8}+\frac{m_b(x_k,x_{k+1})m_b(x_l,x_{l+1})}{2(1+m_b(x_l,x_k))}+\frac{1}{2}N(x_l,x_k)+s\frac{\delta'}{4}.
\end{aligned}$$

On the other hand, note that

$$\frac{m_b(x_k,x_{k+1})}{1+m_b(x_l,x_k)}\leq m_b(x_k,x_{k+1})<\frac{\delta'}{4}<1.$$

Hence,

$$\begin{aligned}
m_b(Tx_l,x_l)-m_{bTx_lx_l} &\leq m_b(Tx_l,x_k)\\
&= \frac{\delta'}{8}+\frac{1}{2}m_b(x_l,x_{l+1})+\frac{1}{2}N(Tx_l,x_k)+s\frac{\delta'}{4}\\
&< \left[\frac{\delta'}{8}+\frac{\delta'}{8}+s\epsilon\right]+s\frac{\delta'}{4}\\
&\leq s(\frac{\delta'}{2}+2\epsilon).
\end{aligned}$$

6

Therefore, for all $m > k$, we have

$$m_b(x_m, x_k) - m_{bx_m,x_k} < s\left(\frac{\delta'}{2} + 2\epsilon\right).$$

Now, for every $m, n \in \mathbb{N}$ such that $m > n > k$, we have

$$\begin{aligned}
m_b(x_m, x_n) - m_{bx_m,x_n} &\leq s\left[(m_b(x_m, x_k) - m_{bx_m,x_k}) + (m_b(x_k, x_n) - m_{bx_k,x_n})\right] \\
&\leq sm_b(x_m, x_k) + sm_b(x_k, x_n) \\
&< s.s(\frac{\delta'}{2} + 2\epsilon) + s.s(\frac{\delta'}{2} + 2\epsilon) \\
&= s^2(4\epsilon + \delta') \leq 5s^2\epsilon
\end{aligned}$$

which implies that $\lim_{n,m\to\infty} m_b(x_m, x_n) - m_{bx_m,x_n}$ exists and finite. Using the same argument it is not difficult to show that $\lim_{n,m\to\infty} M_b(x_m, x_n) - m_{bx_m,x_n}$ exists and finite. Therefore, the sequence $\{x_n\}$ is an m_b-Cauchy sequence and since X is complete, there exists $u \in X$ such that $\lim_{n\to\infty} M_{bx_n,u} - m_{bx_n,u} = 0$.
Finally, we show that is a fixed point for T; that is $Tu = u$.

$$\begin{aligned}
&\lim_{n\to\infty} (M_{bx_n,u} - m_{bx_{n,u}}) = 0 \\
&\lim_{n\to\infty} (M_{bx_{n+1},u} - m_{bx_{n+1,u}}) = 0 \\
&\lim_{n\to\infty} (M_{bTx_n,u} - m_{bTx_{n,u}}) = 0 \\
&M_b(Tu, u) - m_{b_{Tu,u}} = 0
\end{aligned}$$

Then, $M_{bTu,u} = m_{bTu,u}$, and similarly by the convergence of x_n we obtain that $m_b(Tu, u) = m_{bTu,u}$, which implies that $Tu = u$ as required. □

Definition 9. *Let (X, m_b) be an m_b-metric space and let T be a self mapping on X. T is called m_b-orbitally continuous if whenever*

$$\lim_{n\to+\infty} m_b(T_x^n, z) = m_b(z, z) \Rightarrow \lim_{n\to+\infty} m_b(TT_x^n, T_z) = m_b(T_z, T_z) \forall x, z \in X. \tag{7}$$

Remark 2. *Note that, continuous mappings are m_b-orbitally continuous. But the converse is not necessary true, for example, consider the m_b-metric space defined by $m_b(x, y) = [max(x, y)]^q$ $(q \geq 1)$ for all $x, y \in X$ where $X = [0, 1]$ and the map $T : X \longrightarrow X$ defined by*

$$T = \begin{cases} \frac{x}{2} & if \quad 0 \leq x < 1 \\ 0 & if \quad x = 1 \end{cases}$$

It is not difficult to see that T is not continuous, but T is m_b-orbitally continuous.

Theorem 3. *Let (X, m_b) be a complete m_b-metric space with coefficient s and $T : X \longrightarrow T$ be a mapping. Suppose that the following conditions hold:*

7

a) T is an m_b-orbitally continuous generalized Meir-Keeler contraction of type (I),

b) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$, $\alpha(Tx_0, x_0) \geq 1$,

c) if $\{x_n\}$ is a sequence in X such that $x_n \longrightarrow z$ as $n \longrightarrow \infty$ and $\alpha(x_n, x_m) \geq 1$ forall $n, m \in \mathbb{N}$, then $\alpha(z, z) \geq 1$,

d) $s > 1$ or β is a continuous function.
then, T has a fixed point in X.

Proof. Let $x_0 \in X$ be such that condition b) holds and define $\{x_n\}$ in X so that $x_1 = Tx_0$, $x_{n+1} = Tx_0$ $\forall n \in \mathbb{N}$. Without lose of generality, we may suppose that $x_{n+1} \neq x_n$ $\forall n \in \mathbb{N} \cup 0$. Since T is α-admissible, then $\alpha(x_n, x_{n+1}) \geq 1$ $\forall n \in \mathbb{N}$.
As T is a generalized Meir-Keeler contraction of type (I), then by replacing x by x_n and y by y_n in (4), we observe that for every $\epsilon > 0$ there exists $\delta > 0$ such that

$$\epsilon \leq \beta(m_b(x_n, x_{n+1}))M(x_n, x_{n+1}) < \epsilon + \delta \Longrightarrow \alpha(x_n, x_{n+1})m_b(Tx_n, Tx_{n+1}) < \epsilon \tag{8}$$

where

$$M(x_n, x_{n+1}) = max[m_b(x_n, x_{n+1}), m_b(x_{n+2}, x_{n+1})]. \tag{9}$$

Next, we distinguish two following cases:
`Case 1.` Assume that $M(x_n, x_{n+1}) = m_b(x_{n+2}, x_{n+1})$.
In this case, equation (8) becomes

$$\epsilon \leq \beta(m_b(x_n, x_{n+1}))m_b(x_{n+2}, x_{n+1}) < \epsilon + \delta \Longrightarrow \alpha(x_n, x_{n+1})m_b(Tx_n, Tx_{n+1}) < \epsilon$$

and using that $\alpha(x_n, x_{n+1}) \geq 1$ $\forall n \in \mathbb{N}$, we have

$$m_b(Tx_n, Tx_{n+1})m_b(x_{n+1}, x_{n+2}) < \epsilon \leq \beta(m_b(x_n, x_{n+1}))m_b(x_{n+2}, x_{n+1}).$$

Then $m_b(x_{n+1}, x_{n+2}) < m_b(x_{n+2}, x_{n+1})$ $\forall n \in \mathbb{N}$ which gives a contradiction.
`Case 2.` Assume that $M(x_n, x_{n+1}) = m_b(x_n, x_{n+1})$.
Since $M(x_n, x_{n+1}) > 0$ $\forall n \in \mathbb{N}$ due to Remark 1, we get

$$\begin{aligned} m_b(x_{n+1}, x_{n+2}) &\leq \alpha(x_n, x_{n+1})m_b(Tx_n, Tx_{n+1}) \\ &< \epsilon \\ &\leq \beta(m_b(x_n, x_{n+1}))m_b(x_n, x_{n+1}) \\ &< \frac{1}{s}m_b(x_n, x_{n+1}) \leq m_b(x_n, x_{n+1}). \end{aligned} \tag{10}$$

That is $\{m_b(x_n, x_{n+1})\}$ is a strictly decreasing positive sequence in $\mathbb{R}^+$ and it converges to some $r \geq 0$. Let prove that $r = 0$.
Let be untrue, the we have $r > 0$. We assert that $0 < r \leq m_b(x_n, x_{n+1})$ $\forall n \in \mathbb{N}$.
First, suppose that $s > 1$. Applying equation (10), we have $m_b(x_{n+1}, x_{n+2}) < \dfrac{1}{s}m_b(x_n, x_{n+1})$.
By taking the limit as n tends to infinity, we get $r \leq \dfrac{1}{s}r < r$ which is a contradiction and so $r = 0$.
Next, suppose that β is a continuous function. We prove in the following claim that $\{\beta(m_b(x_n, x_{n+1}))m_b(x_n, x_{n+1})\}$ is a strictly decreasing positive sequence in $\mathbb{R}^+$.

Claim 1. *Let* $\beta : [0,\infty[\longrightarrow [0,\frac{1}{s})$ *a continuous function. Then,* $\{\beta(m_b(x_n,x_{n+1}))m_b(x_n,x_{n+1})\}$ *is strictly decreasing positive sequence in* $\mathbb{R}^+$.

First, note that

$$\begin{aligned}\beta(m_b(x_{n+1},x_{n+2}))m_b(x_{n+1},x_{n+2}) &< m_b(x_{n+1},x_{n+2})\\ &\leq \alpha(x_n,x_{n+1})m_b(Tx_n,Tx_{n+1})\\ &< \beta(m_b(x_n,x_{n+1}))M(x_n,x_{n+1}).\end{aligned}$$

If $M(x_n,x_{n+1}) = m_b(x_n,x_{n+1})$, we obtain
$\beta(m_b(x_{n+1},x_{n+2}))m_b(x_{n+1},x_{n+2}) < \beta(m_b(x_n,x_{n+1}))m_b(x_n,x_{n+1})$.
If $M(x_n,x_{n+1}) = m_b(x_{n+2},x_{n+1})$, we have $m_b(x_{n+2},x_{n+1}) < m_b(x_n,x_{n+1})$ (as $m_b(x_n,x_{n+1})$ is a strictly decerasing). Then, $\beta(m_b(x_{n+1},x_{n+2}))m_b(x_{n+1},x_{n+2}) < \beta(m_b(x_n,x_{n+1}))m_b(x_n,x_{n+1})$.
Thus, $\{\beta(m_b(x_n,x_{n+1}))m_b(x_n,x_{n+1})\}$ is strictly decreasing positive sequence in $\mathbb{R}^+$ which prove our claim as desired.
From Claim 1, we have $\{\beta(m_b(x_n,x_{n+1}))m_b(x_n,x_{n+1})\}$ converges to some $r' \geq 0$. We consider the two follwing cases:
`Case 1.` $r' = 0$
Since $\lim_{n\to\infty} m_b(x_n,x_{n+1}) \neq 0$ so we have

$$\exists\epsilon > 0, \forall k \in \mathbb{N}, \exists n_k \geq k, m_b(x_{n_k},x_{n_{k+1}}) \geq \epsilon.$$

Now, let $\epsilon' > 0$ be given. Since $\lim_{n\to\infty} \beta(m_b(x_{n_k},x_{n_{k+1}}))m_b(x_{n_k},x_{n_{k+1}}) = 0$.
Therefore, using (4), we derive

$$\exists k' \in \mathbb{N}, \forall k \geq k', \epsilon\beta(m_b(x_{n_k},x_{n_{k+1}})) \leq \beta(m_b(x_{n_k},x_{n_{k+1}}))m_b(x_{n_k},x_{n_{k+1}}) < \epsilon'.$$

It enforces that $\lim_{n\to\infty} \beta(m_b(x_{n_k},x_{n_{k+1}})) = 0$.
By continuity of β, we obtain $\beta(r) = 0 \Longrightarrow r = 0$ which is a contradiction.

`Case 2.` $r' > 0$
we can distinguish two subcases: $r < r'$ and $r > r'$.
If $r < r'$, then $\beta(m_b(x_n,x_{n+1}))m_b(x_n,x_{n+1}) < \frac{1}{s}m_b(x_n,x_{n+1})$ and by taking the limit as n tends to infinity we get $r' \leq \frac{r}{s} \leq r$ which is a contradiction with $r' > 0$.

If $r > r'$, let $\delta > 0$ be such that satisfying (4) whenever $\epsilon = r'$. We know there exists $N_0 \in \mathbb{N}$ such that

$$r' \leq \beta(m_b(x_{N_0},x_{N_0+1}))m_b(x_{N_0},x_{N_0+1}) < r' + \delta.$$

Thus

$$\begin{aligned} r < m_b(x_{N_0+1},x_{N_0+2}) &\leq \alpha(x_{N_0},x_{N_0+1})m_b(Tx_{N_0},Tx_{N_0+1})\\ &< r' \leq r\end{aligned}$$

which leads to contradiction with $0 < r \leq m_b(x_n,x_{n+1})\ \forall n \in \mathbb{N}$.

9

Thus, $r = 0$ and so $\lim_{n\to\infty} m_b(x_n, x_{n+1})$.

Next, we intend to show that the sequence $\{x_n\}$ is an m_b- Cauch sequence. For this purpose, we will prove that for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $\lim_{n,m\to\infty} m_b(x_n, x_m) - m_{bx_{n,m}} < \infty$. We will prove that for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$m_b(x_l, x_{l+k}) - m_{bx_{l,l+k}} < \epsilon. \tag{11}$$

Since the sequence $\{m_b(x_n, x_{n+1})\} \longrightarrow 0, n \longrightarrow \infty$, for every $\delta > 0$ there exists $N \in \mathbb{N}$ such that $m_b(x_n, x_{n+1}) < \delta$ for all $n \geq N$. Choose $\delta < \epsilon$. We will prove equation (11) by using induction on k.

• for $k = 1$, we have $m_b(x_l, x_{l+1}) < \epsilon \Rightarrow m_b(x_l, x_{l+1}) - m_{bx_{l,l+1}} < \epsilon$ so, (11) it clearly holds for all $l \geq N$ (due to the choice of δ).

• Assume that the inequality (11) holds for some $k = m$, that is $m_b(x_l, x_{l+m}) - m_{bx_{l,l+m}} < \epsilon$ $\forall l \geq N$.

For $k = m + 1$, we have to show that

$$m_b(x_l, x_{l+m+1}) - m_{bx_{l,l+m+1}} < \epsilon \ \ \forall \ \ l \geq N \tag{12}$$

Employing condition (4) of the definition fo the M_b-metric space, we get

$$\begin{aligned} m_b(x_{l-1}, x_{l+m}) - m_{bx_{l,l+m}} &< s[m_b(x_{l-1}, x_l) - m_{bx_{l-1,l}} + m_b(x_l, x_{l+m}) - m_{bx_{l,l+m}} - m_b(x_l, x_l)] \\ &\leq s[m_b(x_{l-1}, x_l) + m_b(x_l, x_{l+m})] \\ &\leq s[\delta + \epsilon] \ \ \forall l \geq N. \end{aligned}$$

If $\beta(m_b(x_{l-1}, x_{l+m}))m_b(x_{l-1}, x_{l+m}) \geq \epsilon$, then we deduce

$$\begin{aligned} \epsilon &\leq \beta(m_b(x_{l-1}, x_{l+m}))m_b(x_{l-1}, x_{l+m}) \\ &\leq \beta(m_b(x_{l-1}, x_{l+m}))M(x_{l-1}, x_{l+m}) \\ &= \beta(m_b(x_{l-1}, x_{l+m}))max[m_b(x_{l-1}, x_{l+m}), m_b(x_l, x_{l-1}), m_b(x_{l+m+1}, x_{l+m})] \\ &< \beta(m_b(x_{l-1}, x_{l+m}))max[s(\delta + \epsilon), \delta, \delta] \\ &< \delta + \epsilon. \end{aligned}$$

Using (8) with $x = x_{l-1}$, $y = x_{l+m}$, we find

$$\epsilon \leq \beta(m_b(x_{l-1}, x_{l+m}))M(x_{l-1}, x_{l+m}) < \epsilon + \delta,$$

then

$$\alpha(x_{l-1}, x_{l+m})m_b(Tx_{l-1}, Tx_{l+m}) < \epsilon$$

which gives $m_b(x_l, x_{l+m+1}) < \epsilon$. Hence, (8) holds for $k = m + 1$.

If $\beta(m_b(x_{l-1}, x_{l+m}))m_b(x_{l-1}, x_{l+m}) < \epsilon$, then

$$\begin{aligned} \beta(m_b(x_{l-1}, x_{l+m}))M(x_{l-1}, x_{l+m}) &= \beta(m_b(x_{l-1}, x_{l+m}))max[m_b(x_{l-1}, x_{l+m}), \\ &\quad m_b(x_l, x_{l-1}), m_b(x_{l+m+1}, x_{l+m})] \\ &< \beta(m_b(x_{l-1}, x_{l+m}))max[m_b(x_{l-1}, x_{l+m}), \delta, \delta] \\ &< \epsilon. \end{aligned}$$

10

From Remark 1, we get

$$\alpha(x_{l-1}, x_{l+m})m_b(Tx_{l-1}, Tx_{l+m}) < \beta(m_b(x_{l-1}, x_{l+m}))M(x_{l-1}, x_{l+m}) < \epsilon$$

then

$$\alpha(x_{l-1}, x_{l+m})m_b(x_l, x_{l+m+1}) < \epsilon.$$

So

$$m_b(x_l, x_{l+m+1}) < \alpha(x_{l-1}, x_{l+m})m_b(x_l, x_{l+m+1}) < \epsilon$$

that is (11) holds for $k = m + 1$.
Note that $M(x_{l-1}, x_{l+m}) > 0$, otherwise $m_b(x_l, x_{l-1}) = 0$ and hence $x_l = x_{l-1}$, which is contradiction. Thus, $m_b(x_l, x_{l+k}) < \epsilon \; \forall l \geq N$ and $k \geq 1$, it means

$$m_b(x_n, x_m) < \epsilon \;\; \forall \;\; m \geq n \geq N. \tag{13}$$

Hence, it is easy to deduce that $\{x_n\}$ is an m_b-Cauchy sequence. Since X is a complete m_b-metric space, there exists $u \in X$ such that $\lim_{n\to\infty}(M_{bx_n,u} - m_{bx_{n,u}}) = 0$.
Now, we will show that $Tu = u$ for any $n \in \mathbb{N}$. We have

$$\begin{aligned}
&\lim_{n\to\infty}(M_{bx_n,u} - m_{bx_{n,u}}) = 0\\
&\lim_{n\to\infty}(M_{bx_{n+1},u} - m_{bx_{n+1,u}}) = 0\\
&\lim_{n\to\infty}(M_{bTx_n,u} - m_{bTx_{n,u}}) = 0\\
&M_b(Tu, u) - m_{b_{Tu,u}} = 0
\end{aligned}$$

Then, $M_{bTu,u} = m_{bTu,u}$, and similarly by the convergence we obtain that $m_b(Tu, u) = m_{bTu,u}$, which implies that $Tu = u$ as desired. □

Next, we prove the same result for a self mapping T on X which is an m_b-orbitally continuous generalized Meir-Keeler contraction of type (II).

Theorem 4. *Let (X, m_b) be a complete m_b-metric space, $T : X \longrightarrow X$ be a mapping. Assume that the following conditions are satisfied:*

a) T is an m_b-orbitally continuous generalized Meir-Keeler contraction of type (II),

b) there exists $x_0 \in X$ such that $\alpha(x_0, T_{x_0}) \geq 1$, $\alpha(T_{x_0}, x_0)$,

c) If $\{x_n\}$ is a sequence in X such that $x_n \longrightarrow z$ as $n \longrightarrow \infty$ and $\alpha(x_n, x_m) \geq 1$ for all $n, m \in \mathbb{N}$, then $\alpha(z, z) \geq 1$,

d) $s > 1$ or β is a continuous function, then T has a unique fixed point in X.

Proof. By remark 1 , we have $N(x, y) \leq M(x, y)$. Hence, similarly to the proof of theorem 3, the result of our theorem will follow as desired. □

Theorem 5. *Let (X, m_b) be a complete m_b-metric space with coefficient s and satisfies the following conditions:*

11

a) if $\{x_n\}$ is a sequence in X which converges to z with respect to τ_{m_b} and satisfies $\alpha(x_{n+1}, x_n) \geq 1$ and $\alpha(x_n, x_{n+1}) \geq 1$ for all n, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(x_z, x_{n_k}) \geq 1$ and $\alpha(x_{n_k}, x_z) \geq 1$ for all k,

b) $T : X \longrightarrow X$ is a generalized Meir-Keeler contraction of type (II),

c) there exists $x_0 \in X$ such that $\alpha(x_0, T_{x_0}) \geq 1$, $\alpha(T_{x_0}, x_0) \geq 1$

d) $s > 1$ or β is a continuous function then, T has a fixed point in X.

Proof. By the proof of theorem 2, one can easly deduce that $\{x_n\}$ defined by $x_1 = T_{x_0}$ and $x_{n+1} = T_{x_n}$ $(n \in \mathbb{N})$ converges to some $z \in X$ with $m_b(z, z) = 0$, by condition a), there exist a subsequence $\{x_{n_k}\}$ of x_n such that $\alpha(z, x_{n_k}) \geq 1$ and $\alpha(x_{n_k}, z) \geq 1$ for all k.
Note that, if $N(z, x_{n_k}) = 0$, then $T_z = z$ and we are done.
Now, by remark 1 for all $k \in \mathbb{N}$ we have

$$\begin{aligned} m_b(T_z, x_{n+1}) = m_b(T_z, Tx_n) &\leq \alpha(z, x_{n_k}) m_b(T_z, T_{x_{n_k}}) \\ &< \beta(m_b(z, x_{n_k})) N(z, x_{n_k}). \end{aligned}$$

Taking the limit $k \longrightarrow \infty$ we obtain $\lim_{k\to\infty} N(z, x_{n_k}) = \max\{0, \frac{1}{2} m_b(T_z, z)\} = \frac{1}{2} m_b(T_z, z)$.
Thus, $\lim_{n\to\infty} m_b(T_z, x_{n_{k+1}}) \leq \frac{1}{2s} m_b(T_z, z)$. Hence,

$$m_b(T_z, z) \leq s m_b(T_z, x_{n_{k+1}}) + s m_b(x_{n_{k+1}}, z).$$

Taking the limit $k \longrightarrow \infty$ we obtain

$$m_b(T_z, z) \leq \frac{1}{2} m_b(T_z, z).$$

which implies $m_b(T_z, z) = 0$, similarly we can show that $M_{bT_z,z} = 0$ and therefore, $T_z = z$ as desired. □

3 Acknowledgement

The authors would like to thank Prince Sultan University for funding this work through research group Nonlinear Analysis Methods in Applied Mathematics (NAMAM) group number RG-DES-2017-01-17.

References

[1] Asadi et al: "New extention of p-metric spaces with some fixed point results on M-metric spaces." *Journal of Inequalities and Applications.* 2014, 2014:18

[2] N. Gholamian, M. Khanehgir, *Fixed points of generalized Meir-Keeler contraction mappings in b-metric-like spaces*, Fixed Point Theory and Applications, (2016) 2016:34 DOI: 10.1186/s13663-016-0507-6.

[3] A. Meir, E. Keeler, *A Theorem on Contraction Mappings*, Journal of Mathematical Analysis and Applications, **28** (1969), 326-369.

[4] W. Shatanawi, MB. Hani, *A Coupled Fixed Point Theorem in b-metric spaces*, International Journal of Pure and Applied Mathematics, **4** 109, 889-897.

[5] T. Abdeljawad, *Meir-Keeler α-contractive fixed and common fixed point theorems*, Fixed Point Theorem and Apllications, **2013**, 2013:19, DOI: 10.1186/1687-1812-2013-19.

[6] T. Abeljawad, K. Abodayeh, N. Mlaiki,*On Fixed Point Generalizations to Partial b-metric Spaces*, Journal of Computational Analysis & Applications, **19** (2015), 883-891.

[7] N. Mlaiki, A. Zarrad, N. Souayah, A. Mukheimer, T. Abdeljawed, Fixed Point Theorems in M_b-metric spaces, *Journal of Mathematical Analysis*, **7** (2016),1-9.

[8] N. Souayah and N. Mlaiki, *A coincident point principle for two weakly compatible mappings in partial S-metric spaces*, Journal of Nonlinear science and applications, **9** (2016), 2217-2223.

[9] N. Mlaiki, A. Zarrad, N. Souayah, A. Mukheimer, T. Abdeljawed, *Fixed point theorems in M_b-metric spaces*, Journal of Mathematical Analysis, **7** (5)(2016), 1-9.

[10] N. Souayah, *A fixed point in partial S_b-metric spaces*, An. St. Univ. Ovidius Constanta, **24**(3) (2016), 351-362.

[11] N. Mlaiki, M. Souayah, K. Abodayeh, T. Abdeljawad, *Contraction principles in M_s-metric spaces*, Journal of Nonlinear Sciences and Applications, **10** (2017), 575-582

[12] N. Souayah and N. Mlaiki, *A fixed point theorem in S_b-metric spaces*, J. Math. Computer Sci. **16** (2016), 131-139.

Generalized Ulam-Hyers Stability for Generalized types of $(\gamma-\psi)-$Meir-Keeler Mappings via Fixed Point Theory in $S-$metric spaces

Mi Zhou[1], Xiao-lan Liu[2,3]*, Arslan Hojat Ansari [4], Yeol Je Cho [5], Stojan Radenović[6]

Abstract: In this paper, we introduce several extensions of Meir-Keeler contractive mappings in the structure of $S-$metric spaces. Then we investigate some existence, uniqueness, and generalized Ulam-Hyers stability results for the classes of MKC mappings via fixed point theory. Besides the theoretical results, we also present some illustrative examples to verify the effectiveness and applicability of our main results.

MSC: 47H10;54H25

Keywords: Generalized Ulam-Hyers stability; $(\gamma-\psi)-$Meir-Keeler contraction mappings; $S-$metric space; fixed point theory.

1. Introduction

1.1. $S-$metric spaces

Very recently, Sedghi et al.[1] have introduced the notion of an $S-$metric space and proved that this notion is a generalization of a $G-$metric space and D^*-metric space. Also, they have proved some properties of an $S-$metric and some fixed point results for a self-map on $S-$metric spaces. After that, many interesting results were obtained by transporting certain results in metric spaces and known generalizes metric spaces to $S-$metric spaces, see ([2]-[10]).

First, we recall the definition of an $S-$metric space and some useful notions and lemmas for the following discussion.

In the sequel, the letters $\mathbb{N}, \mathbb{R}^+$ and $\mathbb{R}$ will denote the sets of positive integers, nonnegative real numbers and real numbers, respectively.

Definition 1.1. **[1]** Let X be a nonempty set. An $S-$metric on X is a function $S : X^3 \mapsto [0,\infty)$ that satisfies the following conditions for $\forall x, y, z, a \in X$:

$(S1)$ $S(x,y,z) = 0$ if and only if $x = y = z$;

$(S2)$ $S(x,y,z) \leq S(x,x,a) + S(y,y,a) + S(z,z,a)$.

*Correspondence: stellalwp@163.com

[2] College of Science, Sichuan University of Science and Engineering, Zigong, Sichuan 643000, China

[3] Sichuan Province University Key Laboratory of Bridge Non-destruction Detecting and Engineering Computing, Zigong, Sichuan 643000, China

Full list of author information is available at the end of the article

The pair (X, S) is called an $S-$metric space.

Immediate examples of such $S-$metric spaces are:

(1) Let $X = \mathbb{R}^+$ and $\|\cdot\|$ be a norm on X, then $S(x, y, z) = \|2x + y - 3z\| + \|x - z\|$ is an $S-$metric on X, for $\forall x, y, z \in X$.

(2) Let X be a nonempty set, d is ordinary metric on X, the $S_d(x, y, z) = d(x, z) + d(y, z)$ is an $S-$metric on X, for $\forall x, y, z \in X$.

Lemma 1.1. [1] Let (X, S) be an $S-$metric space. Then

$$S(x, x, z) \leq 2S(x, x, y) + S(y, y, z), and \;\; S(x, x, z) \leq 2S(x, x, y) + S(z, z, y),$$

for $\forall x, y, z \in X$.

Lemma 1.2. [1] Let (X, S) be an $S-$metric space. Then $S(x, x, y) = S(y, y, x)$, for $\forall x, y \in X$.

Lemma 1.3. Let (X, S) be an $S-$metric space. Then, for $\forall x, y, z \in X$, it follows that

(1) $S(x, y, y) \leq S(x, x, y)$.

(2) $S(x, y, x) \leq S(x, x, y)$.

(3) $S(x, y, z) \leq S(x, x, z) + S(y, y, z)$.

(4) $S(x, y, z) \leq S(y, y, z) + S(x, x, z)$.

(5) $S(x, y, z) \leq S(y, y, x) + S(z, z, x)$.

(6) $S(x, x, z) \leq \frac{3}{2}[S(y, y, z) + S(y, y, x)]$.

(7) $S(x, y, z) \leq \frac{2}{3}[S(x, x, y) + S(y, y, z) + S(z, z, x)]$.

Proof. It follows from $(S2)$ and Lemma 1.2, one can easily obtain $(1) - (5)$. Now, we prove (6) and (7) also hold true.
By Lemma 1.1 and Lemma 1.2, we have

$$\begin{aligned} 2S(x, x, z) &= S(x, x, z) + S(z, z, x) \\ &\leq [2S(x, x, y) + S(y, y, z)] + [2S(z, z, y) + S(x, x, y)] \\ &= 3[S(y, y, z) + S(y, y, x)]. \end{aligned}$$

Hence, $S(x, x, z) \leq \frac{3}{2}[S(y, y, z) + S(y, y, x)]$. Then (6) holds true.
By virtue of $(3) - (5)$ and Lemma 1.2, we have $3S(x, y, z) = 2[S(x, x, y) + S(y, y, z) + S(z, z, x)]$, which implies (7) holds true. ☐

Definition 1.2. [1] Let (X, S) be an $S-$metric space.

(1) A sequence $\{x_n\} \subset X$ is said to convergent to $x \in X$ if $S(x_n, x_n, x) \to 0$ as $n \to \infty$. That is, for each $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for $\forall n \geq n_0$, we have $S(x_n, x_n, x) < \epsilon$.

(2) A sequence $\{x_n\} \subset X$ is said to be a Cauchy sequence if $S(x_n, x_n, x_m) \to 0$ as $n, m \to \infty$. That is, for each $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for $\forall n, m \geq n_0$, we have $S(x_n, x_n, x_m) < \epsilon$, or for each $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for $\forall l, m, n \geq n_0$, we have $S(x_l, x_m, x_n) < \epsilon$.

(3) The $S-$metric space (X, S) is said to be complete if every Cauchy sequence is a convergent sequence.

(4) A mapping $T : X \mapsto X$ is said to be $S-$continuous if $\{Tx_n\}$ is $S-$convergent to Tx, where $\{x_n\}$ is an $S-$convergent sequence converging to x.

Lemma 1.4. **[1]** Let (X, S) be an $S-$metric space. If there exist sequences $\{x_n\}$ and $\{y_n\}$ such that $x_n \to x$ and $y_n \to y$ as $n \to \infty$, then $S(x_n, x_n, y_n) \to S(x, x, y)$.

Lemma 1.5. **[1]** Let (X, S) be an $S-$metric space. If the sequences $\{x_n\}$ in X such that $x_n \to x$, then x is unique.

1.2. The generalized Ulam-Hyers Stability

The stability problem of functional equations, originated from a question of Ulam [11], in 1940, concerns the stability of group homomorphism which stated as follows:
Let G_1 be a group and G_2 be a metric group with a metric $d(\cdot, \cdot)$. Given $\epsilon > 0$, does there exist $\delta > 0$ such that if a function $h : G_1 \mapsto G_2$ satisfies the inequality

$$d(h(xy), h(x)h(y)) < \delta,$$

$\forall x, y \in G_1$, then there is a homomorphism $H : G_1 \mapsto G_2$ with $d(h(x), H(x)) < \epsilon$, $\forall x \in G_1$?
If the answer is affirmative, then we say that the equation of homomorphism $H(xy) = H(x)H(y)$ is stable. The first affirmative partial answer to the equation of Ulam for Banach spaces was given by Hyers [12] in 1941. Thereafter, this type of stability is called the Ulam-Hyers stability and has attracted attentions of many mathematicians.
In particular, Ulam-Hyers stability results in fixed point theory and remarkable results on the stability of certain classes of functional equation via fixed point approach have been studied densely, see ([13]-[16]).

Definition 1.3. Let (X, S) be an $S-$metric space and $T : X \mapsto X$ be a mapping. By definition, the fixed point equation

$$x = Tx, \quad x \in X \tag{1}$$

is said to be generalized Ulam-Hyers stable in the framework of an $S-$metric space if there exists an increasing operator $\varphi : [0, \infty) \mapsto [0, \infty)$, continuous at 0 and $\varphi(0) = 0$, such that for each $\epsilon > 0$ and an $\epsilon-$solution $w^* \in X$, that is

$$S(w^*, Tw^*, Tw^*) \leq \epsilon, \tag{2}$$

there is a solution $x^* \in X$ of the fixed point equation (1) such that

$$S(w^*, x^*, x^*) \leq \varphi(\epsilon). \tag{3}$$

If $\varphi(t) = ct, \forall t \geq 0$, where $c > 0$, then (1) is said to be Ulam-Hyers stable in the framework of an $S-$metric space.

1.3. The generalized $(\gamma-\psi)-$Meir-Keeler contractive mappings

In 1969, Meir and Keeler [17] established a fixed point theorem in a metric space (X,d) for mappings satisfying the condition that for each $\epsilon>0$ there exists $\delta(\epsilon)>0$ such that

$$\epsilon \leq d(x,y) < \epsilon+\delta \quad implies \quad d(Tx,Ty)<\epsilon, \tag{4}$$

$\forall x,y\in X$. This condition is called the Meri-Keeler contractive (MKC, for short) type condition. Since then, many authors extended and improved this condition and established fixed point results for new generalized conditions, see Maiti and Pal [18], Park and Rhoades [19], Mongkolkeha and Kuman [20] and so on. On the other hand, Samet et al.[21] introduced the concepts of $\alpha-\psi-$contractive mapping and $\alpha-$admissible mapping in metric spaces. Also they proved a fixed point theorem for $\alpha-\psi$ contractive mappings in complete metric spaces using the concept of $\alpha-$addmissible mappings. Motivated by Samet's work, Latif et al.[22] introduced a new type of a generalized $(\alpha-\psi)-$Meir-Keeler contractive mapping and established some interesting theorems on the existence of fixed points for such mappings via admissible mappings.
Admissible mappings in the setting of $S-$metric spaces can be defined as follows.

Definition 1.4. A mapping $T:X\mapsto X$ is called $\gamma-$admissible if for $\forall x,y,z\in X$, we have

$$\gamma(x,y,z)\geq 1 \Rightarrow \gamma(Tx,Ty,Tz)\geq 1,$$

where, $\gamma: X^3\mapsto[0,\infty)$ is a given function. If in addition, $\begin{cases}\gamma(x,y,y)\geq 1\\ \gamma(y,z,z)\geq 1\end{cases}$ implies $\gamma(x,z,z)\geq 1, \quad \forall x,y,z\in X$. Then T is called triangular $\gamma-$admissible.

Example 1.1. Let $X=[1,\infty)$ and $T:X\mapsto X$. Define $Tx=x^2$ and $\gamma(x,y,z)=\begin{cases}2, & if \quad x\geq y\geq z;\\ 0, & otherwise.\end{cases}$
Then T is $\gamma-$admissible.

Definition 1.5. We say that:

(1) A sequence $\{x_n\}$ in X is $(T,\gamma)-$orbital if $x_n=T^n x_0$ and $\gamma(x_n,x_{n+1},x_{n+1})\geq 1$, $\forall n\in\{0\}\cup\mathbb{N}$.

(2) T is $\gamma-$orbital continuous if, for every $(T,\gamma)-$orbital sequence $\{x_n\}$ in X such that $x_n\to x$ as $n\to\infty$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $Tx_{n_k}\to Tx$ as $k\to\infty$.

(3) X is $(T,\gamma)-$regular if, for every $(T,\gamma)-$orbital sequence $\{x_n\}$ in X such that $x_n\to x$ as $n\to\infty$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\gamma(x_{n_k},x,x)\geq 1$, $\forall k\in\mathbb{N}$.

(4) X is $\gamma-$regular if, for every sequence $\{x_n\}$ in X such that $x_n\to x$ as $n\to\infty$ and $\gamma(x_n,x_{n+1},x_{n+1})\geq 1$, $\forall n\in\{0\}\cup\mathbb{N}$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\gamma(x_{n_k},x,x)\geq 1$, $\forall k\in\mathbb{N}$.

(5) X is $(T,\gamma)-$limit if, for every sequence $\{x_n\}$ in X such that $x_n\to x$ as $n\to\infty$ and $\gamma(x_n,x_{n+1},x_{n+1})\geq 1$, $\forall n\in\{0\}\cup\mathbb{N}$, then $\gamma(x,Tx,Tx)\geq 1$.

Remark 1.1. **(1)** If T is continuous, then T is $\gamma-$orbital continuous (for any γ).

(2) If X is $\gamma-$regular, then X is also $(T,\gamma)-$regular (for any γ).

Lemma 1.6. Let $\gamma : X^3 \mapsto [0,\infty)$ and $T : X \mapsto X$ be $\gamma-$admissible with triangular admissibility. Assume that there exists $x_0 \in X$ such that $\gamma(x_0, Tx_0, Tx_0) \geq 1$. Define a sequence $\{x_n\}$ by $x_n = T^n x_0$. Then $\gamma(x_m, x_n, x_n) \geq 1$, for $\forall m, n \in \mathbb{N}$ with $m < n$.

Proof. Since there exists $x_0 \in X$ such that $\gamma(x_0, Tx_0, Tx_0) \geq 1$, then from the definition of $\gamma-$admissibility, we deduce that $\gamma(x_1, x_2, x_2) = \gamma(Tx_0, Tx_1, Tx_1) \geq 1$.
By continuing this process, we get $\gamma(x_n, x_{n+1}, x_{n+1}) \geq 1$, $\forall n \in 0 \cup \mathbb{N}$.
Suppose that $m < n$. Since $\begin{cases} \gamma(x_m, x_{m+1}, x_{m+1}) \geq 1 \\ \gamma(x_{m+1}, x_{m+2}, x_{m+2}) \geq 1, \end{cases}$
by the definition of triangular $\gamma-$ admissibility, we deduce that $\gamma(x_m, x_{m+2}, x_{m+2}) \geq 1$. By continuing this process, we get $\gamma(x_m, x_n, x_n) \geq 1$, $\forall m, n \in \mathbb{N}$ with $m < n$. □

Let Ψ stand for the family of nondecreasing functions $\psi : [0,\infty) \mapsto [0,\infty)$ satisfying conditions:

(Ψ1) $\Sigma_{n=1}^{\infty} \psi^n(t) < \infty$, $\forall t > 0$, where ψ^n is the n^{th} iterate of ψ;

(Ψ2) $\psi(0) = 0$.

Remark 1.2. For every function $\psi : [0,\infty) \mapsto [0,\infty)$ the following holds:
if ψ is nondecreasing, then for each $t > 0$,

$$\lim_{n\to\infty} \psi^n(t) = 0 \Rightarrow \psi(t) < t \Rightarrow \psi(0) = 0.$$

Therefore, if $\psi \in \Psi$, then for every $t > 0$, $\psi(t) < t$ and ψ is continuous at 0.

Definition 1.6. Let (X, S) be an $S-$metric space and $T : X \mapsto X$. The mapping T is called a $(\gamma - \psi)-$Meir-Keeler contractive mapping if there exist two functions $\psi \in \Psi$ and $\gamma : X^3 \mapsto [0,\infty)$ satisfying the following condition: for each $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ such that

$$\epsilon \leq \psi(S(x,y,y)) < \epsilon + \delta(\epsilon) \;\; implies \;\; \gamma(x,y,y)S(Tx,Ty,Ty) < \epsilon, \forall x, y \in X.$$

Remark 1.3. It is easily shown that if $T : X \mapsto X$ is a $(\gamma - \psi)-$Meir-Keeler contractive mapping, then

$$\gamma(x,y,y)S(Tx,Ty,Ty) < \psi(S(x,y,y)),$$

$\forall x, y \in X$, when $x \neq y$.

Definition 1.7. Let (X, S) be an $S-$metric space and $T : X \mapsto X$. The mapping T is called a $(\gamma - \psi)-$Meir-Keeler contractive mapping of dim3 if there exist two functions $\psi \in \Psi$ and $\gamma : X^3 \mapsto [0,\infty)$ satisfying the following condition: for each $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ such that

$$\epsilon \leq \psi(S(x,y,z)) < \epsilon + \delta(\epsilon) \;\; implies \;\; \gamma(x,Tx,Tx)\gamma(y,Ty,Ty)\gamma(z,Tz,Tz)S(Tx,Ty,Tz) < \epsilon.$$

Remark 1.4. It is easily shown that if $T : X \mapsto X$ is a $(\gamma - \psi)-$Meir-Keeler contractive mapping of dim3, then

$$\gamma(x,Tx,Tx)\gamma(y,Ty,Ty)\gamma(z,Tz,Tz)S(Tx,Ty,Tz) < \psi(S(x,y,z)),$$

$\forall x, y, z \in X$ when $x \neq y \neq z$.

Definition 1.8. Let (X,S) be an $S-$metric space and $T:X\mapsto X$. The mapping T is called a generalized $(\gamma-\psi)-$Meir-Keeler contractive mapping of type A if there exist two functions $\psi\in\Psi$ and $\gamma:X^3\mapsto[0,\infty)$ satisfying the following condition: for each $\epsilon>0$ there exists $\delta(\epsilon)>0$ such that

$$\epsilon\le\psi(M_1(x,y))<\epsilon+\delta(\epsilon)\ \ implies\ \ \gamma(x,y,y)S(Tx,Ty,Ty)<\epsilon,$$

where $M_1(x,y)=\max\{S(x,y,y),S(x,Tx,Tx),S(y,Ty,Ty)\}$, $\forall x,y\in X$.

Definition 1.9. Let (X,S) be an $S-$metric space and $T:X\mapsto X$. The mapping T is called a generalized $(\gamma-\psi)-$Meir-Keeler contractive mapping of type B if there exist two functions $\psi\in\Psi$ and $\gamma:X^3\mapsto[0,\infty)$ satisfying the following condition: for each $\epsilon>0$ there exists $\delta(\epsilon)>0$ such that

$$\epsilon\le\psi(M_2(x,y))<\epsilon+\delta(\epsilon)\ \ implies\ \ \gamma(x,y,y)S(Tx,Ty,Ty)<\epsilon,$$

where $M_2(x,y)=\max\{S(x,y,y),\frac{1}{2}[S(x,Tx,Tx)+S(y,Ty,Ty)]\}$, $\forall x,y\in X$.

Remark 1.5. **(1)** It is obviously that $M_2(x,y)\le M_1(x,y),\forall x,y\in X$, where $M_1(x,y),M_2(x,y)$ are defined in Definition 1.8 and Definition 1.9, respectively.

(2) Let $T:X\mapsto X$ be a generalized $(\gamma-\psi)-$Meir-Keeler contractive mapping of type A (resp., type B). Then $\gamma(x,y,y)S(Tx,Ty,Ty)<\psi(M_1(x,y)),(resp.,\psi(M_2(x,y)))$, $\forall x,y\in X$.

Definition 1.10. Let (X,S) be an $S-$metric space and $T:X\mapsto X$. The mapping T is called a generalized $(\gamma-\psi)-$Meir-Keeler contractive mapping of dim3 of type A if there exist two functions $\psi\in\Psi$ and $\gamma:X^3\mapsto[0,\infty)$ satisfying the following condition: for each $\epsilon>0$ there exists $\delta(\epsilon)>0$ such that

$$\epsilon\le\psi(M_1^{'}(x,y,z))<\epsilon+\delta(\epsilon)\ \ implies\ \ \gamma(x,Tx,Tx)\gamma(y,Ty,Ty)\gamma(z,Tz,Tz)S(Tx,Ty,Tz)<\epsilon,$$

where

$$M_1^{'}(x,y,z)=\max\{S(x,y,y),S(y,z,z),S(z,x,x),S(x,Tx,Tx),S(y,Ty,Ty)S(z,Tz,Tz)\},\forall x,y,z\in X.$$

Definition 1.11. Let (X,S) be an $S-$metric space and $T:X\mapsto X$. The mapping T is called a generalized $(\gamma-\psi)-$Meir-Keeler contractive mapping of dim3 of type B if there exist two functions $\psi\in\Psi$ and $\gamma:X^3\mapsto[0,\infty)$ satisfying the following condition: for each $\epsilon>0$ there exists $\delta(\epsilon)>0$ such that

$$\epsilon\le\psi(M_2^{'}(x,y,z))<\epsilon+\delta(\epsilon)\ \ implies\ \ \gamma(x,Tx,Tx)\gamma(y,Ty,Ty)\gamma(z,Tz,Tz)S(Tx,Ty,Ty)<\epsilon,$$

where

$$M_2^{'}(x,y,z)=\max\{S(x,y,y),S(y,z,z),S(z,x,x),\frac{1}{2}[S(x,Tx,Tx)+S(y,Ty,Ty)],\\ \frac{1}{2}[S(y,Ty,Ty)+S(z,Tz,Tz)],\frac{1}{2}[S(z,Tz,Tz)+S(x,Tx,Tx)]\},$$

$\forall x,y,z\in X$.

Remark 1.6. **(1)** It is obviously that $M_2^{'}(x,y,z)\le M_1^{'}(x,y,z),\forall x,y,z\in X$, where $M_1^{'}(x,y,z),M_2^{'}(x,y,z)$ are defined in Definition 1.10 and Definition 1.11, respectively.

(2) Let $T:X\mapsto X$ be a generalized $(\gamma-\psi)-$Meir-Keeler contractive mapping of dim3 of type A (resp., type B). Then $\gamma(x,Tx,Tx)\gamma(y,Ty,Ty)\gamma(z,Tz,Tz)S(Tx,Ty,Tz)<\psi(M_1^{'}(x,y,z)),(resp.,\psi(M_2^{'}(x,y,z)))$, $\forall x,y,z\in X$.

2. Fixed point theorems for several types of $(\gamma-\psi)-$Meir-Keeler contractive mappings in $S-$metric spaces

In this section, by introducing the class of $(\gamma-\psi)-$Meir-Keeler contractive mapping and the classes of generalized $(\gamma-\psi)-$Meir-Keeler contractive mappings, we study the existence and uniqueness of fixed points for these contractive mappings via $\gamma-$admissible mappings.

Proposition 2.1. Assume that T is $\gamma-$admissible and $(\gamma-\psi)-$Meir-Keeler contractive. Let $x,y\in X$ such that $\gamma(x,y,y)\geq 1$. Then

$$\gamma(T^nx,T^ny,T^ny)\geq 1,\quad \forall n\in\mathbb{N}, \tag{5}$$

the sequence $\{S(T^nx,T^ny,T^ny)\}$ is non-increasing, bounded and $S(T^nx,T^ny,T^ny)\to 0$ as $n\to\infty$.

Proof. Since T is $\gamma-$admissible and $\gamma(x,y,y)\geq 1$, then (5) follows directly by induction on n. Next, let $n\in\mathbb{N}$. If $T^nx\neq T^ny$, by (5) and Remark 1.3, it follows that

$$\begin{aligned}S(T^{n+1}x,T^{n+1}y,T^{n+1}y)\leq&\gamma(T^nx,T^ny,T^ny)S(T^{n+1}x,T^{n+1}y,T^{n+1}y)\\ &=\gamma(T^nx,T^ny,T^ny)S(T(T^nx),T(T^ny),T(T^ny))\\ &<\psi(S(T^nx,T^ny,T^ny))\\ &<S(T^nx,T^ny,T^ny).\end{aligned}$$

Else, if $T^nx=T^ny$, then $S(T^nx,T^ny,T^ny)=S(T^{n+1}x,T^{n+1}y,T^{n+1}y)$.
Eventually, we conclude that $\{S(T^nx,T^ny,T^ny)\}$ is a non-increasing and bounded sequence.
Hence, there exists $r\in[0,\infty)$ such that $\lim\limits_{n\to\infty}S(T^nx,T^ny,T^ny)=r$.
In what follows, we will prove that $r=0$. Suppose, on the contrary, that $r>0$. Since T is a $(\gamma-\psi)-$Meir-Keeler contractive mapping, for $\epsilon=\psi(r)>0$, there exists $\delta>0$ and a $p\in\mathbb{N}$ such that

$$\epsilon\leq\psi(S(T^px,T^py,T^py))<\epsilon+\delta\ \ \textit{implies}\ \ \gamma(T^px,T^py,T^py)S(T^{p+1}x,T^{p+1}y,T^{p+1}y)<\epsilon.$$

By taking (5) into account, we get that

$$S(T^{p+1}x,T^{p+1}y,T^{p+1}y)<\epsilon=\psi(r)<r,$$

which is a contradiction, since $r=\inf\{S(T^nx,T^ny,T^ny)\}_{n=1}^{\infty}$.
Consequently, we have $\lim\limits_{n\to\infty}S(T^nx,T^ny,T^ny)=0$. □

Proposition 2.2. Assume that T is $\gamma-$admissible and $(\gamma-\psi)-$Meir-Keeler contractive of dim3. Let $x,y,z\in X$ such that $\gamma(x,Tx,Tx)\geq 1$, $\gamma(y,Ty,Ty)\geq 1$, $\gamma(z,Tz,Tz)\geq 1$. Then

$$\gamma(T^nx,T^ny,T^nz)\geq 1,\quad \forall n\in\mathbb{N}, \tag{6}$$

the sequence $\{S(T^nx,T^ny,T^nz)\}$ is non-increasing, bounded and $S(T^nx,T^ny,T^nz)\to 0$ as $n\to\infty$.

Proof. Using similar process to the proof of Proposition 2.1, one can safely draw the conclusion. □

Theorem 2.1. Let (X,S) be a complete $S-$metric space and $T:X\mapsto X$ be a $(\gamma-\psi)-$MKC mapping. Assume that

$(A1)$ T is $\gamma-$admissible;

$(A2)$ there exists $x_0 \in X$ such that $\gamma(x_0, Tx_0, Tx_0) \geq 1$;

$(A3)$ T is $\gamma-$orbital continuous.

Then, there exists $x^* \in X$ such that $Tx^* = x^*$.

Proof. Due to assumption $(A2)$, there exists $x_0 \in X$ such that $\gamma(x_0, Tx_0, Tx_0) \geq 1$. Define an iterative sequence $\{x_n\}$ in X by $x_{n+1} = Tx_n, \forall n \in \{0\} \cup \mathbb{N}$. Note that if $x_{n_0} = x_{n_0+1}$ for some n_0, then $x^* = x_{n_0}$ is a fixed point of T. So we suppose that $x_n \neq x_{n+1}$ for $\forall n \in \{0\} \cup \mathbb{N}$. Since T is $\gamma-$admissible, we have that

$$\gamma(x_0, x_1, x_1) = \gamma(x_0, Tx_0, Tx_0) \geq 1 \Rightarrow \gamma(Tx_0, Tx_1, Tx_1) = \gamma(x_1, x_2, x_2) \geq 1.$$

By induction, we get that

$$\gamma(x_n, x_{n+1}, x_{n+1}) \geq 1, \quad \forall n \in \{0\} \cup \mathbb{N}. \tag{7}$$

From (7) together with the assumption of the theorem that T is a $(\gamma - \psi)-$MKC mapping, it follows that for $\forall n \in \mathbb{N}$, we have that

$$\begin{aligned} S(x_n, x_{n+1}, x_{n+1}) &= S(Tx_{n-1}, Tx_n, Tx_n) \\ &\leq \gamma(x_{n-1}, x_n, x_n) S(Tx_{n-1}, Tx_n, Tx_n) \\ &\leq \psi(S(x_{n-1}, x_n, x_n)). \end{aligned}$$

Since $\psi \in \Psi$, by induction, we have that

$$S(x_n, x_{n+1}, x_{n+1}) < \psi^n(S(x_0, x_1, x_1)), \quad \forall n \in \mathbb{N}. \tag{8}$$

Using $(S2)$ and (8), for $\forall m, n \in \mathbb{N}$ with $m < n$, we have that

$$\begin{aligned} S(x_m, x_n, x_n) &\leq 2 \sum_{k=m}^{n-2} S(x_k, x_{k+1}, x_{k+1}) + S(x_{n-1}, x_n, x_n) \\ &\leq 2 \sum_{k=m}^{n-2} \psi^k(S(x_0, x_1, x_1)) + \psi^{n-1}(S(x_0, x_1, x_1)). \end{aligned}$$

Since $\psi \in \Psi$ and $S(x_0, x_1, x_1) > 0$, by Remark 1.2, we get that

$$\lim_{n,m \to \infty} S(x_m, x_n, x_n) = 0.$$

This implies that $\{x_n\}$ is a Cauchy sequence in the $S-$metric space (X, S).
As (X, S) is complete, then there exists $x^* \in X$ such that

$$\lim_{n \to \infty} S(x_n, x_n, x^*) = 0. \tag{9}$$

Since T is $\gamma-$orbital continuous, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that Tx_{n_k} converges to Tx^* as $k \to \infty$. By the uniqueness of this limit, we get $x^* = Tx^*$, that is x^* is a fixed point of T. $\square$

Theorem 2.2. Let (X, S) be a complete $S-$metric space and $T : X \mapsto X$ be a $(\gamma - \psi)-$MKC mapping of dim3. Assume that

$(A1)$ T is $\gamma-$admissible;

$(A2)$ there exists $x_0 \in X$ such that $\gamma(x_0, Tx_0, Tx_0) \geq 1$;

$(A3)$ T is $\gamma-$orbital continuous.

Then, there exists $x^* \in X$ such that $Tx^* = x^*$.

Proof. Due to assumption $(A2)$, there exists $x_0 \in X$ such that $\gamma(x_0, Tx_0, Tx_0) \geq 1$. Define an iterative sequence $\{x_n\}$ in X by $x_{n+1} = Tx_n$ for all $n \in \{0\} \cup \mathbb{N}$. Note that if $x_{n_0} = x_{n_0+1}$ for some n_0, then $x^* = x_{n_0}$ is a fixed point of T. So we suppose that $x_n \neq x_{n+1}$ for all $n \in \{0\} \cup \mathbb{N}$. Since T is $\gamma-$admissible, we have that

$$\gamma(x_0, x_1, x_1) = \gamma(x_0, Tx_0, Tx_0) \geq 1 \Rightarrow \gamma(Tx_0, Tx_1, Tx_1) = \gamma(x_1, x_2, x_2) \geq 1.$$

By induction, we get that

$$\gamma(x_n, x_{n+1}, x_{n+1}) \geq 1, \quad \forall n \in \{0\} \cup \mathbb{N}. \tag{10}$$

From (10) together with the assumption of the theorem that T is a $(\gamma - \psi)-$MKC mapping of dim3, it follows that for $\forall n \in \mathbb{N}$, we have that

$$\begin{aligned} S(x_n, x_{n+1}, x_{n+1}) &= S(Tx_{n-1}, Tx_n, Tx_n) \\ &\leq \gamma(x_{n-1}, x_n, x_n)\gamma(x_n, x_{n+1}, x_{n+1})\gamma(x_n, x_{n+1}, x_{n+1})S(Tx_{n-1}, Tx_n, Tx_n) \\ &\leq \psi(S(x_{n-1}, x_n, x_n)). \end{aligned}$$

Since $\psi \in \Psi$, by induction, we have that

$$S(x_n, x_{n+1}, x_{n+1}) < \psi^n(S(x_0, x_1, x_1)), \quad \forall n \in \mathbb{N}.$$

Using Lemma 1.3 and (10), for $l, m, n \in \mathbb{N}$ with $l < m < n$, we have that

$$\begin{aligned} S(x_l, x_m, x_n) &\leq S(x_l, x_l, x_m) + S(x_m, x_m, x_n) \\ &\leq 2\sum_{k=l}^{m-2} S(x_k, x_{k+1}, x_{k+1}) + S(x_{m-1}, x_m, x_m) + 2\sum_{k=m}^{n-2} S(x_k, x_{k+1}, x_{k+1}) + S(x_{n-1}, x_n, x_n) \\ &\leq 2\sum_{k=l}^{m-2} \psi^k(S(x_0, x_1, x_1)) + \psi^{m-1}(S(x_0, x_1, x_1)) + 2\sum_{k=m}^{n-2} \psi^k(S(x_0, x_1, x_1)) + \psi^{n-1}(S(x_0, x_1, x_1)). \end{aligned}$$

Since $\psi \in \Psi$ and $S(x_0, x_1, x_1) > 0$, by Remark 1.2, we get that

$$\lim_{l,m,n \to \infty} S(x_l, x_m, x_n) = 0.$$

This implies that $\{x_n\}$ is a Cauchy sequence in the $S-$metric space (X, S).
As (X, S) is complete, then there exists $x^* \in X$ such that

$$\lim_{n \to \infty} S(x_n, x_n, x^*) = 0.$$

Since T is $\gamma-$orbital continuous, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that Tx_{n_k} converges to Tx^* as $k \to \infty$. By the uniqueness of this limit, we get $x^* = Tx^*$, that is x^* is a fixed point of T. $\square$

In the next theorems, we replace the $\gamma-$orbital continuity of T by a regularity condition or $(T-\gamma)-$limit condition over the $S-$metric spaces (X,S).

Theorem 2.3. Let (X,S) be a complete $S-$metric space and $T:X\mapsto X$ be a $(\gamma-\psi)-$MKC mapping. Assume that

$(A1)$ T is $\gamma-$admissible;

$(A2)$ there exists $x_0\in X$ such that $\gamma(x_0,Tx_0,Tx_0)\geq 1$;

$(A3)$ (X,S) is $(T,\gamma)-$ regular.

Then, there exists $x^*\in X$ such that $Tx^*=x^*$.

Proof. Following the line of the proof of Theorem 2.1, it follows that the sequence $\{x_n\}$ defined by $x_{n+1}=Tx_n$, $\forall n\in\{0\}\cup\mathbb{N}$ is a Cauchy sequence in the complete $S-$metric space (X,S), that is convergent to $x^*\in X$.
Since $\{x_n\}$ is a $(T,\gamma)-$orbital sequence, by $(A3)$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\gamma(x_{n_k},x^*,x^*)\geq 1,\quad \forall k\in\mathbb{N}. \tag{11}$$

Using Remark 1.3 and (11), we have that

$$\begin{aligned} S(x_{n_k+1},Tx^*,Tx^*) &= S(Tx_{n_k},Tx^*,Tx^*)\\ &\leq \gamma(x_{n_k},x^*,x^*)S(Tx_{n_k},Tx^*,Tx^*)\\ &\leq \psi(S(x_{n_k},x^*,x^*)).\end{aligned}$$

Letting $k\to\infty$, since ψ is continuous at $t=0$, it follows that $S(x^*,Tx^*,Tx^*)=0$, then $x^*=Tx^*$. □

Theorem 2.4. Let (X,S) be a complete $S-$metric space and $T:X\mapsto X$ be a $(\gamma-\psi)-$MKC mapping of dim3. Assume that

$(A1)$ T is $\gamma-$admissible;

$(A2)$ there exists $x_0\in X$ such that $\gamma(x_0,Tx_0,Tx_0)\geq 1$;

$(A3)$ (X,S) is $(T,\gamma)-$limit.

Then, there exists $x^*\in X$ such that $Tx^*=x^*$.

Proof. Following the line of the proof of Theorem 2.1, it follows that the sequence $\{x_n\}$ defined by $x_{n+1}=Tx_n$, for all $n\in\{0\}\cup\mathbb{N}$ is a Cauchy sequence in the complete $S-$metric space (X,S), that is convergent to $x^*\in X$.
By $(A3)$, we have

$$\gamma(x^*,Tx^*,Tx^*)\geq 1. \tag{12}$$

Using Remark 1.4 and (12), we have that

$$\begin{aligned} S(x_{n+1},Tx^*,Tx^*) =&S(Tx_n,Tx^*,Tx^*)\\ &\leq \gamma(x_n,x_{n+1},x_{n+1})\gamma(x^*,Tx^*,Tx^*)\gamma(x^*,Tx^*,Tx^*)S(Tx_n,Tx^*,Tx^*)\\ &\leq \psi(S(x_n,x^*,x^*)).\end{aligned}$$

Letting $n\to\infty$, since ψ is continuous at $t=0$, it follows that $S(x^*,Tx^*,Tx^*)=0$, then $x^*=Tx^*$. □

Example 2.1. Let $X = [0, \infty)$ be an $S-$metric space with the $S-$metric defined by $S(x, y, z) = |x - z| + |y - z|, \forall x, y, z \in X$. For $\forall k > 1$, consider the self-mapping $T : X \mapsto X$ given by
$Tx=\begin{cases} e^{x-1}, & x \geq 1, \\ \frac{x^2}{4}, & 0 \leq x < 1. \end{cases}$
Also, define $\gamma : X^3 \mapsto [0, 1)$ as
$\gamma(x,y,z)=\begin{cases} 1, & x, y, z \in [0, 1), \\ 0, & otherwise. \end{cases}$
Let $\psi(t) = \frac{t}{2}$ for $t \geq 0$.
Clearly, T is not continuous at $x = 1$. Then we will claim that T is a $(\gamma - \psi)-$MKC.
Let $\epsilon > 0$ be given. Take $\delta = \epsilon$ and suppose that $\epsilon \leq \frac{1}{2}|x - y| < \epsilon + \delta$, we want to show that $\gamma(x, y, y)S(Tx, Ty, Ty) < \epsilon$.
Suppose that $\gamma(x, y, y) = 1$, then $x, y \in [0, \infty)$ and $|x + y| < 2$. So $Tx = \frac{x^2}{4} \in [0, 1)$, $Ty = \frac{y^2}{4} \in [0, 1)$.
Hence, $S(Tx, Ty, Ty) = |\frac{x^2}{4} - \frac{y^2}{4}| = \frac{|x^2-y^2|}{4} = \frac{|x+y||x-y|}{4} < \frac{|x-y|}{2} < \frac{\epsilon+\delta}{2} < \epsilon$.
Also, T is $\gamma-$admissible. To see that, let $x, y, z \in X$ such that $\gamma(x, y, z) \geq 1$, which implies that $x, y, z \in [0, 1)$. Due to the definitions of γ and T, we have that

$$Tx = \frac{x^2}{4} \in [0, 1), \quad Ty = \frac{y^2}{4} \in [0, 1), \quad Tz = \frac{z^2}{4} \in [0, 1).$$

Hence, $\gamma(Tx, Ty, Ty) \geq 1$. Moreover, there exists $x_0 \in X$ such that $\gamma(x_0, Tx_0, Tx_0) \geq 1$. Indeed, for any $x_0 \in [0, 1)$, we have $\gamma(x_0, \frac{x_0^2}{4}, \frac{x_0^2}{4}) \geq 1$.
Finally, let $\{x_n\}$ be a $(T, \gamma)-$orbital sequence such that $x_n \to x$ as $n \to \infty$. By the definition of γ, we have that $x_n \in [0, 1)$. Then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\gamma(x_{n_k}, x, x) \geq 1$, $\forall k \in \mathbb{N}$.
So we conclude that all the hypotheses of Theorem 2.3 are fulfilled. In fact, 0 and 1 are two fixed points of T.

Example 2.2. Let $X = [0, \infty)$ be an $S-$metric space with the $S-$metric defined by $S(x, y, z) = |x - z| + |y - z|, \forall x, y, z \in X$. For $\forall k > 1$, consider the self-mapping $T : X \mapsto X$ given by
$Tx=\begin{cases} x^{x-1}, & x \geq 1, \\ \frac{x^2}{4}, & 0 \leq x < 1. \end{cases}$
Also, define $\gamma : X^3 \mapsto [0, 1)$ as
$\gamma(x,y,z)=\begin{cases} 1, & x, y, z \in [0, 1), \\ 0, & otherwise. \end{cases}$
Let $\psi(t) = \frac{t}{2}$ for $t \geq 0$.
Clearly, T is not continuous at $x = 1$. Then we will claim that T is a $(\gamma - \psi)-$MKC mapping of dim3.
Let $\epsilon > 0$ be given. Take $\delta = \epsilon$ and suppose that $\epsilon \leq \frac{1}{2}(|x - y| + |y - z|) < \epsilon + \delta$, we want to show that $\gamma(x, Tx, Tx)\gamma(y, Ty, Ty)\gamma(z, Tz, Tz)S(Tx, Ty, Tz) < \epsilon$.
Suppose that $\gamma(x, Tx, Tx) = \gamma(y, Ty, Ty) = \gamma(z, Tz, Tz) = 1$, then $x, y, z, Tx, Ty, Tz \in [0, 1)$ and $|x + y| < 2, |y + z| < 2$. So $Tx = \frac{x^2}{4} \in [0, 1)$, $Ty = \frac{y^2}{4} \in [0, 1)$, $Tz = \frac{z^2}{4} \in [0, 1)$.

Hence,

$$\begin{aligned} S(Tx,Ty,Tz) &= |\frac{x^2}{4}-\frac{y^2}{4}|+|\frac{y^2}{4}-\frac{z^2}{4}| \\ &= \frac{|x^2-y^2|}{4}+\frac{|y^2-z^2|}{4} \\ &= \frac{|x+y||x-y|}{4}+\frac{|y+z||y-z|}{4} \\ &< \frac{|x-y|}{2}+\frac{|y-z|}{2} \\ &< \frac{\epsilon+\delta}{2} \\ &= \epsilon. \end{aligned}$$

Also, T is $\gamma-$admissible. To see that, let $x,y,z\in X$ such that $\gamma(x,y,z)\geq 1$, which implies that $x,y,z\in[0,1)$. Due to the definitions of γ and T, we have that

$$Tx=\frac{x^2}{4}\in[0,1),\quad Ty=\frac{y^2}{4}\in[0,1),\quad Tz=\frac{z^2}{4}\in[0,1).$$

Hence, $\gamma(Tx,Ty,Tz)\geq 1$. Moreover, there exists $x_0\in X$ such that $\gamma(x_0,Tx_0,Tx_0)\geq 1$. Indeed, for any $x_0\in[0,1)$, we have $\gamma(x_0,\frac{x_0^2}{4},\frac{x_0^2}{4})\geq 1$.
Finally, let $\{x_n\}$ be a sequence such that $x_n\to x$ as $n\to\infty$ with $\gamma(x_n,x_{n+1},x_{n+1})\geq 1$. By the definition of γ, we have that $x,Tx\in[0,1)$. Then $\gamma(x,Tx,Tx)\geq 1$.
So we conclude that all the hypotheses of Theorem 2.4 are fulfilled. In fact, 0 and 1 are two fixed points of T.

Now, we propose the following conditions for the uniqueness of a fixed point of a $(\gamma-\psi)-$MKC mapping and a $(\gamma-\psi)-$MKC mapping of dim3. Let $Fix(T)$ denote the set of fixed points of the mapping T.

$(U1)$ For $\forall x,y\in Fix(T)$, there exists $z\in X$ such that $\gamma(x,z,z)\geq 1$ and $\gamma(y,z,z)\geq 1$.

Theorem 2.5. Adding the condition $(U1)$ to the hypotheses of Theorem 2.1(resp.Theorem 2.3), we obtain the uniqueness of a fixed point T.

Proof. Let $u,v\in X$ be two fixed points of T. By $(U1)$, there exists $z\in X$ such that $\gamma(u,z,z)\geq 1$ and $\gamma(v,z,z)\geq 1$.
Since T is $\gamma-$admissible, we get by induction that

$$\gamma(u,u,T^nz)\geq 1\quad and\quad \gamma(v,v,T^nz)\geq 1,\quad \forall n\in\mathbb{N}. \tag{13}$$

From (13), we have that

$$\begin{aligned} S(u,u,T^nz) &= S(Tu,Tu,T(T^{n-1}z)) \\ &\leq \gamma(u,u,T^{n-1}z)S(Tu,Tu,T(T^{n-1}z)) \\ &< \psi(S(u,u,T^{n-1}z)). \end{aligned}$$

Iteratively, we get

$$S(u,u,T^nz)<\psi^n(S(u,u,z)).$$

Letting $n \to \infty$, and since $\psi \in \Psi$, we have that

$$\lim_{n\to\infty} S(u,u,T^n z) = 0. \tag{14}$$

Similarly, we also can get

$$\lim_{n\to\infty} S(v,v,T^n z) = 0. \tag{15}$$

Combining (14) and (15), it follows that $T^n z \to u$ and $T^n z \to v$, as $n \to \infty$.
By Lemma 1.5, we get $u = v$, that is, fixed point of T is unique. □

As an alternative uniqueness condition for fixed points of $(\gamma - \psi)-$MKC mappings, we suggest the following hypothesis:

$(U2)$ For $\forall x, y \in Fix(T)$, then $\gamma(x,y,y) \geq 1$.

Theorem 2.6. Adding the condition $(U2)$ to the hypotheses of Theorem 2.1(resp.Theorem 2.3), we obtain the uniqueness of a fixed point T.

Proof. Let u, v be two distinct fixed points of T. Then $\gamma(u,v,v) > 0$.
Due to the property of ψ, we get that

$$\psi(S(u,v,v)) > 0.$$

Let $\epsilon = \psi(S(u,v,v)) > 0$; then, for any $\delta > 0$, we find that

$$\epsilon = \psi(S(u,v,v)) < \epsilon + \delta.$$

Considering $(U2)$ and the assumption of theorem that T is a $(\gamma - \psi)-$MKC mapping, we obtain that

$$S(u,v,v) \leq \gamma(u,v,v)S(Tu,Tv,Tv) < \psi(S(u,v,v)) < S(u,v,v),$$

which is a contradiction. Then $u = v$. □

As a uniqueness condition for fixed points of $(\gamma - \psi)-$MKC mappings of dim3, we suggest the following hypothesis:

$(U3)$ For $\forall x \in Fix(T)$, then $\gamma(x,x,x) \geq 1$.

Theorem 2.7. Adding the condition $(U3)$ to the hypotheses of Theorem 2.2(resp.Theorem 2.4), we obtain the uniqueness of a fixed point T.

Proof. Let u, v be two distinct fixed points of T.
Due to the property of ψ, we get that $\psi(S(u,v,v)) > 0$.
Let $\epsilon = \psi(S(u,v,v)) > 0$; then, for any $\delta > 0$, we find that

$$\epsilon = \psi(S(u,v,v)) < \epsilon + \delta.$$

Considering $(U3)$ and the assumption of theorem that T is a $(\gamma - \psi)-$MKC mapping of dim3, we obtain that

$$S(u,v,v) \leq \gamma(u,Tu,Tu)\gamma(v,Tv,Tv)\gamma(v,Tv,Tv)S(Tu,Tv,Tv) < \psi(S(u,v,v)) < S(u,v,v),$$

which is a contradiction. Then $u = v$. □

Theorem 2.8. Let (X, S) be a complete $S-$metric space and $T : X \mapsto X$ be a generalized $(\gamma - \psi)-$MKC mapping of type A. Assume also that:

(A1) T is triangular $\gamma-$admissible;

(A2) there exists $x_0 \in X$ such that $\gamma(x_0, Tx_0, Tx_0) \geq 1$;

(A3) (X, S) is $(T, \gamma)-$regular.

Then, there exists $x^* \in X$ such that $Tx^* = x^*$.

Proof. In view of assumption $(A2)$, let $x_0 \in X$ be such that $\gamma(x_0, Tx_0, Tx_0) \geq 1$.
Define the sequence $\{x_n\}$ in X by $x_{n+1} = Tx_n$, $\forall n \in \{0\} \cup \mathbb{N}$. Without loss of generality, we assume that $x_n \neq x_{n+1}$, for $\forall n \in \{0\} \cup \mathbb{N}$, then

$$S(x_n, x_{n+1}, x_{n+1}) > 0, \quad \forall n\{0\} \cup \mathbb{N}. \tag{16}$$

Indeed, if there exists some $n_0 \in \mathbb{N}$ such that $x_{n_0} = x_{n_0+1}$, then the proof is complete, since $x^* = x_{n_0+1} = Tx_{n_0} = Tx^*$. Since T is triangular $\gamma-$admissible, by Lemma 1.6, we have that

$$\gamma(x_n, x_m, x_m) \geq 1, \quad \forall n, m \in \mathbb{N} \;\; with \;\; n < m. \tag{17}$$

Step1. We will prove that

$$\lim_{n\to\infty} S(x_n, x_{n+1}, x_{n+1}) = 0. \tag{18}$$

Taking (16) and (17) into account together with the fact that T is generalized $(\gamma - \psi)-$MKC mapping of type A, for each $n \in \{0\} \cup \mathbb{N}$, we get

$$\begin{aligned} S(x_n, x_{n+1}, x_{n+1}) &= S(Tx_{n-1}, Tx_n, Tx_n) \\ &\leq \gamma(x_{n-1}, x_n, x_n) S(Tx_{n-1}, Tx_n, Tx_n) \\ &\leq \psi(M_1(x_{n-1}, x_n)) \\ &< \psi(M_1(x_{n-1}, x_n)), \end{aligned}$$

where

$$\begin{aligned} M_1(x_{n-1}, x_n) &= \max\{S(x_{n-1}, x_n, x_n), S(x_{n-1}, Tx_{n-1}, Tx_{n-1}), S(x_n, Tx_n, Tx_n)\} \\ &= \max\{S(x_{n-1}, x_n, x_n), S(x_n, x_n, x_{n+1})\}. \end{aligned}$$

If $M_1(x_{n-1}, x_n) = S(x_n, x_{n+1}, x_{n+1})$. Since ψ is nondecreasing, from the inequality above, we have that

$$S(x_n, x_{n+1}, x_{n+1}) \leq \psi(S(x_n, x_{n+1}, x_{n+1})) < S(x_n, x_{n+1}, x_{n+1}), \quad \forall n \in \mathbb{N},$$

which is a contradiction. Thus, $M_1(x_{n-1}, x_n) = S(x_{n-1}, x_n, x_n)$ and we also have that

$$S(x_n, x_{n+1}, x_{n+1}) \leq \psi(S(x_{n-1}, x_n, x_n)) < S(x_{n-1}, x_n, x_n), \quad \forall n \in \mathbb{N}. \tag{19}$$

So, we deduce that the sequence $\{S(x_n, x_{n+1}, x_{n+1})\}$ is non-increasing and bounded below by zero. Hence, there exists $t \in [0, \infty)$ such that

$$\lim_{n\to\infty} S(x_n, x_{n+1}, x_{n+1}) = t. \tag{20}$$

Iteratively, we derive from (19) that

$$S(x_n, x_{n+1}, x_{n+1}) \le \psi^n(S(x_0, x_1, x_1)), \quad \forall n \in \mathbb{N}. \tag{21}$$

On account of (21) and Remark 1.2, we obtain

$$\lim_{n\to\infty} S(x_n, x_{n+1}, x_{n+1}) = 0. \tag{22}$$

Step2. We will show that $\{x_n\}$ is a Cauchy sequence.
Suppose, on the contrary, that there exist $\epsilon > 0$ and a subsequence $\{x_{n(i)}\}$ of $\{x_n\}$ such that

$$S(x_{n(i)}, x_{n(i+1)}, x_{n(i+1)}) > 2\epsilon. \tag{23}$$

First, we will show that the existence of $k \in \mathbb{N}$ such that $n(i) < k \le n(i+1)$. Later, we will prove that for given $\epsilon > 0$ above, there exists $\delta > 0$ such that

$$\frac{\epsilon}{2} < \psi(M_1(x_{n(i)}, x_k)) < \frac{\epsilon+\delta}{2},$$

but

$$\gamma(x_{n(i)}, x_k, x_k) S(Tx_{n(i)}, Tx_k, Tx_k) \ge \epsilon,$$

which contradicts (23), where $M_1(x_{n(i)}, x_k) = \max\{S(x_{n(i)}, x_k, x_k), S(x_{n(i)}, x_{n(i)+1}, x_{n(i)+1}), S(x_k, x_{k+1}, x_{k+1})\}$
Let $r = \min\{\epsilon, \frac{\delta}{2}\}$. Taking Step1 into account, we will choose $n_0 \in \mathbb{N}$ such that

$$S(x_n, x_{n+1}, x_{n+1}) < \frac{r}{8}, \tag{24}$$

for all $n > n_0$. Let $n(i) > n_0$. According to our construction, we have $n(i) \le n(i+1) - 1$.
If $S(x_{n(i)}, x_{n(i+1)-1}, x_{n(i+1)-1}) < \frac{\epsilon+r}{2}$, then by Lemma 1.1, we have

$$\begin{aligned} S(x_{n(i)}, x_{n(i+1)}, n_{n(i+1)}) &\le 2S(x_{n(i)}, x_{n(i+1)-1}, n_{n(i+1)-1}) + S(x_{n(i+1)-1}, x_{n(i+1)}, n_{n(i+1)}) \\ &\le \epsilon + r + \frac{r}{8} \\ &= \epsilon + \frac{7r}{8} \\ &< 2\epsilon, \end{aligned}$$

which contradicts (23). Consequently, there exist values of $k \in \mathbb{N}$ such that $n(i) \le k \le n(i+1)$ and $S(x_{n(i)}, x_k, x_k) > \frac{\epsilon+r}{2}$.
Indeed, if $S(x_{n(i)}, x_{n(i)+1}, x_{n(i)+1}) \ge \frac{\epsilon+r}{2}$, then we have $S(x_{n(i)}, x_{n(i)+1}, x_{n(i)+1}) \ge \frac{r}{8}$, which contradicts (24).
Hence, we can choose the smallest integer $k > n(i)$ such that

$$S(x_{n(i)}, x_k, x_k) \ge \frac{\epsilon+r}{2}.$$

So, necessarily, we also have $S(x_{n(i)}, x_{k-1}, x_{k-1}) < \frac{\epsilon+r}{2}$.
Therefore, we find that

$$\begin{aligned} S(x_{n(i)}, x_k, x_k) &\le 2S(x_k, x_{k-1}, x_{k-1}) + S(x_{n(i)}, x_{k-1}, x_{k-1}) \\ &< 2 \cdot \frac{r}{8} + \frac{\epsilon+r}{2} \\ &= \frac{\epsilon}{2} + \frac{3r}{4}. \end{aligned}$$

Hence, we get the following approximation:

$$\frac{\epsilon + r}{2} \leq S(x_{n(i)}, x_k, x_k) \leq \frac{\epsilon}{2} + \frac{3r}{4}, \tag{25}$$

for a integer k satisfying $n(i) \leq k \leq n(i+1)$.
On the other hand, the three terms of $M_1(x_{n(i)}, x_k)$ are bounded above by $\frac{\epsilon}{2} + r$, that is

$$\begin{aligned} S(x_{n(i)}, x_k, x_k) &< \frac{\epsilon}{2} + \frac{3r}{4} < \frac{\epsilon}{2} + r. \\ S(x_{n(i)}, x_{n(i)+1}, x_{n(i)+1}) &< \frac{r}{8} < \frac{\epsilon}{2} + r. \\ S(x_k, x_{k+1}, x_{k+1}) &< \frac{r}{8} < \frac{\epsilon}{2} + r. \end{aligned}$$

Combing these estimations presented above, we conclude that

$$\psi(M_1(x_{n(i)}, x_k)) < M_1(x_{n(i)}, x_k) < \frac{\epsilon}{2} + r < \frac{\epsilon + \delta}{2}.$$

Since T is generalized $(\gamma - \psi)-$MKC mapping of type of A and it is $\gamma-$triangular admissible mapping, we have that

$$S(x_{n(i)+1}, x_{k+1}, x_{k+1}) \leq \gamma(x_{n(i)}, x_k, x_k) S(x_{n(i)+1}, x_{k+1}, x_{k+1}) < \frac{\epsilon}{2}.$$

At the same time, by Lemma 1.1, we have that

$$\begin{aligned} S(x_{n(i)}, x_k, x_k) &\leq 2S(x_{n(i)}, x_{n(i)+1}, x_{n(i)+1}) + S(x_{n(i)+1}, x_k, x_k) \\ &\leq 2S(x_{n(i)}, x_{n(i)+1}, x_{n(i)+1}) + 2S(x_k, x_{k+1}, x_{k+1}) + S(x_{n(i)+1}, x_{k+1}, x_{k+1}) \\ &< 2 \cdot \frac{r}{8} + 2 \cdot \frac{r}{8} + \frac{\epsilon}{2} \\ &= \frac{\epsilon + r}{2}, \end{aligned}$$

which contradicts (25).
Thus, claim (23) is false and the sequence $\{x_n\}$ is a Cauchy sequence, that is

$$\lim_{n,m\to\infty} S(x_n, x_m, x_m) = 0. \tag{26}$$

Since (X, S) is a complete $S-$metric space, then there exists $x^* \in X$ such that

$$\lim_{n\to\infty} S(x_n, x^*, x^*) = \lim_{n,m\to\infty} S(x_n, x_m, x^*) = 0. \tag{27}$$

We will prove that $x^* = Tx^*$. Suppose, on the contrary, that $S(x^*, Tx^*, Tx^*) > 0$.
From (27) and assumption $(A3)$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\gamma(x_{n_k}, x^*, x^*) \geq 1, \quad \forall k \in \mathbb{N}. \tag{28}$$

By using Lemma 1.1 and (28) together with the assumption of the theorem that T is a generalized $(\gamma - \psi)-$MKC mapping of type A, we get that

$$\begin{aligned} S(x^*, Tx^*, Tx^*) &\leq 2S(Tx_{n_k}, Tx^*, Tx^*) + S(Tx_{n_k}, x^*, x^*) \\ &\leq 2\gamma(x_{n_k}, x^*, x^*) S(Tx_{n_k}, Tx^*, Tx^*) + S(x_{n_k+1}, x^*, x^*) \\ &\leq \psi(M_1(x_{n_k}, x^*)) + S(x_{n_k+1}, x^*, x^*), \end{aligned}$$

where, $M_1(x_{n_k}, x^*) = \max\{S(x_{n_k}, x^*, x^*), S(x_{n_k}, x_{n_k+1}, x_{n_k+1}), S(x^*, Tx^*, Tx^*)\}$.
Suppose that $M_1(x_{n_k}, x^*) = S(x_{n_k}, x^*, x^*)$, then from the above inequality, we get that

$$\begin{aligned}S(x^*, Tx^*, Tx^*) &\leq \psi(S(x_{n_k}, x^*, x^*)) + S(x_{n_k+1}, x^*, x^*)\\ &< S(x_{n_k}, x^*, x^*) + S(x_{n_k+1}, x^*, x^*).\end{aligned}$$

Taking $k \to \infty$ in the inequality above, we have

$$S(x^*, Tx^*, Tx^*) < 2S(x^*, x^*, x^*) = 0,$$

which is a contradiction.
Next, we suppose that $M_1(x_{n_k}, x^*) = S(x_{n_k}, x_{n_k+1}, x_{n_k+1})$, then we have that

$$\begin{aligned}S(x^*, Tx^*, Tx^*) &\leq \psi(S(x_{n_k}, x_{n_k+1}, x_{n_k+1})) + S(x_{n_k+1}, x^*, x^*)\\ &< S(x_{n_k}, x_{n_k+1}, x_{n_k+1}) + S(x_{n_k+1}, x^*, x^*).\end{aligned}$$

Taking $k \to \infty$ in the inequality above, this implies that

$$S(x^*, Tx^*, Tx^*) < \lim_{k\to\infty}[S(x_{n_k}, x_{n_k+1}, x_{n_k+1}) + S(x_{n_k+1}, x^*, x^*)] = 0,$$

which is again a contradiction.
Finally, we suppose that $M_1(x_{n_k}, x^*) = S(x^*, Tx^*, Tx^*)$, then we obtain that

$$S(x^*, Tx^*, Tx^*) \quad < \psi(S(x^*, Tx^*, Tx^*)) + S(x_{n_k+1}, x^*, x^*). \tag{29}$$

Letting $k \to \infty$ in (29), we get that

$$\begin{aligned}S(x^*, Tx^*, Tx^*) &< \psi(S(x^*, Tx^*, Tx^*)) + S(x^*, x^*, x^*)\\ &< S(x^*, Tx^*, Tx^*) + S(x^*, x^*, x^*)\\ &= S(x^*, Tx^*, Tx^*),\end{aligned}$$

so we also have a contradiction. Thus, we have $S(x^*, Tx^*, Tx^*) = 0$, and by $(S1)$ in Definition 1.1, we have $x^* = Tx^*$. □

Theorem 2.9. Let (X, S) be a complete $S-$metric space and $T : X \mapsto X$ be a generalized $(\gamma - \psi)-$MKC mapping of dim3 of type A. Assume also that:

(A1) T is triangular $\gamma-$admissible;

(A2) there exists $x_0 \in X$ such that $\gamma(x_0, Tx_0, Tx_0) \geq 1$;

(A3) (X, S) is $(T, \gamma)-$limit.

Then, there exists $x^* \in X$ such that $Tx^* = x^*$.

Proof. In view of assumption $(A2)$, let $x_0 \in X$ be such that $\gamma(x_0, Tx_0, Tx_0) \geq 1$.
Define the sequence $\{x_n\}$ in X by $x_{n+1} = Tx_n$, for all $n \in \{0\} \cup \mathbb{N}$. Without loss of generality, we assume that $x_n \neq x_{n+1}$, for $\forall n \in \{0\} \cup \mathbb{N}$, then

$$S(x_n, x_{n+1}, x_{n+1}) > 0, \quad \forall n \in \{0\} \cup \mathbb{N}. \tag{30}$$

Indeed, if there exists some $n_0 \in \mathbb{N}$ such that $x_{n_0} = x_{n_0+1}$, then the proof is complete, since $x^* = x_{n_0+1} = Tx_{n_0} = Tx^*$. Since T is triangular $\gamma-$admissible, by Lemma 1.6, we have that

$$\gamma(x_m, x_n, x_n) \geq 1, \quad \forall m, n \in \mathbb{N} \quad with \quad m < n. \tag{31}$$

Step1. We will prove that

$$\lim_{n\to\infty} S(x_n, x_{n+1}, x_{n+1}) = 0.$$

Taking (30) and (31) into account together with the fact that T is generalized $(\gamma - \psi)-$MKC mapping of dim3 of type A, for each $n \in \{0\} \cup \mathbb{N}$, we get

$$\begin{aligned} S(x_n, x_{n+1}, x_{n+1}) &= S(Tx_{n-1}, Tx_n, Tx_n) \\ &\leq \gamma(x_{n-1}, x_n, x_n)\gamma(x_n, x_{n+1}, x_{n+1})\gamma(x_n, x_{n+1}, x_{n+1})S(Tx_{n-1}, Tx_n, Tx_n) \\ &\leq \psi(M_1^{'}(x_{n-1}, x_n, x_n)) \\ &< M_1^{'}(x_{n-1}, x_n, x_n), \end{aligned}$$

where

$$\begin{aligned} M_1^{'}(x_{n-1}, x_n, x_n) = \max\{&S(x_{n-1}, x_n, x_n), S(x_n, x_n, x_n), S(x_n, x_{n-1}, x_{n-1}), \\ &S(x_{n-1}, Tx_{n-1}, Tx_{n-1}), S(x_n, Tx_n, Tx_n), S(x_n, Tx_n, Tx_n)\} \\ = \max\{&S(x_{n-1}, x_n, x_n), S(x_n, x_n, x_{n+1})\}. \end{aligned}$$

If $M_1^{'}(x_{n-1}, x_n, x_n) = S(x_n, x_{n+1}, x_{n+1})$. Since ψ is nondecreasing, from the inequality above, we have that

$$S(x_n, x_{n+1}, x_{n+1}) \leq \psi(S(x_n, x_{n+1}, x_{n+1})) < S(x_n, x_{n+1}, x_{n+1}), \quad \forall n \in \mathbb{N},$$

which is a contradiction. Thus, $M_1^{'}(x_{n-1}, x_n, x_n) = S(x_{n-1}, x_n, x_n)$ and we also have that

$$S(x_n, x_{n+1}, x_{n+1}) \leq \psi(S(x_{n-1}, x_n, x_n)) < S(x_{n-1}, x_n, x_n), \quad \forall n \in \mathbb{N}. \tag{32}$$

So, we deduce that the sequence $\{S(x_n, x_{n+1}, x_{n+1})\}$ is non-increasing and bounded below by zero. Hence, there exists $t \in [0, \infty)$ such that

$$\lim_{n\to\infty} S(x_n, x_{n+1}, x_{n+1}) = t.$$

Iteratively, we derive from (32) that

$$S(x_n, x_{n+1}, x_{n+1}) \leq \psi^n(S(x_0, x_1, x_1)), \quad \forall n \in \mathbb{N}. \tag{33}$$

On account of (33) and Remark 1.2, we obtain

$$\lim_{n\to\infty} S(x_n, x_{n+1}, x_{n+1}) = 0.$$

Step2. We will show that $\{x_n\}$ is a Cauchy sequence.
We will prove that for each $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for $\forall m, n \geq n_0$,

$$S(x_m, x_m, x_n) < \epsilon. \tag{34}$$

Taking Step1 into account, for each $\epsilon > 0$, we can choose $n_0 \in \mathbb{N}$ such that

$$S(x_n, x_{n+1}, x_{n+1}) < \frac{\epsilon - \psi(\epsilon)}{2}, \forall n \geq n_0. \tag{35}$$

We prove (34) by induction on n. (34) holds for $m = n$ and $n = n+1$ by using (35) and the fact that $\frac{\epsilon-\psi(\epsilon)}{2} < \epsilon$. Assume (34) holds for $n = k$. For $n = k+1$, we have

$$\begin{aligned} S(x_m, x_{k+1}, x_{k+1}) &\leq 2S(x_m, x_{m+1}, x_{m+1}) + S(x_{m+1}, x_{k+1}, x_{k+1}) \\ &\leq \epsilon - \psi(\epsilon) + \psi(S(x_m, x_k, x_k)) \\ &\leq \epsilon - \psi(\epsilon) + \psi(\epsilon) \\ &= \epsilon. \end{aligned}$$

By induction on n, we conclude that (34) holds for all $n \geq m \geq n_0$. So $\{x_n\}$ is a Cauchy sequence that is

$$\lim_{n,m\to\infty} S(x_n, x_m, x_m) = 0.$$

Since (X, S) is a complete $S-$metric space, then there exists $x^* \in X$ such that

$$\lim_{n\to\infty} S(x_n, x^*, x^*) = \lim_{n,m\to\infty} S(x_n, x_m, x^*) = 0.$$

We will prove that $x^* = Tx^*$. Suppose, on the contrary, that $S(x^*, Tx^*, Tx^*) > 0$. From assumption $(A3)$, we have that

$$\gamma(x^*, Tx^*, Tx^*) \geq 1, \quad \forall k \in \mathbb{N}.$$

By using Lemma 1.1 and above inequality together with the assumption of the theorem that T is a generalized $(\gamma - \psi)-$MKC mapping of dim3 of type A, we get that

$$\begin{aligned} S(x^*, Tx^*, Tx^*) &\leq 2S(Tx_{n_k}, Tx^*, Tx^*) + S(Tx_{n_k}, x^*, x^*) \\ &\leq 2\gamma(x_{n_k}, Tx_{n_k}, Tx_{n_k})\gamma(x^*, Tx^*, Tx^*)\gamma(x^*, Tx^*, Tx^*)S(Tx_{n_k}, Tx^*, Tx^*) + S(x_{n_k+1}, x^*, x^*) \\ &\leq 2\psi(M_1^{'}(x_{n_k}, x^*, x^*)) + S(x_{n_k+1}, x^*, x^*), \end{aligned}$$

where, $M_1^{'}(x_{n_k}, x^*, x^*) = \max\{S(x_{n_k}, x^*, x^*), S(x_{n_k}, x_{n_k+1}, x_{n_k+1}), S(x^*, Tx^*, Tx^*)\}$. Suppose that $M_1^{'}(x_{n_k}, x^*, x^*) = S(x_{n_k}, x^*, x^*)$, then from the above inequality, we get that

$$\begin{aligned} S(x^*, Tx^*, Tx^*) &\leq 2\psi(S(x_{n_k}, x^*, x^*)) + S(x_{n_k+1}, x^*, x^*) \\ &< 2S(x_{n_k}, x^*, x^*) + S(x_{n_k+1}, x^*, x^*). \end{aligned}$$

Taking $k \to \infty$ in the inequality above, we have

$$S(x^*, Tx^*, Tx^*) < 3S(x^*, x^*, x^*) = 0,$$

which is a contradiction.
Next, we suppose that $M_1^{'}(x_{n_k}, x^*, x^*) = S(x_{n_k}, x_{n_k+1}, x_{n_k+1})$, then we have that

$$\begin{aligned} S(x^*, Tx^*, Tx^*) &\leq \psi(S(x_{n_k}, x_{n_k+1}, x_{n_k+1})) + S(x_{n_k+1}, x^*, x^*) \\ &< S(x_{n_k}, x_{n_k+1}, x_{n_k+1}) + S(x_{n_k+1}, x^*, x^*). \end{aligned}$$

Taking $k \to \infty$ in the inequality above, this implies that

$$S(x^*, Tx^*, Tx^*) < \lim_{k\to\infty} [S(x_{n_k}, x_{n_k+1}, x_{n_k+1}) + S(x_{n_k+1}, x^*, x^*)] = 0,$$

which is again a contradiction.
Finally, we suppose that $M_1^{'}(x_{n_k}, x^*, x^*) = S(x^*, Tx^*, Tx^*)$, then we obtain that

$$S(x^*, Tx^*, Tx^*) < \psi(S(x^*, Tx^*, Tx^*)) + S(x_{n_k+1}, x^*, x^*).$$

Letting $k \to \infty$ in above inequality, we get that

$$\begin{aligned} S(x^*, Tx^*, Tx^*) &< \psi(S(x^*, Tx^*, Tx^*)) + S(x^*, x^*, x^*) \\ &< S(x^*, Tx^*, Tx^*) + S(x^*, x^*, x^*) \\ &= S(x^*, Tx^*, Tx^*), \end{aligned}$$

so we also have a contradiction. Thus, we have $S(x^*, Tx^*, Tx^*) = 0$, and by $(S1)$ in Definition 1.1, we have $x^* = Tx^*$. □

Example 2.3. Let $X = [0, \infty)$ and $S(x, y, z) = |x - y| + |x - z|, \forall x, y, z \in X$. Then (X, S) is a complete $S-$metric space.
Define $T : X \mapsto X$ and $\gamma : X^3 \mapsto [0, \infty)$ as follow:
$Tx = \begin{cases} kx - (k-1), & k > 1, \quad x \geq 1; \\ \frac{x}{4}, & x \in [0, 1). \end{cases}$ and $\gamma(x, y, z) = \begin{cases} 1, & if \quad x, y, z \in [0, 1); \\ 0, & otherwise. \end{cases}$
Let $\psi(t) = \frac{t}{2}$, $t \geq 0$.
We first show that T is a triangular $\gamma-$admissible mapping. Let $x, y, z \in X$, if $\gamma(x, y, z) \geq 1$, the $x, y, z \in [0, 1)$. On the other hand, for $\forall x, y, z \in [0, 1)$, we have $Tx = \frac{x}{4} \in [0, 1)$, $Ty = \frac{y}{4} \in [0, 1)$, $Tz = \frac{z}{4} \in [0, 1)$. It follows that $\gamma(Tx, Ty, Tz) \geq 1$. Also, if $\gamma(x, y, y) \geq 1$ and $\gamma(y, y, z) \geq 1$, then $x, y, z \in [0, 1)$ and hence $\gamma(x, z, z) \geq 1$. Thus, the first assertion holds. Notice that $\gamma(0, 0, 0) = 1$.
Next, if $\{x_n\}$ is a $(T, \gamma)-$orbital sequence such that $x_n \to x$ as $n \to \infty$. By the definition of γ, we have that $x_n \in [0, 1)$ and $x \in [0, 1)$. Then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\gamma(x_{n_k}, x, x) \geq 1$, $\forall k \in \mathbb{N}$.
Finally, we will show that T is generalized $(\gamma - \psi)-$MKC mapping of type A.
If $\gamma(x, y, y) = 0$, it is obviously to verify the assertion.
If $\gamma(x, y, y) \neq 0$, it follows that $x, y \in [0, 1)$ and $\gamma(x, y, y) = 1$.
For $\epsilon > 0$,
Case 1. If $M_1(x, y) = 2|x - y|$, taking $\delta = \epsilon$, then $\epsilon \leq \psi(M_1(x, y)) = |x - y| < 2\epsilon$ implies that $\gamma(x, y, y)S(Tx, Ty, Ty) = \frac{|x-y|}{2} < \varepsilon$.
Case 2. If $M_1(x, y) = \frac{3|x|}{2}$, taking $\delta = \frac{\epsilon}{3}$, then $\epsilon \leq \psi(M_1(x, y)) = \frac{3|x|}{4} < \epsilon + \frac{\epsilon}{3}$ implies that $\gamma(x, y, y)S(Tx, Ty, Ty) = \frac{1}{2}|x - y| \leq \frac{1}{2}(|x| + |y|) < \frac{1}{2}(|x| + |x|) = |x| < \epsilon$.
Case 3. If $M_1(x, y) = \frac{3|y|}{2}$, taking $\delta = \frac{\epsilon}{3}$, then $\epsilon \leq \psi(M_1(x, y)) = \frac{3|y|}{4} < \epsilon + \frac{\epsilon}{3}$ implies that $\gamma(x, y, y)S(Tx, Ty, Ty) = \frac{1}{2}|x - y| \leq \frac{1}{2}(|x| + |y|) < \frac{1}{2}(|y| + |y|) = |y| < \epsilon$.
Therefore, conditions of Theorem 2.8 hold and T has a fixed point. Indeed, $x^* = 0$ and $x^* = 1$ are two fixed points.

In what follows, we present an existence theorem for fixed point of a generalized $(\gamma - \psi)-$MKC mapping of type B and a generalized $(\gamma - \psi)-$MKC mapping of dims3 of type B. Taking Remark 1.5 and Remark 1.6 into account, we observe that the proof of this theorem is similar to the proof of Theorem 2.8 and Theorem 2.9.

Theorem 2.10. Let (X, S) be a complete $S-$metric space and $T : X \mapsto X$ be a generalized $(\gamma - \psi)-$MKC mapping of type B. Assume also that:

(A1) T is triangular $\gamma-$admissible;

(A2) there exists $x_0 \in X$ such that $\gamma(x_0, Tx_0, Tx_0) \geq 1$;

(A3) (X, S) is $(T, \gamma)-$regular.

Then, there exists $x^* \in X$ such that $Tx^* = x^*$.

Theorem 2.11. Let (X, S) be a complete $S-$metric space and $T : X \mapsto X$ be a generalized $(\gamma - \psi)-$MKC mapping of dim3 of type B. Assume also that:

(A1) T is triangular $\gamma-$admissible;

(A2) there exists $x_0 \in X$ such that $\gamma(x_0, Tx_0, Tx_0) \geq 1$;

(A3) (X, S) is $(T, \gamma)-$limit.

Then, there exists $x^* \in X$ such that $Tx^* = x^*$.

Definition 2.1. Let (X, S) be an $S-$metric space and $T : X \mapsto X$. The mapping T is called a generalized $(\gamma - \psi)-$Meir-Keeler contractive mapping of type C if there exist two functions $\psi \in \Psi$ and $\gamma : X^3 \mapsto [0, \infty)$ satisfying the following condition: for each $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ such that

$$\epsilon \leq \psi(M_3(x, y)) < \epsilon + \delta(\epsilon) \quad implies \quad \gamma(x, y, y)S(Tx, Ty, Ty) < \epsilon, \tag{36}$$

where $M_3(x, y) = \max\{S(x, y, y), S(x, Tx, Tx), S(y, Ty, Ty), \frac{1}{8}[S(x, Ty, Ty) + S(y, Tx, Tx)]\}$, $\forall x, y \in X$.

Theorem 2.12. Let (X, S) be a complete $S-$metric space and $T : X \mapsto X$ be a generalized $(\gamma - \psi)-$MKC mapping of type C. Assume also that:

(A1) T is triangular $\gamma-$admissible;

(A2) there exists $x_0 \in X$ such that $\gamma(x_0, Tx_0, Tx_0) \geq 1$;

(A3) (X, S) is $(T, \gamma)-$regular.

Then, there exists $x^* \in X$ such that $Tx^* = x^*$.

Proof. In view of assumption $(A2)$, let $x_0 \in X$ be such that $\gamma(x_0, Tx_0, Tx_0) \geq 1$.
Define the sequence $\{x_n\}$ in X by $x_{n+1} = Tx_n$, $\forall n \in \{0\} \cup \mathbb{N}$.
Since T is triangular $\gamma-$admissible, by Lemma 1.6, we have that

$$\gamma(x_n, x_m, x_m) \geq 1, \quad \forall n, m \in \mathbb{N} \quad with \quad n < m.$$

If there exists some $n_0 \in \mathbb{N}$ such that $x_{n_0} = x_{n_0+1}$, then the proof is complete, since $x^* = x_{n_0+1} = Tx_0 = Tx^*$. For this, we assume that $x_n \neq x_{n+1}$, $\forall n \in \{0\} \cup \mathbb{N}$, then

$$S(x_n, x_{n+1}, x_{n+1}) > 0, \quad \forall n \in \{0\} \cup \mathbb{N}. \tag{37}$$

Step1. We will prove that

$$\lim_{n \to \infty} S(x_n, x_{n+1}, x_{n+1}) = 0. \tag{38}$$

Taking (36) and (38) into account together with the fact that T is generalized $(\gamma-\psi)-$MKC mapping of type C, for each $n \in \{0\} \cup \mathbb{N}$, we get

$$\begin{aligned} S(x_n, x_{n+1}, x_{n+1}) &= S(Tx_{n-1}, Tx_n, Tx_n) \\ &\leq \gamma(x_{n-1}, x_n, x_n) S(Tx_{n-1}, Tx_n, Tx_n) \\ &\leq \psi(M_3(x_{n-1}, x_n)) \\ &< \psi(M_3(x_{n-1}, x_n)), \end{aligned}$$

where

$$\begin{aligned} M_3(x_{n-1}, x_n) = \max\{&S(x_{n-1}, x_n, x_n), S(x_{n-1}, Tx_{n-1}, Tx_{n-1}), S(x_n, Tx_n, Tx_n), \\ &\frac{1}{8}[S(x_{n-1}, Tx_n, Tx_n) + S(x_n, Tx_{n-1}, Tx_{n-1})]\} \\ = \max\{&S(x_{n-1}, x_n, x_n), S(x_n, x_n, x_{n+1}), \frac{1}{8}[S(x_{n-1}, x_{n+1}, x_{n+1}) + S(x_n, x_n, x_n)]\}. \end{aligned}$$

Regarding Lemma 1.1, we estimate the last term in the expression of $M_3(x_{n-1}, x_n)$ as follows:

$$\begin{aligned} &\frac{1}{8}[S(x_{n-1}, x_{n+1}, x_{n+1}) + S(x_n, x_n, x_n)] \\ &= \frac{1}{8} S(x_{n-1}, x_{n+1}, x_{n+1}) \\ &\leq \frac{1}{8}[2S(x_{n-1}, x_n, x_n) + S(x_n, x_{n+1}, x_{n+1}) \\ &= \frac{1}{4} S(x_{n-1}, x_n, x_n) + \frac{1}{8} S(x_n, x_{n+1}, x_{n+1}) \\ &\leq \max\{S(x_{n-1}, x_{n+1}, x_{n+1}), S(x_n, x_n, x_n)\}. \end{aligned}$$

Consequently, we get that

$$M_3(x_{n-1}, x_n) = \max\{S(x_{n-1}, x_n, x_n), S(x_n, x_{n+1}, x_{n+1})\}. \tag{39}$$

Let us consider the two cases. If $M_3(x_{n-1}, x_n) = S(x_n, x_{n+1}, x_{n+1})$. Since ψ is nondecreasing, then we have that

$$S(x_n, x_{n+1}, x_{n+1}) \leq \psi(S(x_n, x_{n+1}, x_{n+1})) < S(x_n, x_{n+1}, x_{n+1}), \tag{40}$$

which is a contradiction. Thus, $M_3(x_{n-1}, x_n) = S(x_{n-1}, x_n, x_n)$ and we also have that

$$S(x_n, x_{n+1}, x_{n+1}) \leq \psi(S(x_{n-1}, x_n, x_n)) < S(x_{n-1}, x_n, x_n), \quad \forall n \in \mathbb{N}. \tag{41}$$

So, we derive that the sequence $\{S(x_n, x_{n+1}, x_{n+1})\}$ is non-increasing and bounded below by zero. Hence, there exists $t \in [0, \infty)$ such that

$$\lim_{n\to\infty} S(x_n, x_{n+1}, x_{n+1}) = t. \tag{42}$$

Recursively, we deduce from (41) that

$$S(x_n, x_{n+1}, x_{n+1}) \leq \psi^n(S(x_0, x_1, x_1)), \quad \forall n \in \mathbb{N}. \tag{43}$$

On account of (43) and Remark 1.2, we obtain

$$\lim_{n\to\infty} S(x_n, x_{n+1}, x_{n+1}) = 0. \tag{44}$$

Step2. We will show that $\{x_n\}$ is a Cauchy sequence.
Suppose, on the contrary, that there exist $\epsilon > 0$ and a subsequence $\{x_{n(i)}\}$ of $\{x_n\}$ such that

$$S(x_{n(i)}, x_{n(i+1)}, x_{n(i+1)}) > 2\epsilon. \tag{45}$$

First, we will show that the existence of $k \in \mathbb{N}$ such that $n(i) < k \leq n(i+1)$. Later, we will prove that for given $\epsilon > 0$ above, there exists $\delta > 0$ such that

$$\frac{\epsilon}{2} < \psi(M_3(x_{n(i)}, x_k)) < \frac{\epsilon + \delta}{2},$$

but

$$\gamma(x_{n(i)}, x_k, x_k) S(Tx_{n(i)}, Tx_k, Tx_k) \geq \epsilon,$$

which contradicts (45), where

$$\begin{aligned} M_3(x_{n(i)}, x_k) = &\max\{S(x_{n(i)}, x_k, x_k), S(x_{n(i)}, Tx_{n(i)}, Tx_{n(i)1}), S(x_k, Tx_k, Tx_k), \\ &\frac{1}{8}[S(x_{n(i)}, Tx_k, Tx_k) + S(x_k, Tx_{n(i)}, Tx_{n(i)})]\}. \end{aligned}$$

Let $r = \min\{\epsilon, \frac{\delta}{2}\}$. Taking Step1 into account, we will choose $n_0 \in \mathbb{N}$ such that

$$S(x_n, x_{n+1}, x_{n+1}) < \frac{r}{8}, \tag{46}$$

for all $n > n_0$. Let $n(i) > n_0$. According to our construction, we have $n(i) \leq n(i+1) - 1$.
If $S(x_{n(i)}, x_{n(i+1)-1}, x_{n(i+1)-1}) < \frac{\epsilon + r}{2}$, then by Lemma 1.1, we have

$$\begin{aligned} S(x_{n(i)}, x_{n(i+1)}, n_{n(i+1)}) &\leq 2S(x_{n(i)}, x_{n(i+1)-1}, n_{n(i+1)-1}) + S(x_{n(i+1)-1}, x_{n(i+1)}, n_{n(i+1)}) \\ &\leq \epsilon + r + \frac{r}{8} \\ &= \epsilon + \frac{7r}{8} \\ &< 2\epsilon, \end{aligned}$$

which contradicts (45). Therefore, there exist values of $k \in \mathbb{N}$ such that $n(i) \leq k \leq n(i+1)$ and $S(x_{n(i)}, x_k, x_k) > \frac{\epsilon + r}{2}$.
Indeed, if $S(x_{n(i)}, x_{n(i)+1}, x_{n(i)+1}) \geq \frac{\epsilon + r}{2}$, then we have $S(x_{n(i)}, x_{n(i)+1}, x_{n(i)+1}) \geq \frac{r}{8}$, which contradicts (46).
Hence, we can choose the smallest integer $k > n(i)$ such that

$$S(x_{n(i)}, x_k, x_k) \geq \frac{\epsilon + r}{2}.$$

So, necessarily, we also have $S(x_{n(i)}, x_{k-1}, x_{k-1}) < \frac{\epsilon + r}{2}$.
Therefore, we find that

$$\begin{aligned} S(x_{n(i)}, x_k, x_k) &\leq 2S(x_k, x_{k-1}, x_{k-1}) + S(x_{n(i)}, x_{k-1}, x_{k-1}) \\ &< 2 \cdot \frac{r}{8} + \frac{\epsilon + r}{2} \\ &= \frac{\epsilon}{2} + \frac{3r}{4}. \end{aligned}$$

Hence, we get the following approximation:

$$\frac{\epsilon + r}{2} \leq S(x_{n(i)}, x_k, x_k) \leq \frac{\epsilon}{2} + \frac{3r}{4}, \tag{47}$$

for a integer k satisfying $n(i) \leq k \leq n(i+1)$.
On the other hand, the first three terms of $M_3(x_{n(i)}, x_k)$ are bounded above by $\frac{\epsilon}{2} + r$, that is

$$S(x_{n(i)}, x_k, x_k) < \frac{\epsilon}{2} + \frac{3r}{4} < \frac{\epsilon}{2} + r.$$
$$S(x_{n(i)}, x_{n(i)+1}, x_{n(i)+1}) < \frac{r}{8} < \frac{\epsilon}{2} + r.$$
$$S(x_k, x_{k+1}, x_{k+1}) < \frac{r}{8} < \frac{\epsilon}{2} + r.$$

Eventually, the last term of $M_3(x_{n(i)}, x_k)$ can be estimated as follows:

$$\begin{aligned}
&\frac{1}{8}[S(x_{n(i)}, Tx_k, Tx_k) + S(x_k, Tx_{n(i)}, Tx_{n(i)})] \\
&\leq \frac{1}{8}[2S(x_{n(i)}, x_k, x_k) + S(x_k, x_{k+1}, x_{k+1}) + 2S(x_{n(i)}, x_k, x_k) + S(x_{n(i)}, x_{n(i)+1}, x_{n(i)+1})] \\
&= \frac{1}{8}[4S(x_{n(i)}, x_k, x_k) + S(x_k, x_{k+1}, x_{k+1}) + S(x_{n(i)}, x_{n(i)+1}, x_{n(i)+1})] \\
&< \frac{\epsilon}{4} + \frac{3r}{8} + \frac{r}{32} \\
&= \frac{\epsilon}{4} + \frac{13r}{32} \\
&< \frac{\epsilon}{2} + r.
\end{aligned}$$

Combing these estimations presented above, we conclude that

$$\psi(M_3(x_{n(i)}, x_k)) < M_3(x_{n(i)}, x_k) < \frac{\epsilon}{2} + r < \frac{\epsilon + \delta}{2}.$$

Since T is generalized $(\gamma - \psi)-$MKC mapping of type of C and it is $\gamma-$triangular admissible mapping, we have that

$$S(x_{n(i)+1}, x_{k+1}, x_{k+1}) \leq \gamma(x_{n(i)}, x_k, x_k) S(x_{n(i)+1}, x_{k+1}, x_{k+1}) < \frac{\epsilon}{2}.$$

At the same time, by Lemma 1.1, we have that

$$\begin{aligned}
S(x_{n(i)}, x_k, x_k) &\leq 2S(x_{n(i)}, x_{n(i)+1}, x_{n(i)+1}) + S(x_{n(i)+1}, x_k, x_k) \\
&\leq 2S(x_{n(i)}, x_{n(i)+1}, x_{n(i)+1}) + 2S(x_k, x_{k+1}, x_{k+1}) + S(x_{n(i)+1}, x_{k+1}, x_{k+1}) \\
&< 2 \cdot \frac{r}{8} + 2 \cdot \frac{r}{8} + \frac{\epsilon}{2} \\
&= \frac{\epsilon + r}{2},
\end{aligned}$$

which contradicts (47).
Thus, claim (45) is false and the sequence $\{x_n\}$ is a Cauchy sequence, that is

$$\lim_{n,m \to \infty} S(x_n, x_m, x_m) = 0. \tag{48}$$

Since (X, S) is a complete $S-$metric space, then there exists $x^* \in X$ such that

$$\lim_{n \to \infty} S(x_n, x^*, x^*) = \lim_{n,m \to \infty} S(x_n, x_m, x^*) = 0.$$

We will prove that $x^* = Tx^*$. Suppose, on the contrary, that $S(x^*, Tx^*, Tx^*) > 0$.
From (42) and assumption $(A3)$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\gamma(x_{n_k}, x^*, x^*) \geq 1, \quad \forall k \in \mathbb{N}. \tag{49}$$

By using Lemma 1.1 and (48) together with the assumption of the theorem that T is a generalized $(\gamma-\psi)-$MKC mapping of type C, we get that

$$\begin{aligned}S(x^*,Tx^*,Tx^*) &\leq 2S(Tx_{n_k},Tx^*,Tx^*)+S(Tx_{n_k},x^*,x^*)\\ &\leq 2\gamma(x_{n_k},x^*,x^*)S(Tx_{n_k},Tx^*,Tx^*)+S(x_{n_k+1},x^*,x^*)\\ &\leq \psi(M_3(x_{n_k},x^*))+S(x_{n_k+1},x^*,x^*),\end{aligned}$$

where,

$$\begin{aligned}M_3(x_{n_k},x^*) =& \max\{S(x_{n_k},x^*,x^*),S(x_{n_k},x_{n_k+1},x_{n_k+1}),S(x^*,Tx^*,Tx^*),\\ &\frac{1}{8}[S(x_{n_k},Tx^*,Tx^*)+S(x^*,x_{n_k+1},x_{n_k+1})]\}.\end{aligned}$$

Notice that as $S(x^*,Tx^*,Tx^*)>0$, then we have that $M_3(x_{n_k},x^*)>0$.
From Lemma 1.1, it follows that

$$\begin{aligned}&\frac{1}{8}[S(x_{n_k},Tx^*,Tx^*)+S(x^*,x_{n_k+1},x_{n_k+1})]\\ &\leq \frac{1}{8}[2S(x_{n_k},x^*,x^*)+S(x^*,Tx^*,Tx^*)+2S(x^*,x_{n_k},x_{n_k})+S(x_{n_k},x_{n_k+1},x_{n_k+1})]\\ &= \frac{1}{2}S(x_{n_k},x^*,x^*)+\frac{1}{8}S(x^*,Tx^*,Tx^*)+\frac{1}{8}S(x_{n_k},x_{n_k+1},x_{n_k+1})\\ &\leq \max\{S(x_{n_k},x^*,x^*),S(x^*,Tx^*,Tx^*),S(x_{n_k},x_{n_k+1},x_{n_k+1})\}.\end{aligned}$$

By the above inequality, we have that

$$M_3(x_{n_k},x^*)= \quad \max\{S(x_{n_k},x^*,x^*),S(x^*,Tx^*,Tx^*),S(x_{n_k},x_{n_k+1},x_{n_k+1})\}.$$

Suppose that $M_3(x_{n_k},x^*)=S(x_{n_k},x^*,x^*)$, then, we get that

$$\begin{aligned}S(x^*,Tx^*,Tx^*) &\leq \psi(S(x_{n_k},x^*,x^*))+S(x_{n_k+1},x^*,x^*)\\ &< S(x_{n_k},x^*,x^*)+S(x_{n_k+1},x^*,x^*).\end{aligned}$$

Taking $k\to\infty$ in the inequality above, we have

$$S(x^*,Tx^*,Tx^*)<2S(x^*,x^*,x^*)=0,$$

which is a contradiction.
Next, we suppose that $M_3(x_{n_k},x^*)=S(x_{n_k},x_{n_k+1},x_{n_k+1})$, then we have that

$$\begin{aligned}S(x^*,Tx^*,Tx^*) &\leq \psi(S(x_{n_k},x_{n_k+1},x_{n_k+1}))+S(x_{n_k+1},x^*,x^*)\\ &< S(x_{n_k},x_{n_k+1},x_{n_k+1})+S(x_{n_k+1},x^*,x^*).\end{aligned}$$

Taking $k\to\infty$ in the inequality above, this implies that

$$S(x^*,Tx^*,Tx^*)<\lim_{k\to\infty}[S(x_{n_k},x_{n_k+1},x_{n_k+1})+S(x_{n_k+1},x^*,x^*)]=0,$$

which is again a contradiction.
Finally, we suppose that $M_3(x_{n_k},x^*)=S(x^*,Tx^*,Tx^*)$, then we obtain that

$$S(x^*,Tx^*,Tx^*) \quad <\psi(S(x^*,Tx^*,Tx^*))+S(x_{n_k+1},x^*,x^*)<S(x^*,Tx^*,Tx^*)+S(x_{n_k+1},x^*,x^*).$$

Letting $k \to \infty$ in above inequality, we get that

$$\begin{aligned} S(x^*, Tx^*, Tx^*) &< S(x^*, Tx^*, Tx^*) + S(x^*, x^*, x^*) \\ &= S(x^*, Tx^*, Tx^*), \end{aligned}$$

so we also have a contradiction. Thus, we have $S(x^*, Tx^*, Tx^*) = 0$, and by $(S1)$ in Definition 1.1, we have $x^* = Tx^*$. $\square$

Definition 2.2. Let (X, S) be an $S-$metric space and $T : X \mapsto X$. The mapping T is called a generalized $(\gamma - \psi)-$Meir-Keeler contractive mapping of dim3 of type C if there exist two functions $\psi \in \Psi$ and $\gamma : X^3 \mapsto [0, \infty)$ satisfying the following condition: for each $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ such that

$$\epsilon \leq \psi(M_3^{'}(x, y, z)) < \epsilon + \delta(\epsilon) \quad implies \quad \gamma(x, Tx, Tx)\gamma(y, Ty, Ty)\gamma(z, Tz, Tz)S(Tx, Ty, Tz) < \epsilon,$$

where

$$\begin{aligned} M_3^{'}(x, y, z) = \max\{&S(x, y, y), S(y, z, z), S(z, x, x), S(x, Tx, Tx), S(y, Ty, Ty), S(z, Tz, Tz), \\ &\frac{1}{8}[S(x, Ty, Ty) + S(y, Tx, Tx)], \frac{1}{8}[S(y, Tz, Tz) + S(z, Ty, Ty)], \\ &\frac{1}{8}[S(z, Tx, Tx) + S(x, Tz, Tz)]\}, \end{aligned}$$

$\forall x, y, z \in X$.

Theorem 2.13. Let (X, S) be a complete $S-$metric space and $T : X \mapsto X$ be a generalized $(\gamma - \psi)-$MKC mapping of dim3 of type C. Assume also that:

(A1) T is $\gamma-$admissible;

(A2) there exists $x_0 \in X$ such that $\gamma(x_0, Tx_0, Tx_0) \geq 1$;

(A3) (X, S) is $(T, \gamma)-$limit.

Then, there exists $x^* \in X$ such that $Tx^* = x^*$.

Proof. In view of assumption $(A2)$, let $x_0 \in X$ be such that $\gamma(x_0, Tx_0, Tx_0) \geq 1$.
Define the sequence $\{x_n\}$ in X by $x_{n+1} = Tx_n$, for all $n \in \{0\} \cup \mathbb{N}$.
Since T is triangular $\gamma-$admissible, by Lemma 1.6, we have that

$$\gamma(x_n, x_{n+1}, x_{n+1}) \geq 1, \quad \forall n \in \mathbb{N} \quad . \tag{50}$$

If there exists some $n_0 \in \mathbb{N}$ such that $x_{n_0} = x_{n_0+1}$, then the proof is complete, since $x^* = x_{n_0+1} = Tx_{n_0} = Tx^*$. For this, we assume that $x_n \neq x_{n+1}$, for all $n \in \mathbb{N}$, then

$$S(x_n, x_{n+1}, x_{n+1}) > 0, \quad \forall n \in \{0\} \cup \mathbb{N}. \tag{51}$$

Step1. We will prove that

$$\lim_{n\to\infty} S(x_n, x_{n+1}, x_{n+1}) = 0.$$

Taking (50) and (51) into account together with the fact that T is generalized $(\gamma-\psi)-$MKC mapping of dim3 of type C, for each $n\in\mathbb{N}$, we get

$$\begin{aligned}
S(x_n,x_{n+1},x_{n+1}) &= S(Tx_{n-1},Tx_n,Tx_n)\\
&\le \gamma(x_{n-1},x_n,x_n)\gamma(x_n,x_{n+1},x_{n+1})\gamma(x_n,x_{n+1},x_{n+1})S(Tx_{n-1},Tx_n,Tx_n)\\
&\le \psi(M_3^{'}(x_{n-1},x_n,,x_n))\\
&< M_3^{'}(x_{n-1},x_n,x_n),
\end{aligned}$$

where

$$\begin{aligned}
M_3^{'}(x_{n-1},x_n,x_n) = \max\{&S(x_{n-1},x_n,x_n),S(x_n,x_n,x_n),S(x_n,x_{n-1},x_{n-1}),S(x_{n-1},Tx_{n-1},Tx_{n-1}),\\
&S(x_n,Tx_n,Tx_n),S(x_n,Tx_n,Tx_n),\frac{1}{8}[S(x_{n-1},Tx_n,Tx_n)+S(x_n,Tx_{n-1},Tx_{n-1})],\\
&\frac{1}{8}[S(x_n,Tx_n,Tx_n)+S(x_n,Tx_n,Tx_n)],\frac{1}{8}[S(x_n,Tx_{n-1},Tx_{n-1})+S(x_{n-1},Tx_n,Tx_n)]\}\\
&=\max\{S(x_{n-1},x_n,x_n),S(x_n,x_n,x_{n+1}),\frac{1}{8}[S(x_{n-1},x_{n+1},x_{n+1})+S(x_n,x_n,x_n)],\\
&\frac{1}{8}[2S(x_n,x_{n+1},x_{n+1})],\frac{1}{8}[S(x_{n-1},x_{n+1},x_{n+1})+S(x_n,x_n,x_n]\}.
\end{aligned}$$

Regarding Lemma 1.1, we estimate the last term in the expression of $M_3^{'}(x_{n-1},x_n,x_n)$ as follows:

$$\begin{aligned}
&\frac{1}{8}[S(x_{n-1},x_{n+1},x_{n+1})+S(x_n,x_n,x_n)]\\
&=\frac{1}{8}S(x_{n-1},x_{n+1},x_{n+1})\\
&\le\frac{1}{8}[2S(x_{n-1},x_n,x_n)+S(x_n,x_{n+1},x_{n+1})\\
&=\frac{1}{4}S(x_{n-1},x_n,x_n)+\frac{1}{8}S(x_n,x_{n+1},x_{n+1})\\
&\le\max\{S(x_{n-1},x_{n+1},x_{n+1}),S(x_n,x_n,x_n)\}.
\end{aligned}$$

Consequently, we get that

$$M_3(x_{n-1},x_n,x_n)=\max\{S(x_{n-1},x_n,x_n),S(x_n,x_{n+1},x_{n+1})\}.$$

Let us consider the two cases. If $M_3^{'}(x_{n-1},x_n,x_n)=S(x_n,x_{n+1},x_{n+1})$. Since ψ is nondecreasing, then we have that

$$S(x_n,x_{n+1},x_{n+1})\le\psi(S(x_n,x_{n+1},x_{n+1}))<S(x_n,x_{n+1},x_{n+1}),$$

which is a contradiction. Thus, $M_3^{'}(x_{n-1},x_n,x_n)=S(x_{n-1},x_n,x_n)$ and we also have that

$$S(x_n,x_{n+1},x_{n+1})\le\psi(S(x_{n-1},x_n,x_n))<S(x_{n-1},x_n,x_n),\quad\forall n\in\mathbb{N}. \tag{52}$$

So, we derive that the sequence $\{S(x_n,x_{n+1},x_{n+1})\}$ is non-increasing and bounded below by zero. Hence, there exists $t\in[0,\infty)$ such that

$$\lim_{n\to\infty}S(x_n,x_{n+1},x_{n+1})=t.$$

Recursively, we deduce from (52) that

$$S(x_n,x_{n+1},x_{n+1})\le\psi^n(S(x_0,x_1,x_1)),\quad\forall n\in\mathbb{N}. \tag{53}$$

On account of (53) and Remark 1.2, we obtain

$$\lim_{n\to\infty} S(x_n, x_{n+1}, x_{n+1}) = 0.$$

Step2. We will show that $\{x_n\}$ is a Cauchy sequence.
We will prove that for each $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for $\forall m, n \geq n_0$,

$$S(x_m, x_m, x_n) < \epsilon. \tag{54}$$

Taking Step1 into account, for each $\epsilon > 0$, we can choose $n_0 \in \mathbb{N}$ such that

$$S(x_n, x_{n+1}, x_{n+1}) < \frac{\epsilon - \psi(\epsilon)}{2}, \forall n \geq n_0. \tag{55}$$

We prove (54) by induction on n. (54) holds for $m = n$ and $n = n+1$ by using (55) and the fact that $\frac{\epsilon-\psi(\epsilon)}{2} < \epsilon$. Assume (54) holds for $n = k$. For $n = k+1$, we have

$$\begin{aligned} S(x_m, x_{k+1}, x_{k+1}) &\leq 2S(x_m, x_{m+1}, x_{m+1}) + S(x_{m+1}, x_{k+1}, x_{k+1}) \\ &\leq \epsilon - \psi(\epsilon) + \psi(S(x_m, x_k, x_k)) \\ &\leq \epsilon - \psi(\epsilon) + \psi(\epsilon) \\ &= \epsilon. \end{aligned}$$

By induction on n, we conclude that (54) holds for all $n \geq m \geq n_0$. So $\{x_n\}$ is a Cauchy sequence that is

$$\lim_{n,m\to\infty} S(x_n, x_m, x_m) = 0.$$

Since (X, S) is a complete $S-$metric space, then there exists $x^* \in X$ such that

$$\lim_{n\to\infty} S(x_n, x^*, x^*) = \lim_{n,m\to\infty} S(x_n, x_m, x^*) = 0.$$

We will prove that $x^* = Tx^*$. Suppose, on the contrary, that $S(x^*, Tx^*, Tx^*) > 0$.
From assumption $(A3)$, we have that

$$\gamma(x^*, Tx^*, Tx^*) \geq 1, \quad \forall k \in \mathbb{N}.$$

By using Lemma 1.1 and above inequality together with the assumption of the theorem that T is a generalized $(\gamma - \psi)-$MKC mapping of dim3 of type C,

$$\begin{aligned} S(x^*, Tx^*, Tx^*) &\leq 2S(Tx_{n_k}, Tx^*, Tx^*) + S(Tx_{n_k}, x^*, x^*) \\ &\leq 2\gamma(x_{n_k}, Tx_{n_k}, Tx_{n_k})\gamma(x^*, Tx^*, Tx^*)\gamma(x^*, Tx^*, Tx^*)S(Tx_{n_k}, Tx^*, Tx^*) + S(x_{n_k+1}, x^*, x^*) \\ &\leq \psi(M_3^{'}(x_{n_k}, x^*, x^*)) + S(x_{n_k+1}, x^*, x^*), \end{aligned}$$

where,

$$\begin{aligned} M_3^{'}(x_{n_k}, x^*, x^*) = \max\{ & S(x_{n_k}, x^*, x^*), S(x^*, x^*, x^*), S(x^*, x_{n_k}, x_{n_k}), S(x_{n_k}, x_{n_k+1}, x_{n_k+1}), \\ & S(x^*, Tx^*, Tx^*), S(x^*, Tx^*, Tx^*), S(x^*, Tx^*, Tx^*), \\ & \frac{1}{8}[S(x_{n_k}, Tx^*, Tx^*) + S(x^*, x_{n_k+1}, x_{n_k+1})]\}. \end{aligned}$$

Notice that as $S(x^*, Tx^*, Tx^*) > 0$, then we have that $M_3^{'}(x_{n_k}, x^*, x^*) > 0$.
From Lemma 1.1, it follows that

$$\begin{aligned}
&\frac{1}{8}[S(x_{n_k}, Tx^*, Tx^*) + S(x^*, x_{n_k+1}, x_{n_k+1})] \\
&\le \frac{1}{8}[2S(x_{n_k}, x^*, x^*) + S(x^*, Tx^*, Tx^*) + 2S(x^*, x_{n_k}, x_{n_k}) + S(x_{n_k}, x_{n_k+1}, x_{n_k+1})] \\
&= \frac{1}{2}S(x_{n_k}, x^*, x^*) + \frac{1}{8}S(x^*, Tx^*, Tx^*) + \frac{1}{8}S(x_{n_k}, x_{n_k+1}, x_{n_k+1}) \\
&\le \max\{S(x_{n_k}, x^*, x^*), S(x^*, Tx^*, Tx^*), S(x_{n_k}, x_{n_k+1}, x_{n_k+1})\}.
\end{aligned}$$

By the above inequality, we have that

$$M_3^{'}(x_{n_k}, x^*, x^*) = \max\{S(x_{n_k}, x^*, x^*), S(x^*, Tx^*, Tx^*), S(x_{n_k}, x_{n_k+1}, x_{n_k+1})\}.$$

Suppose that $M_3^{'}(x_{n_k}, x^*, x^*) = S(x_{n_k}, x^*, x^*)$, then, we get that

$$\begin{aligned}
S(x^*, Tx^*, Tx^*) &\le \psi(S(x_{n_k}, x^*, x^*)) + S(x_{n_k+1}, x^*, x^*) \\
&< S(x_{n_k}, x^*, x^*) + S(x_{n_k+1}, x^*, x^*).
\end{aligned}$$

Taking $k \to \infty$ in the inequality above, we have

$$S(x^*, Tx^*, Tx^*) < 2S(x^*, x^*, x^*) = 0,$$

which is a contradiction.
Next, we suppose that $M_3^{'}(x_{n_k}, x^*, x^*) = S(x_{n_k}, x_{n_k+1}, x_{n_k+1})$, then we have that

$$\begin{aligned}
S(x^*, Tx^*, Tx^*) &\le \psi(S(x_{n_k}, x_{n_k+1}, x_{n_k+1})) + S(x_{n_k+1}, x^*, x^*) \\
&< S(x_{n_k}, x_{n_k+1}, x_{n_k+1}) + S(x_{n_k+1}, x^*, x^*).
\end{aligned}$$

Taking $k \to \infty$ in the inequality above, this implies that

$$S(x^*, Tx^*, Tx^*) < \lim_{k\to\infty}[S(x_{n_k}, x_{n_k+1}, x_{n_k+1}) + S(x_{n_k+1}, x^*, x^*)] = 0,$$

which is again a contradiction.
Finally, we suppose that $M_3^{'}(x_{n_k}, x^*, x^*) = S(x^*, Tx^*, Tx^*)$, then we obtain that

$$S(x^*, Tx^*, Tx^*) < \psi(S(x^*, Tx^*, Tx^*)) + S(x_{n_k+1}, x^*, x^*) < S(x^*, Tx^*, Tx^*) + S(x_{n_k+1}, x^*, x^*).$$

Letting $k \to \infty$ in above inequality, we get that

$$\begin{aligned}
S(x^*, Tx^*, Tx^*) &< S(x^*, Tx^*, Tx^*) + S(x^*, x^*, x^*) \\
&= S(x^*, Tx^*, Tx^*),
\end{aligned}$$

so we also have a contradiction. Thus, we have $S(x^*, Tx^*, Tx^*) = 0$, and by $(S1)$ in Definition 1.1, we have $x^* = Tx^*$. □

In what follows, we propose the condition for the uniqueness of a fixed point of a generalized $(\gamma - \psi)-$MKC of type C mappings:
$(U1')$ For $\forall x^*, y^* \in Fix(T)$, there exists $z^* \in X$ such that $\gamma(x^*, z^*, z^*) \ge 1$, $\gamma(y^*, z^*, z^*) \ge 1$ and $\gamma(z^*, Tz^*, Tz^*) \ge 1$.
$(U2')$ Let $x^*, y^* \in Fix(T)$. If there exists a sequence $\{x_n\}$ in X such that $\gamma(x^*, x_n, x_n) \ge 1$, $\gamma(y^*, x_n, x_n) \ge 1$, then $S(x_n, x_{n+1}, x_{n+1}) \le \inf\{S(x^*, x_n, x_n), S(y^*, x_n, x_n)\}$.

Theorem 2.14. Adding conditions $(U1'), (U2')$ to the statements of Theorem 2.12, one has that T has the unique fixed point.

Proof. Let x^*, y^* be two distinct fixed points of T. Form condition $(U1')$, there exists $z^* \in X$ such that

$$\gamma(x^*, z^*, z^*) \geq 1, \gamma(y^*, z^*, z^*) \geq 1, \gamma(z^*, Tz^*, Tz^*) \geq 1.$$

Owing to the fact that T is triangular $\gamma-$admissible and $\gamma(z^*, Tz^*, Tz^*) \geq 1$, we have

$$\gamma(Tz^*, T^2z^*, T^2z^*) \geq 1.$$

Inductively, we find

$$\gamma(T^{n-1}z^*, T^nz^*, T^nz^*) \geq 1, \quad \forall n \in \mathbb{N}.$$

Since $\gamma(x^*, z^*, z^*) \geq 1$ and $\gamma(z^*, Tz^*, Tz^*) \geq 1$, then by the triangular $\gamma-$admissibility of T, we have

$$\gamma(x^*, Tz^*, Tz^*) \geq 1.$$

Again, since $\gamma(x^*, Tz^*, Tz^*) \geq 1$ and $\gamma(Tz^*, T^2z^*, T^2z^*) \geq 1$, we derive

$$\gamma(x^*, T^2z^*, T^2z^*) \geq 1.$$

Inductively, we get

$$\gamma(x^*, T^nz^*, T^nz^*) \geq 1, \quad \forall n \in \mathbb{N}. \tag{56}$$

In the similar way, we also have that

$$\gamma(y^*, T^nz^*, T^nz^*) \geq 1, \quad \forall n \in \mathbb{N}. \tag{57}$$

Define an iterative sequence $\{z_n\}$ by $z_{n+1} = Tz_n$, $\forall n \in \{0\} \cup \mathbb{N}$ and $z_0 = z^*$.
Step1. We will prove that $\lim\limits_{n\to\infty} S(x^*, z_n, z_n) = 0$.
By (56) and the statement of the theorem that T is generalized $(\gamma - \psi)-$MKC mapping of type C. we have

$$\begin{aligned} S(x^*, z_{n+1}, z_{n+1}) &\leq \gamma(x^*, z_n, z_n) S(Tx^*, Tz_n, Tz_n) \\ &\leq \psi(M_3(x^*, z_n)). \end{aligned}$$

If $\psi(M_3(x^*, z_n)) = 0$, then

$$\lim_{n\to\infty} S(x^*, z_n, z_n) = 0.$$

Now, suppose that $\psi(M_3(x^*, z_n)) > 0$, then $M_3(x^*, z_n) > 0$.
Since T is a generalized $(\gamma - \psi)-$MKC mapping of type C, we get

$$\begin{aligned} S(x^*, z_{n+1}, z_{n+1}) &\leq \gamma(x^*, z_n, z_n) S(Tx^*, Tz_n, Tz_n) \\ &\leq \psi(M_3(x^*, z_n)) \\ &< M_3(x^*, z_n), \end{aligned}$$

where $M_3(x^*, z_n) = \max\{S(x^*, z_n, z_n), S(x^*, Tx^*, Tx^*), S(z_n, Tz_n, Tz_n), \frac{1}{8}[S(x^*, Tz_n, Tz_n)+S(z_n, Tx^*, Tx^*)]\}$. Taking $(U2')$ and Lemma 1.1 into account, we have

$$M_3(x^*, z_n) = S(x^*, z_n, z_n).$$

Thus, $S(x^*, z_{n+1}, z_{n+1}) < S(x^*, z_n, z_n)$.
Letting $n \to \infty$ in the inequality above, we obtain

$$\lim_{n\to\infty} S(x^*, z_{n+1}, z_{n+1}) < \lim_{n\to\infty} S(x^*, z_n, z_n),$$

which is a contradiction. Then,

$$M_3(x^*, z_n) = S(x^*, z_n, z_n) = 0.$$

Hence, we get that

$$\lim_{n\to\infty} S(x^*, z_n, z_n) = 0.$$

Step2. We will prove that $\lim_{n\to\infty} S(y^*, z_n, z_n) = 0$.
In a analogous way of Step1., we can complete the proof of $\lim_{n\to\infty} S(y^*, z_n, z_n) = 0$.
By Lemma 1.1,

$$S(x^*, y^*, y^*) \le 2S(x^*, z_n, z_n) + S(y^*, z_n, z_n).$$

Letting $n \to \infty$ in the above inequality, we get

$$S(x^*, y^*, y^*) = 0,$$

therefore, we have $x^* = y^*$. □

As a uniqueness condition for fixed points of $(\gamma - \psi)-$MKC mappings of dim3 of type C, we suggest the following hypothesis:

$(U3')$ For $\forall x^*, y^* \in Fix(T)$, $\gamma(x^*, x^*, x^*) \ge 1$, $\gamma(y^*, y^*, y^*) \ge 1$.

Theorem 2.15. Adding condition $(U3')$ to the statements of Theorem 2.13, one has that T has the unique fixed point.

Proof. Let x^*, y^* be two distinct fixed points of T. Form condition $(U3')$

$$\gamma(x^*, x^*, x^*) \ge 1, \gamma(y^*, y^*, y^*) \ge 1. \tag{58}$$

By (58) and the statement of the theorem that T is generalized $(\gamma - \psi)-$MKC mapping of dim3 of type C. we have

$$\begin{aligned} S(x^*, y^*, y^*) &\le \gamma(x^*, Tx^*, Tx^*)\gamma(y^*, Ty^*, Ty^*)\gamma(y^*, Ty^*, Ty^*)S(Tx^*, Ty^*, Ty^*) \\ &\le \psi(M_3^{'}(x^*, y^*, y^*)) < M_3^{'}(x^*, y^*, y^*). \end{aligned}$$

but

$$\begin{aligned} M_3^{'}(x^*, y^*, y^*) = \max\{&S(x^*, y^*, y^*), S(y^*, y^*, y^*), S(y^*, x^*, x^*), S(x^*, Tx^*, Tx^*), S(y^*, Ty^*, Ty^*), \\ &S(y^*, Ty^*, Ty^*), \frac{1}{8}[S(x^*, Ty^*, Ty^*) + S(y^*, Tx^*, Tx^*)], \\ &\frac{1}{8}[S(y^*, Ty^*, Ty^*) + S(y^*, Ty^*, Ty^*)], \frac{1}{8}[S(y^*, Tx^*, Tx^*) + S(x^*, Ty^*, Ty^*)]\} \\ &= S(x^*, y^*, y^*). \end{aligned}$$

so,

$$S(x^*, y^*, y^*) < S(x^*, y^*, y^*)$$

which is again a contradiction.therefore, we have $x^* = y^*$. □

3. Generalized Ulam-Hyers Stability for MKC mappings

In the following section, by introducing the generalized Ulam-Hyers stability in the framework of $S-$metric spaces, we study the stability for MKC mappings.

Theorem 3.1. Let (X, S) be a complete $S-$metric and $T : X \to X$ be a self-mapping. Suppose that all the hypotheses of Theorem 2.12 hold. In addition, assume that

(A1) the function $\beta : [0, \infty) \to [0, \infty), \beta(r) = r - \psi(r)$ is strictly increasing and onto.

(A2) for any $\epsilon-$solution $w^* \in X$ of (2), one has $\gamma(w^*, x^*, x^*) \geq 1$, where $x^* \in Fix(T)$.

Then, the fixed point problem (1) is generalized Ulam-Hyers stable.

Proof. From the conclusion of Theorem 2.12, it follows that there exists $x^* \in Fix(T)$ such that $S(x^*, Tx^*, Tx^*) = 0$. Let $\epsilon > 0$ and w^* be a $\epsilon-$solution of (2).
From $(A2)$, we have $\gamma(x^*, w^*, w^*) \geq 1$. Since T is triangular $\gamma-$admissible, we can obtain that $\gamma(Tx^*, Tw^*, Tw^*) = \gamma(x^*, Tw^*, Tw^*) \geq 1$.
Thus, we also get that

$$\begin{aligned}
S(x^*, w^*, w^*) &= S(Tx^*, w^*, w^*) \\
&\leq S(Tx^*, Tw^*, Tw^*) + 2S(w^*, Tw^*, Tw^*) \\
&\leq \gamma(Tx^*, Tw^*, Tw^*)S(Tx^*, Tw^*, Tw^*) + 2S(w^*, Tw^*, Tw^*) \\
&< \psi(M_3(x^*, w^*)) + 2\epsilon,
\end{aligned}$$

where $M_3(x^*, w^*) = \max\{S(x^*, w^*, w^*), S(x^*, Tx^*, Tx^*), S(w^*, Tw^*, Tw^*), \frac{1}{8}[S(x^*, Tw^*, Tw^*)+S(w^*, Tx^*, Tx^*)]\}$.
We also get

$$\begin{aligned}
&\frac{1}{8}[S(x^*, Tw^*, Tw^*) + S(w^*, Tx^*, Tx^*)] \\
&\leq \frac{1}{8}[2S(x^*, w^*, w^*) + S(w^*, Tw^*, Tw^*) + 2S(w^*, x^*, x^*) + S(x^*, Tx^*, Tx^*)] \\
&= \frac{1}{8}[4S(x^*, w^*, w^*) + S(w^*, Tw^*, Tw^*)] \\
&= \frac{1}{2}S(x^*, w^*, w^*) + \frac{1}{8}S(w^*, Tw^*, Tw^*) \\
&\leq \frac{1}{2}S(x^*, w^*, w^*) + \frac{1}{8}\epsilon \\
&< \max\{S(x^*, w^*, w^*), \epsilon\}.
\end{aligned}$$

From the inequality above, we have that

$$M_3(x^*, w^*) < \max\{S(x^*, w^*, w^*), \epsilon\}.$$

It is obviously that if $S(x^*, w^*, w^*) < \epsilon$, then the proof is complete.
Suppose that $\max\{S(x^*, w^*, w^*), \epsilon\} = S(x^*, w^*, w^*)$. Then, we have

$$M_3(x^*, w^*) < S(x^*, w^*, w^*).$$

So, we can deduce that

$$S(x^*, w^*, w^*) \le \psi(S(x^*, w^*, w^*)) + 2\epsilon,$$
$$S(x^*, w^*, w^*) - \psi(S(x^*, w^*, w^*)) \le 2\epsilon.$$

From assumption $(A1)$, we get that

$$\beta(S(x^*, w^*, w^*)) \le 2\epsilon.$$

Hence, $S(x^*, w^*, w^*) \le \beta^{-1}(2\epsilon)$.
Therefore, (1) is generalized Ulam-Hyers stable. □

Theorem 3.2. Let (X, S) be a complete $S-$metric and $T : X \to X$ be a self-mapping. Suppose that all the hypotheses of Theorem 2.13 hold. In addition, assume that

(A1) the function $\beta : [0, \infty) \to [0, \infty), \beta(r) = r - \psi(r)$ is strictly increasing and onto.

(A2) for any $\epsilon-$solution $w^* \in X$ of (2), one has $\gamma(w^*, x^*, x^*) \ge 1$, where $x^* \in Fix(T)$.

Then, the fixed point problem (1) is generalized Ulam-Hyers stable.

Proof. From the conclusion of Theorem 2.13, it follows that there exists $x^* \in Fix(T)$ such that $S(x^*, Tx^*, Tx^*) = 0$. Let $\epsilon > 0$ and w^* be a $\epsilon-$solution of (2).
From $(A2)$, we have $\gamma(x^*, w^*, w^*) \ge 1$. Since T is triangular $\gamma-$admissible, we can obtain that $\gamma(Tx^*, Tw^*, Tw^*) = \gamma(x^*, Tw^*, Tw^*) \ge 1$.
Thus, we also get that

$$\begin{aligned} S(x^*, w^*, w^*) &= S(Tx^*, w^*, w^*) \\ &\le S(Tx^*, Tw^*, Tw^*) + 2S(w^*, Tw^*, Tw^*) \\ &\le \gamma(Tx^*, Tw^*, Tw^*)S(Tx^*, Tw^*, Tw^*) + 2S(w^*, Tw^*, Tw^*) \\ &< \psi(M_3^{'}(x^*, w^*, w^*)) + 2\epsilon, \end{aligned}$$

where

$$\begin{aligned} M_3^{'}(x^*, w^*) = \max\{&S(x^*, w^*, w^*), S(w^*, w^*, w^*), S(w^*, x^*, x^*), \\ &S(x^*, Tx^*, Tx^*), S(w^*, Tw^*, Tw^*), \frac{1}{8}[S(x^*, Tw^*, Tw^*) + S(w^*, Tx^*, Tx^*)] \\ &\frac{1}{8}[S(w^*, Tw^*, Tw^*) + S(w^*, Tw^*, Tw^*)], \frac{1}{8}[S(w^*, Tx^*, Tx^*) + S(x^*, Tw^*, Tw^*)]\} \\ &= \max\{S(x^*, w^*, w^*), S(w^*, Tw^*, Tw^*), \frac{1}{8}[S(x^*, Tw^*, Tw^*) + S(w^*, Tx^*, Tx^*)]\} \end{aligned}$$

We also get

$$\begin{aligned} &\frac{1}{8}[S(x^*, Tw^*, Tw^*) + S(w^*, Tx^*, Tx^*)] \\ &\le \frac{1}{8}[2S(x^*, w^*, w^*) + S(w^*, Tw^*, Tw^*) + 2S(w^*, x^*, x^*) + S(x^*, Tx^*, Tx^*)] \\ &= \frac{1}{8}[4S(x^*, w^*, w^*) + S(w^*, Tw^*, Tw^*)] \\ &= \frac{1}{2}S(x^*, w^*, w^*) + \frac{1}{8}S(w^*, Tw^*, Tw^*) \\ &\le \frac{1}{2}S(x^*, w^*, w^*) + \frac{1}{8}\epsilon \\ &< \max\{S(x^*, w^*, w^*), \epsilon\}. \end{aligned}$$

From the inequality above, we have that

$$M_3^{'}(x^*, w^*, w^*) < \max\{S(x^*, w^*, w^*), \epsilon\}.$$

It is obviously that if $S(x^*, w^*, w^*) < \epsilon$, then the proof is complete.
Suppose that $\max\{S(x^*, w^*, w^*), \epsilon\} = S(x^*, w^*, w^*)$. Then, we have

$$M_3^{'}(x^*, w^*, w^*) < S(x^*, w^*, w^*).$$

So, we can deduce that

$$S(x^*, w^*, w^*) \leq \psi(S(x^*, w^*, w^*)) + 2\epsilon,$$
$$S(x^*, w^*, w^*) - \psi(S(x^*, w^*, w^*)) \leq 2\epsilon.$$

From assumption $(A1)$, we get that

$$\beta(S(x^*, w^*, w^*)) \leq 2\epsilon.$$

Hence, $S(x^*, w^*, w^*) \leq \beta^{-1}(2\epsilon)$.
Therefore, (1) is generalized Ulam-Hyers stable. □

Corollary 3.1. Let (X, S) be a complete $S-$metric and $T : X \to X$ be a self-mapping. Suppose that all the hypotheses of Theorem 2.8(resp., Theorem 2.10) hold. In addition, assume that

(A1) the function $\beta : [0, \infty) \to [0, \infty), \beta(r) = r - \psi(r)$ is strictly increasing and onto.

(A2) for any $\epsilon-$solution $w^* \in X$ of (2), one has $\gamma(w^*, x^*, x^*) \geq 1$, where $x^* \in Fix(T)$.

Then, the fixed point problem (1) is generalized Ulam-Hyers stable.

Proof. The proof is an analog of the proof of Theorem 3.1. □

Corollary 3.2. Let (X, S) be a complete $S-$metric and $T : X \to X$ be a self-mapping. Suppose that all the hypotheses of Theorem 2.9(resp., Theorem 2.11) hold. In addition, assume that

(A1) the function $\beta : [0, \infty) \to [0, \infty), \beta(r) = r - \psi(r)$ is strictly increasing and onto.

(A2) for any $\epsilon-$solution $w^* \in X$ of (2), one has $\gamma(w^*, x^*, x^*) \geq 1$, where $x^* \in Fix(T)$.

Then, the fixed point problem (1) is generalized Ulam-Hyers stable.

Proof. The proof is an analog of the proof of Theorem 3.2. □

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

Author details

[1] School of Science and Technology, Sanya University, Sanya, Hainan 572000, China.
[2] College of Science, Sichuan University of Science and Engineering, Zigong, Sichuan 643000, China.
[3] Sichuan Province University Key Laboratory of Bridge Non-destruction Detecting and Engineering Computing, Zigong, Sichuan 643000, China.

[4] Department of Mathematics, Karaj Branch Islamic Azad University, Karaj, Iran.

[5] Department of Mathematics Education and RINS, Gyeongsang National University, Gajwa-dong,900, 52828 Jinju, Korea.

[6] Faculty of Mechanical Engineering, University of Belgrade, Kraljice Marije 16, 11120 Belgrade, Serbia.

Acknowledgements

Mi Zhou was supported by Scientific Research Fund of Hainan Province Education Department (Grant No.Hnjg2016ZD-20).

Xiao-lan Liu was partially supported by National Natural Science Foundation of China (Grant No.61573010), Artificial Intelligence of Key Laboratory of Sichuan Province(No.2015RZJ01), Science Research Fund of Science and Technology Department of Sichuan Province(N0.2017JY0125), Scientific Research Fund of Sichuan Provincial Education Department(No.16ZA0256), Scientific Research Fund of Sichuan University of Science and Engineering (No.2014RC01, No.2014RC03, No.2017RCL54).

Yeol Je Cho was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Science, ICT and future Planning (2014R1A2A2A01002100).

Stojan Radenović was supported by the Ministry of Education, Science and Technological Development of Serbia.

The authors thank the editor and the referees for their useful comments and suggestions.

References

[1] S. Sedghi, N. Shobe, A. Aliouche, *A generalized of fixed point theorems in $S-$metric spaces*, Mat. Vesn., **64**(2012), 258–266.

[2] J. M. Afra, *Fixed point type theorem for weak contraction in $S-$metric spaces*, Int. J. Res. Rer. Appl. Sci., **22**(2015),11–14.

[3] P. Chouhan, N. Malviya, *A common unique fixed point theorem for expansive type mappings in $S-$metric spaces*, Int. Math. Forum, **8**(2013), 1287–1293.

[4] N. T. Hieu, N. T. Thanhly, N. V. Dung, *A generalization of Ćirić quasi-contractions for maps on $S-$metric spaces*, Thai. J. Math., **13** (2015), 369–380.

[5] S. Sedghi, N. V. Dung, *Fixed point theorems in $S-$metric spaces*, Mat. Vesn., **66**(2014), 113–124.

[6] N. V. Dung, N. T. Hieu, S. Radojević, *Fixed point theorems for $g-$monotone maps on partially orderded $S-$metric spaces*, Filomat, **28**(2014),1685–1898.

[7] M. Zhou, X. L. Liu, *On coupled common fixed point theorems for nonlinear contractions with the mixed weakly monotone property in partially ordered $S-$metric spaces*, Journal of Function Spaces, **2016**(2016), Article ID 7529523, 9 pages.

[8] M. Zhou, X. L. Liu, D. D. Diana, B. Damjanović, *Coupled coincidence point results for Geraghty-type contraction by using monotone property in partially ordered $S-$metric spaces*, J. Nonlinear Sci. Appl., **9**(2016), 5950-5969.

[9] S. Sedghi, A. Gholidahneh, T. Došenović, J. Esfahani, S. Radenović, *Common fixed point of four maps in S_b-metric spaces*, to appear in Journal of Linear and Topol. Algebra.

[10] A. Gholidahneh, S. Sedghi, T. Došenović, S. Radenović, *Ordered $S-$metric spaces and coupled common fixed point theorems of integral type contraction*, to appear in Mathematics Interdisciplinary Research.

[11] S. M. Ulam, *Problems in Modern Mathematics*, John Wiley & Sons, New York, NY, USA, 1964.

[12] D. H. Hyers, *On the stability of linear functional equation*, Proceedings of the National Academy of Sciences of the United States of America, **27**(1941), 222–224.

[13] M. F. Bota-Boriceanu, A. Petursel, *Ulam-Hyers stability for operatorial equations*, Analele Stiintifice ale Universitatii, **57**(1) (2011), 65–74.

[14] V. L. Lazăr, *Ulam-Hyers stability for partial differential inclusions*, Electronic Journal of Qualitative Theory of Differential Equations, **21**(2012), 1–19.

[15] J. Brzdek, K. Ciepliński, *A fixed point theorems and the Hyers-Ulam stability in non-Archimedean spaces*, J. Math. Anal. Appl., **400**(1)(2013), 68–75.

[16] L. Cadariu, L. Gavruta, and P. Gavruta, *Fixed points and generalized Hyers-Ulam stability*, Abstract and Applied Analysis, **2012**(2012), Article ID 712743, 10 pages.

[17] A. Meir, E. Keeler, *A theorem on contraction mappings*, J.Math.Anal.Appl., **28**(1969), 326–329.

[18] M. Maiti, T. K. Pal, *Generalizations of two fixed points theorems*, Bull. Calcutta Math. Soc., **70**(1978), 57–61.

[19] S. Park, B. E. Rhoades, *Meir-Keeler type contractive conditions*, Math. Japn., **26**(1)(1981), 13–20.

[20] C. Mongkolkeha, P. Kumam, *Best proximity points for asympototic proximal pointwise weaker Meir-Keeler-type $\psi-$contraction mappings*, J. Egypt. Math. Soc., **26**(1)(1981), 13–20.

[21] B. Samet, C. Vetro, P. Vetro, *Fixed point theorem for $\alpha-\psi$ contractive type mappings*, Nonlinear Anal., **75**(2012), 2154–2165.

[22] A. Latif, M. E. Gordji, E. Karapinar, W. Sintunavarat, *Fixed point results for generalized $(\alpha-\psi)-$Meir-Keeler contractive mappings and applications*, Journal of Inequalities and Applications, **68**(2014), 11 pages.

New oscillation criteria for second-order nonlinear delay dynamic equations with nonpositive neutral coefficients on time scales

Ming Zhang[a,b]*, Wei Chen[a]†, M.M.A. El-Sheikh[c]‡, R.A. Sallam[c]§, A.M. Hassan[d]¶ and Tongxing Li[b]‖

[a]*School of Information Engineering, Wuhan University of Technology, Wuhan, Hubei 430070, P. R. China*
[b]*School of Information Science and Engineering, Linyi University, Linyi, Shandong 276005, P. R. China*
[c]*Department of Mathematics, Faculty of Science, Menoufia University, Shebin El-Koom 32511, Egypt*
[d]*Department of Mathematics, Faculty of Science, Benha University, Benha-Kalubia 13518, Egypt*

Abstract

We analyze the oscillatory behavior of solutions to a nonlinear second-order neutral delay dynamic equation with a nonpositive neutral coefficient under the assumptions that allow applications to Emden–Fowler type dynamic equations. New theorems complement and improve related contributions to the subject. An example is included.

Keywords: Oscillation, second-order delay dynamic equation, neutral type equation, Emden–Fowler type equation.
Mathematics Subject Classification 2010: 34K11, 34N05.

1 Introduction

In this paper, we study the oscillation of a class of second-order neutral dynamic equations

$$\left[r(t)(z^{\Delta}(t))^{\alpha}\right]^{\Delta} + q(t)f(x(\delta(t))) = 0. \tag{1.1}$$

Here $t \in [t_0, \infty)_{\mathbb{T}}$, $\alpha \geq 1$ is a quotient of odd natural numbers, and $z(t) = x(t) - p(t)x(\tau(t))$. The increasing interest in oscillation of solutions to various classes of equations is motivated by their applications in natural sciences, engineering, and control; see, for instance, [1–30] and the references cited therein. Analysis of qualitative properties of (1.1) is important not only for the sake of further development of the oscillation theory, but for practical reasons too. As a matter of fact, a particular case of (1.1), an Emden–Fowler dynamic equation

$$\left[r(t)(x^{\Delta}(t))^{\alpha}\right]^{\Delta} + q(t)x^{\beta}(\delta(t)) = 0,$$

has applications in mathematical, theoretical, and chemical physics; see Li and Rogovchenko [15–18].

Throughout the paper, we assume that the following assumptions are satisfied:

*e–mail: zhangming@lyu.edu.cn
†e–mail: greatchen@whut.edu.cn
‡e–mail: msheikh_1999@yahoo.com
§e–mail: ragaasallam@yahoo.com
¶e–mail: ahmed.mohamed@fsc.bu.edu.eg
‖e–mail: litongx2007@163.com (Corresponding author)

(H_1) $r \in \mathrm{C_{rd}}([t_0,\infty)_{\mathbb{T}},(0,\infty))$, $\int_{t_0}^{\infty} r^{-\frac{1}{\alpha}}(t)\Delta t = \infty$, $R(t) = \int_{t_1}^{t} r^{-\frac{1}{\alpha}}(s)\Delta s$, where $t_1 \in [t_0,\infty)_{\mathbb{T}}$ is sufficiently large;

(H_2) $p, q \in \mathrm{C_{rd}}([t_0,\infty)_{\mathbb{T}},\mathbb{R})$, $0 \le p(t) \le p_0 < 1$, $q(t) \ge 0$, and $q(t)$ is not identically zero for large t;

(H_3) $\tau, \delta \in \mathrm{C_{rd}}([t_0,\infty)_{\mathbb{T}},\mathbb{T})$, $\tau(t) \le t$, $\delta(t) \le t$, and $\lim_{t\to\infty}\tau(t) = \lim_{t\to\infty}\delta(t) = \infty$;

(H_4) $f \in \mathrm{C}(\mathbb{R},\mathbb{R})$, $uf(u) > 0$ for all $u \ne 0$, and there exists a positive constant k such that $f(u)/u^{\beta} \ge k$ for all $u \ne 0$, where $\beta \ge \alpha$ is a quotient of odd natural numbers.

By a solution to (1.1) we mean a function $x \in \mathrm{C_{rd}}[T_x,\infty)_{\mathbb{T}}$, $T_x \in [t_0,\infty)_{\mathbb{T}}$, such that $r(z^{\Delta})^{\alpha} \in \mathrm{C^1_{rd}}[T_x,\infty)_{\mathbb{T}}$ and x satisfies (1.1) on $[T_x,\infty)_{\mathbb{T}}$. We consider only those solutions x of (1.1) which satisfy $\sup\{|x(t)| : t \in [T,\infty)_{\mathbb{T}}\} > 0$ for all $T \in [T_x,\infty)_{\mathbb{T}}$ and assume that (1.1) possesses such solutions. As usual, a solution of (1.1) is said to be oscillatory if it is not of the same sign eventually; otherwise, it is called nonoscillatory.

Recently, a great deal of interest in oscillatory properties of solutions to various classes of equations with nonnegative neutral coefficients has been shown; see, for instance, [2,4,5,14–17,19,20,22,27,28] and the references cited therein. However, there are relatively fewer results for equations with nonpositive neutral coefficients; see [3, 4, 7, 13, 21, 23–25, 29]. In the papers by Arul and Shobha [3] and Li et al. [21], a particular case of (1.1), a neutral differential equation

$$[r(t)(z'(t))^{\alpha}]' + q(t)f(x(\delta(t))) = 0$$

was studied. Seghar et al. [23] investigated the neutral difference equation

$$\Delta(a_n\Delta(x_n - p_n x_{n-k})) + q_n f(x_{n-l}) = 0.$$

Bohner and Li [7] and Karpuz [13] established oscillation results for neutral dynamic equations

$$\left(r(t)|z^{\Delta}(t)|^{p-2}z^{\Delta}(t)\right)^{\Delta} + q(t)|x(\delta(t))|^{p-2}x(\delta(t)) = 0, \quad z(t) = x(t) - p(t)x(\tau(t))$$

and

$$(x(t) - p(t)x(\tau(t)))^{\Delta\Delta} + q(t)x(\delta(t)) = 0,$$

whereas Zhang et al. [29] explored (1.1) assuming that $\alpha = \beta$.

It should be noted that research in this paper was strongly motivated by the paper [29]. Our principal goal is to analyze the oscillatory behavior of solutions to (1.1) in the case where $\beta \ge \alpha$. As customary for papers on oscillation, all functional inequalities are supposed to hold eventually. Without loss of generality, we can deal only with positive solutions of (1.1).

2 Main results

For the proofs of our oscillation criteria we need the following lemmas. The first lemma is extracted from the monograph by Bohner and Peterson [9, Theorem 1.93], and the latter lemmas can be obtained by similar techniques to those used in [3, 21].

Lemma 2.1. *Assume that $v : \mathbb{T} \to \mathbb{R}$ is strictly increasing and $\tilde{\mathbb{T}} := v(\mathbb{T})$ is a time scale. Let $y : \tilde{\mathbb{T}} \to \mathbb{R}$. If $v^{\Delta}(t)$ and $y^{\tilde{\Delta}}(v(t))$ exist for $t \in \mathbb{T}^{\kappa}$, then*

$$(y(v(t)))^{\Delta} = y^{\tilde{\Delta}}(v(t))v^{\Delta}(t).$$

Lemma 2.2. *Let x be a positive solution of (1.1). Then z has the following two possible cases:*

(I) $z(t) > 0, \quad z^{\Delta}(t) > 0, \quad (r(t)(z^{\Delta}(t))^{\alpha})^{\Delta} \le 0;$

(II) $z(t) < 0, \quad z^{\Delta}(t) > 0, \quad (r(t)(z^{\Delta}(t))^{\alpha})^{\Delta} \le 0$

for $t \in [t_1,\infty)_{\mathbb{T}}$, where $t_1 \in [t_0,\infty)_{\mathbb{T}}$ is sufficiently large.

Lemma 2.3. *Let x be a positive solution of (1.1) and assume that the corresponding z has property (II) of Lemma 2.2. Then*

$$\lim_{t\to\infty} x(t) = 0.$$

Lemma 2.4. *If x is a positive solution of (1.1) such that case (I) of Lemma 2.2 is satisfied, then $x(t) \geq z(t)$ and $z(t)/R(t)$ is strictly decreasing for large t.*

Theorem 2.1. *Assume that $\delta([t_0,\infty)_{\mathbb{T}}) = [\delta(t_0),\infty)_{\mathbb{T}}$ and $\delta^{\Delta}(t) > 0$. If for any $M > 0$,*

$$\limsup_{t\to\infty} \left[Q(t) + \alpha\int_t^{\infty} \delta^{\Delta}(s) r^{-\frac{1}{\alpha}}(\delta(s)) Q^{\frac{\alpha+1}{\alpha}}(\sigma(s))\Delta s\right]\left(\int_{t_0}^{\delta(t)} r^{-\frac{1}{\alpha}}(s)\Delta s\right)^{\alpha} > 1, \tag{2.1}$$

where $Q(t) = kM^{\beta-\alpha}\int_t^{\infty} q(u)\Delta u$, then solutions of (1.1) are either oscillatory or converge to zero as $t \to \infty$.

Proof. Let x be a nonoscillatory solution of (1.1) such that $x(t) > 0$, $x(\tau(t)) > 0$, and $x(\delta(t)) > 0$ for $t \in [t_1,\infty)_{\mathbb{T}}$. It follows from Lemma 2.2 that z satisfies either (I) or (II) for $t \in [t_1,\infty)_{\mathbb{T}}$.

Case 1. Suppose first that z satisfies case (I). By virtue of the definition of z,

$$x(t) = z(t) + p(t)x(\tau(t)) \geq z(t)$$

and so we can write (1.1) in the form

$$\left[r(t)(z^{\Delta}(t))^{\alpha}\right]^{\Delta} \leq -kq(t)z^{\beta}(\delta(t)).$$

Defining the Riccati transformation

$$\nu(t) = \frac{r(t)(z^{\Delta}(t))^{\alpha}}{z^{\alpha}(\delta(t))}, \tag{2.2}$$

then $\nu(t) > 0$ and there exists a constant $M > 0$ such that

$$\begin{aligned}\nu^{\Delta}(t) &= \frac{\left[r(t)(z^{\Delta}(t))^{\alpha}\right]^{\Delta}}{z^{\alpha}(\delta(t))} + \left[r(t)(z^{\Delta}(t))^{\alpha}\right]^{\sigma}\left[\frac{1}{z^{\alpha}(\delta(t))}\right]^{\Delta} \\ &\leq -kM^{\beta-\alpha}q(t) - \alpha\delta^{\Delta}(t)\nu(\sigma(t))\frac{z^{\Delta}(\delta(t))}{z(\delta(\sigma(t)))}.\end{aligned} \tag{2.3}$$

Taking into account that $\nu^{\frac{1}{\alpha}}(\sigma(t)) = r^{\frac{1}{\alpha}}(\sigma(t))z^{\Delta}(\sigma(t))/z(\delta(\sigma(t)))$, $r(t)(z^{\Delta}(t))^{\alpha} \leq 0$, and $\delta(t) \leq t \leq \sigma(t)$, we conclude that

$$\frac{z^{\Delta}(\delta(t))}{z(\delta(\sigma(t)))} \geq \frac{\nu^{\frac{1}{\alpha}}(\sigma(t))}{r^{\frac{1}{\alpha}}(\delta(t))}. \tag{2.4}$$

Combining (2.3) and (2.4), we arrive at

$$\nu^{\Delta}(t) \leq -kM^{\beta-\alpha}q(t) - \alpha\delta^{\Delta}(t)r^{-\frac{1}{\alpha}}(\delta(t))\nu^{\frac{\alpha+1}{\alpha}}(\sigma(t)). \tag{2.5}$$

Integrating (2.5) from t to s, we deduce that

$$\nu(s) - \nu(t) \leq -kM^{\beta-\alpha}\int_t^s q(u)\Delta u - \alpha\int_t^s \delta^{\Delta}(u)r^{-\frac{1}{\alpha}}(\delta(u))\nu^{\frac{\alpha+1}{\alpha}}(\sigma(u))\Delta u,$$

which yields

$$\nu(t) \geq kM^{\beta-\alpha}\int_t^s q(u)\Delta u + \alpha\int_t^s \delta^{\Delta}(u)r^{-\frac{1}{\alpha}}(\delta(u))\nu^{\frac{\alpha+1}{\alpha}}(\sigma(u))\Delta u.$$

Passing to the limit as $s \to \infty$, we have

$$\nu(t) \geq Q(t) + \alpha\int_t^{\infty} \delta^{\Delta}(u)r^{-\frac{1}{\alpha}}(\delta(u))\nu^{\frac{\alpha+1}{\alpha}}(\sigma(u))\Delta u. \tag{2.6}$$

An application of (2.6) implies that

$$\nu(t) \geq Q(t) + \alpha\int_t^{\infty} \delta^{\Delta}(u)r^{-\frac{1}{\alpha}}(\delta(u))Q^{\frac{\alpha+1}{\alpha}}(\sigma(u))\Delta u. \tag{2.7}$$

By virtue of (2.2), we conclude that

$$\begin{aligned}\frac{1}{\nu(t)} &= \frac{1}{r(t)}\left(\frac{z(\delta(t))}{z^{\Delta}(t)}\right)^{\alpha}\\ &= \frac{1}{r(t)}\left(\frac{z(t_2)+\int_{t_2}^{\delta(t)} r^{\frac{1}{\alpha}}(s)z^{\Delta}(s)r^{-\frac{1}{\alpha}}(s)\Delta s}{z^{\Delta}(t)}\right)^{\alpha}\\ &\geq \frac{1}{r(t)}\left(\frac{r^{\frac{1}{\alpha}}(t)z^{\Delta}(t)\int_{t_2}^{\delta(t)} r^{-\frac{1}{\alpha}}(s)\Delta s}{z^{\Delta}(t)}\right)^{\alpha},\end{aligned}$$

that is,

$$\nu(t)\left(\int_{t_2}^{\delta(t)} r^{-\frac{1}{\alpha}}(s)\Delta s\right)^{\alpha} \leq 1. \tag{2.8}$$

Using (2.7) and (2.8), we deduce that

$$\limsup_{t\to\infty}\left[Q(t)+\alpha\int_t^{\infty}\delta^{\Delta}(s)r^{-\frac{1}{\alpha}}(\delta(s))Q^{\frac{\alpha+1}{\alpha}}(\sigma(s))\Delta s\right]\left(\int_{t_2}^{\delta(t)} r^{-\frac{1}{\alpha}}(s)\Delta s\right)^{\alpha}\leq 1,$$

which contradicts (2.1).

Case 2. Suppose now that z satisfies case (II). It follows from Lemma 2.3 that $\lim_{t\to\infty} x(t)=0$. This completes the proof. □

Theorem 2.2. *If there exists a positive function $\beta\in C^1_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}},\mathbb{R})$ such that for all sufficiently large $t_1\in[t_0,\infty)_{\mathbb{T}}$, for some $t_2\in[t_1,\infty)_{\mathbb{T}}$, and for any $M>0$,*

$$\int_{t_2}^{\infty}\left[kM^{\beta-\alpha}q(s)\beta(s)\left(\frac{R(\delta(s))}{R(s)}\right)^{\alpha}-\frac{1}{(\alpha+1)^{\alpha+1}}\frac{(\beta^{\Delta}(s))^{\alpha+1}r(s)}{\beta^{\alpha}(s)}\right]\Delta s=\infty, \tag{2.9}$$

then conclusion of Theorem 2.1 *remains intact.*

Proof. Assume that x is a nonoscillatory solution of (1.1) on $[t_0,\infty)_{\mathbb{T}}$ that satisfies $x(t)>0$, $x(\tau(t))>0$, and $x(\delta(t))>0$ for $t\in[t_1,\infty)_{\mathbb{T}}$. By virtue of Lemma 2.2, z satisfies either (I) or (II) for $t\in[t_1,\infty)_{\mathbb{T}}$.

Case 1. Suppose that z satisfies case (I). Define the Riccati transformation

$$\omega(t)=\beta(t)\frac{r(t)(z^{\Delta}(t))^{\alpha}}{z^{\alpha}(t)}.$$

Then $\omega(t)>0$ and there exists a constant $M>0$ such that

$$\begin{aligned}\omega^{\Delta}(t) &= \left[r(t)(z^{\Delta}(t))^{\alpha}\right]^{\Delta}\frac{\beta(t)}{z^{\alpha}(t)}+\left[r(t)(z^{\Delta}(t))^{\alpha}\right]^{\sigma}\left[\frac{\beta(t)}{z^{\alpha}(t)}\right]^{\Delta}\\ &\leq -kM^{\beta-\alpha}q(t)\beta(t)\frac{z^{\alpha}(\delta(t))}{z^{\alpha}(t)}+\frac{\beta^{\Delta}(t)}{\beta(\sigma(t))}\omega(\sigma(t))-\alpha\frac{\beta(t)}{\beta^{\sigma}(t)}\frac{z^{\Delta}(t)}{z(t)}\omega(\sigma(t))\\ &\leq -kM^{\beta-\alpha}q(t)\beta(t)\frac{z^{\alpha}(\delta(t))}{z^{\alpha}(t)}+\frac{\beta^{\Delta}(t)}{\beta(\sigma(t))}\omega(\sigma(t))-\alpha\frac{\beta(t)}{\beta^{\frac{\alpha+1}{\alpha}}(\sigma(t))r^{\frac{1}{\alpha}}(t)}\omega^{\frac{\alpha+1}{\alpha}}(\sigma(t)).\end{aligned}$$

In view of Lemma 2.4, we obtain

$$\omega^{\Delta}(t)\leq -kM^{\beta-\alpha}q(t)\beta(t)\left(\frac{R(\delta(t))}{R(t)}\right)^{\alpha}+\frac{\beta^{\Delta}(t)}{\beta(\sigma(t))}\omega(\sigma(t))-\alpha\frac{\beta(t)}{\beta^{\frac{\alpha+1}{\alpha}}(\sigma(t))r^{\frac{1}{\alpha}}(t)}\omega^{\frac{\alpha+1}{\alpha}}(\sigma(t)). \tag{2.10}$$

Applying the inequality

$$B\omega-A\omega^{\frac{\alpha+1}{\alpha}}\leq\frac{\alpha^{\alpha}}{(\alpha+1)^{\alpha+1}}\frac{B^{\alpha+1}}{A^{\alpha}},\quad A>0$$

with

$$B=\frac{\beta^{\Delta}(t)}{\beta(\sigma(t))}\quad\text{and}\quad A=\alpha\frac{\beta(t)}{\beta^{\frac{\alpha+1}{\alpha}}(\sigma(t))r^{\frac{1}{\alpha}}(t)},$$

and using (2.10), we deduce that

$$\omega^{\Delta}(t) \leq -kM^{\beta-\alpha}q(t)\beta(t)\left(\frac{R(\delta(t))}{R(t)}\right)^{\alpha} + \frac{1}{(\alpha+1)^{\alpha+1}}\frac{(\beta^{\Delta}(t))^{\alpha+1}r(t)}{\beta^{\alpha}(t)}. \tag{2.11}$$

Integrating (2.11) from t_2 ($t_2 \in [t_1, \infty)_{\mathbb{T}}$) to t, we arrive at

$$\int_{t_2}^{t}\left[kM^{\beta-\alpha}q(s)\beta(s)\left(\frac{R(\delta(s))}{R(s)}\right)^{\alpha} - \frac{1}{(\alpha+1)^{\alpha+1}}\frac{(\beta^{\Delta}(s))^{\alpha+1}r(s)}{\beta^{\alpha}(s)}\right]\Delta s \leq \omega(t_2),$$

which contradicts (2.9).

Case 2. If z satisfies case (II), then $\lim_{t\to\infty} x(t) = 0$ due to Lemma 2.3. The proof is complete. □

Remark 2.1. *On the basis of Theorem* 2.2*, one can obtain Philos-type oscillation criteria for equation* (1.1)*. The details are left to the reader.*

Example 2.1. For $t \in [1, \infty)_{\mathbb{T}}$, consider the second-order superlinear Emden–Fowler neutral delay dynamic equation

$$\left(x(t) - \frac{1}{3}x\left(\frac{t}{2}\right)\right)^{\Delta\Delta} + \frac{\gamma}{t}x^{\beta}\left(\frac{t}{4}\right) = 0, \quad \beta > 1,\ \gamma > 0. \tag{2.12}$$

Let $\beta(t) = 1$. It follows from Theorem 2.2 that every solution x of equation (2.12) is either oscillatory or satisfies $\lim_{t\to\infty} x(t) = 0$.

Remark 2.2. *For a class of second-order neutral delay dynamic equations* (1.1)*, we derived two new oscillation results which complement and improve those obtained by Zhang et al.* [29]*. A distinguishing feature of our criteria is that we do not impose specific restriction $\alpha = \beta$. Since the sign of the derivative z^{Δ} is not known, it is difficult to establish sufficient conditions which ensure that every solution x of* (1.1) *is just oscillatory and does not satisfy $\lim_{t\to\infty} x(t) = 0$. Neither is it possible to use the technique exploited in this paper for proving that all solutions of* (1.1) *approach zero at infinity. As mentioned in the paper by Zhang et al.* [29]*, it would be of interest to study* (1.1) *in the case where $\int_{t_0}^{\infty} r^{-\frac{1}{\alpha}}(t)\Delta t < \infty$.*

Acknowledgements

This research is supported by Project of Shandong Province Independent Innovation and Achievement Transformation (Grant No. 2014ZZCX02702), Shandong Province Key Research and Development Project (Grant No. 2016GGX109001), Shandong Provincial Natural Science Foundation (Grant Nos. ZR2017MF050, ZR2014FL008, and ZR2015FL014), and Project of Shandong Province Higher Educational Science and Technology Program (Grant No. J17KA049).

References

[1] R. P. Agarwal, M. Bohner, T. Li, and C. Zhang. Oscillation criteria for second-order dynamic equations on time scales. *Applied Mathematics Letters*, 31 (2014) 34–40.

[2] R. P. Agarwal, D. O'Regan, and S. H. Saker. Oscillation criteria for second-order nonlinear neutral delay dynamic equations. *Journal of Mathematical Analysis and Applications*, 300 (2004) 203–217.

[3] R. Arul and V. S. Shobha. Improvement results for oscillatory behavior of second order neutral differential equations with nonpositive neutral term. *British Journal of Mathematics & Computer Science*, 12 (2016) 1–7.

[4] B. Baculíková and J. Džurina. Oscillation of third-order neutral differential equations. *Mathematical and Computer Modelling*, 52 (2010) 215–226.

[5] M. Bohner, S. R. Grace, and I. Jadlovská. Oscillation criteria for second-order neutral delay differential equations. *Electronic Journal of Qualitative Theory of Differential Equations*, 2017 (2017) 1–12.

[6] M. Bohner, T. S. Hassan, and T. Li. Fite–Hille–Wintner-type oscillation criteria for second-order half-linear dynamic equations with deviating arguments. *Indagationes Mathematicae*, (2018) in press.

[7] M. Bohner and T. Li. Oscillation of second-order p-Laplace dynamic equations with a nonpositive neutral coefficient. *Applied Mathematics Letters*, 37 (2014) 72–76.

[8] M. Bohner and T. Li. Kamenev-type criteria for nonlinear damped dynamic equations. *Science China Mathematics*, 58 (2015) 1445–1452.

[9] M. Bohner and A. Peterson. *Dynamic Equations on Time Scales: An Introduction with Applications*. Birkhäuser, Boston, 2001.

[10] J. Džurina and I. Jadlovská. A note on oscillation of second-order delay differential equations. *Applied Mathematics Letters*, 69 (2017) 126–132.

[11] L. Erbe, A. Peterson, and S. H. Saker. Oscillation criteria for second-order nonlinear delay dynamic equations. *Journal of Mathematical Analysis and Applications*, 333 (2007) 505–522.

[12] S. Hilger. Analysis on measure chain–a unified approach to continuous and discrete calculus. *Results in Mathematics*, 18 (1990) 18–56.

[13] B. Karpuz. Sufficient conditions for the oscillation and asymptotic behaviour of higher-order dynamic equations of neutral type. *Applied Mathematics and Computation*, 221 (2013) 453–462.

[14] T. Li, R. P. Agarwal, and M. Bohner. Some oscillation results for second-order neutral dynamic equations. *Hacettepe Journal of Mathematics and Statistics*, 41 (2012) 715–721.

[15] T. Li and Yu. V. Rogovchenko. Asymptotic behavior of higher-order quasilinear neutral differential equations. *Abstract and Applied Analysis*, 2014 (2014) 1–11.

[16] T. Li and Yu. V. Rogovchenko. Oscillation of second-order neutral differential equations. *Mathematische Nachrichten*, 288 (2015) 1150–1162.

[17] T. Li and Yu. V. Rogovchenko. Oscillation criteria for second-order superlinear Emden–Fowler neutral differential equations. *Monatshefte für Mathematik*, 184 (2017) 489–500.

[18] T. Li and Yu. V. Rogovchenko. On asymptotic behavior of solutions to higher-order sublinear Emden–Fowler delay differential equations. *Applied Mathematics Letters*, 67 (2017) 53–59.

[19] T. Li, Yu. V. Rogovchenko, and C. Zhang. Oscillation of second-order neutral differential equations. *Funkcialaj Ekvacioj*, 56 (2013) 111–120.

[20] T. Li and S. H. Saker. A note on oscillation criteria for second-order neutral dynamic equations on isolated time scales. *Communications in Nonlinear Science and Numerical Simulation*, 19 (2014) 4185–4188.

[21] Q. Li, R. Wang, F. Chen, and T. Li. Oscillation of second-order nonlinear delay differential equations with nonpositive neutral coefficients. *Advances in Difference Equations*, 2015 (2015) 1–7.

[22] T. Li, C. Zhang, and E. Thandapani. Asymptotic behavior of fourth-order neutral dynamic equations with noncanonical operators. *Taiwanese Journal of Mathematics*, 18 (2014) 1003–1019.

[23] D. Seghar, E. Thandapani, and S. Pinelas. Oscillation theorems for second order difference equations with negative neutral term. *Tamkang Journal of Mathematics*, 46 (2015) 441–451.

[24] E. Thandapani, V. Balasubramanian, and J. R. Graef. Oscillation criteria for second order neutral difference equations with negative neutral term. *International Journal of Pure and Applied Mathematics*, 87 (2013) 283–292.

[25] E. Thandapani and K. Mahalingam. Necessary and sufficient conditions for oscillation of second order neutral difference equations. *Tamkang Journal of Mathematics*, 34 (2003) 137–145.

[26] J. Wang, M. M. A. El-Sheikh, R. A. Sallam, D. I. Elimy, and T. Li. Oscillation results for nonlinear second-order damped dynamic equations. *Journal of Nonlinear Sciences and Applications*, 8 (2015) 877–883.

[27] C. Zhang, R. P. Agarwal, M. Bohner, and T. Li. New oscillation results for second-order neutral delay dynamic equations. *Advances in Difference Equations*, 2012 (2012) 1–14.

[28] C. Zhang, R. P. Agarwal, M. Bohner, and T. Li. Oscillation of second-order nonlinear neutral dynamic equations with noncanonical operators. *Bulletin of the Malaysian Mathematical Sciences Society*, 38 (2015) 761–778.

[29] M. Zhang, W. Chen, M. M. A. El-Sheikh, R. A. Sallam, A. M. Hassan, and T. Li. Oscillation criteria for second-order nonlinear delay dynamic equations of neutral type. *Advances in Difference Equations*, (2018) in press.

[30] C. Zhang and T. Li. Some oscillation results for second-order nonlinear delay dynamic equations. *Applied Mathematics Letters*, 26 (2013) 1114–1119.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 27, NO.4, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC

A Consistency Reaching Approach for Probability-interval Valued Hesitant Fuzzy Preference Relations

Jiuping Xu[1,*] , Kang Xu[1,2], Zhibin Wu[1]

1. Business School, Sichuan University, Chengdu 610065, P R China
2. School of Economics and Management, Hubei University of Automotive Technology, Shiyan, Hubei, 442002, P R China

Abstract

In a group decision making (GDM) situation with qualitative settings and complex environments, experts may require intervals with corresponding possibility values, rather than only interval-valued hesitant fuzzy sets (IVHFSs) or probability-hesitant fuzzy sets (P-HFSs), to express their preferences. In this paper, in line with such situations, probability-interval valued hesitant fuzzy sets (P-IVHFSs) are presented to address GDM problems with hesitant fuzzy intervals and the corresponding possibility values. A P-IVHFS can serve as an extension of both a P-HFS and an IVHFS. As important tools in GDM, P-IVHFSs can describe the actual preferences of decision-makers and better reflect their uncertainty, hesitancy, and inconsistency, thus enhancing the modeling abilities of HFSs. Firstly, the concept of P-IVHFSs is defined, and then some properties of P-IVHFSs are presented. Furthermore, probability-interval valued hesitant fuzzy preference relations (P-IVHFPRs) are defined and the consistency of P-IVHFPRs is discussed. Then, based on related research, a decomposition method is developed to deal with the consistency of P-IVHFPRs. Finally an example is provided to illustrate the proposed approach.

Keywords:
Decision making, Fuzzy sets, P-IVHFS, Preference relation, Consistency

1. Introduction

Torra initiated the notable concept of HFSs, which represented a new generalization for fuzzy sets, as this method permits an element to have not just one but a set of several possible membership values. Consequently, HFSs can describe the hesitancy experienced by decision makers (DMs) in the decision-making process. As a result of this innovation, the HFS has attracted an increasing amount of attention in academia since its introduction. In recent years, there have been a number of developments regarding the theory of HFSs. For example, Xu and Xia defined the concept of the hesitant fuzzy element (HFE), which can be considered to be the basic unit of a HFS. Moreover, Rodríguez et al. proposed the hesitant fuzzy linguistic term set to deal with linguistic decision making. Chen et al. extended HFSs to IVHFSs, which represent the membership degrees of an element to a set with several possible interval values. Farhadinia proposed a series of score functions for HFSs and Wei, Zhang, Yu, and Ai et al. studied their aggregation operators. Farhadinia, Xu and Xia, Peng et al., and Chen et al. discussed the information measures of HFSs. Wang et al. studied the interval-valued hesitant fuzzy linguistic set, which can serve as an extension of both a linguistic term set and an interval-valued hesitant fuzzy set. Finally, Wu and Xu presented the concept of possibility distribution for a hesitant fuzzy linguistic term set and Zhu and Xu extended HFSs to P-HFSs.

* Corresponding author. Tel: +86 28 85418191; Fax: +86 28 85415143.
E-mail address: xujiuping@scu.edu.cn(J. Xu)

In group decision making (GDM) problems with fuzzy preference relations, some of the experts' preference properties are often assumed and it is desirable to avoid contradictions or, in other words, inconsistent opinions. One of these properties is associated with the pairwise comparison transitivity between any three alternatives. For fuzzy preference relations, the transitivity has been modeled in many different ways depending on the role of the preference intensities. The purpose of consistency control is to measure the level of consistency of each individual preference relation so as to identify the expert, alternative and preference values that are the most inconsistent within the GDM problem. This inconsistency identification is also used to suggest possible new consistent preference valuee.

In the process of GDM, preference relations are very popular tools for expressing the DM's preferences when they compare a set of alternatives. Various types of preference relations have been suggested for different environments. For example, Orlovsky proposed the concept of fuzzy preference relations, and Xu introduced the concept of interval fuzzy preference relations to express uncertainty and vagueness. In many practical decision making problems, due to a lack of available information, it may be difficult for DMs to quantify their opinions precisely with a crisp number; however they can be represented by an interval number within $[0, 1]$. This means that it is vital to introduce the concept of IVHFSs, which permit the membership degrees of an element to a given set to have some different interval values. Chen et al. introduced interval-valued hesitant preference relations and their applications to GDM. Moreover, Farhadinia discussed the information measures of IVHFSs and Wang et al. developed interval-valued hesitant fuzzy linguistic sets, and discussed their applications in multi-criteria decision-making problems.

However, in a GDM situation with qualitative settings and in complex environments, experts may require intervals with corresponding possibility values rather than only IVHFSs or P-HFSs, to express their preferences. Consider the following case for example: the DMs of a large organization discuss the membership of x into a set A; forty percent of them want to assign values between 0.3 and 0.4, while the remaining sixty percent wish to assign values between 0.5 and 0.6. At this point, interval numbers with probability values can be used, i.e., $\{[0.3, 0.4](40\%), [0.5, 0.6](60\%)\}$, or $\{[0.3, 0.4](0.4), [0.5, 0.6](0.6)\}$, to represent the preferences of the large organization. In accordance with such cases, in this paper, P-IVHFSs are presented to address GDM problems with hesitant fuzzy intervals and the corresponding possibility values. A P-IVHFS can serve as an extension of both a P-HFS and an IVHFS. Furthermore, as a powerful tool in GDM, P-IVHFSs can describe the actual preferences of decision-makers flexibly and better reflect their uncertainty, hesitancy, and inconsistency , and thus enhance the modeling abilities of HFSs. The consistency of preference relations has become a research topic of great interest in recent years. For example, Liao et al. defined the concept of the multiplicative consistent hesitant fuzzy preference relation. Furthermore, Wu and Xu developed separate consistency and consensus processes to deal with the hesitant fuzzy linguistic preference relations of individual rationality and group rationality. Zhu and Xu proposed the concept of the probability-hesitant fuzzy preference relation. As mentioned earlier, to date there has been a great deal of research into preference relations and interval preference relations. Nevertheless, in a probability-interval valued hesitant fuzzy environment, it is still not known how to calculate or improve the consistency of preference relations. Therefore, this study focuses on resolving this problem.

In this paper, based on the P-HFS and IVHFS, a definition of P-IVHFS is provided, and the relationship between the P-HFS, IVHFS and P-IVHFS is illustrated. Furthermore, motivated by the comparison method of HFEs, the comparison method of P-IVHFEs is defined. Additionally, inspired by the operations of IVHFEs, the complement, union and intersection and operational laws of P-IVHFEs are provided. Moreover, based on related studies, the definition of P-IVHFPRs is also provided. Subsequently, the consistency of P-IVHFPRs is discussed, using the multiplicative transitivity to verify the consistency of a P-IVHFPR. Finally, based on the method in a hesitant fuzzy environment, some definitions related to multiplicative consistent P-

IVHFPRs are provided, and a decomposition method to repair the consistency of P-IVHFPRs is proposed.

The rest of this paper is organized as follows. In Section 2, some concepts and properties associated with the topic are briefly reviewed. In Section 3, P-IVHFSs are proposed and some of their properties are discussed. In Section 4, P-IVHFPRs are proposed and in Section 5, the consistency of P-IVHFPRs is discussed. In Section 6, based on the multiplicative consistency of hesitant fuzzy preference relations, a decomposition method to deal with the consistency of P-IVHFPRs is proposed. Finally an example is provided to illustrate the algorithm.

2. Preliminaries

In this section, some concepts and properties associated with the topic are briefly reviewed.

Definition 1. *Let $\tilde{a} = [a^L, a^U] = \{x|a^L \le x \le a^U\}$, and then $\tilde{a}$ is called an interval number. For convenience, interval numbers are sometimes also called interval values. In particular, if $a^L = a^U$, $\tilde{a}$ is a real number. If $a^L \ge 0$, then $\tilde{a}$ is called a positive interval number.*

For any two positive interval numbers $\tilde{a} = [a^L, a^U], \tilde{b} = [b^L, b^U]$ and $\lambda \ge 0, \delta > 0$, then
(1) $\tilde{a} = \tilde{b}$ if $a^L = b^L$ and $a^U = b^U$;
(2) $\tilde{a} + \tilde{b} = [a^L + b^L, a^U + b^U]$;
(3) $\tilde{a} \cdot \tilde{b} = [a^L \cdot b^L, a^U \cdot b^U]$;
(4) $\lambda\tilde{a} = [\lambda a^L, \lambda a^U]$;
(5) $\tilde{a}^\delta = [a^L, a^U]^\delta = [(a^L)^\delta, (a^U)^\delta]$;
(6) $\delta^{\tilde{a}} = \delta^{[a^L,a^U]} = [\min\{\delta^{a^L}, \delta^{a^U}\}, \max\{\delta^{a^L}, \delta^{a^U}\}] = \begin{cases} [\delta^{a^U}, \delta^{a^L}], & if\ 0 < \delta < 1; \\ [\delta^{a^L}, \delta^{a^U}], & if\ \delta \ge 1. \end{cases}$

Definition 2. *[29] Let $\tilde{a}_1 = [a_1^L, a_1^U]$ and $\tilde{a}_2 = [a_2^L, a_2^U]$ be two interval numbers, and $len(\tilde{a}_1) = a_1^U - a_1^L$, $len(\tilde{a}_2) = a_2^U - a_2^L$, then the degree of possibility of $\tilde{a}_1 \ge \tilde{a}_2$ is defined as follows:*

$$p(\tilde{a}_1 \ge \tilde{a}_2) = \max\{1 - \max\{\frac{a_2^U - a_1^L}{len(\tilde{a}_1) + len(\tilde{a}_2)}, 0\}, 0\} \tag{1}$$

Similarly, the degree of possibility of $\tilde{a}_2 \ge \tilde{a}_1$ is defined as follows:

$$p(\tilde{a}_2 \ge \tilde{a}_1) = \max\{1 - \max\{\frac{a_1^U - a_2^L}{len(\tilde{a}_1) + len(\tilde{a}_2)}, 0\}, 0\} \tag{2}$$

Based on Definition 2, the following results hold:
(1) $0 \le p(\tilde{a}_1 \ge \tilde{a}_2) \le 1,\ 0 \le p(\tilde{a}_2 \ge \tilde{a}_1) \le 1$.
(2) $p(\tilde{a}_1 \ge \tilde{a}_2) + p(\tilde{a}_2 \ge \tilde{a}_1) = 1$. Especially, $p(\tilde{a}_1 \ge \tilde{a}_1) = p(\tilde{a}_2 \ge \tilde{a}_2) = 1$.

Definition 3. *[15, 16] Let X be a universal set, a hesitant fuzzy set (HFS) on X is in terms of a function that when applied to X returns a subset of $[0, 1]$.*

To be easily understood, the HFS can be expressed by a mathematical symbol [21]:

$$\tilde{A} = \left\{ \left\langle x, \tilde{h}_{\tilde{A}}(x) \right\rangle |x \in X \right\}$$

where $\tilde{h}_{\tilde{A}}(x)$ is a set of some values in $[0, 1]$, denoting the possible membership degrees of the element $x \in X$ to the set $\tilde{A}$. $\tilde{h}_{\tilde{A}}(x)$ is called a hesitant fuzzy element (HFE) and Θ the set of all HFEs [22].

For three HFEs h, h_1 and h_2, Torra and Narukawa [15, 16] defined the corresponding complement, union and intersection, namely

(1) $h^c = \cup_{\gamma \in h}\{1 - \gamma\}$;
(2) $h_1 \cup h_2 = \cup_{\gamma_1 \in h_1, \gamma_2 \in h_2} \max\{\gamma_1, \gamma_2\}$;
(3) $h_1 \cap h_2 = \cap_{\gamma_1 \in h_1, \gamma_2 \in h_2} \min\{\gamma_1, \gamma_2\}$.

Operational laws on the HFEs h, h_1 and h_2 have been given as follows [22]:

(1) $h^\lambda = \cup_{\gamma \in h}\{\gamma^\lambda\}, \lambda > 0$;
(2) $\lambda h = \cup_{\gamma \in h}\{1-(1-\gamma)^\lambda\}, \lambda > 0$;
(3) $h_1 \oplus h_2 = \cup_{\gamma_1 \in h_1, \gamma_2 \in h_2}\{\gamma_1 + \gamma_2 - \gamma_1\gamma_2\}$;
(4) $h_1 \otimes h_2 = \cup_{\gamma_1 \in h_1, \gamma_2 \in h_2}\{\gamma_1\gamma_2\}$.

Definition 4. *[2] Let X be a universal set, and $D[0,1]$ be the set of all closed subintervals of $[0,1]$. An interval-valued hesitant fuzzy set (IVHFS) on X is*

$$\tilde{A} = \left\{ \left\langle x_i, \tilde{h}_{\tilde{A}}(x_i) \right\rangle | x_i \in X, i = 1, 2, \ldots, n \right\}$$

where $\tilde{h}_{\tilde{A}}(x_i) : X \to D[0,1]$ denotes all possible interval-valued membership degrees of the element $x_i \in X$ to the set $\tilde{A}$. For convenience, we call $\tilde{h}_{\tilde{A}}(x_i)$ an interval-valued hesitant fuzzy element (IVHFE), which is denoted by

$$\tilde{h}_{\tilde{A}}(x_i) = \left\{ \tilde{\gamma} \middle| \tilde{\gamma} \in \tilde{h}_{\tilde{A}}(x_i) \right\}$$

Here $\tilde{\gamma} = [\tilde{\gamma}^L, \tilde{\gamma}^U]$ is an interval number. $\tilde{\gamma}^L = \inf \tilde{\gamma}$ and $\tilde{\gamma}^U = \sup \tilde{\gamma}$ represent the lower and upper limits of $\tilde{\gamma}$, respectively. When the lower and upper limits of the interval numbers are identical, IVHFS reduces to HFS [15]. Namely HFS is a special case of IVHFS.

Example 1. *Let $X = \{x_1, x_2\}$ be a universal set, and the two IVHFEs $\tilde{h}_{\tilde{A}}(x_1) = \{[0.1, 0.3], [0.4, 0.5]\}$ and $\tilde{h}_{\tilde{A}}(x_2) = \{[0.1, 0.2], [0.4, 0.6], [0.7, 0.8]\}$ denote the membership degrees of $x_i (i = 1, 2)$ to the set $\tilde{A}$. $\tilde{A}$ is an IVHFS, where*

$$\tilde{A} = \{\langle x_1, \{[0.1, 0.3], [0.4, 0.5]\}\rangle, \langle x_2, \{[0.1, 0.2], [0.4, 0.6], [0.7, 0.8]\}\rangle\}$$

Definition 5. *[2] For an IVHFE $\tilde{h}$, $s(\tilde{h}) = \frac{1}{l_{\tilde{h}}} \sum_{\tilde{\gamma} \in \tilde{h}} \tilde{\gamma}$ is called the score function of $\tilde{h}$ where $l_{\tilde{h}}$ is the number of the interval values in $\tilde{h}$, and $s(\tilde{h})$ is an interval value belonging to $[0,1]$. For two IVHFEs $\tilde{h}_1$ and $\tilde{h}_2$, if $s(\tilde{h}_1) \geq s(\tilde{h}_2)$, then $\tilde{h}_1 \geq \tilde{h}_2$.*

Definition 6. *[2] For three IVHFEs h, h_1 and h_2, the corresponding complement, union and intersection and operational laws have been given as follows. If $\tilde{\gamma}^L = \tilde{\gamma}^U$, then the following operations reduce to those of HFEs:*

(1) $\tilde{h}^c = \{[1-\tilde{\gamma}^U, 1-\tilde{\gamma}^L] \,\big|\, \tilde{\gamma} \in \tilde{h}\}$;
(2) $\tilde{h}_1 \cup \tilde{h}_2 = \{[\max(\tilde{\gamma}_1^L, \tilde{\gamma}_2^L), \max(\tilde{\gamma}_1^U, \tilde{\gamma}_2^U)] \,\big|\, \tilde{\gamma}_1 \in \tilde{h}_1, \tilde{\gamma}_2 \in \tilde{h}_2\}$;
(3) $\tilde{h}_1 \cap \tilde{h}_2 = \{[\min(\tilde{\gamma}_1^L, \tilde{\gamma}_2^L), \min(\tilde{\gamma}_1^U, \tilde{\gamma}_2^U)] \,\big|\, \tilde{\gamma}_1 \in \tilde{h}_1, \tilde{\gamma}_2 \in \tilde{h}_2\}$;
(4) $\tilde{h}^\lambda = \{[(\tilde{\gamma}^L)^\lambda, (\tilde{\gamma}^U)^\lambda] \,\big|\, \tilde{\gamma} \in \tilde{h}\}, \lambda > 0$;
(5) $\lambda\tilde{h} = \{[1-(1-\tilde{\gamma}^L)^\lambda, 1-(1-\tilde{\gamma}^U)^\lambda] \,\big|\, \tilde{\gamma} \in \tilde{h}\}, \lambda > 0$;
(6) $\tilde{h}_1 \oplus \tilde{h}_2 = \{[\tilde{\gamma}_1^L + \tilde{\gamma}_2^L - \tilde{\gamma}_1^L \cdot \tilde{\gamma}_2^L, \tilde{\gamma}_1^U + \tilde{\gamma}_2^U - \tilde{\gamma}_1^U \cdot \tilde{\gamma}_2^U] \,\big|\, \tilde{\gamma}_1 \in \tilde{h}_1, \tilde{\gamma}_2 \in \tilde{h}_2\}$;
(7) $\tilde{h}_1 \otimes \tilde{h}_2 = \{[\tilde{\gamma}_1^L \cdot \tilde{\gamma}_2^L, \tilde{\gamma}_1^U \cdot \tilde{\gamma}_2^U] \,\big|\, \tilde{\gamma}_1 \in \tilde{h}_1, \tilde{\gamma}_2 \in \tilde{h}_2\}$.

3. P-IVHFS

Inspired by the P-HFS [19, 37] and IVHFS [2], the definition of a P-IVHFS is provided.

Definition 7. *Let X be a universal set, and $D[0,1]$ be the set of all closed subintervals of $[0,1]$. A P-IVHFS on X is*

$$\tilde{A} = \left\{ \left\langle x_i, \tilde{h}_{\tilde{A}}(x_i, p_{ij}) \right\rangle | x_i \in X, i = 1, 2, \ldots, n,\ j = 1, 2, \ldots, m_i \right\}$$

where $\sum_{j=1}^{m_i} p_{ij} = 1$, m_i denotes the number of the interval values in $\tilde{h}_{\tilde{A}}(x_i, p_{ij})$, p_{ij} denotes the corresponding probability of the jth interval value in $\tilde{h}_{\tilde{A}}(x_i, p_{ij})$, and $\tilde{h}_{\tilde{A}}(x_i, p_{ij}) : X \to D[0,1]$ denotes all possible interval-valued membership degrees of the element $x_i \in X$ to the set $\tilde{A}$. For convenience, $\tilde{h}_{\tilde{A}}(x_i, p_{ij})$ is called a probability-interval valued hesitant fuzzy element(P-IVHFE), which is denoted by

$$\tilde{h}_{\tilde{A}}(x_i, p_{ij}) = \left\{ \tilde{\gamma} \middle| \tilde{\gamma} \in \tilde{h}_{\tilde{A}}(x_i, p_{ij}) \right\}$$

Here $\tilde{\gamma}$ is an interval number with a corresponding possibility. For simplicity, P-IVHFE can be denoted by $\tilde{h}_i$, $i = 1, 2, \cdots$. A P-IVHFE is the basic unit of a P-IVHFS, and the former can be considered as a special case of the latter. The relationship between a P-IVHFE and a P-IVHFS is similar to that between an IVHFE and an IVHFS [2].

Suppose that $\tilde{\gamma}^L = \inf \tilde{\gamma}$ and $\tilde{\gamma}^U = \sup \tilde{\gamma}$ represent the lower and upper limits of $\tilde{\gamma}$, respectively. When the lower and upper limits of the interval numbers are identical, the interval numbers are reduced to crisp numbers, and a P-IVHFS is reduced to a P-HFS. Thus, a P-HFS is a special case of a P-IVHFS. Meanwhile, it is clear that without the probability description p_{ij}, that is the probability values $p_{ij} (j = 1, 2, \cdots)$ are identical, a P-IVHFE is reduced to an IVHFE, and a P-IVHFS is reduced to an IVHFS. Thus, an IVHFE is a special case of a P-IVHFE, and an IVHFS is a special case of a P-IVHFS.

Example 2. *Let $X = \{x_1, x_2\}$ be a universal set, and the two P-IVHFEs*

$$\tilde{h}_{\tilde{A}}(x_1, p_{1j}) = \{[0.2, 0.3](p_{11} = 0.4),\ [0.5, 0.6](p_{12} = 0.6)\}$$
$$\tilde{h}_{\tilde{A}}(x_2, p_{2j}) = \{[0.1, 0.2](p_{21} = 0.3), [0.3, 0.5](p_{22} = 0.5),\ [0.6, 0.7](p_{23} = 0.2)\}$$

denote the membership degrees of $x_i (i = 1, 2)$ to the set $\tilde{A}$. $\tilde{A}$ is a P-IVHFS, where

$$\tilde{A} = \left\{ \begin{array}{l} \langle x_1, \{[0.2, 0.3](p_{11} = 0.4),\ [0.5, 0.6](p_{12} = 0.6)\}\rangle, \\ \langle x_2, \{[0.1, 0.2](p_{21} = 0.3), [0.3, 0.5](p_{22} = 0.5),\ [0.6, 0.7](p_{23} = 0.2)\}\rangle \end{array} \right\}$$

Based on the comparison method of HFEs [21], the following comparison method of P-IVHFEs is defined:

Definition 8. *For a P-IVHFE, $s(\tilde{h}) = \sum_{\tilde{\gamma} \in \tilde{h}} \tilde{\gamma} p_{\tilde{\gamma}}$ is called the score of $\tilde{h}$, where $p_{\tilde{\gamma}}$ is the corresponding probability of $\tilde{\gamma}$. It is clear that $s(\tilde{h})$ is also an interval number.*

Then by Eqs. (1) and (2), we can get the possibilities of $s(\tilde{h}_1) \geq s(\tilde{h}_2)$ and $s(\tilde{h}_2) \geq s(\tilde{h}_1)$, namely $p(\tilde{h}_1 \geq \tilde{h}_2)$ and $p(\tilde{h}_2 \geq \tilde{h}_1)$.

If $p(s(\tilde{h}_1) \geq s(\tilde{h}_2)) > 0.5$, then $s(\tilde{h}_1)$ is superior to $s(\tilde{h}_2)$, and thus $\tilde{h}_1$ is superior to $\tilde{h}_2$, denoted by $\tilde{h}_1 > \tilde{h}_2$ or $\tilde{h}_2 < \tilde{h}_1$.

If $p(s(\tilde{h}_1) \geq s(\tilde{h}_2)) < 0.5$, then $s(\tilde{h}_2)$ is superior to $s(\tilde{h}_1)$, and thus $\tilde{h}_2$ is superior to $\tilde{h}_1$, denoted by $\tilde{h}_2 > \tilde{h}_1$ or $\tilde{h}_1 < \tilde{h}_2$.

In particular, if $p(s(\tilde{h}_1) \geq s(\tilde{h}_2)) = 0.5$, then $\tilde{h}_1$ is indifferent to $\tilde{h}_2$, denoted by $\tilde{h}_1 \sim \tilde{h}_2$.

Example 3. *In Example 2, for the two P-IVHFEs*

$$\tilde{h}_1 = \{[0.2, 0.3](p_{11} = 0.4),\ [0.5, 0.6](p_{12} = 0.6)\}$$
$$\tilde{h}_2 = \{[0.1, 0.2](p_{21} = 0.3), [0.3, 0.5](p_{22} = 0.5),\ [0.6, 0.7](p_{23} = 0.2)\}$$

according to Definition 1, we have

$$s(\tilde{h}_1) = [0.2, 0.3] \times 0.4 + [0.5, 0.6] \times 0.6 = [0.38, 0.48]$$

$$s(\tilde{h}_2) = [0.1, 0.2] \times 0.3 + [0.3, 0.5] \times 0.5 + [0.6, 0.7] \times 0.2 = [0.3, 0.45]$$

Using Definition 2, we obtain

$$p(s(\tilde{h}_1) \geq s(\tilde{h}_2)) = \max\{1 - \max\{\frac{0.45 - 0.38}{0.15 + 0.1}, 0\}, 0\} \ = 0.72$$

which indicates that $\tilde{h}_1 > \tilde{h}_2$.

To be easily formulated, a P-IVHFE can be denoted by $\tilde{h} = \{[\tilde{\gamma}_i^L, \tilde{\gamma}_i^U](p_{[\tilde{\gamma}_i^L, \tilde{\gamma}_i^U]}) \Big| \tilde{\gamma}_i \in \tilde{h}\}$, for simplicity, denoted by $\tilde{h} = \{[\tilde{\gamma}_i^L, \tilde{\gamma}_i^U](p_{\tilde{\gamma}_i}) \Big| \tilde{\gamma}_i \in \tilde{h}\}$, where $p_{[\tilde{\gamma}_i^L, \tilde{\gamma}_i^U]}$ (or $p_{\tilde{\gamma}_i}$) denotes the corresponding probability value of $[\tilde{\gamma}_i^L, \tilde{\gamma}_i^U]$ (i.e., $\tilde{\gamma}_i$). Based on the operations of IVHFEs [2], the complement, union and intersection and operational laws of P-IVHFEs can be provided as follows:

Definition 9. *Let $\tilde{h}$, $\tilde{h}_1$ and $\tilde{h}_2$ be three P-IVHFEs, then*

(1) $\tilde{h}^c = \{[1 - \tilde{\gamma}_i^U, 1 - \tilde{\gamma}_i^L](p_{\tilde{\gamma}_i}) \Big| \tilde{\gamma}_i \in \tilde{h}\}$;
(2) $\tilde{h}_1 \cup \tilde{h}_2 = \{[\max(\tilde{\gamma}_1^L, \tilde{\gamma}_2^L), \max(\tilde{\gamma}_1^U, \tilde{\gamma}_2^U)](p_{\tilde{\gamma}_1} \cdot p_{\tilde{\gamma}_2}) \Big| \tilde{\gamma}_1 \in \tilde{h}_1, \tilde{\gamma}_2 \in \tilde{h}_2\}$;
(3) $\tilde{h}_1 \cap \tilde{h}_2 = \{[\min(\tilde{\gamma}_1^L, \tilde{\gamma}_2^L), \min(\tilde{\gamma}_1^U, \tilde{\gamma}_2^U)](p_{\tilde{\gamma}_1} \cdot p_{\tilde{\gamma}_2}) \Big| \tilde{\gamma}_1 \in \tilde{h}_1, \tilde{\gamma}_2 \in \tilde{h}_2\}$;
(4) $\tilde{h}^\lambda = \{[(\tilde{\gamma}_i^L)^\lambda, (\tilde{\gamma}_i^U)^\lambda](p_{\tilde{\gamma}_i}) \Big| \tilde{\gamma}_i \in \tilde{h}\}$;
(5) $\lambda\tilde{h} = \{[1 - (1 - \tilde{\gamma}_i^L)^\lambda, 1 - (1 - \tilde{\gamma}_i^U)^\lambda](p_{\tilde{\gamma}_i}) \Big| \tilde{\gamma}_i \in \tilde{h}\}, \lambda > 0$;
(6) $\tilde{h}_1 \oplus \tilde{h}_2 = \{[\tilde{\gamma}_1^L + \tilde{\gamma}_2^L - \tilde{\gamma}_1^L \cdot \tilde{\gamma}_2^L, \tilde{\gamma}_1^U + \tilde{\gamma}_2^U - \tilde{\gamma}_1^U \cdot \tilde{\gamma}_2^U](p_{\tilde{\gamma}_1} \cdot p_{\tilde{\gamma}_2}) \Big| \tilde{\gamma}_1 \in \tilde{h}_1, \tilde{\gamma}_2 \in \tilde{h}_2\}$;
(7) $\tilde{h}_1 \otimes \tilde{h}_2 = \{[\tilde{\gamma}_1^L \cdot \tilde{\gamma}_2^L, \tilde{\gamma}_1^U \cdot \tilde{\gamma}_2^U](p_{\tilde{\gamma}_1} \cdot p_{\tilde{\gamma}_2}) \Big| \tilde{\gamma}_1 \in \tilde{h}_1, \tilde{\gamma}_2 \in \tilde{h}_2\}$.

It is clear that without the probability description p_{ij}, that is the probability values $p_{ij}(j = 1, 2, \cdots)$ are identical, then the operational laws of P-IVHFEs are reduced to those of the IVHFEs.

Theorem 1. *When IVHFEs are extended to P-IVHFEs, the following operational laws [2] still are true in the P-IVHFS environment. Let $\tilde{h}$, $\tilde{h}_1$ and $\tilde{h}_2$ be three P-IVHFEs, then*

(1) $\tilde{h}_1 \oplus \tilde{h}_2 = \tilde{h}_2 \oplus \tilde{h}_1$;
(2) $\tilde{h}_1 \otimes \tilde{h}_2 = \tilde{h}_2 \otimes \tilde{h}_1$;
(3) $\lambda(\tilde{h}_1 \oplus \tilde{h}_2) = \lambda\tilde{h}_1 \oplus \lambda\tilde{h}_2, \lambda > 0$;
(4) $(\tilde{h}_1 \otimes \tilde{h}_2)^\lambda = \tilde{h}_1^\lambda \otimes h_2^\lambda, \lambda > 0$;
(5) $\lambda_1\tilde{h} \oplus \lambda_2\tilde{h} = (\lambda_1 + \lambda_2)\tilde{h}, \lambda_1, \lambda_2 > 0$;
(6) $\tilde{h}^{\lambda_1} \oplus \tilde{h}^{\lambda_2} = \tilde{h}^{(\lambda_1 + \lambda_2)}, \lambda_1, \lambda_2 > 0$.
(7) $\tilde{h}_1^c \cup \tilde{h}_2^c = (\tilde{h}_1 \cap \tilde{h}_2)^c$;
(8) $\tilde{h}_1^c \cap \tilde{h}_2^c = (\tilde{h}_1 \cup \tilde{h}_2)^c$;
(9) $(\tilde{h}^c)^\lambda = (\lambda\tilde{h})^c$;
(10) $\lambda(\tilde{h}^c) = (\tilde{h}^\lambda)^c$;
(11) $\tilde{h}_1^c \oplus \tilde{h}_2^c = (\tilde{h}_1 \otimes \tilde{h}_2)^c$;
(12) $\tilde{h}_1^c \otimes \tilde{h}_2^c = (\tilde{h}_1 \oplus \tilde{h}_2)^c$.

Since they can be proven analogously, like those in an IVHFS environment, they are just listed without any proof. Meanwhile, according to Definition 9, the following operational laws also hold:

Theorem 2. *Let $\tilde{h}$, $\tilde{h}_1$ and $\tilde{h}_2$ be three P-IVHFEs, then*

(1) $(\tilde{h} \cup \tilde{h}_1) \cup \tilde{h}_2 = \tilde{h} \cup (\tilde{h}_1 \cup \tilde{h}_2)$;
(2) $(\tilde{h} \cap \tilde{h}_1) \cap \tilde{h}_2 = \tilde{h} \cap (\tilde{h}_1 \cap \tilde{h}_2)$;
(3) $(\tilde{h} \oplus \tilde{h}_1) \oplus \tilde{h}_2 = \tilde{h} \oplus (\tilde{h}_1 \oplus \tilde{h}_2)$;
(4) $(\tilde{h} \otimes \tilde{h}_1) \otimes \tilde{h}_2 = \tilde{h} \otimes (\tilde{h}_1 \otimes \tilde{h}_2)$.

Proof. *In the following, only (3) is proven; others can be obtained directly by Definition 9. Suppose that*

$$\tilde{h} = \{[\tilde{\gamma}^L, \tilde{\gamma}^U](p_{\tilde{\gamma}}) \Big| \tilde{\gamma} \in \tilde{h}\};\ \tilde{h}_1 = \{[\tilde{\gamma}_1^L, \tilde{\gamma}_1^U](p_{\tilde{\gamma}_1}) \Big| \tilde{\gamma}_1 \in \tilde{h}_1\};\ \tilde{h}_2 = \{[\tilde{\gamma}_2^L, \tilde{\gamma}_2^U](p_{\tilde{\gamma}_2}) \Big| \tilde{\gamma}_2 \in \tilde{h}_2\},$$

then according to Definition 9,

$$\tilde{h} \oplus \tilde{h}_1 = \{[\tilde{\gamma}^L + \tilde{\gamma}_1^L - \tilde{\gamma}^L \cdot \tilde{\gamma}_1^L, \tilde{\gamma}^U + \tilde{\gamma}_1^U - \tilde{\gamma}^U \cdot \tilde{\gamma}_1^U](p_{\tilde{\gamma}} \cdot p_{\tilde{\gamma}_1}) \Big| \tilde{\gamma} \in \tilde{h}, \tilde{\gamma}_1 \in \tilde{h}_1\}$$

is obtained, therefore,

$$\begin{aligned}
&(\tilde{h} \oplus \tilde{h}_1) \oplus \tilde{h}_2 \\
&= \{[(\tilde{\gamma}^L + \tilde{\gamma}_1^L - \tilde{\gamma}^L \cdot \tilde{\gamma}_1^L) + \tilde{\gamma}_2^L - (\tilde{\gamma}^L + \tilde{\gamma}_1^L - \tilde{\gamma}^L \cdot \tilde{\gamma}_1^L) \cdot \tilde{\gamma}_2^L, (\tilde{\gamma}^U + \tilde{\gamma}_1^U - \tilde{\gamma}^U \cdot \tilde{\gamma}_1^U) + \tilde{\gamma}_2^U \\
&\quad -(\tilde{\gamma}^U + \tilde{\gamma}_1^U - \tilde{\gamma}^U \cdot \tilde{\gamma}_1^U) \cdot \tilde{\gamma}_2^U]((p_{\tilde{\gamma}} \cdot p_{\tilde{\gamma}_1}) \cdot p_{\tilde{\gamma}_2}) \Big| \tilde{\gamma} \in \tilde{h}, \tilde{\gamma}_1 \in \tilde{h}_1, \tilde{\gamma}_2 \in \tilde{h}_2\} \\
&= \{[\tilde{\gamma}^L + \tilde{\gamma}_1^L + \tilde{\gamma}_2^L - \tilde{\gamma}^L \cdot \tilde{\gamma}_1^L - \tilde{\gamma}^L \cdot \tilde{\gamma}_2^L - \tilde{\gamma}_1^L \cdot \tilde{\gamma}_2^L + \tilde{\gamma}^L \cdot \tilde{\gamma}_1^L \cdot \tilde{\gamma}_2^L, \tilde{\gamma}^U + \tilde{\gamma}_1^U + \tilde{\gamma}_2^U - \tilde{\gamma}^U \cdot \tilde{\gamma}_1^U \\
&\quad -\tilde{\gamma}^U \cdot \tilde{\gamma}_2^U - \tilde{\gamma}_1^U \cdot \tilde{\gamma}_2^U + \tilde{\gamma}^U \cdot \tilde{\gamma}_1^U \cdot \tilde{\gamma}_2^U](p_{\tilde{\gamma}} \cdot p_{\tilde{\gamma}_1} \cdot p_{\tilde{\gamma}_2}) \Big| \tilde{\gamma} \in \tilde{h}, \tilde{\gamma}_1 \in \tilde{h}_1, \tilde{\gamma}_2 \in \tilde{h}_2\}
\end{aligned}$$

Likewise,

$$\begin{aligned}
&\tilde{h} \oplus (\tilde{h}_1 \oplus \tilde{h}_2) \\
&= \{[\tilde{\gamma}^L + (\tilde{\gamma}_1^L + \tilde{\gamma}_2^L - \tilde{\gamma}_1^L \cdot \tilde{\gamma}_2^L) - \tilde{\gamma}^L \cdot (\tilde{\gamma}_1^L + \tilde{\gamma}_2^L - \tilde{\gamma}_1^L \cdot \tilde{\gamma}_2^L), \tilde{\gamma}^U + (\tilde{\gamma}_1^U + \tilde{\gamma}_2^U - \tilde{\gamma}_1^U \cdot \tilde{\gamma}_2^U) \\
&\quad -\tilde{\gamma}^U \cdot (\tilde{\gamma}_1^U + \tilde{\gamma}_2^U - \tilde{\gamma}_1^U \cdot \tilde{\gamma}_2^U)](p_{\tilde{\gamma}} \cdot (p_{\tilde{\gamma}_1} \cdot p_{\tilde{\gamma}_2})) \Big| \tilde{\gamma} \in \tilde{h}, \tilde{\gamma}_1 \in \tilde{h}_1, \tilde{\gamma}_2 \in \tilde{h}_2\} \\
&= \{[\tilde{\gamma}^L + \tilde{\gamma}_1^L + \tilde{\gamma}_2^L - \tilde{\gamma}^L \cdot \tilde{\gamma}_1^L - \tilde{\gamma}^L \cdot \tilde{\gamma}_2^L - \tilde{\gamma}_1^L \cdot \tilde{\gamma}_2^L + \tilde{\gamma}^L \cdot \tilde{\gamma}_1^L \cdot \tilde{\gamma}_2^L, \tilde{\gamma}^U + \tilde{\gamma}_1^U + \tilde{\gamma}_2^U - \tilde{\gamma}^U \cdot \tilde{\gamma}_1^U \\
&\quad -\tilde{\gamma}^U \cdot \tilde{\gamma}_2^U - \tilde{\gamma}_1^U \cdot \tilde{\gamma}_2^U + \tilde{\gamma}^U \cdot \tilde{\gamma}_1^U \cdot \tilde{\gamma}_2^U](p_{\tilde{\gamma}} \cdot p_{\tilde{\gamma}_1} \cdot p_{\tilde{\gamma}_2}) \Big| \tilde{\gamma} \in \tilde{h}, \tilde{\gamma}_1 \in \tilde{h}_1, \tilde{\gamma}_2 \in \tilde{h}_2\}
\end{aligned}$$

can be obtained. Therefore, we have

$$(\tilde{h} \oplus \tilde{h}_1) \oplus \tilde{h}_2 = \tilde{h} \oplus (\tilde{h}_1 \oplus \tilde{h}_2)$$

which completes the proof. □

4. P-IVHFPRs and Consistency

In this section, we present P-IVHFPRs and discuss their consistency.

4.1. P-IVHFPRs

In the GDM process, preference relations are very popular tools for expressing the DM's preferences when they compare a set of alternatives. Various types of preference relations have been given for different environments [2].

In order to represent preference relations more objectively, suppose that DMs are allowed to provide several possible interval fuzzy preference values and the associated probability values when they compare two alternatives, then we get the following P-IVHFPR:

Definition 10. *Let $X = \{x_1, x_2, \ldots x_n\}$ be a universal set. A P-IVHFPR on X is denoted by a matrix $\tilde{R} = (\tilde{h}_{ij})_{n\times n} \subset X \times X$, where $\tilde{h}_{ij} = \{\tilde{h}^t_{ij}(p^t_{ij}), t = 1, 2, \cdots, m_{ij}\}$ is a P-IVHFE, indicating all possible degrees to which x_i is preferred to x_j and the corresponding probability values with m_{ij} representing the number of intervals in the P-IVHFE. In addition, $\tilde{h}^t_{ij}$ should satisfy*

$$\begin{aligned}
&\inf \tilde{h}^{\sigma(t)}_{ij} + \sup \tilde{h}^{\sigma(m_{ij}+1-t)}_{ji} = \sup \tilde{h}^{\sigma(t)}_{ij} + \inf \tilde{h}^{\sigma(m_{ij}+1-t)}_{ji} = 1,\\
&p^{\sigma(t)}_{ij} = p^{\sigma(m_{ij}+1-t)}_{ji},\\
&\tilde{h}_{ii} = \{[0.5, 0.5](p = 1)\}, \quad i, j = 1, 2, \cdots, n
\end{aligned} \tag{3}$$

where we arrange the intervals in $\tilde{h}_{ij}$ in an increasing order, and let $\tilde{h}^{\sigma(t)}_{ij}$ be the tth smallest interval in $\tilde{h}_{ij}$. $\inf \tilde{h}^{\sigma(t)}_{ij}$ and $\sup \tilde{h}^{\sigma(t)}_{ij}$ denote the lower and upper limits of $\tilde{h}^{\sigma(t)}_{ij}$ respectively, $p^{\sigma(t)}_{ij}$ and $p^{\sigma(m_{ij}+1-t)}_{ji}$ denote the corresponding values of $\tilde{h}^{\sigma(t)}_{ij}$ and $\tilde{h}^{\sigma(m_{ij}+1-t)}_{ji}$ respectively, $p = 1$ denotes the corresponding value is equal to 1.

Example 4. *The following matrix in which every element is a P-IVHFE can represent a probability-interval valued hesitant fuzzy preference relation:*

$$\tilde{R}_e = (\tilde{h}_{ij})_{3\times 3} = \begin{pmatrix} \{[0.5, 0.5](1)\} & \{[0.4, 0.5](0.6), [0.7, 0.8](0.4)\} & \{[0.5, 0.6](1)\} \\ \{[0.2, 0.3](0.4), [0.5, 0.6](0.6)\} & \{[0.5, 0.5](1)\} & \{[0.3, 0.4](0.2), [0.5, 0.7](0.5), [0.8, 0.9](0.3)\} \\ \{[0.4, 0.5](1)\} & \{[0.1, 0.2](0.3), [0.3, 0.5](0.5), [0.6, 0.7](0.2)\} & \{[0.5, 0.5](1)\} \end{pmatrix}$$

where $\tilde{h}_{ij}$ denotes the group preference degree that the alternative x_i is superior to the alternative x_j.

Motivated by [2, 35], it can be explained how the elements in the matrix are obtained. Take $\tilde{h}_{23}$ as an example. Since $\tilde{h}_{23}$ represents all possible probability-interval valued preference degrees to which x_3 is preferred to x_2, its values come from $\tilde{h}^1_{23} = [0.1, 0.2]$, $\tilde{h}^2_{23} = [0.3, 0.5]$, $\tilde{h}^3_{23} = [0.6, 0.7]$ which is provided by a DM. The DM is sure that the preference value is the interval $[0.1, 0.2]$ with a probability of 30%, and the interval $[0.3, 0.5]$ with a probability of 50%, and the interval $[0.6, 0.7]$ with a probability of 20%. Therefore, the $\tilde{h}_{23}$ can be denoted by $\{[0.1, 0.2](0.3), [0.3, 0.5](0.5), [0.6, 0.7](0.2)\}$. Similarly the symmetric element of $\tilde{h}_{23}$, i.e., $\tilde{h}_{32}$ can be denoted by $\{[0.3, 0.4](0.2), [0.5, 0.7](0.5), [0.8, 0.9](0.3)\}$. Other symmetric elements $\tilde{h}_{ij}$ and $\tilde{h}_{ji}$ in $\tilde{R}_e$ are obtained in an analogous way, and satisfy the complementary properties defined in Eq.(3). In addition, when $i = j$, $\tilde{h}_{ij}$ represents the preference degree to which x_i is preferred to itself; namely, it is preferred equally , therefore $\tilde{h}_{ii} = \{[0.5, 0.5](1)\} (i = 1, 2, 3)$. Through the above procedure, the aforementioned matrix $\tilde{R}_e$ is obtained.

4.2. The Consistency of P-IVHFPRs

Cardinal consistency is a stronger concept than ordinal consistency. In the analytic hierarchy process, Saaty [13] first addressed the issue of consistency, and developed the notions of perfect consistency and acceptable consistency. Ordinal consistency is based on the notion of transitivity, meaning that if A is preferred to B and B is preferred to C, it perceives A to be preferred to C, which is normally referred to as weak transitivity [4, 26]. The weak transitivity is the minimum requirement condition to ensure that the hesitant fuzzy preference relation is consistent. There are further two conditions, named additive transitivity and multiplicative transitivity [14] which are more restrictive than weak transitivity and can imply reciprocity. Even though both additive transitivity and multiplicative transitivity can be used to measure consistency, the additive consistency may produce infeasible results [27]. Thus, the multiplicative transitivity is also used to verify the consistency of a P-IVHFPR.

Let $U = (u_{ij})_{n\times n}$, where u_{ij} denotes a ratio of preference intensity for the alternative A_i to that for A_j. Then the condition of multiplicative transitivity can be rewritten as follows:

$$u_{ij} u_{jk} u_{ki} = u_{ik} u_{kj} u_{ji} \tag{4}$$

Under the assumption of reciprocity, and in the case where $(u_{ik}, u_{kj}) \notin \{(0,1),(1,0)\}$, Eq.(4) can be expressed as follows [4]:

$$u_{ij} = \frac{u_{ik}u_{kj}}{u_{ik}u_{kj} + (1-u_{ik})(1-u_{kj})} \tag{5}$$

in the case where $(u_{ik}, u_{kj}) \in \{(0,1),(1,0)\}$, stipulating $u_{ij} = 0$.

Based on the multiplicative consistency of hesitant fuzzy preference relations, and using a decomposition method, the following definition is obtained:

Definition 11. *Let $\tilde{R} = (\tilde{h}_{ij})_{n\times n}$ be a P-IVHFPR on a fixed set $X = \{x_1, x_2, \cdots, x_n\}$ and $\tilde{h}_{ij} = \{\tilde{h}^t_{ij}(p^t_{ij}),\ t = 1,2,\cdots,m_{ij}\}$ be a P-IVHFE; suppose that $\tilde{h}^t_{ij} = [\#\tilde{h}^t_{ij}(x), \dagger\tilde{h}^t_{ij}(x)]$, where $\#\tilde{h}^t_{ij}(x)$ is the left endpoint of $\tilde{h}^t_{ij}$, and $\dagger\tilde{h}^t_{ij}(x)$ is the right endpoint of $\tilde{h}^t_{ij}$, let*

$$\tilde{R}^A = [\tilde{R}^t_{ij}(p^t_{ij})]_{n\times n} = \begin{bmatrix} \{0.5(1)\} & \{\#\tilde{h}^t_{12}(p^t_{12})\} & \cdots & \{\#\tilde{h}^t_{1n}(p^t_{1n})\} \\ \{\dagger\tilde{h}^t_{21}(p^t_{21})\} & \{0.5(1)\} & \cdots & \{\#\tilde{h}^t_{2n}(p^t_{2n})\} \\ \cdots & \cdots & \cdots & \cdots \\ \{\dagger\tilde{h}^t_{n1}(p^t_{n1})\} & \{\dagger\tilde{h}^t_{n2}(p^t_{n2})\} & \cdots & \{0.5(1)\} \end{bmatrix}$$

namely, $\tilde{R}^t_{ij} = \begin{cases} \#\tilde{h}^t_{ij}, & if\ i<j, \\ 0.5, & if\ i=j, \\ \dagger\tilde{h}^t_{ij}, & if\ i>j \end{cases}$ *which means if $i<j$, taking the left endpoint of $\tilde{h}^t_{ij}$, while if $i>j$, taking the right endpoint of $\tilde{h}^t_{ij}$. And let*

$$\tilde{R}^B = [\tilde{r}^t_{ij}(p^t_{ij})]_{n\times n} = \begin{bmatrix} \{0.5(1)\} & \{\dagger\tilde{h}^t_{12}(p^t_{12})\} & \cdots & \{\dagger\tilde{h}^t_{1n}(p^t_{1n})\} \\ \{\#\tilde{h}^t_{21}(p^t_{21})\} & \{0.5(1)\} & \cdots & \{\dagger\tilde{h}^t_{2n}(p^t_{2n})\} \\ \cdots & \cdots & \cdots & \cdots \\ \{\#\tilde{h}^t_{n1}(p^t_{n1})\} & \{\#\tilde{h}^t_{n2}(p^t_{n2})\} & \cdots & \{0.5(1)\} \end{bmatrix}$$

which means if $i<j$, taking the right endpoint of $\tilde{h}^t_{ij}$, while if $i>j$, taking the left endpoint of $\tilde{h}^t_{ij}$, namely,

$$\tilde{r}^t_{ij} = \begin{cases} \dagger\tilde{h}^t_{ij}, & if\ i<j, \\ 0.5, & if\ i=j, \\ \#\tilde{h}^t_{ij}, & if\ i>j \end{cases}$$

for convenience, $\tilde{R}^A$ and $\tilde{R}^B$ are called the decomposition of $\tilde{R}$, while $\tilde{R}$ is the composition of $\tilde{R}^A$ and $\tilde{R}^B$. Then $\tilde{R} = (\tilde{h}_{ij})_{n\times n}$ is multiplicative consistent if and only if $\tilde{R}^A$ and $\tilde{R}^B$ are both multiplicative consistent, i.e., the following two conditions are satisfied simultaneously:

$$(1)\ \tilde{R}^{\sigma(s)}_{ij} = \begin{cases} 0, & if\ (\tilde{R}_{ik}, \tilde{R}_{kj}) \in \{(\{0\},\{1\}),(\{1\},\{0\})\} \\ \frac{\tilde{R}^{\sigma(s)}_{ik}p^{\sigma(s)}_{ik}\tilde{R}^{\sigma(s)}_{kj}p^{\sigma(s)}_{kj}}{\tilde{R}^{\sigma(s)}_{ik}p^{\sigma(s)}_{ik}\tilde{R}^{\sigma(s)}_{kj}p^{\sigma(s)}_{kj}+(1-\tilde{R}^{\sigma(s)}_{ik})p^{\sigma(s)}_{ik}(1-\tilde{R}^{\sigma(s)}_{kj})p^{\sigma(s)}_{kj}}, & otherwise, \end{cases}$$

$$for\ all\ i\le k\le j$$

$$i.e.,\ \tilde{R}^{\sigma(s)}_{ij} = \begin{cases} 0, & if\ (\tilde{R}_{ik}, \tilde{R}_{kj}) \in \{(\{0\},\{1\}),(\{1\},\{0\})\} \\ \frac{\tilde{R}^{\sigma(s)}_{ik}\tilde{R}^{\sigma(s)}_{kj}}{\tilde{R}^{\sigma(s)}_{ik}\tilde{R}^{\sigma(s)}_{kj}+(1-\tilde{R}^{\sigma(s)}_{ik})(1-\tilde{R}^{\sigma(s)}_{kj})}, & otherwise, \end{cases}$$

$$for\ all\ i\le k\le j$$

$$p^{\sigma(s)}_{ij} = p^{\sigma(s)}_{ik}p^{\sigma(s)}_{kj}$$

$$(2)\ \tilde{r}^{\sigma(s)}_{ij} = \begin{cases} 0, & if\ (\tilde{r}_{ik}, \tilde{r}_{kj}) \in \{(\{0\},\{1\}),(\{1\},\{0\})\} \\ \frac{\tilde{r}^{\sigma(s)}_{ik}\tilde{r}^{\sigma(s)}_{kj}}{\tilde{r}^{\sigma(s)}_{ik}\tilde{r}^{\sigma(s)}_{kj}+(1-\tilde{r}^{\sigma(s)}_{ik})(1-\tilde{r}^{\sigma(s)}_{kj})}, & otherwise, \end{cases}$$

$$for\ all\ i\le k\le j$$

$$p^{\sigma(s)}_{ij} = p^{\sigma(s)}_{ik}p^{\sigma(s)}_{kj}$$

where $\tilde{R}_{ij}^{\sigma(s)}$, $\tilde{R}_{ik}^{\sigma(s)}$ and $\tilde{R}_{kj}^{\sigma(s)}$ are the sth smallest values in $\tilde{R}_{ij}$, $\tilde{R}_{ik}$ and $\tilde{R}_{kj}$ respectively; $p_{ij}^{\sigma(s)}$, $p_{ik}^{\sigma(s)}$ and $p_{kj}^{\sigma(s)}$ are their corresponding probability values, respectively; $\tilde{r}_{ij}^{\sigma(s)}$, $\tilde{r}_{ik}^{\sigma(s)}$ and $\tilde{r}_{kj}^{\sigma(s)}$ are the sth smallest values in $\tilde{r}_{ij}$, $\tilde{r}_{ik}$ and $\tilde{r}_{kj}$ respectively; and $p_{ij}^{\sigma(s)}$, $p_{ik}^{\sigma(s)}$ and $p_{kj}^{\sigma(s)}$ are their corresponding probability values, respectively.

If without the probability description and $\#h_{ik}^t = \dagger h_{ik}^t$, $\#h_{kj}^t = \dagger h_{kj}^t$, then Definition 11 is reduced to that of a hesitant fuzzy preference relation.

It can be proven that any P-IVHFPR $\tilde{R} = (\tilde{h}_{ij})_{2\times 2}$ is multiplicative consistent.

By extending the definitions in a hesitant fuzzy environment, the following definitions in a probability-interval valued hesitant fuzzy environment are obtained:

Definition 12. *Let $\tilde{R} = (\tilde{h}_{ij})_{n\times n}$ be a P-IVHFPR on a fixed set $X = \{x_1, x_2, \cdots, x_n\}$ and $\tilde{h}_{ij} = \{\tilde{h}_{ij}^t(p_{ij}^t),\ t = 1, 2, \cdots, m_{ij}\}$ be a P-IVHFE in which m_{ij} represents the number of intervals suppose that $\tilde{h}_{ij}^t = [\#\tilde{h}_{ij}^t(x), \dagger\tilde{h}_{ij}^t(x)]$, let*

$$\tilde{R}^A = [\tilde{R}_{ij}^t(p_{ij}^t)]_{n\times n} = \begin{bmatrix} \{0.5(1)\} & \{\#\tilde{h}_{12}^t(p_{12}^t)\} & \cdots & \{\#\tilde{h}_{1n}^t(p_{1n}^t)\} \\ \{\dagger\tilde{h}_{21}^t(p_{21}^t)\} & \{0.5(1)\} & \cdots & \{\#\tilde{h}_{2n}^t(p_{2n}^t)\} \\ \cdots & \cdots & \cdots & \cdots \\ \{\dagger\tilde{h}_{n1}^t(p_{n1}^t)\} & \{\dagger\tilde{h}_{n2}^t(p_{n2}^t)\} & \cdots & \{0.5(1)\} \end{bmatrix}$$

$$\tilde{R}^B = [\tilde{r}_{ij}^t(p_{ij}^t)]_{n\times n} = \begin{bmatrix} \{0.5(1)\} & \{\dagger\tilde{h}_{12}^t(p_{12}^t)\} & \cdots & \{\dagger\tilde{h}_{1n}^t(p_{1n}^t)\} \\ \{\#\tilde{h}_{21}^t(p_{21}^t)\} & \{0.5(1)\} & \cdots & \{\dagger\tilde{h}_{2n}^t(p_{2n}^t)\} \\ \cdots & \cdots & \cdots & \cdots \\ \{\#\tilde{h}_{n1}^t(p_{n1}^t)\} & \{\#\tilde{h}_{n2}^t(p_{n2}^t)\} & \cdots & \{0.5(1)\} \end{bmatrix}$$

then we call $\bar{R}$ a prefect multiplicative consistent P-IVHFPR, if $\bar{R}$ is the composition of $\bar{R}^A$ and $\bar{R}^B$, $\bar{R}^A = (\bar{R}_{ij}(x))_{n\times n}(p_{ij}^t))$, $\bar{R}^B = (\bar{r}_{ij}(x))_{n\times n}(p_{ij}^t))$, and

$$\bar{R}_{ij}^{\sigma(s)}(x) = \begin{cases} \frac{1}{j-i-1}\sum\limits_{k=i+1}^{j-1} \frac{\tilde{R}_{ik}^{\sigma(s)}(x)\tilde{R}_{kj}^{\sigma(s)}(x)}{\tilde{R}_{ik}^{\sigma(s)}(x)\tilde{R}_{kj}^{\sigma(s)}(x)+(1-\tilde{R}_{ik}^{\sigma(s)}(x))(1-\tilde{R}_{kj}^{\sigma(s)}(x))}, & i+1<j \\ \tilde{R}_{ij}^{\sigma(s)}(x), \quad i+1=j \\ \{0.5\}, \quad i=j \\ 1-\bar{R}_{ji}^{\sigma(s)}(x), \quad i>j \end{cases} \tag{6}$$

$$\bar{r}_{ij}^{\sigma(s)}(x) = \begin{cases} \frac{1}{j-i-1}\sum\limits_{k=i+1}^{j-1} \frac{\tilde{r}_{ik}^{\sigma(s)}(x)\tilde{r}_{kj}^{\sigma(s)}(x)}{\tilde{r}_{ik}^{\sigma(s)}(x)\tilde{r}_{kj}^{\sigma(s)}(x)+(1-\tilde{r}_{ik}^{\sigma(s)}(x))(1-\tilde{r}_{kj}^{\sigma(s)}(x))}, & i+1<j \\ \tilde{r}_{ij}^{\sigma(s)}(x), \quad i+1=j \\ \{0.5\}, \quad i=j \\ 1-\bar{r}_{ji}^{\sigma(s)}(x), \quad i>j \end{cases} \tag{7}$$

where $\bar{R}_{ij}^{\sigma(s)}(x)$, $\tilde{R}_{ik}^{\sigma(s)}(x)$, $\tilde{R}_{kj}^{\sigma(s)}(x)$, $\bar{r}_{ij}^{\sigma(s)}(x)$, $\tilde{r}_{ik}^{\sigma(s)}(x)$, $\tilde{r}_{kj}^{\sigma(s)}(x)$ denote the sth smallest values in $\bar{R}_{ij}(x)$, $\tilde{R}_{ik}(x)$, $\tilde{R}_{kj}(x)$, $\bar{r}_{ij}(x)$, $\tilde{r}_{ik}(x)$, $\tilde{r}_{kj}(x)$ respectively, and $s = 1, 2, \cdots, l$, $l = \max\{m_{ik}, m_{kj}\}$, in which m_{ik}, m_{kj} represent the number of intervals in $\tilde{h}_{ik}$ and $\tilde{h}_{kj}$ respectively.

If the two endpoints of the intervals are considered, the two corresponding probability-hesitant fuzzy preference relations are multiplicative consistent, thus it is believed that the P-IVHFPR is multiplicative consistent.

Definition 13. *Let $\tilde{R} = (\tilde{h}_{ij})_{n\times n}$, $\tilde{R}^A$, $\tilde{R}^B$, $\bar{R}^A$, $\bar{R}^B$ be as before, then we call $\tilde{R} = (\tilde{h}_{ij})_{n\times n}$ an acceptable multiplicative consistent P-IVHFPR if*

$$\begin{cases} d(\tilde{R}^A, \bar{R}^A) < \theta_0 \\ d(\tilde{R}^B, \bar{R}^B) < \theta_0 \end{cases}$$

where $d(\tilde{R}^A, \bar{R}^A)$ is the distance measure between $\tilde{R}^A$ and $\bar{R}^A$, $d(\tilde{R}^B, \bar{R}^B)$ is the distance measure between $\tilde{R}^B$ and $\bar{R}^B$. $d(\tilde{R}^A, \bar{R}^A)$ and $d(\tilde{R}^B, \bar{R}^B)$ can be calculated by the following Eqs.(8) and (9). θ_0 is the consistency level. We usually take $\theta_0 = 0.1$ in practice.

4.3. An Iterative Algorithm for Improving the Consistency of P-IVHFPR

In general, the P-IVHFPR constructed by the decision maker often has an unacceptable multiplicative consistency which means $d(\tilde{R}^A, \bar{R}^A) \geq \theta_0$ or $d(\tilde{R}^B, \bar{R}^B) \geq \theta_0$. At this time, it is necessary to adjust the elements in the P-IVHFPR in order to improve the consistency. Based on the algorithm in a hesitant fuzzy environment [9], An iterative algorithm is proposed as follows to repair the consistency of the P-IVHFPR.

An Iterative Algorithm for Improving the Consistency of P-IVHFPR

Input: P-IVHFPR $\tilde{R} = (\tilde{h}_{ij})_{n\times n}$; k, the number of iterations; δ, the step size, $0 \leq \lambda = k\delta \leq 1$; θ_0, the consistency level. Hereby we take $\theta_0 = 0.1$.

Output: P-IVHFPR $\tilde{R}^{(k)}$, with satisfactory consistency.

Step 1. Let $k = 1$, and construct a perfect multiplicative consistent P-IVHFPR $\bar{R}$, where $\bar{R}$ is the composition of $\bar{R}^A$ and $\bar{R}^B$, $\bar{R}^A = (\bar{R}_{ij}(x))_{n\times n}$, $\bar{R}^B = (\bar{r}_{ij}(x))_{n\times n}$. $\bar{R}^A$ and $\bar{R}^B$ are defined in Definition 12.

Step 2. Calculate the deviations $d(\tilde{R}^{(k)A}, \bar{R}^A)$ and $d(\tilde{R}^{(k)B}, \bar{R}^B)$. Eqs.(8) and (9) are given as follows:

$$\begin{cases} d_{Ham\min g}(\tilde{R}^{(k)A}, \bar{R}^A) = \frac{1}{(n-1)(n-2)} \sum\limits_{i=1}^{n} \sum\limits_{j=1}^{n} \left[\sum\limits_{s=1}^{m_{ij}} \left| R_{ij}^{(k)\sigma(s)} - \bar{R}_{ij}^{\sigma(s)} \right| p_{ij}^{\sigma(s)} \right] \\ d_{Ham\min g}(\tilde{R}^{(k)B}, \bar{R}^B) = \frac{1}{(n-1)(n-2)} \sum\limits_{i=1}^{n} \sum\limits_{j=1}^{n} \left[\sum\limits_{s=1}^{m_{ij}} \left| r_{ij}^{(k)\sigma(s)} - \bar{r}_{ij}^{\sigma(s)} \right| p_{ij}^{\sigma(s)} \right] \end{cases} \tag{8}$$

or

$$\begin{cases} d_{Euclidean}(\tilde{R}^{(k)A}, \bar{R}^A) = \left[\frac{1}{(n-1)(n-2)} \sum\limits_{i=1}^{n} \sum\limits_{j=1}^{n} \left(\sum\limits_{s=1}^{m_{ij}} \left| \left(R_{ij}^{(k)\sigma(s)} - \bar{R}_{ij}^{\sigma(s)} \right) p_{ij}^{\sigma(s)} \right|^2 \right) \right]^{\frac{1}{2}} \\ d_{Euclidean}(\tilde{R}^{(k)B}, \bar{R}^B) = \left[\frac{1}{(n-1)(n-2)} \sum\limits_{i=1}^{n} \sum\limits_{j=1}^{n} \left(\sum\limits_{s=1}^{m_{ij}} \left| \left(r_{ij}^{(k)\sigma(s)} - \bar{r}_{ij}^{\sigma(s)} \right) p_{ij}^{\sigma(s)} \right|^2 \right) \right]^{\frac{1}{2}} \end{cases} \tag{9}$$

where $R_{ij}^{(k)\sigma(s)}, \bar{R}_{ij}^{\sigma(s)}, r_{ij}^{(k)\sigma(s)}, \bar{r}_{ij}^{\sigma(s)}$ are the sth smallest values in $\tilde{R}^{(k)A}$, $\bar{R}^A$, $\tilde{R}^{(k)B}$, $\bar{R}^B$ respectively. $\tilde{R}^{(k)A}$ and $\tilde{R}^{(k)B}$ are the resolution of $\tilde{R}^{(k)}$. $\bar{R}^A$ and $\bar{R}^B$ are the resolution of $\bar{R}$, which is the corresponding perfect multiplicative relation of $\tilde{R}$. If $d(\tilde{R}^{(k)A}, \bar{R}^A) < \theta_0$ and $d(\tilde{R}^{(k)B}, \bar{R}^B) < \theta_0$, then go to Step 4; Otherwise, go to Step 3.

Step 3. Repair the inconsistent multiplicative P-IVHFPR, transforming $\tilde{R}^{(k)A}$ to $\widehat{R}^{(k)A}$ and $\tilde{R}^{(k)B}$ to $\widehat{R}^{(k)B}$ by using the following equations. We give Eqs.(10) and (11).

$$\widehat{R}_{ij}^{(k)\sigma(s)} = \frac{\left(R_{ij}^{(k)\sigma(s)}\right)^{1-\lambda} \left(\bar{R}_{ij}^{\sigma(s)}\right)^{\lambda}}{\left(R_{ij}^{(k)\sigma(s)}\right)^{1-\lambda} \left(\bar{R}_{ij}^{\sigma(s)}\right)^{\lambda} + \left(1-R_{ij}^{(k)\sigma(s)}\right)^{1-\lambda} \left(1-\bar{R}_{ij}^{\sigma(s)}\right)^{\lambda}} \quad i, j = 1, 2, \cdots, n \tag{10}$$

$$\widehat{r}_{ij}^{(k)\sigma(s)} = \frac{\left(r_{ij}^{(k)\sigma(s)}\right)^{1-\lambda} \left(\bar{r}_{ij}^{\sigma(s)}\right)^{\lambda}}{\left(r_{ij}^{(k)\sigma(s)}\right)^{1-\lambda} \left(\bar{r}_{ij}^{\sigma(s)}\right)^{\lambda} + \left(1-r_{ij}^{(k)\sigma(s)}\right)^{1-\lambda} \left(1-\bar{r}_{ij}^{\sigma(s)}\right)^{\lambda}} \quad i, j = 1, 2, \cdots, n \tag{11}$$

where $\widehat{R}_{ij}^{(k)\sigma(s)}, R_{ij}^{(k)\sigma(s)}, \bar{R}_{ij}^{\sigma(s)}$ are the sth smallest values in $\widehat{R}_{ij}^{(k)}, R_{ij}^{(k)}, \bar{R}_{ij}$ respectively, $\widehat{r}_{ij}^{(k)\sigma(s)}$, $r_{ij}^{(k)\sigma(s)}$, $\bar{r}_{ij}^{\sigma(s)}$ are the sth smallest values in $\widehat{r}_{ij}^{(k)}, r_{ij}^{(k)}, \bar{r}_{ij}$ respectively. Let $R^{(k+1)A} = \widehat{R}^{(k)A}$, $R^{(k+1)B} = \widehat{R}^{(k)B}$ and $k = k + 1$, then go to Step 2.

Step 4. Output $\tilde{R}^{(k)}$.

Step 5. End.

From the calculation process, it can be seen that the iterative process is convergent; for example when we take $\lambda = 1$. Therefore, only the steps are listed without providing any proof.

5. An Illustrative Example and Discussion

In this section, an example is used to illustrate the algorithm.

5.1. Illustrative Example

A large project of Jiudianxia reservoir operation [2, 25] is employed to demonstrate the validity of our approach. The reservoir is designed for many purposes, such as power generation, irrigation, total water supply for industry, agriculture, residents and environment. Because of different requirements for the partition of the amount of water, four reservoir operation schemes x_1, x_2, x_3 and x_4 are suggested.

x_1: maximum plant output, enough supply of water used in the Tao River basin, higher and lower supply for society and economy;

x_2: maximum plant output, enough supply of water used in the Tao River basin, higher and lower supply for society and economy, lower supply for ecosystem;

x_3: maximum plant output, enough supply of water used in the Tao River basin, higher and lower supply for society and economy, total supply for ecosystem and environment, whose 90% is used for flushing sands at low water period;

x_4: maximum plant output, enough supply of water used in the Tao River basin, higher and lower supply for society and economy, total supply for ecosystem and environment, whose 50% is used for flushing sands at low water period.

To select the best scheme, the government assigns a large consultancy organization to evaluate four competing schemes. Due to uncertainties, the DMs give their preference information regarding alternatives in the form of interval values with probabilities. Take schemes x_1 and x_2 as an example; the DMs evaluate the degrees to which x_1 is preferred to x_2, where 40% give a rating of [0.2,0.3] and the remaining 60% give [0.5,0.6]. Assume that these DMs in the consultancy firm cannot be persuaded each other to change their minds, the preference information that x_1 is preferred to x_2 provided by the organization can be considered as a P-IVHFE, i.e., $\{[0.2, 0.3](0.4), [0.5, 0.6](0.6)\}$. The preference information of the organization is listed as a P-IVHFPR $\tilde{R}$.

$$
\tilde{R} = (\tilde{h}_{ij})_{4\times 4} = \begin{bmatrix}
\{[0.5,0.5](1)\} & \{[0.4,0.5](0.6),[0.7,0.8](0.4)\} & \{[0.5,0.6](1)\} & \{[0.4,0.5](1)\} \\
\{[0.2,0.3](0.4),[0.5,0.6](0.6)\} & \{[0.5,0.5](1)\} & \{[0.6,0.7](1)\} & \{[0.4,0.5](0.4),[0.5,0.7](0.6)\} \\
\{[0.4,0.5](1)\} & \{[0.3,0.4](1)\} & \{[0.5,0.5](1)\} & \{[0.1,0.2](0.3),[0.3,0.5](0.5),\ [0.6,0.7](0.2) \\
\{[0.5,0.6](1)\} & \{[0.3,0.5](0.6),[0.5,0.6](0.4)\} & \{[0.3,0.4](0.2),[0.5,0.7](0.5),\ [0.8,0.9](0.3)\} & \{[0.5,0.5](1)\}
\end{bmatrix}
$$

To get the optimal alternative, the following steps are adopted.

Step 1. First of all, let $k = 1$ and construct the perfect multiplicative consistent P-IVHFPR $\bar{R}$. By Definition 12, we get

$$
\tilde{R}^A = \begin{bmatrix}
\{0.5(1)\} & \{0.4(0.6),0.7(0.4)\} & \{0.5(1)\} & \{0.4(1)\} \\
\{0.3(0.4),0.6(0.6)\} & \{0.5(1)\} & \{0.6(1)\} & \{0.4(0.4),0.5(0.6)\} \\
\{0.5(1)\} & \{0.4(1)\} & \{0.5(1)\} & \{0.1(0.3),0.3(0.5),\ 0.6(0.2)\} \\
\{0.6(1)\} & \{0.5(0.6),0.6(0.4)\} & \{0.4(0.2),0.7(0.5),\ 0.9(0.3)\} & \{0.5(1)\}
\end{bmatrix}
$$

$$\tilde{R}^B = \begin{bmatrix} \{0.5(1)\} & \{0.5(0.6), 0.8(0.4)\} & \{0.6(1)\} & \{0.5(1)\} \\ \{0.2(0.4), 0.5(0.6)\} & \{0.5(1)\} & \{0.7(1)\} & \{0.5(0.4), 0.7(0.6)\} \\ \{0.4(1)\} & \{0.3(1)\} & \{0.5(1)\} & \{0.2(0.3), 0.5(0.5),\ 0.7(0.2)\} \\ \{0.5(1)\} & \{0.3(0.6), 0.5(0.4)\} & \{0.3(0.2), 0.5(0.5),\ 0.8(0.3)\} & \{0.5(1)\} \end{bmatrix}$$

Therefore, according to Eq.(6), we have

$$\bar{R}_{13}^{\sigma(1)} = \frac{\tilde{R}_{12}^{\sigma(1)}\tilde{R}_{23}^{\sigma(1)}}{\tilde{R}_{12}^{\sigma(1)}\tilde{R}_{23}^{\sigma(1)}+(1-\tilde{R}_{12}^{\sigma(1)})(1-\tilde{R}_{23}^{\sigma(1)})} = \frac{0.4\times 0.6}{0.4\times 0.6+(1-0.4)\times(1-0.6)} = 0.5$$
$$p_{13}^{\sigma(1)} = 0.6$$

$$\bar{R}_{13}^{\sigma(2)} = \frac{\tilde{R}_{12}^{\sigma(2)}\tilde{R}_{23}^{\sigma(2)}}{\tilde{R}_{12}^{\sigma(2)}\tilde{R}_{23}^{\sigma(2)}+(1-\tilde{R}_{12}^{\sigma(2)})(1-\tilde{R}_{23}^{\sigma(2)})} = \frac{0.7\times 0.6}{0.7\times 0.6+(1-0.7)\times(1-0.6)} = 0.778$$
$$p_{13}^{\sigma(2)} = 0.4$$

where $\tilde{R}_{23}$, i.e., $\{0.6(1)\}$ can be regarded as $\{0.6(0.6), 0.6(0.4)\}$. So,

$$\bar{R}_{13} = \{0.5(0,6), 0.778(0.4)\}$$

hence,

$$\bar{R}_{31} = \{0.222(0.4), 0.5(0,6)\}$$

Analogously, by Eq.(6), we have

$$\bar{R}_{14}^{\sigma(s)} = \frac{1}{2}\left(\frac{\tilde{R}_{12}^{\sigma(s)}\tilde{R}_{24}^{\sigma(s)}}{\tilde{R}_{12}^{\sigma(s)}\tilde{R}_{24}^{\sigma(s)}+(1-\tilde{R}_{12}^{\sigma(s)})(1-\tilde{R}_{24}^{\sigma(s)})} + \frac{\tilde{R}_{13}^{\sigma(s)}\tilde{R}_{34}^{\sigma(s)}}{\tilde{R}_{13}^{\sigma(s)}\tilde{R}_{34}^{\sigma(s)}+(1-\tilde{R}_{13}^{\sigma(s)})(1-\tilde{R}_{34}^{\sigma(s)})}\right) \quad s = 1, 2, \cdots$$

Similar to the previous method to deal with P-IVHFE $\{0.6(1)\}$, in order to facilitate observing the probability values, $\tilde{R}_{34} = \{0.1(0.3), 0.3(0.5), 0.6(0.2)\}$ can be regarded as, or in other words,

$$\begin{aligned}\tilde{R}_{34} &= \{0.1(0.3), 0.3(0.5), 0.6(0.2)\} \\ &= \{0.1(0.3), 0.3(0.1), 0.3(0.2), 0.3(0.2), 0.6(0.2)\}\end{aligned}$$

Similarly,

$$\begin{aligned}\tilde{R}_{12} &= \{0.4(0.6), 0.7(0.4)\} \\ &= \{0.4(0.3), 0.4(0.1), 0.4(0.2), 0.70.2), 0.7(0.2)\} \\ \tilde{R}_{24} &= \{0.4(0.4), 0.5(0.6)\} \\ &= \{0.4(0.3), 0.4(0.1), 0.5(0.2), 0.5(0.2), 0.5(0.2)\} \\ \tilde{R}_{13} &= \{0.5(1)\} \\ &= \{0.5(0.3), 0.5(0.1), 0.5(0.2), 0.5(0.2), 0.5(0.2)\}\end{aligned}$$

therefore,

$$\begin{aligned}\bar{R}_{14}^{\sigma(1)} &= \frac{1}{2}\left(\frac{\tilde{R}_{12}^{\sigma(1)}\tilde{R}_{24}^{\sigma(1)}}{\tilde{R}_{12}^{\sigma(1)}\tilde{R}_{24}^{\sigma(1)}+(1-\tilde{R}_{12}^{\sigma(1)})(1-\tilde{R}_{24}^{\sigma(1)})} + \frac{\tilde{R}_{13}^{\sigma(1)}\tilde{R}_{34}^{\sigma(1)}}{\tilde{R}_{13}^{\sigma(1)}\tilde{R}_{34}^{\sigma(1)}+(1-\tilde{R}_{13}^{\sigma(1)})(1-\tilde{R}_{34}^{\sigma(1)})}\right) \\ &= \frac{1}{2}\left(\frac{0.4\times 0.4}{0.4\times 0.4+(1-0.4)\times(1-0.4)} + \frac{0.5\times 0.1}{0.5\times 0.1+(1-0.5)\times(1-0.1)}\right) \\ &= 0.204 \\ p_{14}^{\sigma(1)} &= 0.3\end{aligned}$$

Similarly, we have

$$\begin{aligned}&\bar{R}_{14}^{\sigma(2)} = 0.304,\ p_{14}^{\sigma(2)} = 0.1, \\ &\bar{R}_{14}^{\sigma(3)} = 0.35,\ p_{14}^{\sigma(3)} = 0.2, \\ &\bar{R}_{14}^{\sigma(4)} = 0.5,\ p_{14}^{\sigma(4)} = 0.2, \\ &\bar{R}_{14}^{\sigma(5)} = 0.65,\ p_{14}^{\sigma(5)} = 0.2\end{aligned}$$

thus,

$$\begin{aligned}
\bar{R}_{14} &= \{0.204(0.3), 0.304(0.1), 0.35(0.2), 0.5(0.2), 0.65(0.2)\} \\
\bar{R}_{41} &= \{0.35(0.2), 0.5(0.2), 0.65(0.2), 0.696(0.1), 0.796(0.3)\}
\end{aligned}$$

Analogously, we get

$$\begin{aligned}
\bar{R}_{24} &= \{0.143(0.3), 0.391(0.5), 0.692(0.2)\} \\
\bar{R}_{42} &= \{0.308(0.2), 0.609(0.5), 0.857(0.3)\}
\end{aligned}$$

hence,

$$\bar{R}^A = \begin{bmatrix}
\{0.5(1)\} & \{0.4(0.6), 0.7(0.4)\} & \{0.5(0,6), 0.778(0.4)\} & \{0.204(0.3), 0.304(0.1), 0.35(0.2), 0.5(0.2), 0.65(0.2)\} \\
\{0.3(0.4), 0.6(0.6)\} & \{0.5(1)\} & \{0.4(0.4), 0.5(0.6)\} & \{0.143(0.3), 0.391(0.5), 0.692(0.2)\} \\
\{0.222(0.4), 0.5(0,6)\} & \{0.4(1)\} & \{0.5(1)\} & \{0.1(0.3), 0.3(0.5),\ 0.6(0.2) \\
\{0.35(0.2), 0.5(0.2), 0.65(0.2), 0.696(0.1), 0.796(0.3)\} & \{0.308(0.2), 0.609(0.5), 0.857(0.3)\} & \{0.4(0.2), 0.7(0.5),\ 0.9(0.3)\} & \{0.5(1)\}
\end{bmatrix}$$

In the similar way, according to Eq.(7), we can obtain

$$\bar{R}^B = \begin{bmatrix}
\{0.5(1)\} & \{0.5(0.6), 0.8(0.4)\} & \{0.7(0.6), 0.903(0.4)\} & \{0.386(0.3), 0.55(0.1), 0.65(0.2), 0.752(0.2), 0.841(0.2)\} \\
\{0.2(0.4), 0.5(0.6)\} & \{0.5(1)\} & \{0.7(1)\} & \{0.368(0.3), 0.7(0.5), 0.845(0.2)\} \\
\{0.097(0.4), 0.3(0.6)\} & \{0.3(1)\} & \{0.5(1)\} & \{0.2(0.3), 0.5(0.5),\ 0.7(0.2)\} \\
\{0.159(0.2), 0.248(0.2), 0.35(0.2), 0.45(0.1), 0.614(0.3)\} & \{0.155(0.2), 0.3(0.5), 0.632(0.3)\} & \{0.3(0.2), 0.5(0.5),\ 0.8(0.3)\} & \{0.5(1)\}
\end{bmatrix}$$

Step 2. Calculate the deviations $d(\tilde{R}^{(k)A}, \bar{R}^A)$ and $d(\tilde{R}^{(k)B}, \bar{R}^B)$.
Using Eq.(8), we get

$$\begin{aligned}
& d_{Ham\min g}(\tilde{R}^A, \bar{R}^A) = \tfrac{1}{(n-1)(n-2)} \sum_{i=1}^{n} \sum_{j=1}^{n} \left[\sum_{s=1}^{m_{ij}} \left| R_{ij}^{(k)\sigma(s)} - \bar{R}_{ij}^{\sigma(s)} \right| p_{ij}^{\sigma(s)} \right] \\
& = \tfrac{1}{6} \sum_{i=1}^{4} \sum_{j=1}^{4} \left[\sum_{s=1}^{m_{ij}} \left| R_{ij}^{\sigma(s)} - \bar{R}_{ij}^{\sigma(s)} \right| p_{ij}^{\sigma(s)} \right] \\
& = \tfrac{1}{6}[(|0.5-0.5| \times 0.6 + |0.5-0.778| \times 0.4) + (|0.204-0.4| \times 0.3 + |0.304-0.4| \times 0.1 \\
& + |0.35-0.4| \times 0.2 + |0.5-0.4| \times 0.2 + |0.65-0.4| \times 0.2) + (|0.143-0.4| \times 0.3 \\
& + |0.391-0.4| \times 0.1 + |0.391-0.5| \times 0.4 + |0.692-0.5| \times 0.2) + (|0.5-0.222| \times 0.4 \\
& + |0.5-0.5| \times 0.6) + (|0.35-0.6| \times 0.2 + |0.5-0.6| \times 0.1 + |0.65-0.6| \times 0.2 \\
& + |0.696-0.6| \times 0.1 + |0.796-0.6| \times 0.3) + (|0.308-0.5| \times 0.2 + |0.609-0.5| \times 0.4 \\
& + |0.609-0.6| \times 0.1 + |0.857-0.6| \times 0.3)] \\
& = \tfrac{0.8092}{6} = 0.135 > \theta_0 = 0.1
\end{aligned}$$

Analogously, by Eq.(8), we can obtain

$$d_{Ham\min g}(\tilde{R}^B, \bar{R}^B) = \frac{0.9152}{6} = 0.153 > \theta_0 = 0.1$$

Therefore, $\tilde{R}^A$ and $\tilde{R}^B$ are both not multiplicative consistent P-IVHFPR. $\tilde{R}^A$ and $\tilde{R}^B$ need to be repaired by Eqs.(10) and (11).
Step 3. Repair the inconsistent multiplicative P-IVHFPR.

Hereby we try to assign a value, such as let $\lambda = 0.7$, then

$$\widehat{R}^{(1)A} = \begin{bmatrix} \{0.5(1)\} & \{0.4(0.6), 0.7(0.4)\} & \{0.5(0.6), 0.706(0.4)\} & \{0.255(0.3), 0.331(0.1), 0.365(0.2), 0.47(0.2), 0.577(0.2)\} \\ \{0.3(0.4), 0.6(0.6)\} & \{0.5(1)\} & \{0.4(0.4), 0.5(0.6)\} & \{0.202(0.3), 0.394(0.1), 0.423(0.4), 0.638(0.2)\} \\ \{0.294(0.4), 0.5(0,6)\} & \{0.4(1)\} & \{0.5(1)\} & \{0.1(0.3), 0.3(0.5),\ 0.6(0.2) \\ \{0.423(0.2), 0.53(0.2), 0.635(0.2), 0.669(0.1), 0.745(0.3)\} & \{0.362(0.2), 0.577(0.4), 0.606(0.1), 0.798(0.3)\} & \{0.4(0.2), 0.7(0.5),\ 0.9(0.3)\} & \{0.5(1)\} \end{bmatrix}$$

$$\widehat{R}^{(1)B} = \begin{bmatrix} \{0.5(1)\} & \{0.5(0.6), 0.8(0.4)\} & \{0.671(0.6), 0.843(0.4)\} & \{0.419(0.3), 0.535(0.1), 0.607(0.2), 0.685(0.2), 0.762(0.2)\} \\ \{0.2(0.4), 0.5(0.6)\} & \{0.5(1)\} & \{0.7(1)\} & \{0.406(0.3), 0.644(0.1), 0.7(0.4), 0.809(0.2)\} \\ \{0.157(0.4), 0.329(0.6)\} & \{0.3(1)\} & \{0.5(1)\} & \{0.2(0.3), 0.5(0.5),\ 0.7(0.2)\} \\ \{0.238(0.2), 0.315(0.2), 0.393(0.2), 0.465(0.1), 0.581(0.3)\} & \{0.191(0.2), 0.3(0.4), 0.356(0.1), 0.594(0.3)\} & \{0.3(0.2), 0.5(0.5),\ 0.8(0.3)\} & \{0.5(1)\} \end{bmatrix}$$

It follows that the normalized Hamming distance

$$d_{Hamming}(\widehat{R}^{(1)A}, \bar{R}^A) = 0.037 < \theta_0 = 0.1$$
$$d_{Hamming}(\widehat{R}^{(1)B}, \bar{R}^B) = 0.038 < \theta_0 = 0.1$$

Let $R^{(2)A} = \widehat{R}^{(1)A}, R^{(2)B} = \widehat{R}^{(1)B}$, then we have

$$d_{Hamming}(R^{(2)A}, \bar{R}^A) = 0.037 < 0.1$$
$$d_{Hamming}(R^{(2)B}, \bar{R}^B) = 0.038 < 0.1$$

the normalized Hamming distances are less than the consistency level 0.1, so $R^{(2)A}$ and $R^{(2)B}$ are the repaired $\tilde{R}^A$ and $\tilde{R}^B$ respectively.

Step 4. Output $\tilde{R}^{(k)}$.

The composition of $R^{(2)A}$ and $R^{(2)B}$, i.e.,

$$\tilde{R}^{(2)} = \begin{bmatrix} \{[0.5, 0.5](1)\} & \{[0.4, 0.5](0.6), [0.7, 0.8](0.4)\} & \{[0.5, 0.671](0.6), [0.706, 0.843](0.4)\} & \{[0.255, 0.419](0.3), [0.33, 0.535](0.1), [0.365, 0.607](0.2), [0.47, 0.685](0.2), [0.577, 0.762](0.2)\} \\ \{[0.2, 0.3](0.4), [0.5, 0.6](0.6)\} & \{[0.5, 0.5](1)\} & \{[0.6, 0.7](1)\} & \{[0.202, 0.406](0.3), [0.394, 0.644](0.1), [0.423, 0.7](0.4), [0.638, 0.809](0.2)\} \\ \{[0.157, 0.294](0.4), [0.329, 0.5](0.6)\} & \{[0.3, 0.4](1)\} & \{[0.5, 0.5](1)\} & \{[0.1, 0.2](0.3), [0.3, 0.5](0.5),\ [0.6, 0.7](0.2) \\ \{[0.238, 0.423](0.2), [0.315, 0.53](0.2), [0.393, 0.635](0.2), [0.465, 0.669](0.1), [0.581, 0.745](0.3)\} & \{[0.191, 0.362](0.2), [0.3, 0.577](0.4), [0.356, 0.606](0.1), [0.594, 0.798](0.3)\} & \{[0.3, 0.4](0.2), [0.5, 0.7](0.5),\ [0.8, 0.9](0.3)\} & \{[0.5, 0.5](1)\} \end{bmatrix}$$

is the repaired multiplicative P-IVHFPR of $\tilde{R}$.

Step 5. The last step is to sort the four schemes (alternatives).

Using Definition 8, let $p_{ij} = p(\tilde{R}_{ij}^{(2)} \geq \tilde{R}_{ji}^{(2)})$, then we get the following complementary matrix:

$$P = \begin{bmatrix} 0.5 & 1 & 1 & 0.454 \\ 0 & 0.5 & 1 & 0.554 \\ 0 & 0 & 0.5 & 0 \\ 0.546 & 0.446 & 1 & 0.5 \end{bmatrix}$$

If critical value λ is allowed to be an appropriate value, such as a value between the largest and the second largest value of $p_{ij}, i, j = 1, 2, 3, 4$ (not including 1), e.g., $\lambda = 0.55$, and

$$\tilde{p}_{ij} = \begin{cases} 1, & if\ p_{ij} \geq \lambda, \\ 0, & if\ p_{ij} < \lambda \end{cases}$$

then further we get

$$\tilde{P} = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

According to $\tilde{p}$, we have

$$x_1 \succ x_2, x_1 \succ x_3, x_2 \succ x_3, x_2 \succ x_4, x_4 \succ x_3$$

namely,

$$x_1 \succ x_2 \succ x_4 \succ x_3$$

which indicates that the first scheme is the most desirable according to the opinion of the large consultancy firm.

5.2. Discussion and Comparison

Having carefully analyzed the calculation process and results, the following conclusions can be drawn:

(1) Since there are probability values in a probability-interval valued hesitant fuzzy environment, after the multiplications, there are decimals which are not the integer multiples of 0.1 in the calculated results, such as $0.696, 0.857, 0.391 \cdots$. If one searches the relevant documents on interval-valued preference relations, it can be found that this inevitably happens in the calculation process. Therefore, future research could try and explain this phenomenon.

(2) It can be seen that there are overlapping intervals in the calculated results. Such as a P-IVHFE,

$$\tilde{R}_{12}^{(2)} = \{[0.238, 0.423](0.2), [0.315, 0.53](0.2), [0.393, 0.635](0.2), [0.465, 0.669](0.1), [0.581, 0.745](0.3)\}$$

where between the intervals $[0.238, 0.423]$ and $[0.315, 0.53]$, there is an overlapping interval $[0.315, 0.423]$. To deal with this problem, without a loss of generality, it is assumed that all the interval values have a uniform distribution, then they can be changed into an equivalent expression in which the intervals are not overlapping. For example, as for $\tilde{R}_{12}^{(2)}$, we have

$$\begin{aligned}
&[0.238, 0.423](0.2) \\
=&\{[0.238, 0.315](0.2 \times \tfrac{0.315-0.238}{0.423-0.238}), [0.315, 0.393](0.2 \times \tfrac{0.393-0.315}{0.423-0.238}), [0.393, 0.423](0.2 \times \tfrac{0.423-0.393}{0.423-0.238})\} \\
=& \{[0.238, 0.315](0.083), [0.315, 0.393](0.084), [0.393, 0.423](0.033)\}
\end{aligned}$$

In a similar way, we get

$$[0.315, 0.53](0.2) = \{[0.315, 0.393](0.073), [0.393, 0.465](0.067), [0.465, 0.53](0.060)\}$$

$$[0.393, 0.635](0.2) = \{[0.393, 0.465](0.059), [0.465, 0.581](0.096), [0.581, 0.635](0.045)\}$$

$$[0.465, 0.669](0.1) = \{[0.465, 0.581](0.057), [0.581, 0.669](0.043)\}$$

therefore,

$$\begin{aligned}
\tilde{R}_{12}^{(2)} &= \{[0.238, 0.315](0.083), [0.315, 0.393](0.084 + 0.073), [0.393, 0.465](0.033 + 0.067 + 0.059), \\
&\quad [0.465, 0.581](0.060 + 0.096 + 0.057), [0.581, 0.745](0.045 + 0.043 + 0.3)\} \\
&= \{[0.238, 0.315](0.083), [0.315, 0.393](0.157), [0.393, 0.465](0.159), \\
&\quad [0.465, 0.581](0.213), [0.581, 0.745](0.388)\}
\end{aligned}$$

There are not any overlapping intervals in this new expression.

It can be seen from the above example that P-IVHFPRs are useful in resolving large GDM problems, because they express intuitively the uncertain and hesitant preference information provided by each DM in a decision-making organization. This differs from the approach of interval-valued fuzzy sets for GDM, where the opinions of the DMs based on a pairwise comparison of alternatives, are first aggregated and, correspondingly, only the average interval-valued preference information is obtained. However, the use of, P-IVHFPRs does not need to perform such an aggregation and, hence, provides a more comprehensive description of the opinions of these DMs [2]. In the above example, if the probability-interval valued preference information is firstly aggregated at the beginning of the calculation, with regard to the probability values as the corresponding weights, using Definition 8, e.g.,

$$
\begin{aligned}
&s(\tilde{h}_{12}) = (\{[0.4, 0.5](0.6), [0.7, 0.8](0.4)\}) \\
&= [0.4 \times 0.6 + 0.7 \times 0.4, 0.5 \times 0.6 + 0.8 \times 0.4] = [0.52, 0.62]
\end{aligned}
$$

then we get

$$
\begin{aligned}
&\tilde{R} = (\tilde{h}_{ij})_{4\times 4} \\
&= \begin{bmatrix}
\{[0.5,0.5](1)\} & \{[0.4,0.5](0.6),[0.7,0.8](0.4)\} & \{[0.5,0.6](1)\} & \{[0.4,0.5](1)\} \\
\{[0.2,0.3](0.4),[0.5,0.6](0.6)\} & \{[0.5,0.5](1)\} & \{[0.6,0.7](1)\} & \{[0.4,0.5](0.4),[0.5,0.7](0.6)\} \\
\{[0.4,0.5](1)\} & \{[0.3,0.4](1)\} & \{[0.5,0.5](1)\} & \{[0.1,0.2](0.3),[0.3,0.5](0.5),\ [0.6,0.7](0.2) \\
\{[0.5,0.6](1)\} & \{[0.3,0.5](0.6),[0.5,0.6](0.4)\} & \{[0.3,0.4](0.2),[0.5,0.7](0.5),\ [0.8,0.9](0.3)\} & \{[0.5,0.5](1)\}
\end{bmatrix}
\end{aligned}
$$

$$
\rightarrow [s(\tilde{h}_{ij})]_{4\times 4} = \begin{bmatrix}
[0.5, 0.5] & [0.52, 0.62] & [0.5, 0.6] & [0.4, 0.5] \\
[0.38, 0.48] & [0.5, 0.5] & [0.6, 0.7] & [0.46, 0.62] \\
[0.4, 0.5] & [0.3, 0.4] & [0.5, 0.5] & [0.3, 0.45] \\
[0.5, 0.6] & [0.38, 0.54] & [0.55, 0.7] & [0.5, 0.5]
\end{bmatrix}
$$

Further, in the same way as before, let $p_{ij} = (s(\tilde{h}_{ij}) \geq s(\tilde{h}_{ji}))$, then the following complementary matrix is obtained:

$$
P' = \begin{bmatrix}
0.5 & 1 & 1 & 0 \\
0 & 0.5 & 1 & 0.75 \\
0 & 0 & 0.5 & 0 \\
1 & 0.25 & 1 & 0.5
\end{bmatrix}
$$

indicating that

$$
x_1 \succ x_2, x_1 \succ x_3, x_2 \succ x_3, x_2 \succ x_4, x_4 \succ x_1, x_4 \succ x_3
$$

which is heavily inconsistent. Let

$$
\tilde{p}'_{ij} = \begin{cases} 1, & if\ p'_{ij} \geq \lambda, \\ 0, & if\ p'_{ij} < \lambda \end{cases}
$$

Only when we let critical value $\lambda > 0.75$, can a consistent result be obtained. At this time,

$$
\tilde{P}' = \begin{bmatrix}
0 & 1 & 1 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0
\end{bmatrix}
$$

which indicates that

$$
x_1 \succ x_2, x_1 \succ x_3, x_2 \succ x_3, x_4 \succ x_1, x_4 \succ x_3
$$

namely,

$x_4 \succ x_1 \succ x_2 \succ x_3$

From the results of the calculations, one can find a difference in the ranking results derived in these two approaches. The reason is that for each decision-making organization composed of multiple DMs, a group's preference value is obtained by aggregating (namely, averaging) individual preference values. Such an aggregation actually amounts to implementing a transformation of P-IVHFEs into an interval-valued fuzzy number. As a result, it leads to the loss of information, which affects the final ranking results. Thus, the comparison clearly shows the benefits of the proposed GDM approach based on P-IVHFPRs [2].

Compared with that in a hesitant fuzzy environment, this method's implementation could be far more sophisticated in a probability-interval valued hesitant fuzzy environment, but has led to some new problems. For example, in order to get the equivalent expression in which the intervals are not overlapping, it is assumed that all the interval values have a uniform distribution. If they are not have a uniform distribution, but some other type, e.g., a normal distribution, it is not known what would happen. Therefore this should be a topic for future research.

In spite of what has been mentioned above, compared with P-HFSs, IVHFSs and a possibility-hesitant fuzzy linguistic term set, P-IVHFSs can describe the actual preferences of decision-makers and better reflect their uncertainty, hesitancy, and inconsistency, and thus enhance the modeling abilities of HFSs. The proposed method using P-IVHFSs has the following advantages.

First, compared with P-HFSs, P-IVHFSs can better depict uncertainty.

Second, compared with IVHFSs, P-IVHFSs can depict hesitancy more accurately and differentiate intervals according to their possibilities.

Third, compared with a possibility-hesitant fuzzy linguistic term set, P-IVHFSs can express the evaluation information more flexibly. Possibility-hesitant fuzzy linguistic term sets can therefore be regarded as a special case of P-IVHFSs.

Although the representation of P-IVHFSs looks complex, they can depict fuzzy information clearly and retain the completeness of original data or the inherent thoughts of decision-makers, which is a prerequisite of guaranteeing the accuracy of final outcomes. Additionally, as far as the applicability of P-IVHFSs is concerned, decision-makers can make a trade-off between the features of P-IVHFSs and the relative computational cost. Moreover, the complexity and amount of computation can be clearly reduced with the assistance of programming software [17].

6. Conclusion

In this paper, P-HFSs and IVHFSs have been extended to P-IVHFSs. As an important tool in GDM, P-IVHFSs can describe the actual preferences of decision-makers and better reflect their uncertainty, hesitancy, and inconsistency, and thus enhance the modeling abilities of HFSs. Based on related research, a decomposition method has been proposed to deal with the consistency of P-IVHFPRs. A simulated example has also been provided to illustrate the use of the proposed approach. The main contributions of this paper are summarized as follows.

(1) The concept of P-IVHFSs has been defined and some desirable properties of P-IVHFSs have been discussed. P-IVHFSs are a natural development to manage the possible preferences in decision making following the introduction of P-HFSs and IVHFSs.

(2) P-IVHFPRs have been proposed and the consistency of P-IVHFPRs has been discussed, using the multiplicative transitivity to verify the consistency of a P-IVHFPR. Moreover, a decomposition method has been proposed to deal with the consistency of P-IVHFPRs.

(3) Based on the multiplicative consistency of hesitant fuzzy preference relations, an iterative algorithm has been proposed for improving the consistency of P-IVHFPR.

In future research, the developed theoretical structure could be extended to the probability distributions of preferences on the intervals. Another potential area of research would be to analyze the hesitant fuzzy information in P-IVHFPRs.

References

[1] Ai, F.Y., Yang, J.Y., Zhang, P.D.: An approach to multiple attribute decision making problems based on hesitant fuzzy set. J. Intell. Fuzzy Syst. 27(6), 2749-2755 (2014)

[2] Chen, N., Xu, Z.S., Xia, M.M.: Interval-valued hesitant preference relations and their applications to group decision making. Knowledge-Based Syst. 37, 528-540 (2013).

[3] Chen, N., Xu, Z.S., Xia, M.M.: Correlation coefficients of hesitant fuzzy sets and their applications to clustering analysis. Appl. Math. Model. 37, 2197-2211 (2013)

[4] Chiclana, F., Herrera-Viedma, E., Alonso, S., Herrera, F.: Cardinal consistency of reciprocal preference relations: a characterization of multiplicative transitivity. IEEE Trans. Fuzzy Syst. 17, 14-23 (2009)

[5] Chiclana, F., Mata, F., Alonso, S., Herrera-viedma, E., Martínez, L.: Group decision making: From consistency to consensus. Lect. Note. Comput. Sci. 2(2), 80-91 (2007)

[6] Farhadinia, B.: A series of score functions for hesitant fuzzy sets. Inf. Sci. 277, 102-110 (2014)

[7] Farhadinia, B.: Information measures for hesitant fuzzy sets and interval-valued hesitant fuzzy sets. Inf. Sci. 240, 129-144 (2013)

[8] Liao, H.C., Xu, Z.S.: A VIKOR-based method for hesitant fuzzy multi-criteria decision making. Fuzzy Optim. Decis. Mak. 12(4), 373-392 (2013)

[9] Liao, H.C., Xu, Z.S., Xia, M.M.: Multiplicative consistency on hesitant fuzzy preference relation and its application in group decision making. Int. J. Inf. Tech. Decis. 13 (1), 47-76 (2014)

[10] Orlovsky, S.A.: Decision-making with a fuzzy preference relation. Fuzzy Sets Syst. 1, 155-167 (1978)

[11] Peng, D.H., Gao, C.Y., Gao, Z.F.: Generalized hesitant fuzzy synergetic weighted distance measures and their application to multiple criteria decision-making. Appl. Math. Model. 37, 5837-5850 (2013)

[12] Rodríguez, Martínez, L., Herrera, F.: Hesitant fuzzy linguistic term sets for decision making. IEEE Trans. Fuzzy Syst. 20(1), 109-119 (2012)

[13] Saaty, T.L.: The Analytic Hierarchy Process. McGraw-Hill, New York, 1980.

[14] Tanino, T.: Fuzzy preference relation in group decision making. Springer Berlin Heidelberg. 301, 54-71 (1988)

[15] Torra, V.: Hesitant fuzzy sets. Int. J. Intell. Syst. 25(6), 529-539 (2010)

[16] Torra, V., Narukawa, Y.: On hesitant fuzzy sets and decision. In: IEEE International Conference on Fuzzy Systems 1378-1382 (2009)

[17] Wang, J.Q., Wu, J.T., Wang, J., Zhang, H.Y., Chen, X.H.: Interval-valued hesitant fuzzy linguistic sets and their applications in multi-criteria decision-making problems. Inf. Sci. 288, 55-72 (2014)

[18] Wei, G.W. Hesitant fuzzy prioritized operators and their application to multiple attribute decision making. Knowledge-Based Syst. 31, 176-182 (2012)

[19] Wu, Z.B., Xu, J.P.: Possibility distribution-based approach for MAGDM with hesitant fuzzy linguistic information. IEEE Trans. Cybernetics 46 (3), 694-705 (2016)

[20] Wu, Z.B., Xu, J.P.: Managing consistency and consensus in group decision making with hesitant fuzzy linguistic preference relations. Omega 65, 28-40 (2016)

[21] Xia, M.M., Xu, Z.S.: Studies on the aggregation of intuitionistic fuzzy and hesitant fuzzy information. Technical Report (2011)

[22] Xia, M.M., Xu, Z.S.: Hesitant fuzzy information aggregation in decision making, Int. J. Approx. Reason. 52, 395-407 (2011)

[23] Xia, M.M., Xu, Z.S.: On distance and correlation measures of hesitant fuzzy information. Int. J. Intell. Syst. 26(5), 410-425 (2011)

[24] Xia, M.M., Xu, Z.S., Chen, N.: Some hesitant fuzzy aggregation operators with their application in group decision making. Group Decis. Negotiation 22(2), 259-279 (2013)

[25] Xu, K., Zhou, J.Z., Gu, R., Qin, H.: Approach for aggregating interval-valued intuitionistic fuzzy information and its application to reservoir operation. Expert Syst. Appl. 38, 9032-9035 (2011)

[26] Xu, Y.J., Herrera, F., Wang, H.M.: A distance-based framework to deal with ordinal and additive inconsistencies for fuzzy reciprocal preference relations. Inf. Sci. 328, 189-205 (2016)

[27] Xu, Z.S.: Hesitant Fuzzy Sets Theory, Studies in Fuzziness and Soft Computing. Springer International Publishing Switzerland. 314, (2014)

[28] Xu, Z.S.: On compatibility of interval fuzzy preference matrices. Fuzzy Optim. Decis. Mak. 3, 217-225 (2004)

[29] Xu,Z.S., Da, Q.L.: The uncertain OWA operator. Int. J. Intell. Syst. 17, 569-575 (2002)

[30] Xu, Z.S., Xia, M.M.: Hesitant fuzzy entropy and cross-entropy and their use in multiattribute decision making. Int. J. Intell. Syst. 27, 799-822 (2012)

[31] Xu, Z.S., Xia, M.M.: Distance and similarity measures for hesitant fuzzy sets. Inf. Sci. 181, 2128-2138 (2011)

[32] Yu, D.J.: Some hesitant fuzzy information aggregation operators based on Einstein operational laws. IEEE Trans. Fuzzy Syst. 29, 320-340 (2014)

[33] Zadeh, L.A.: The concept of a linguistic variable and its application to approximate reasoning-I. Inf. Sci. 8, 199-249 (1975)

[34] Zadeh, L.A.: Fuzzy sets. Inf. Control 8(3), 338-353 (1965)

[35] Zhang, Y.X., Xu, Z.S., Wang, H., Liao, H.C.: Consistency-based risk assessment with probabilistic linguistic preference relation. Appl. Soft Comput. 49 (2016) 817-833.

[36] Zhang, Z.M.: Hesitant fuzzy power aggregation operators and their application to multiple attribute group decision making. Inf. Sci. 234, 150-181 (2013)

[37] Zhu, B., Xu, Z.S.: Probability-hesitant fuzzy sets and the presentation of preference relations. Technological and Economic Development of Economy. In press.

[38] Zhu, B., Xu, Z.S.: Consistency measures for hesitant fuzzy linguistic preference relations. IEEE Trans. Fuzzy Syst. 24(1), 72-85 (2014)

[39] Zhu, B., Xu, Z.S., Xu, J.P.: Deriving a ranking from hesitant fuzzy preference relations under group decision making. IEEE Trans. Cybernetics 44(8), 1328-1337 (2014)

Dynamics and Solutions of Some Recursive Sequences of Higher Order

Asim Asiri[1] and E. M. Elsayed[1,2]

[1]King Abdulaziz University, Faculty of Science,
Mathematics Department, P. O. Box 80203,
Jeddah 21589, Saudi Arabia.
[2]Department of Mathematics, Faculty of Science,
Mansoura University, Mansoura 35516, Egypt.
E-mail: amkasiri@kau.edu.sa, emmelsayed@yahoo.com.

ABSTRACT

In this article we study the existence of solutions and some of their qualitative behavior of the following rational nonlinear difference equation

$$x_{n+1}=\frac{ax_{n-(2k+1)}}{b+cx_{n-k}x_{n-(2k+1)}},\quad n=0,1,...,$$

where a, b and c are real numbers, k is a non-negative integer number and the initial conditions x_{-2k-1}, x_{-2k} ,..., x_{-1}, x_0 are arbitrary non-negative real numbers. Also, the solutions of some special cases of the equation under consideration will be obtained.

Keywords: recursive sequence, periodicity, solutions of difference equations.

Mathematics Subject Classification: 39A10

1. INTRODUCTION

During the last decade, the research on difference equations has been increasing. The fact that difference equations demonstrate themselves as mathematical models representing some real life phenomena is a significant reason of this concern. For example, the are used in probability theory,economics, genetics in biology, geometry, electrical network, quanta in radiation, psychology, sociology, etc. Actually, no doubt that the difference equations play and will play a remarkable role in applicable analysis and in mathematics generally.

Recently, many authors' attention was on studying the global attractivity, boundedness character, periodicity and the solution form of nonlinear difference equations. Now, we write some results in this area: Cinar [3–4] obtained the solutions of the following difference equations

$$x_{n+1}=\frac{x_{n-1}}{1+x_nx_{n-1}},\quad x_{n+1}=\frac{x_{n-1}}{-1+x_nx_{n-1}}.$$

Cinar et al. [5] discussed the solutions and attractivity of the difference equation

$$x_{n+1}=\frac{x_{n-3}}{-1+x_nx_{n-1}x_{n-2}x_{n-3}}.$$

Elabbasy et al. [8–9] looked at the global stability, periodicity character and derive the solution of some special cases of the following difference equations

$$x_{n+1}=ax_n-\frac{bx_n}{cx_n-dx_{n-1}},\quad x_{n+1}=\frac{\alpha x_{n-k}}{\beta+\gamma\prod_{i=0}^{k}x_{n-i}}.$$

Elsayed [13] examined the behavior and found the form of solution of the nonlinear difference equation

$$x_{n+1}=ax_{n-1}+\frac{bx_nx_{n-1}}{cx_n+dx_{n-2}}.$$

In [2], Belhannache et al. investigated the global behavior of the solutions of the difference equation

$$x_{n+1} = \frac{A + Bx_{n-2k-1}}{C + D\prod_{i=1}^{k} x_{n-2i}^{m_i}}.$$

Karatas et al. [29] achieved the solution of the following difference equation

$$x_{n+1} = \frac{ax_{n-(2k+2)}}{-a + \prod_{i=0}^{2k+2} x_{n-i}}.$$

In [35] Simsek and Abdullayev found the solution of the recursive sequence

$$x_{n+1} = \frac{x_{n-(4k+3)}}{1+\prod_{t=0}^{2} x_{n-(k+1)t-k}}.$$

Other related results on rational difference equations can be found in the references. [1-52].

Our aim in this paper is to investigate the dynamics of the solution of the following nonlinear difference equation of higher order

$$x_{n+1} = \frac{ax_{n-(2k+1)}}{b + cx_{n-k}x_{n-(2k+1)}}, \quad n = 0, 1, ..., \tag{1}$$

where a, b and c are real numbers, k a is non negative integer number and the initial conditions x_{-2k-1}, x_{-2k} ,..., x_{-1}, x_0 are arbitrary non-negative real numbers. Also, we obtain the solutions of some special cases of Eq.(1).

Suppose that I is an interval of real numbers and let $f : I^{k+1} \to I$, be a continuously differentiable function. Then for every set of initial conditions $x_{-k}, x_{-k+1}, ..., x_0 \in I$, the difference equation

$$x_{n+1} = f(x_n, x_{n-1}, ..., x_{n-k}), \quad n = 0, 1, ..., \tag{2}$$

has a unique solution $\{x_n\}_{n=-k}^{\infty}$.

Definition 1. (Equilibrium Point)

A point $\overline{x} \in I$ is called an equilibrium point of Eq.(2) if $\overline{x} = f(\overline{x}, \overline{x}, ..., \overline{x})$. That is, $x_n = \overline{x}$ for $n \geq 0$, is a solution of Eq.(2), or equivalently, $\overline{x}$ is a fixed point of f.

Definition 2. (Periodicity)

A sequence $\{x_n\}_{n=-k}^{\infty}$ is said to be periodic with period p if $x_{n+p} = x_n$ for all $n \geq -k$.

Definition 3. (Stability)

(i) The equilibrium point $\overline{x}$ of Eq.(2) is locally stable if for every $\epsilon > 0$, there exists $\delta > 0$ such that for all $x_{-k}, x_{-k+1}, ..., x_{-1}, x_0 \in I$ with

$$|x_{-k} - \overline{x}| + |x_{-k+1} - \overline{x}| + ... + |x_0 - \overline{x}| < \delta,$$

we have

$$|x_n - \overline{x}| < \epsilon \quad \text{for all} \quad n \geq -k.$$

(ii) The equilibrium point $\overline{x}$ of Eq.(2) is locally asymptotically stable if $\overline{x}$ is locally stable solution of Eq.(2) and there exists $\gamma > 0$, such that for all $x_{-k}, x_{-k+1}, ..., x_{-1}, x_0 \in I$ with

$$|x_{-k} - \overline{x}| + |x_{-k+1} - \overline{x}| + ... + |x_0 - \overline{x}| < \gamma,$$

we have $\lim_{n\to\infty} x_n = \overline{x}$.

(iii) The equilibrium point $\overline{x}$ of Eq.(2) is a global attractor if for all $x_{-k}, x_{-k+1}, ..., x_{-1}, x_0 \in I$, we have $\lim_{n\to\infty} x_n = \overline{x}$.

(iv) The equilibrium point $\overline{x}$ of Eq.(2) is globally asymptotically stable if $\overline{x}$ is locally stable, and $\overline{x}$ is also a global attractor of Eq.(2).

(v) The equilibrium point $\overline{x}$ of Eq.(2) is unstable if $\overline{x}$ is not locally stable.

The linearized equation of Eq.(2) about the equilibrium $\overline{x}$ is the linear difference equation

$$y_{n+1} = \sum_{i=0}^{k} \frac{\partial f(\overline{x}, \overline{x}, ..., \overline{x})}{\partial x_{n-i}} y_{n-i}. \tag{3}$$

Theorem A [32]: Assume that $p_i \in R,\ i = 1, 2, ..., k$ and $k \in \{0, 1, 2, ...\}$. Then

$$\sum_{i=1}^{k} |p_i| < 1,$$

is a sufficient condition for the asymptotic stability of the difference equation

$$x_{n+k} + p_1 x_{n+k-1} + ... + p_k x_n = 0, \quad n = 0, 1, ...$$

2. DYNAMICS OF SOLUTIONS OF EQ.(1)

In this section we look at some qualitative behavior of Eq.(1) such as local stability, periodicity and boundedness character of solutions of Eq.(1) when the constants a, b and c are positive real numbers.

2.1. Local Stability of the Equilibrium Points

We now investigate the local stability character of the solutions of Eq.(1).

The equilibrium points of Eq.(1) are given by the relation $\overline{x} = \frac{a\overline{x}}{b+c\overline{x}^2}$, which gives

$$\overline{x} = 0 \quad or \quad \overline{x} = \sqrt{\frac{a-b}{c}}.$$

Note that if $a > b$, then Eq.(1) has a unique positive equilibrium point.

Let $f : (0, \infty)^2 \longrightarrow (0, \infty)$ be a function defined by

$$f(u, v) = \frac{au}{b + cuv}. \tag{4}$$

Therefore it follows that
$$\frac{\partial f(u, v)}{\partial u} = \frac{ab}{(b + cuv)^2}, \quad \frac{\partial f(u, v)}{\partial v} = \frac{-acu^2}{(b + cuv)^2}.$$

THEOREM 2.1. *The following statements are true:*

(1) If $a \leq b$, then $\overline{x} = 0$ is the only equilibrium point of Eq.(1) and it is locally stable.

(2) If $a > b$, then the equilibrium points $\overline{x} = 0$ and $\overline{x} = \sqrt{\frac{a-b}{c}}$ of Eq.(1) are unstable.

Proof. (1) If $a \leq b$, then we see from Eq.(4) that

$$\frac{\partial f(0, 0)}{\partial u} = \frac{a}{b}, \quad \frac{\partial f(0, 0)}{\partial v} = 0.$$

Then the linearized equation associated with Eq.(1) about $\overline{x} = 0$ is

$$y_{n+1} - \frac{a}{b} y_{n-2k-1} = 0, \tag{5}$$

and whose characteristic equation is
$$\lambda^{2k+2} - \frac{a}{b} = 0. \tag{6}$$

It follows by Theorem A that, Eq.(5) is asymptotically stable. Then the equilibrium point $\overline{x} = 0$ of Eq.(1) is locally stable.

(2) Assume that $a > b$. (i) At $\overline{x} = 0$ it follows again from Eq.(6) and Theorem A that $\overline{x} = 0$ is unstable. (ii) At $\overline{x} = \sqrt{\frac{a-b}{c}}$ we see from Eq.(4) that

$$\frac{\partial f(\overline{x},\overline{x})}{\partial u} = \frac{b}{a}, \qquad \frac{\partial f(\overline{x},\overline{x})}{\partial v} = \frac{-(a-b)}{a}.$$

Then the linearized equation of Eq.(1) about $\overline{x} = \sqrt{\frac{a-b}{c}}$ is

$$y_{n+1} + \frac{a-b}{a} y_{n-k} - \frac{b}{a} y_{n-2k-1} = 0, \tag{7}$$

and whose characteristic equation is

$$\lambda^{2k+2} + \frac{a-b}{a}\lambda^{k+1} - \frac{b}{a} = 0. \tag{8}$$

Therefore $\lambda^{k+1} = -1$ or $\lambda^{k+1} = \frac{b}{a}$. Then it follows by Theorem A that the equilibrium point $\overline{x} = \sqrt{\frac{a-b}{c}}$ of Eq.(1) is unstable. The proof is complete.

2.2. Existence of Period (2k+2) Solutions

In this section we look at the existence of period $(2k+2)$ solutions of Eq.(1).

Remark: The initial values $\{x_{-2k-1}, x_{-2k}, x_{-2k+1}, ..., x_{-1}, x_0\}$ of Eq.(1) have not to be equal zero at the same time, otherwise Eq.(1) will have only the zero solution.

In the sequel we assume that any element of the set $\{x_{-2k-1}, x_{-2k}, x_{-2k+1}, ..., x_{-1}, x_0\}$ doesn't equal zero.

Theorem 2.2. *Eq.(1) has positive prime period $(2k+2)$ solutions if and only if*

$$(b + cA_i - a) = 0, \tag{9}$$

where $A_i = x_{-k+i}x_{-2k-1+i}$ (for $i = 0, 1, 2, ..., k$) and $A_{k+1+i} = A_i$.

Proof. Firstly, we suppose that there exists a prime period $(2k+2)$ solution of Eq.(1) of the form

$$..., x_{-2k-1}, x_{-2k}, x_{-2k+1}, ..., x_{-1}, x_0, x_{-2k-1}, x_{-2k}, x_{-2k+1}, ..., x_{-1}, x_0, ...$$

That is $x_{N+1} = x_{N-2k-1}$ *for* $N \geq 0$. We now will show that (9) holds. We see from Eq.(1) that

$$\begin{aligned}
x_{-2k-1} &= x_1 = \frac{ax_{-2k-1}}{b+cA_0}, \quad x_{-2k} = x_2 = \frac{ax_{-2k}}{b+cA_1}, \quad x_{-2k+1} = x_3 = \frac{ax_{-2k+1}}{b+cA_2}, ...,\\
x_{-k-2} &= x_k = \frac{ax_{-k-2}}{b+cA_{k-1}}, \quad x_{-k-1} = x_{k+1} = \frac{ax_{-k-1}}{b+cA_k},\\
x_{-k} &= x_{k+2} = \frac{ax_{-k}}{b+cA_{k+1}} = \frac{ax_{-k}}{b+cA_0}, ..., x_{-2} = x_{2k} = \frac{ax_{-2}}{b+cA_{2k-1}} = \frac{ax_{-2}}{b+cA_{k-2}},\\
x_{-1} &= x_{2k+1} = \frac{ax_{-1}}{b+cA_{2k}} = \frac{ax_{-1}}{b+cA_{k-1}}, \quad x_0 = x_{2k+2} = \frac{ax_0}{b+cA_{2k+1}} = \frac{ax_0}{b+cA_k}.
\end{aligned}$$

Then it is easy to see that

$$\begin{aligned}
x_{-2k-1}(b+cA_0) &= ax_{-2k-1} \Rightarrow x_{-2k-1}(b+cA_0-a) = 0,\\
x_{-2k}(b+cA_1) &= ax_{-2k} \Rightarrow x_{-2k}(b+cA_1-a) = 0,\\
x_{-2k+1}(b+cA_2) &= ax_{-2k+1} \Rightarrow x_{-2k+1}(b+cA_2-a) = 0, ...,\\
x_{-1}(b+cA_{k-1}) &= ax_{-1} \Rightarrow x_{-1}(b+cA_{k-1}-a) = 0,\\
x_0(b+cA_k) &= ax_0 \Rightarrow x_0(b+cA_k-a) = 0.
\end{aligned}$$

Since $x_j \neq 0$ for all $-(2k+1) \leq j \leq 0$, then Condition (9) is satisfied.
Secondly, we suppose that (9) is true. We will prove that Eq.(1) has a prime period $(2k+2)$ solution. It follows from Eq.(1) and Eq.(9) that

$$\begin{aligned} x_1 &= \frac{ax_{-2k-1}}{b+cA_0} = x_{-2k-1}, \quad x_2 = \frac{ax_{-2k}}{b+cA_1} = x_{-2k}, \quad x_3 = \frac{ax_{-2k+1}}{b+cA_2} = x_{-2k+1}, ..., \\ x_k &= \frac{ax_{-k-2}}{b+cA_{k-1}} = x_{-k-2}, \quad x_{k+1} = \frac{ax_{-k-1}}{b+cA_k} = x_{-k-1}, \quad x_{k+2} = \frac{ax_{-k}}{b+cA_{k+1}} = x_{-k}, ..., \\ x_{2k} &= \frac{ax_{-2}}{b+cA_{2k-1}} = x_{-2}, \quad x_{2k+1} = \frac{ax_{-1}}{b+cA_{2k}} = x_{-1}, \quad x_{2k+2} = \frac{ax_0}{b+cA_{2k+1}} = x_0, \end{aligned}$$

which completes the proof.

2.3. Boundedness and Global Stability of Solutions

Here we examine the boundedness nature of the solutions of Eq.(1). In addition, we deal with the global stability of the equilibrium point $\overline{x} = 0$.

THEOREM 2.3. *Every solution of Eq.(1) is bounded.*

Proof. Let $\{x_n\}_{n=-2k-1}^{\infty}$ be a solution of Eq.(1), we have to look at the following two cases

(1) If $a \leq b$. It follows from Eq.(1) that

$$x_{n+1} = \frac{ax_{n-(2k+1)}}{b + cx_{n-k}x_{n-(2k+1)}} \leq \frac{ax_{n-(2k+1)}}{b} \leq x_{n-(2k+1)}.$$

Then the subsequences $\{x_{(2k+2)n-2k-1}\}_{n=0}^{\infty}$, $\{x_{(2k+2)n-2k}\}_{n=0}^{\infty}$, ..., $\{x_{(2k+2)n-1}\}_{n=0}^{\infty}$, $\{x_{(2k+2)n}\}_{n=0}^{\infty}$ are decreasing and so are bounded from above by $M = \max\left\{x_{-2k-1}, x_{-2k}, x_{-2k+1}, ..., x_{-1}, x_0, \sqrt{\frac{a}{c}}\right\}$.

(2) If $a > b$. For the sake of contradiction, we suppose that there exists a subsequence $\{x_{(2k+2)n-2k-1}\}_{n=0}^{\infty}$ and it is not bounded from above. Then we obtain from Eq.(1), for sufficiently large n, that

$$\begin{aligned} \infty &= \lim_{n\to\infty} x_{(2k+2)n+1} = \lim_{n\to\infty} \frac{ax_{(2k+2)n-(2k+1)}}{b + cx_{(2k+2)n-k}x_{(2k+2)n-(2k+1)}} \\ &< \lim_{n\to\infty} \frac{ax_{(2k+2)n-(2k+1)}}{cx_{(2k+2)n-k}x_{(2k+2)n-(2k+1)}} = \lim_{n\to\infty} \frac{a}{cx_{(2k+2)n-k}}. \end{aligned} \tag{10}$$

It follows that the limit of the right hand side of (10) is bounded which is a contradiction, and so the proof of the theorem is complete.

THEOREM 2.4. *If $a \leq b$, then every solution of Eq.(1) converges to the equilibrium point $\overline{x} = 0$.*

Proof. It was shown in Theorem 2.1 that $\overline{x} = 0$ is local stable and then it suffices to show that $\overline{x} = 0$ is global attractor of the solutions of Eq.(1).
We claim that each one of the subsequences $\{x_{(2k+2)n-2k-1}\}_{n=0}^{\infty}$, $\{x_{(2k+2)n-2k}\}_{n=0}^{\infty}$, ..., $\{x_{(2k+2)n-1}\}_{n=0}^{\infty}$, $\{x_{(2k+2)n}\}_{n=0}^{\infty}$ has limit equal to zero. For the sake of contradiction, suppose that there exists a subsequence $\{x_{(2k+2)n-2k-1}\}_{n=0}^{\infty}$ with limit doesn't zero. Now we see from Eq.(1) that

$$bx_{(2k+2)n+1} + cx_{(2k+2)n-k}x_{(2k+2)n-(2k+1)} = ax_{(2k+2)n-(2k+1)},$$

or

$$x_{(2k+2)n-(2k+1)} = \frac{bx_{(2k+2)n+1}}{a - cx_{(2k+2)n+1}x_{(2k+2)n-k}}.$$

Now it follows from the boundedness of the solution that

$$\lim_{n\to\infty} x_{(2k+2)n-(2k+1)} = \lim_{n\to\infty} \frac{bx_{(2k+2)n+1}}{a - cx_{(2k+2)n+1}x_{(2k+2)n-k}} < \frac{bM}{a - cM^2} < 0,$$

where $M \geq \sqrt{\frac{a}{c}}$ which is a contradiction and this completes the proof of the theorem.

Numerical Examples

For confirming the results of this section, we present some numerical examples which show the behavior of solutions of Eq.(1). See Figures 1, 2 and 3 below.

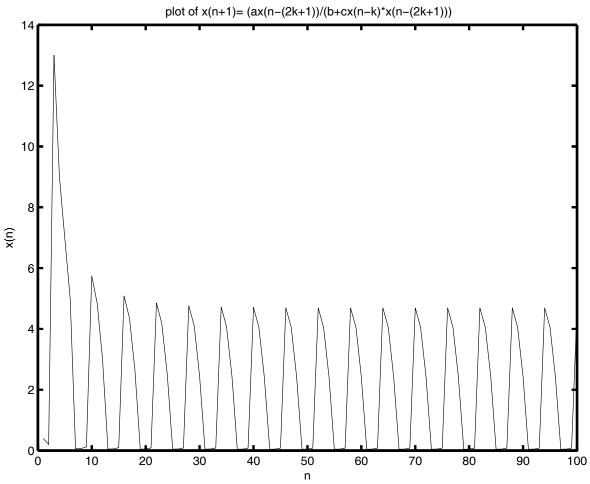


Figure 1: $a=3$, $b=2$, $c=5$, $k=2$, $x_{-5}=0.4$, $x_{-4}=0.2$, $x_{-3}=13$, $x_{-2}=9$, $x_{-1}=7$, $x_0=5$.

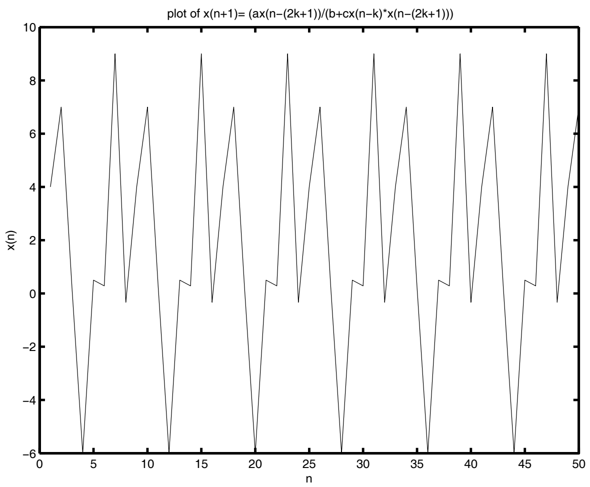


Figure 2: $a=10$, $b=6$, $c=2$, $k=3$, $x_{-7}=4$, $x_{-6}=7$, $x_{-5}=2/9$, $x_{-4}=-6$, $x_{-3}=0.5$, $x_{-2}=2/7$, $x_{-1}=9$, $x_0=-2/6$.

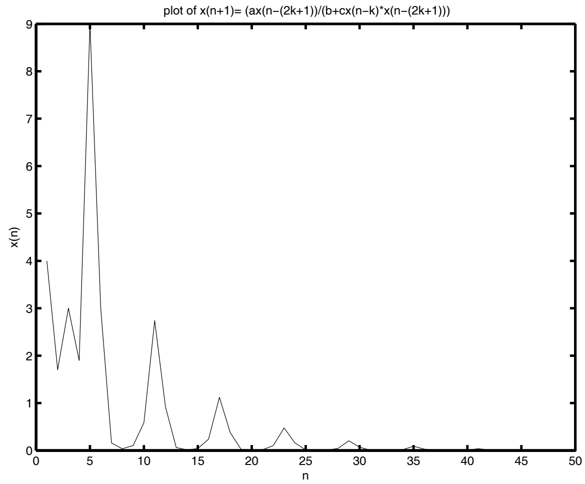


Figure 3: $a = 3$, $b = 7$, $c = 9$, $k = 2$, $x_{-5} = 4$, $x_{-4} = 1.7$, $x_{-3} = 3$, $x_{-2} = 1.9$, $x_{-1} = 9$, $x_0 = 3$.

3. THE SOLUTIONS FORM OF SOME SPECIAL CASES OF EQ.(1)

Our goal in this section is to find a specific form of the solutions of some special cases of Eq.(1) and give numerical examples in each case when the constants a, b and c are integer numbers.

3.1. On the Difference Equation $x_{n+1} = \dfrac{x_{n-(2k+1)}}{-1 + x_{n-k}x_{n-(2k+1)}}$

In this section we obtain the solution of the following equation

$$x_{n+1} = \frac{x_{n-(2k+1)}}{-1 + x_{n-k}x_{n-(2k+1)}}, \quad n = 0, 1, ..., \tag{11}$$

where the initial values are arbitrary non zero real numbers with $x_{-k+i}x_{-2k-1+i} \neq 1$ $(for\ i = 0, 1, 2, ..., k)$.

THEOREM 3.1. *Let $\{x_n\}_{n=-2k-1}^{\infty}$ be a solution of Eq.(11). Then for $n = 1, 2, ...$*

$$\begin{aligned}
x_{(2k+2)n-2k-1} &= \frac{x_{-2k-1}}{(-1 + x_{-k}x_{-2k-1})^n}, \quad x_{(2k+2)n-2k} = \frac{x_{-2k}}{(-1 + x_{-k+1}x_{-2k})^n}, \\
x_{(2k+2)n-2k+1} &= \frac{x_{-2k+1}}{(-1 + x_{-k+2}x_{-2k+1})^n}, \ ..., \\
x_{(2k+2)n-k-1} &= \frac{x_{-k-1}}{(-1 + x_0x_{-k-1})^n}, \quad x_{(2k+2)n-k} = x_{-k}(-1 + x_{-k}x_{-2k-1})^n, \\
x_{(2k+2)n-k+1} &= x_{-k+1}(-1 + x_{-k+1}x_{-2k})^n, ..., \\
x_{(2k+2)n-1} &= x_{-1}(-1 + x_{-1}x_{-k-2})^n, \qquad x_{(2k+2)n} = x_0(-1 + x_0x_{-k-1})^n.
\end{aligned}$$

Proof: For $n = 1$ the result holds. Now suppose that $n > 1$ and that our assumption holds for $n - 1$. That is;

$$\begin{aligned}
x_{(2k+2)n-4k-3} &= \frac{x_{-2k-1}}{(-1 + x_{-k}x_{-2k-1})^{n-1}}, \quad x_{(2k+2)n-4k-2} = \frac{x_{-2k}}{(-1 + x_{-k+1}x_{-2k})^{n-1}}, \\
x_{(2k+2)n-4k-1} &= \frac{x_{-2k+1}}{(-1 + x_{-k+2}x_{-2k+1})^{n-1}}, \ ..., \\
\\
x_{(2k+2)n-3k-3} &= \frac{x_{-k-1}}{(-1 + x_0x_{-k-1})^{n-1}}, \quad x_{(2k+2)n-3k-2} = x_{-k}(-1 + x_{-k}x_{-2k-1})^{n-1}, \\
x_{(2k+2)n-3k-1} &= x_{-k+1}(-1 + x_{-k+1}x_{-2k})^{n-1}, ...,
\end{aligned}$$

$$x_{(2k+2)n-2k-3} = x_{-1}\left(-1 + x_{-1}x_{-k-2}\right)^{n-1}, \quad x_{(2k+2)n-2k-2} = x_0\left(-1 + x_0 x_{-k-1}\right)^{n-1}.$$

Now, it follows from Eq.(11) that

$$\begin{aligned} x_{(2k+2)n-2k-1} &= \frac{x_{(2k+2)n-(4k+3)}}{-1 + x_{(2k+2)n-3k-2}x_{(2k+2)n-(4k+3)}} \\ &= \frac{\frac{x_{-2k-1}}{(-1+x_{-k}x_{-2k-1})^{n-1}}}{-1+x_{-k}(-1+x_{-k}x_{-2k-1})^{n-1}\frac{x_{-2k-1}}{(-1+x_{-k}x_{-2k-1})^{n}-1}} = \frac{\frac{x_{-2k-1}}{(-1+x_{-k}x_{-2k-1})^{n-1}}}{-1+x_{-k}x_{-2k-1}}. \end{aligned}$$

Hence, we have

$$x_{(2k+2)n-2k-1} = \frac{x_{-2k-1}}{(-1+x_{-k}x_{-2k-1})^{n}}.$$

Also, we see from Eq.(11) that

$$\begin{aligned} x_{(2k+2)n-k-1} &= \frac{x_{(2k+2)n-(3k+3)}}{-1 + x_{(2k+2)n-2k-2}x_{(2k+2)n-(3k+3)}} \\ &= \frac{\frac{x_{-k-1}}{(-1+x_0x_{-k-1})^{n-1}}}{-1+x_0(-1+x_0x_{-k-1})^{n-1}\frac{x_{-k-1}}{(-1+x_0x_{-k-1})^{n-1}}} = \frac{\frac{x_{-k-1}}{(-1+x_0x_{-k-1})^{n-1}}}{1+x_0x_{-k-1}}. \end{aligned}$$

Thus

$$x_{(2k+2)n-k-1} = \frac{x_{-k-1}}{(-1+x_0x_{-k-1})^{n}}.$$

Similarly

$$\begin{aligned} x_{(2k+2)n-1} &= \frac{x_{(2k+2)n-(2k+3)}}{-1 + x_{(2k+2)n-k-2}x_{(2k+2)n-(2k+3)}} = \frac{x_{-1}(-1+x_{-1}x_{-k-2})^{n-1}}{-1+\frac{x_{-k-2}}{(-1+x_{-1}x_{-k-2})^{n}}x_{-1}(-1+x_{-1}x_{-k-2})^{n-1}} \\ &= \frac{x_{-1}(-1+x_{-1}x_{-k-2})^{n-1}}{-1+\frac{x_{-1}x_{-k-2}}{(-1+x_{-1}x_{-k-2})}}\left(\frac{-1+x_{-1}x_{-k-2}}{-1+x_{-1}x_{-k-2}}\right) = x_{-1}\left(-1+x_{-1}x_{-k-2}\right)^{n-1}\left(-1+x_{-1}x_{-k-2}\right). \end{aligned}$$

Then, we get

$$x_{(2k+2)n-1} = x_{-1}\left(-1+x_{-1}x_{-k-2}\right)^{n}.$$

Similarly, one can obtain the other relations. Thus, the proof is completed.

Note that the equilibrium points of Eq.(11) are given by the equation $\overline{x} = \dfrac{\overline{x}}{-1+\overline{x}^2}$. Then we have $\overline{x}(\overline{x}^2 - 2) = 0$. Thus Eq.(11) has the equilibrium points $0, \sqrt{2}, -\sqrt{2}$.

THEOREM 3.2. *The following statements are true:*

(a) If $x_{-k+i}x_{-2k-1+i} \neq 2$ $(for\ i = 0, 1, 2, ..., k)$, then all the solutions of Eq.(11) are unbounded.

(b) Eq.(11) has a periodic solutions of period $(2k+2)$ iff $x_{-k+i}x_{-2k-1+i} = 2$ $(for\ i = 0, 1, 2, ..., k)$ and will be take the form $\{x_{-2k-1}, x_{-2k}, ..., x_{-1}, x_0, x_{-2k-1}, x_{-2k}, ..., x_{-1}, x_0, ...\}$.

Proof: (a) The proof in this case follows directly from the form of the solution as given in Theorem 3.1.
(b) First suppose that there exists a prime period $(2k+2)$ solution of Eq.(11) of the form

$$x_{-2k-1}, x_{-2k}, ..., x_{-1}, x_0, x_{-2k-1}, x_{-2k}, ..., x_{-1}, x_0, ...$$

Then we see from the form of solution of Eq.(11) that

$$\begin{aligned} x_{-2k-1} &= \frac{x_{-2k-1}}{\left(-1+x_{-k}x_{-2k-1}\right)^{n}}, \quad x_{-2k}=\frac{x_{-2k}}{\left(-1+x_{-k+1}x_{-2k}\right)^{n}}, \\ x_{-2k+1} &= \frac{x_{-2k+1}}{\left(-1+x_{-k+2}x_{-2k+1}\right)^{n}}, \ ..., \\ x_{-k-1} &= \frac{x_{-k-1}}{\left(-1+x_{0}x_{-k-1}\right)^{n}}, \quad x_{-k}=x_{-k}\left(-1+x_{-k}x_{-2k-1}\right)^{n}, \\ x_{-k+1} &= x_{-k+1}\left(-1+x_{-k+1}x_{-2k}\right)^{n}, ..., \\ x_{-1} &= x_{-1}\left(-1+x_{-1}x_{-k-2}\right)^{n}, \qquad x_{0}=x_{0}\left(-1+x_{0}x_{-k-1}\right)^{n}, \end{aligned}$$

Then

$$\begin{aligned} x_{-k}x_{-2k-1} &= x_{-k+1}x_{-2k}=x_{-k+2}x_{-2k+1}=...=-1+x_{-k}x_{-2k-1}= \\ x_{-k}x_{-2k-1} &= x_{-k+1}x_{-2k}=...=x_{0}x_{-k-1}=2, \end{aligned}$$

or $x_{-k+i}x_{-2k-1+i}=2$. $(for\ i=0,1,2,...,k)$.

Second suppose that

$$\begin{aligned} x_{-k}x_{-2k-1} &= x_{-k+1}x_{-2k}=x_{-k+2}x_{-2k+1}=...=-1+x_{-k}x_{-2k-1}= \\ x_{-k}x_{-2k-1} &= x_{-k+1}x_{-2k}=...=x_{0}x_{-k-1}=2. \end{aligned}$$

Then we see from Eq.(11) that

$$\begin{aligned} x_{(2k+2)n-2k-1} &= x_{-2k-1}, \ \ x_{(2k+2)n-2k}=x_{-2k}, \ \ x_{(2k+2)n-2k+1}=x_{-2k+1}, \ ..., \\ x_{(2k+2)n-k-1} &= x_{-k-1}, \ \ x_{(2k+2)n-k}=x_{-k}, \ \ x_{(2k+2)n-k+1}=x_{-k+1}, ..., \\ x_{(2k+2)n-1} &= x_{-1}, \qquad x_{(2k+2)n}=x_{0}. \end{aligned}$$

Thus we have a period $(2k+2)$ solution and the proof is complete.

In the following we give some numerical examples to confirm the obtained results for Eq.(11). See Figures 4 and 5 below.

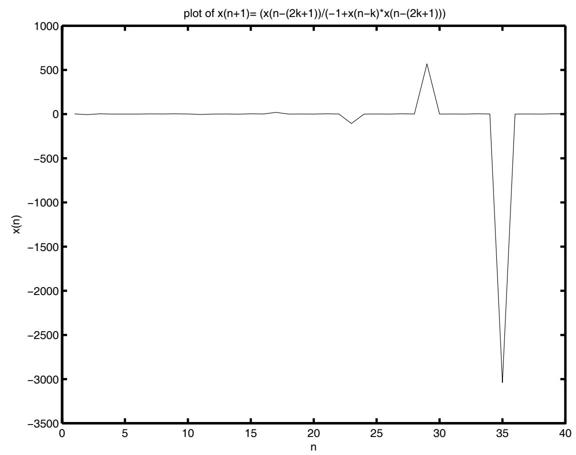


Figure 4: $k=2$, $x_{-5}=2.4$, $x_{-4}=-6.2$, $x_{-3}=4$, $x_{-2}=0.9$, $x_{-1}=0.7$, $x_{0}=0.5$.

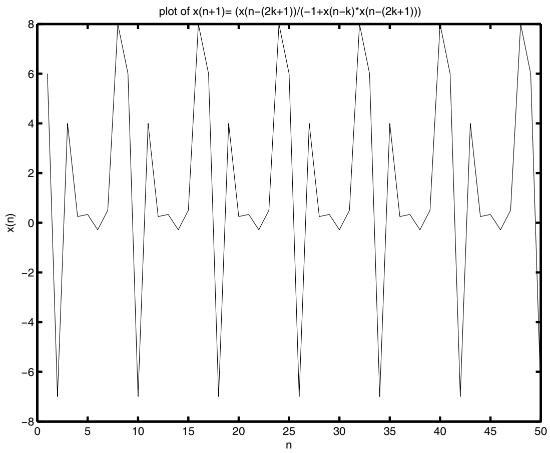


Figure 5: $k=3$, $x_{-7}=6$, $x_{-6}=-7$, $x_{-5}=4$, $x_{-4}=1/4$, $x_{-3}=2/6$, $x_{-2}=-2/7$, $x_{-1}=2/4$, $x_0=8$.

3.2. On the Difference Equation $x_{n+1}=\dfrac{x_{n-(2k+1)}}{1-x_{n-k}x_{n-(2k+1)}}$

In this section we get the solution form of the difference equation

$$x_{n+1}=\frac{x_{n-(2k+1)}}{1-x_{n-k}x_{n-(2k+1)}},\quad n=0,1,..., \tag{12}$$

where the initial values are arbitrary non zero real numbers.

THEOREM 3.3. *Let $\{x_n\}_{n=-2k-1}^{\infty}$ be a solution of Eq.(12). Then for $n=1,2,...$*

$$\begin{aligned}
x_{(2k+2)n-2k-1} &= x_{-2k-1}\prod_{i=0}^{n-1}\left(\frac{1-2ix_{-k}x_{-2k-1}}{1-(2i+1)\,x_{-k}x_{-2k-1}}\right),\ x_{(2k+2)n-2k}=x_{-2k}\prod_{i=0}^{n-1}\left(\frac{1-2ix_{-k+1}x_{-2k}}{1-(2i+1)\,x_{-k+1}x_{-2k}}\right),...,\\
x_{(2k+2)n-k-1} &= x_{-k-1}\prod_{i=0}^{n-1}\left(\frac{1-2ix_0x_{-k-1}}{1-(2i+1)\,x_0x_{-k-1}}\right),\ x_{(2k+2)n-k}=x_{-k}\prod_{i=0}^{n-1}\left(\frac{1-(2i+1)x_{-k}x_{-2k-1}}{1-(2i+2)\,x_{-k}x_{-2k-1}}\right),...,\\
x_{(2k+2)n-1} &= x_{-1}\prod_{i=0}^{n-1}\left(\frac{1-(2i+1)x_{-1}x_{-k-2}}{1-(2i+2)\,x_{-1}x_{-k-2}}\right),\ x_{(2k+2)n}=x_0\prod_{i=0}^{n-1}\left(\frac{1-(2i+1)x_0x_{-k-1}}{1-(2i+2)\,x_0x_{-k-1}}\right).
\end{aligned}$$

Proof: For $n=1$ the result holds. Now suppose that $n>1$ and that our assumption holds for $n-1$. That is;

$$x_{(2k+2)n-4k-3}=x_{-2k-1}\prod_{i=0}^{n-2}\left(\frac{1-2ix_{-k}x_{-2k-1}}{1-(2i+1)\,x_{-k}x_{-2k-1}}\right),\ x_{(2k+2)n-4k-2}=x_{-2k}\prod_{i=0}^{n-2}\left(\frac{1-2ix_{-k+1}x_{-2k}}{1-(2i+1)\,x_{-k+1}x_{-2k}}\right),...,$$

$$x_{(2k+2)n-3k-3}=x_{-k-1}\prod_{i=0}^{n-2}\left(\frac{1-2ix_0x_{-k-1}}{1-(2i+1)\,x_0x_{-k-1}}\right),\ x_{(2k+2)n-3k-2}=x_{-k}\prod_{i=0}^{n-2}\left(\frac{1-(2i+1)x_{-k}x_{-2k-1}}{1-(2i+2)\,x_{-k}x_{-2k-1}}\right),$$

$$\begin{aligned}
x_{(2k+2)n-3k-1} &= x_{-k+1}\prod_{i=0}^{n-2}\left(\frac{1-(2i+1)x_{-k+1}x_{-2k}}{1-(2i+2)\,x_{-k+1}x_{-2k}}\right),...,x_{(2k+2)n-2k-3}=x_{-1}\prod_{i=0}^{n-2}\left(\frac{1-(2i+1)x_{-1}x_{-k-2}}{1-(2i+2)\,x_{-1}x_{-k-2}}\right),\\
x_{(2k+2)n-2k-2} &= x_0\prod_{i=0}^{n-2}\left(\frac{1-(2i+1)x_0x_{-k-1}}{1-(2i+2)\,x_0x_{-k-1}}\right).
\end{aligned}$$

Now, it follows from Eq.(12) that

$$\begin{aligned}
x_{(2k+2)n-2k-1} &= \frac{x_{(2k+2)n-(4k+3)}}{1-x_{(2k+2)n-3k-2}x_{(2k+2)n-(4k+3)}}\\
&= \frac{x_{-2k-1}\prod_{i=0}^{n-2}\left(\frac{1-2ix_{-k}x_{-2k-1}}{1-(2i+1)x_{-k}x_{-2k-1}}\right)}{1-x_{-k}\prod_{i=0}^{n-2}\left(\frac{1-(2i+1)x_{-k}x_{-2k-1}}{1-(2i+2)x_{-k}x_{-2k-1}}\right)x_{-2k-1}\prod_{i=0}^{n-2}\left(\frac{1-2ix_{-k}x_{-2k-1}}{1-(2i+1)x_{-k}x_{-2k-1}}\right)}\\
&= \frac{x_{-2k-1}\prod_{i=0}^{n-2}\left(\frac{1-2ix_{-k}x_{-2k-1}}{1-(2i+1)x_{-k}x_{-2k-1}}\right)}{1-\left(\frac{x_{-k}x_{-2k-1}}{1-(2n-2)x_{-k}x_{-2k-1}}\right)}\left(\frac{1-(2n-2)x_{-k}x_{-2k-1}}{1-(2n-2)x_{-k}x_{-2k-1}}\right)\\
&= x_{-2k-1}\prod_{i=0}^{n-2}\left(\frac{1-2ix_{-k}x_{-2k-1}}{1-(2i+1)x_{-k}x_{-2k-1}}\right)\left(\frac{1-(2n-2)x_{-k}x_{-2k-1}}{1-(2n-1)x_{-k}x_{-2k-1}}\right).
\end{aligned}$$

Hence, we have

$$x_{(2k+2)n-2k-1}=x_{-2k-1}\prod_{i=0}^{n-1}\left(\frac{1-2ix_{-k}x_{-2k-1}}{1-(2i+1)x_{-k}x_{-2k-1}}\right).$$

Similarly

$$\begin{aligned}
x_{(2k+2)n-k-1} &= \frac{x_{(2k+2)n-(3k+3)}}{1-x_{(2k+2)n-2k-2}x_{(2k+2)n-(3k+3)}}\\
&= \frac{x_{-k-1}\prod_{i=0}^{n-2}\left(\frac{1-2ix_0x_{-k-1}}{1-(2i+1)x_0x_{-k-1}}\right)}{1-x_0\prod_{i=0}^{n-2}\left(\frac{1-(2i+1)x_0x_{-k-1}}{1-(2i+2)x_0x_{-k-1}}\right)x_{-k-1}\prod_{i=0}^{n-2}\left(\frac{1-2ix_0x_{-k-1}}{1-(2i+1)x_0x_{-k-1}}\right)}\\
&= \frac{x_{-k-1}\prod_{i=0}^{n-2}\left(\frac{1-2ix_0x_{-k-1}}{1-(2i+1)x_0x_{-k-1}}\right)}{1-x_0x_{-k-1}\prod_{i=0}^{n-2}\left(\frac{1-2ix_0x_{-k-1}}{1-(2i+2)x_0x_{-k-1}}\right)}.\\
&= x_{-k-1}\prod_{i=0}^{n-2}\left(\frac{1-2ix_0x_{-k-1}}{1-(2i+1)x_0x_{-k-1}}\right)\left(\frac{1-(2n-2)x_0x_{-k-1}}{1-(2n-1)x_0x_{-k-1}}\right).
\end{aligned}$$

Hence, we have

$$x_{(2k+2)n-k-1}=x_{-k-1}\prod_{i=0}^{n-1}\left(\frac{1-2ix_0x_{-k-1}}{1-(2i+1)x_0x_{-k-1}}\right).$$

Similarly, we can easily get the other relations. Thus, the proof is completed.

THEOREM 3.4. *Eq.(12) has the unique equilibrium point* $\overline{x}=0$.

Proof: For the equilibrium points of Eq.(12), we can write $\overline{x}=\dfrac{\overline{x}}{1-\overline{x}^2}$. Then we have $\overline{x}^3=0$. Thus the equilibrium point of Eq.(12) is $\overline{x}=0$.

The following figures show the behavior of the solutions of Eq.(12) with a fixed order and some numerical values of the initial values.

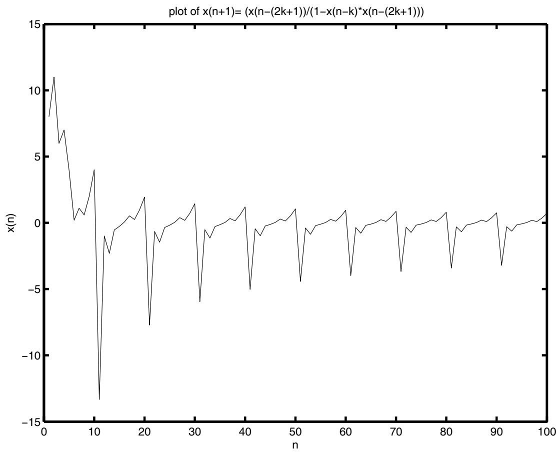


Figure 6: $k = 4$, $x_{-9} = 8$, $x_{-8} = 11$, $x_{-7} = 6$, $x_{-6} = 7$, $x_{-5} = 4$, $x_{-4} = 0.2$, $x_{-3} = 1.1$, $x_{-2} = 0.6$, $x_{-1} = 2$, $x_0 = 4$.

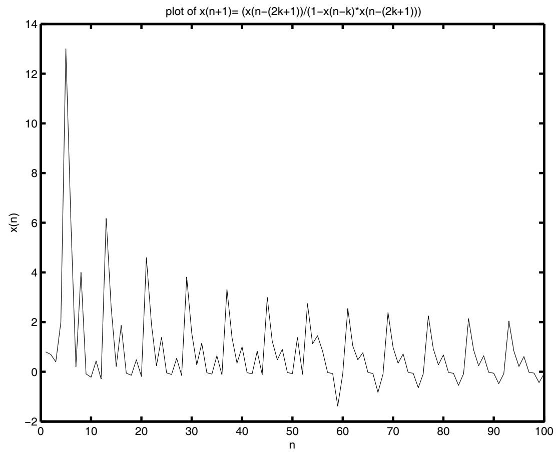


Figure 7: $k = 3$, $x_{-7} = 0.8$, $x_{-6} = 0.7$, $x_{-5} = 0.4$, $x_{-4} = 2$, $x_{-3} = 13$, $x_{-2} = 6$, $x_{-1} = 0.2$, $x_0 = 4$.

Notice: The proofs of the theorems in the following section are similar to that are presented in the previous sections and so they will be omitted.

3.3. On the Difference Equation $x_{n+1} = \dfrac{x_{n-(2k+1)}}{-1 - x_{n-k}x_{n-(2k+1)}}$

Here we obtain the form of the solutions of the following equation

$$x_{n+1} = \frac{x_{n-(2k+1)}}{-1 - x_{n-k}x_{n-(2k+1)}}, \quad n = 0, 1, ..., \tag{13}$$

where the initial values are arbitrary non zero real numbers with $x_{-k+i}x_{-2k-1+i} \neq -1$ $(for\ i = 0, 1, 2, ..., k)$.

Theorem 3.5. *Let $\{x_n\}_{n=-2k-1}^{\infty}$ be a solution of Eq.(13). Then for $n = 1, 2, ...$*

$$\begin{aligned}
x_{(2k+2)n-2k-1} &= \frac{(-1)^n x_{-2k-1}}{(1+x_{-k}x_{-2k-1})^n}, \quad x_{(2k+2)n-2k} = \frac{(-1)^n x_{-2k}}{(1+x_{-k+1}x_{-2k})^n}, \\
x_{(2k+2)n-2k+1} &= \frac{(-1)^n x_{-2k+1}}{(1+x_{-k+2}x_{-2k+1})^n}, \ ..., \\
x_{(2k+2)n-k-1} &= \frac{(-1)^n x_{-k-1}}{(1+x_0x_{-k-1})^n}, \quad x_{(2k+2)n-k} = (-1)^n x_{-k}(1+x_{-k}x_{-2k-1})^n, \\
x_{(2k+2)n-k+1} &= (-1)^n x_{-k+1}(1+x_{-k+1}x_{-2k})^n, ..., \\
x_{(2k+2)n-1} &= (-1)^n x_{-1}(1+x_{-1}x_{-k-2})^n, \quad x_{(2k+2)n} = (-1)^n x_0(1+x_0x_{-k-1})^n.
\end{aligned}$$

Theorem 3.6. *Eq.(13) has a unique equilibrium point which is zero.*

Theorem 3.7. *Let $\{x_n\}_{n=-2k-1}^{\infty}$ be a solution of Eq.(13). Then the following statements are true:*

(1) If $x_{-k+i}x_{-2k-1+i} \neq -2$ $(for\ i = 0, 1, 2, ..., k)$, Then $\{x_n\}_{n=-2k-1}^{\infty}$ is unbounded.

(2) Eq.(13) has a periodic solutions of period $(2k+2)$ iff $x_{-k+i}x_{-2k-1+i} = -2$ $(for\ i = 0, 1, 2, ..., k)$ and will be take the form $\{x_{-2k-1}, x_{-2k}, ..., x_{-1}, x_0, x_{-2k-1}, x_{-2k}, ..., x_{-1}, x_0, ...\}$.

Acknowledgements

This Project was funded by the Deanship of Scientific Research (DSR) at King Abdulaziz University, Jeddah, Saudi Arabia under grant no. G–214–130–38. The authors, therefore, acknowledge with thanks DSR for technical and financial support. Last, but not least, sincere appreciations are dedicated to all our colleagues in the Faculty of Science for their nice wishes.

REFERENCES

1. M. Akbar, N. Ali, The improved F-expansion method with Riccati equation and its applications in mathematical physics, Cogent Mathematics, 4 (1) (2017), No. 1282577, 1-19.
2. F. Belhannache, N. Touafek and R. Abo-zeid, On a higher order rational difference equation, J. Appl. Math. & Informatics, 34 (5-6) (2016), 369-382.
3. C. Cinar, On the positive solutions of the difference equation $x_{n+1} = \frac{x_{n-1}}{1+x_nx_{n-1}}$, Appl. Math. Comp., 150 (2004), 21-24.
4. C. Cinar, On the difference equation $x_{n+1} = \frac{x_{n-1}}{-1+x_nx_{n-1}}$, Appl. Math. Comp., 158 (2004), 813-816.
5. C. Cinar, R. Karatas and I. Yalcinkaya, On solutions of the difference equation $x_{n+1} = \frac{x_{n-3}}{-1+x_nx_{n-1}x_{n-2}x_{n-3}}$, Mathematica Bohemica, 132 (3) (2007), 257-261.
6. Q. Din, Qualitative nature of a discrete predator-prey system, Contemporary Methods in Mathematical Physics and Gravitation, 1 (1) (2015), 27-42.
7. N. Dobashi, E. Suzuki, S. Watanabe, Some polynomials defined by generating functions and differential equations, Cogent Mathematics, 4 (1) (2017), No. 1278830, 1-14.
8. E. M. Elabbasy, H. El-Metwally and E. M. Elsayed, On the difference equation $x_{n+1} = ax_n - \frac{bx_n}{cx_n - dx_{n-1}}$, Adv. Differ. Equ., Volume 2006 (2006), Article ID 82579,1–10.
9. E. M. Elabbasy, H. El-Metwally and E. M. Elsayed, On the difference equations $x_{n+1} = \frac{\alpha x_{n-k}}{\beta+\gamma\prod_{i=0}^{k} x_{n-i}}$, J. Conc. Appl. Math., 5(2) (2007), 101-113.
10. M. M. El-Dessoky, and E. M. Elsayed, On the solutions and periodic nature of some systems of rational difference equations, Journal of Computational Analysis and Applications, 18 (2) (2015), 206-218.

11. H. El-Metwally and E. M. Elsayed, Form of solutions and periodicity for systems of difference equations, Journal of Computational Analysis and Applications, 15 (5) (2013), 852-857.
12. H. El-Metwally and E. M. Elsayed, Qualitative Behavior of some Rational Difference Equations, Journal of Computational Analysis and Applications, 20 (2) (2016), 226-236.
13. E. M. Elsayed, Solution and attractivity for a rational recursive sequence, Discrete Dynamics in Nature and Society, 2011 (2011), Article ID 982309, 17 pages.
14. E. M. Elsayed, Solutions of Rational Difference System of Order Two, Mathematical and Computer Modelling, 55 (2012), 378–384.
15. E. M. Elsayed, Behavior and expression of the solutions of some rational difference equations, Journal of Computational Analysis and Applications, 15 (1) (2013), 73-81.
16. E. M. Elsayed, Solution for systems of difference equations of rational form of order two, Computational and Applied Mathematics, 33 (3) (2014), 751-765.
17. E. M. Elsayed, On the solutions and periodic nature of some systems of difference equations, International Journal of Biomathematics, 7 (6) (2014), 1450067, (26 pages).
18. E. M. Elsayed, Dynamics and Behavior of a Higher Order Rational Difference Equation, The Journal of Nonlinear Science and Applications, 9 (4) (2016), 1463-1474.
19. E. M. Elsayed and A. M. Ahmed, Dynamics of a three-dimensional systems of rational difference equations, Mathematical Methods in The Applied Sciences, 39 (5) (2016), 1026–1038.
20. E. M. Elsayed and A. Alghamdi, Dynamics and Global Stability of Higher Order Nonlinear Difference Equation, Journal of Computational Analysis and Applications, 21 (3) (2016), 493-503.
21. E. M. Elsayed and H. El-Metwally, Stability and solutions for rational recursive sequence of order three, Journal of Computational Analysis and Applications, 17 (2) (2014), 305-315.
22. E. M. Elsayed and H. El-Metwally, Global behavior and periodicity of some difference equations, Journal of Computational Analysis and Applications, 19 (2) (2015), 298-309.
23. E. M. Elsayed and T. F. Ibrahim, Solutions and periodicity of a rational recursive sequences of order five, Bulletin of the Malaysian Mathematical Sciences Society, 38 (1) (2015), 95-112.
24. E. M. Elsayed and T. F. Ibrahim, Periodicity and solutions for some systems of nonlinear rational difference equations, Hacettepe Journal of Mathematics and Statistics, 44 (6) (2015), 1361–1390.
25. Y. Halim, Global character of systems of rational difference equations, Electronic Journal of Mathematical Analysis and Applications, 3 (1) (2015), 204-214.
26. S. Hassan, E. Chatterjee, Dynamics of the equation in the complex plane, Cogent Mathematics, 2 (1) (2015), No. 1122276, 1-12.
27. T. F. Ibrahim, Periodicity and global attractivity of difference equation of higher order, J. Comput. Anal. Appl., 16 (2014), 552-564.
28. D. Jana and E. M. Elsayed, Interplay between strong Allee effect, harvesting and hydra effect of a single population discrete - time system, International Journal of Biomathematics, 9 (1) (2016), 1650004, (25 pages).
29. R. Karatas and C. Cinar, On the solutions of the difference equation $x_{n+1} = \frac{ax_{n-(2k+2)}}{-a+\prod_{i=0}^{2k+2} x_{n-i}}$, Int. J. Contemp. Math. Sciences, 2 (13) (2007), 1505-1509.
30. A. Khaliq, and E. M. Elsayed, The Dynamics and Solution of some Difference Equations, Journal of Nonlinear Sciences and Applications, 9 (3) (2016), 1052-1063.
31. A. Q. Khan, Q. Din, M. N. Qureshi, and T. F. Ibrahim, Global behavior of an anti-competitive system of fourth-order rational difference equations, Computational Ecology and Software, 4 (1) (2014), 35-46.
32. M. R. S. Kulenovic and G. Ladas, Dynamics of Second Order Rational Difference Equations with Open Problems and Conjectures, Chapman & Hall / CRC Press, 2001.
33. M. Phong, A note on a system of two nonlinear difference equations, Electronic Journal of Mathematical Analysis and Applications, 3 (1) (2015), 170 -179.
34. M. N. Qureshi, and A. Q. Khan, Local stability of an open-access anchovy fishery model, Computational Ecology and Software, 5 (1) (2015), 48-62.

35. D. Simsek and F. Abdullayev, On the recursive sequence $x_{n+1} = \dfrac{x_{n-(4k+3)}}{1+\prod_{t=0}^{2} x_{n-(k+1)t-k}}$, Journal of Mathematical Sciences, 6 (222) (2017), 376–387.
36. D. Tollu, Y. Yazlik, and N. Taskara, The Solutions of Four Riccati Difference Equations Associated with Fibonacci Numbers, Balkan journal of Mathematics, 2 (2014), 163-172.
37. N. Touafek, On a second order rational difference equation, Hacet. J. Math. Stat., 41 (2012), 867-874.
38. N. Touafek and E. M. Elsayed, On the periodicity of some systems of nonlinear difference equations, Bull. Math. Soc. Sci. Math. Roumanie, Tome 55 (103) (2) (2012), 217—224.
39. N. Touafek and E. M. Elsayed, On the solutions of systems of rational difference equations, Math. Comput. Mod., 55 (2012), 1987—1997.
40. N. Touafek and E. M. Elsayed, On a second order rational systems of difference equation, Hokkaido Mathematical Journal, 44 (1) (2015), 29–45.
41. N. Touafek, and N. Haddad, On a mixed max-type rational system of difference equations, Electronic Journal of Mathematical Analysis and Applications, 3 (1) (2015), 164 - 169.
42. S. Ufuk Değer, Y. Bolat, Stability conditions a class of linear delay difference systems, Cogent Mathematics, 4 (1) (2017), No. 1294445, 1-13.
43. W. Wang, and H. Feng, On the dynamics of positive solutions for the difference equation in a new population model, J. Nonlinear Sci. Appl., 9 (2016), 1748–1754.
44. Q. Wang and Q. Zhang, Dynamics of a higher-order rational difference equation, Journal of Applied Analysis and Computation, 7 (2) (2017), 770–787.
45. J. Williams, On a class of nonlinear max-type difference equations, Cogent Mathematics, 3 (1) (2016), No. 1269597, 1-11.
46. Y. Yazlik, On the solutions and behavior of rational difference equations, J. Comp. Anal. Appl., 17 (2014), 584–594.
47. Y. Yazlik, E. M. Elsayed and N. Taskara, On the Behaviour of the Solutions of Difference Equation Systems, Journal of Computational Analysis and Applications, 16 (5) (2014), 932–941.
48. Y. Yazlik, D. Tollu, and N. Taskara, On the Solutions of Difference Equation Systems with Padovan Numbers, Applied Mathematics, 4 (2013), 15-20.
49. D. Zhang, J. Huang, L. Wang, and W. Ji, Global Behavior of a Nonlinear Difference Equation with Applications, Open Journal of Discrete Mathematics, 2 (2012), 78-81.
50. D. Zhang, X. Li, L. Wang, S. Cui, On a Max-Type Difference System, Applied Mathematics, 5 (2014), 2959-2967.
51. Q. Zhang, W. Zhang, J. Liu, Y. Shao, On a Fuzzy Logistic Difference Equation, WSEAS Transactions on Mathematics, 13 (2014), 282-290.
52. Q. Zhang, W. Zhang, On the System of Nonlinear Rational Difference Equations, International Journal of Mathematical, Computational, Physical, Electrical and Computer Engineering, 8 (4) (2014), 692-695.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 27, NO.4, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC

Extremal solutions for a coupled system of nonlinear fractional differential equations with p-Laplacian operator *

Ying He†

School of Mathematics and Statistics, Northeast Petroleum University, Daqing163318, P.R.China.

Abstract. This paper studies the existence of extremal solutions for nonlinear fractional differential coupled systems with p-Laplacian operator. The monotone iterative method combined with lower and upper solutions is applied. As an application, an example is presented to illustrate the main result.

Key words. Fractional differential system; p-Laplacian operator; Extremal solution; Monotone iterative technique;

MR(2000) Subject Classifications: 34B15.

1. Introduction

In recent years, fractional differential equations have been of great interest due to the intensive development of the theory of fractional calculus itself and its applications. The study of coupled systems involving fractional-order differential equation is also very significant as such systems appear in a variety of problems of applied nature, especially in bioscience. For details and examples the reader is referred to the papers $[1-4]$ and the reference therein.

In addition, much effort has been made towards the study of the existence of solutions for fractional differential equations involving the p-Laplacian operator based on different fractional derivatives$[5-9]$. In [10], Li and Lin considered a Hadamard fractional boundary value problem with p-Laplacian operator as below:

$$\begin{cases} D^{\beta}(\varphi_p(D^{\alpha}x(t))) = f(t,x(t)), & 0<t<e, \\ x(1)=x'(1)=x'(e)=0, D^{\alpha}x(1)=D^{\alpha}x(e)=0 \end{cases}$$

where $2<\alpha\leq 3$, $1<\beta\leq 2, \varphi_p(s)=|s|^{p-2}s, p>1$, and $f:[1,e]\times[0,+\infty)\longrightarrow[0,+\infty)$ is a positive continuous function. By using the Leray-Schauder type alternative and the Guo-

*This work is supported by the Guiding Innovation Foundation of Northeast Petroleum University (No.2016YDL-02) and Fostering Foundation of Northeast Petroleum University (No.2017PYYL-08).

†Corresponding author. E-mail adress:heying65332015@163.com;

Krasnoselskii fixed point theorem, the existence and the uniqueness of the positive solutions were established.

To best of our knowledge, only few papers considered the method of upper and lower solutions for a coupled system of fractional p-Laplacian equation. Motivated by [11 − 12], in this paper, we use the monotone iterative technique, combined with the method of upper and lower solution to study the coupled system of fractional differential equations with p-Laplacian operator, which is given by

$$\begin{cases} D^{\beta}(\phi_p(D^{\alpha}x(t))) = f(t, x(t), y(t), D^{\alpha}x(t), D^{\alpha}y(t)), & t \in [0,1], \\ D^{\beta}(\phi_p(D^{\alpha}y(t))) = g(t, y(t), x(t), D^{\alpha}y(t), D^{\alpha}x(t)), & t \in [0,1], \\ D^{\alpha}x(t)|_{t=0} = 0, \qquad t^{1-\alpha}x(t)|_{t=0} = r_1, \\ D^{\alpha}y(t)|_{t=0} = 0, \qquad t^{1-\alpha}y(t)|_{t=0} = r_2, \end{cases} \tag{1.1}$$

where $J = [0,1],\ f, g \in C(J \times R^4, R),\ r_1,\ r_2 \in R$ and $r_1 \le r_2$, D^{α}, D^{β} are the standard Riemann-Liouville fractional derivatives, satisfying $0 < \alpha, \beta < 1$, $1 < \alpha + \beta < 2$, $\phi_p(t) = |t|^{p-2}t,\ p > 1$, is the p-Laplacian operator and $(\phi_p)^{-1} = \phi_q$, $\frac{1}{p} + \frac{1}{q} = 1$.

The rest of this paper is organized as follows. In section 2, we give some necessary definitions and lemmas. In section 3, the main result and proof are given. Finally, an example is presented to illustrate the main result.

2. Preliminaries

In this section, we establish some preliminary results that will be used in the next section to attain existence results for the nonlinear system (1.1)

Let $C[0,1]$ denote the Banach space of continuous functions from $[0,1]$ into R with the norm $\|u\|_C = max_{t\in[0,1]}|u(t)|$. Denote $C_{1-\alpha}[0,1]$ by

$$C_{1-\alpha}[0,1] = \{x \in C(0,1] : \ t^{1-\alpha}x \in C[0,1]\}.$$

Then, $C_{1-\alpha}[0,1]$ is a Banach spaces with the norm $\|x\|_{1-\alpha} = \|t^{1-\alpha}x(t)\|_C$. It is clear that $C[0,1] := C_0[0,1] \subset C_{1-\alpha}[0,1]$ with $\|x\|_{C1-\alpha} \le \|x\|_C$ for $0 < \alpha \le 1$ and $C_{1-\alpha}[0,1] \subset L[0,1]$ (note $L[0,1]$ is the space of Lebesgue integrable functions defined on $[0,1]$). Denote $C^{\alpha}[0,1]$ by

$$C^{\alpha}[0,1] = \{x(t) \subset C[0,1] : (D^{\alpha}x)(t) \subset C[0,1] \text{ and } D^{\alpha}x(t)|t = 0 = 0\}$$

.

Lemma 2.1: Let $0 < \beta < 1,\ \sigma \in C[0,1]$,$M \ge 0$ and $M\Gamma(1-\beta) < 1$, then the problem

$$\begin{cases} D^{\beta}u(t) + Mu(t) = \sigma(t), & 0 \le t \le 1, \\ u(0) = 0, \end{cases} \tag{2.1}$$

has a unique solution.

Proof. Equation (2.1) is equivalent to the following integral equation

$$u(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1}(\sigma(s) - Mu(s))ds, \quad \forall\ t \in J$$

2

Let

$$Au(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1}(\sigma(s) - Mu(s))ds, \quad \forall\, t \in J$$

By $M \geq 0$ and $M\Gamma(1-\beta) < 1$, for any $u, v \in C[0,1]$, we have

$$\begin{aligned} \|Au(t) - Av(t)\|_C &\leq \frac{M}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} ds \|u - v\|_C \\ &\leq \frac{M}{\Gamma(\beta)\beta} \|u - v\|_C \\ &< \frac{1}{\Gamma(\beta)} \cdot \frac{1}{\beta} \cdot \frac{1}{\Gamma(1-\beta)} \|u - v\|_C \\ &= \frac{\sin \Pi\beta}{\Pi\beta} \|u - v\|_C \\ &< \|u - v\|_C \end{aligned}$$

So

$$\|Au - Av\|_C < \|u - v\|_C.$$

By the Banach fixed point theorem, the operator A has a unique fixed point. That is (2.1) has a unique solution.

Lemma 2.2: Let $0 < \alpha < 1,\ h \in C_{1-\alpha}[0,1]$, then the problem

$$\begin{cases} D^\alpha x(t) = h(t), & t \in (0,1], \\ t^{1-\alpha} x(t)|_{t=0} = r \end{cases} \tag{2.2}$$

has a unique solution

$$x(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s) ds + rt^{\alpha-1}.$$

Proof. The conclusion is obvious, so we omit it.

Lemma 2.3: Assume that $0 < \alpha, \beta < 1$, $x(t), y(t) \in C^\alpha[0,1]$, $\sigma_1, \sigma_2 \in C[0,1]$, $M,\ N$ be nonnegative constants, satisfying $M \geq N$ and $(M+N)\Gamma(1-\beta) < 1$, then the following fractional differential system

$$\begin{cases} D^\beta(\phi_p(D^\alpha x(t))) = \sigma_1(t) - M\phi_p(D^\alpha x(t)) - N\phi_p(D^\alpha y(t)), \\ D^\beta(\phi_p(D^\alpha y(t))) = \sigma_2(t) - M\phi_p(D^\alpha y(t)) - N\phi_p(D^\alpha x(t)) \\ D^\alpha x(t)|_{t=0} = 0, & t^{1-\alpha} x(t)|_{t=0} = r_1, \\ D^\alpha y(t)|_{t=0} = 0, & t^{1-\alpha} y(t)|_{t=0} = r_2, \end{cases} \tag{2.3}$$

has a unique solution in $C^\alpha[0,1] \times C^\alpha[0,1]$.

Proof. Let

$$\phi_p(D^\alpha x(t)) = \frac{u(t) + v(t)}{2} \quad \text{and} \quad \phi_p(D^\alpha y(t)) = \frac{u(t) - v(t)}{2}, \ \forall\, t \in [0,1].$$

Using (2.3), we have that

$$\begin{cases} D^\beta u(t) = \sigma_1(t) + \sigma_2(t) - (M+N)u(t) \\ u(t)|_{t=0} = \phi_p(D^\alpha x(t))|_{t=0} + \phi_p(D^\alpha y(t))|_{t=0} = 0, \end{cases} \tag{2.4}$$

3

and

$$\begin{cases} D^{\beta}v(t) = \sigma_1(t) - \sigma_2(t) - (M-N)v(t) \\ v(t)|_{t=0} = \phi_p(D^{\alpha}x(t))|_{t=0} - \phi_p(D^{\alpha}y(t))|_{t=0} = 0, \end{cases} \tag{2.5}$$

Since, M, N are nonnegative constants, and $M \geq N$, we have

$$(M-N)\Gamma(1-\beta) \leq (M+N)\Gamma(1-\beta) < 1. \tag{2.6}$$

In view of $x(t), y(t) \in C^{\alpha}[0,1]$, we have $D^{\alpha}x(t), D^{\alpha}y(t) \in C[0,1]$. By (2.6) and Lemma 2.1, we know that (2.4) and (2.5) have a unique solution. In consequence, $\phi_p(D^{\alpha}x(t))$ and $\phi_p(D^{\alpha}y(t))$ are also unique. That is

$$\phi_p(D^{\alpha}x(t)) = \omega_1(t) \in C[0,1], \qquad \phi_p(D^{\alpha}y(t)) = \omega_2(t) \in C[0,1],$$

then,

$$D^{\alpha}x(t) = \phi_q(\omega_1(t)), \qquad D^{\alpha}y(t) = \phi_q(\omega_2(t)),$$

In view of the initial condition $t^{1-\alpha}x(t)|_{t=0} = r_1,\ t^{1-\alpha}y(t)|_{t=0} = r_2$, we obtain

$$\begin{cases} D^{\alpha}x(t) = \phi_q(\omega_1(t)), & t \in [0,1] \\ D^{\alpha}y(t) = \phi_q(\omega_2(t)), & t \in [0,1] \\ t^{1-\alpha}x(t)|_{t=0} = r_1, \\ t^{1-\alpha}y(t)|_{t=0} = r_2, \end{cases} \tag{2.7}$$

Let

$$x(t) = \frac{p(t)+q(t)}{2} \quad \text{and} \quad y(t) = \frac{p(t)-q(t)}{2}.$$

Using (2.7), we have

$$\begin{cases} D^{\alpha}p(t) = \phi_q(\omega_1(t)) + \phi_q(\omega_2(t)), & t \in (0,1] \\ t^{1-\alpha}p(t)|_{t=0} = r_1 + r_2, \end{cases} \tag{2.8}$$

and

$$\begin{cases} D^{\alpha}q(t) = \phi_q(\omega_1(t)) - \phi_q(\omega_2(t)), & t \in (0,1] \\ t^{1-\alpha}q(t)|_{t=0} = r_1 - r_2, \end{cases} \tag{2.9}$$

By Lemma 2.2, we know that both (2.8) and (2.9) have a unique solution in $C^{\alpha}[0,1]$. Hence, x and y are unique too.

Lemma 2.4: Let $0 < \beta < 1$, M be nonnegative constant and $w \in C[0,1]$ satisfies

$$\begin{cases} D^{\beta}w(t) + Mw(t) \geq 0, & 0 < t < 1, \\ w(0) \geq 0, \end{cases}$$

then, $w(t) \geq 0, \ \ \forall\ t \in [0,1]$.

Proof. We assume that $w(t) \geq 0$ is not true. Then there exist $t^*,\ t_* \in (0,1]$ such that $w(t^*) = 0,\ w(t_*) < 0$ and $w(t) \geq 0,\ \forall\ t \in (0,t^*)$, $w(t) < 0,\ \forall\ t \in (t^*, t_*)$. Since $M \geq 0$, we have $D^{\beta}w(t) \geq 0,\ \forall\ t \in (t^*, t_*)$. This together with $D^{\beta}w(t) = \frac{d}{dt}I^{1-\beta}w(t)$ implies $I^{1-\beta}w(t)$ is nondecreasing on (t^*, t_*).

4

Hence, for any $t \in (t^*, t_*)$, we get

$$I^{1-\beta}w(t) - I^{1-\beta}w(t^*) \geq 0.$$

On the other hand

$$\begin{aligned} I^{1-\beta}w(t) - I^{1-\beta}w(t^*) &= \frac{1}{\Gamma(1-\beta)}\int_0^t (t-s)^{-\beta}w(s)ds - \frac{1}{\Gamma(1-\beta)}\int_0^{t^*}(t^*-s)^{-\beta}w(s)ds \\ &= \frac{1}{\Gamma(1-\beta)}\int_0^{t^*}[(t-s)^{-\beta}-(t^*-s)^{-\beta}]w(s)ds + \frac{1}{\Gamma(1-\beta)}\int_{t^*}^t (t-s)^{-\beta}w(s)ds \\ &< 0, \qquad \forall\, t \in (t^*, t_*), \end{aligned}$$

which is a contradiction. Thus the conclusion of Lemma 2.4 holds.

Lemma 2.5: If $x(t) \in C_{1-\alpha}[0,1]$ satisfies

$$\begin{cases} D^\alpha x(t) \geq 0, & t \in (0,1], \\ t^{1-\alpha}x(t)|_{t=0} \geq 0, \end{cases} \tag{2.10}$$

then $x(t) \geq 0$, for $t \in (0,1]$.

Proof. By Lemma 2.2, we know that problem (2.10) has a unique solution

$$x(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}h(s)ds + rt^{\alpha-1}$$

Let $h(t) \geq 0$ and $r \geq 0$, then we obtain (2.10) and the conclusion of lemma 2.5.

Lemma 2.6: Let M, N be nonnegative constants, and $M \geq N$. If $u, v \in C[0,1]$ satisfy the inequalities

$$\begin{cases} D^\beta u(t) \geq -Mu(t) + Nv(t), & t \in [0,1] \\ D^\beta v(t) \geq -Mv(t) + Nu(t), & t \in [0,1] \\ u(t)|_{t=0} \geq 0, \\ v(t)|_{t=0} \geq 0 \end{cases} \tag{2.11}$$

then $u(t) \geq 0,\ v(t) \geq 0,\ \ \forall\, t \in [0,1]$.

Proof. Let $p(t) = u(t) + v(t),\ \ \forall\, t \in [0,1]$. Then by (2.11), we have

$$\begin{cases} D^\beta p(t) \geq -(M-N)p(t), & t \in [0,1] \\ p(t)|_{t=0} \geq 0 \end{cases} \tag{2.12}$$

Thus, by (2.12) and Lemma 2.4, we have that

$$p(t) \geq 0,\ \ \forall\, t \in [0,1]\ \ i.e.\ \ u(t) + v(t) \geq 0,\ \ \forall\, t \in [0,1]. \tag{2.13}$$

Next, we show that $u(t) \geq 0,\ v(t) \geq 0,\ \ \forall\, t \in [0,1]$.
Using (2.11) and (2.13), we find that

$$\begin{cases} D^\beta u(t) \geq -(M+N)u(t), & t \in [0,1] \\ u(t)|_{t=0} \geq 0 \end{cases} \tag{2.14}$$

5

which, in view of (2.14) and Lemma 2.4, yield $u(t) \geq 0, \quad \forall\, t \in [0,1]$. In a similar manner, it can be shown that $v(t) \geq 0, \quad \forall\, t \in [0,1]$.

3. Main Results

In this section, we prove the existence of extremal solutions of nonlinear system (1.1). For convenience, we list the following conditions:
(H_1): There exist $x_0, y_0 \in C^\alpha[0,1]$ and $x_0(t) \leq y_0(t)$ such that

$$\begin{cases} D^\beta(\phi_p(D^\alpha x_0(t))) \leq f(t, x_0(t), y_0(t), D^\alpha x_0(t), D^\alpha y_0(t)), & t \in [0,1], \\ D^\alpha x_0(t)|_{t=0} = 0, \qquad t^{1-\alpha} x(t)|_{t=0} \leq r_1, \\ D^\beta(\phi_p(D^\alpha y_0(t))) \geq g(t, y_0(t), x_0(t), D^\alpha y_0(t), D^\alpha x_0(t)), & t \in [0,1], \\ D^\alpha y_0(t)|_{t=0} = 0, \qquad t^{1-\alpha} y_0(t)|_{t=0} \geq r_2, \end{cases}$$

(H_2): There exist nonnegative constant $M,\ N$ satisfying $M \geq N$ and $(M+N)\Gamma(1-\beta) < 1$, such that

$$\begin{aligned} & f(t, \overline{x(t)}, \overline{y(t)}, D^\alpha \overline{x(t)}, D^\alpha \overline{y(t)}) - f(t, x(t), y(t), D^\alpha x(t), D^\alpha y(t)) \\ & \leq M[\phi_p(D^\alpha x(t)) - \phi_p(D^\alpha \overline{x(t)})] + N[\phi_p(D^\alpha y(t)) - \phi_p(D^\alpha \overline{y(t)})] \\ & g(t, \overline{x(t)}, \overline{y(t)}, D^\alpha \overline{x(t)}, D^\alpha \overline{y(t)}) - g(t, x(t), y(t), D^\alpha x(t), D^\alpha y(t)) \\ & \leq M[\phi_p(D^\alpha x(t)) - \phi_p(D^\alpha \overline{x(t)})] + N[\phi_p(D^\alpha y(t)) - \phi_p(D^\alpha \overline{y(t)})] \end{aligned}$$

where $x_0(t) \leq \overline{x} \leq x \leq y_0(t)$, $x_0(t) \leq y \leq \overline{y} \leq y_0(t)$, and

$$\begin{aligned} & f(t, x(t), y(t), D^\alpha x(t), D^\alpha y(t)) - g(t, y(t), x(t), D^\alpha y(t), D^\alpha x(t)) \\ & \leq M[\phi_p(D^\alpha y(t)) - \phi_p(D^\alpha x(t))] + N[\phi_p(D^\alpha x(t)) - \phi_p(D^\alpha y(t))] \end{aligned}$$

with $x_0(t) \leq x \leq y \leq y_0(t)$.

Theorem 3.1:Suppose that conditions (H_1) and (H_2) hold. Then there is $(x^*, y^*) \in [x_0, y_0] \times [x_0, y_0]$ an extremal solution of the nonlinear problem (1.1). Moreover there exist monotone iterative sequences $\{x_n\}, \{y_n\} \subset C^\alpha$ such that $x_n \to x^*, y_n \to y^* (n \to \infty)$ uniformly on $t \in (0,1]$, and

$$x_0 \leq x_1 \leq \cdots \leq x_n \leq \cdots \leq x^* \leq y^* \leq \cdots \leq y_n \leq \cdots \leq y_1 \leq y_0,$$

moreover, we have

$$D^\alpha x_0 \leq D^\alpha x_1 \leq \cdots \leq D^\alpha x_n \leq \cdots \leq D^\alpha x^* \leq D^\alpha y^* \leq \cdots \leq D^\alpha y_n \leq \cdots \leq D^\alpha y_1 \leq D^\alpha y_0.$$

where

$$[x_0, y_0] = \{x \in C^\alpha[0,1] :\ x_0(t) \leq x(t) \leq y_0(t), t \in [0,1]\}$$

Proof. For any $x_{n-1}, y_{n-1} \in C^\alpha[0,1],\ n \geq 1$, we define

$$\begin{cases} \sigma_n^1(t) = f(t, x_{n-1}(t), y_{n-1}(t), D^\alpha x_{n-1}(t), D^\alpha y_{n-1}(t)) + M\phi_p(D^\alpha x_{n-1}(t)) + N\phi_p(D^\alpha y_{n-1}(t)), \\ \sigma_n^2(t) = g(t, y_{n-1}(t), x_{n-1}(t), D^\alpha y_{n-1}(t), D^\alpha x_{n-1}(t)) + M\phi_p(D^\alpha y_{n-1}(t)) + N\phi_p(D^\alpha x_{n-1}(t)), \end{cases}$$

and consider (2.3) as follows

$$\begin{cases} D^{\beta}(\phi_p(D^{\alpha}x_n(t))) = \sigma_n^1(t) - M\phi_p(D^{\alpha}x_n(t)) - N\phi_p(D^{\alpha}y_n(t)) & t \in (0,1], \\ D^{\beta}(\phi_p(D^{\alpha}y_n(t))) = \sigma_n^2(t) - M\phi_p(D^{\alpha}y_n(t)) - N\phi_p(D^{\alpha}x_n(t)) & t \in (0,1],, \\ D^{\alpha}x_n(t)|_{t=0} = 0, \qquad t^{1-\alpha}x_n(t)|_{t=0} = r_1, \\ D^{\alpha}y_n(t)|_{t=0} = 0, \qquad t^{1-\alpha}y_n(t)|_{t=0} = r_2. \end{cases} \tag{3.1}$$

In view of Lemma 2.3, the problem (3.1) has a unique solution in $C^{\alpha}[0,1] \times C^{\alpha}[0,1]$.

Now, we show that $\{x_n(t)\}$, $\{y_n(t)\}$ satisfy the relation

$$x_{n-1} \le x_n \le y_n \le y_{n-1}, \text{and } D^{\alpha}x_{n-1} \le D^{\alpha}x_n \le D^{\alpha}y_n \le D^{\alpha}y_{n-1}, \quad n = 1, 2, \dots \tag{3.2}$$

Let $u(t) = \phi_p(D^{\alpha}x_1(t)) - \phi_p(D^{\alpha}x_0(t)),\ v(t) = \phi_p(D^{\alpha}y_0(t)) - \phi_p(D^{\alpha}y_1(t))$.
Thus, by condition (3.1) and (H_1), we have

$$\begin{cases} D^{\beta}u(t) \ge -Mu(t) + Nv(t), \\ D^{\beta}v(t) \ge -Mv(t) + Nu(t), \\ u(t)|_{t=0} = \phi_p(D^{\alpha}x_1(t))|_{t=0} - \phi_p(D^{\alpha}x_0(t))|_{t=0} = 0, \\ v(t)|_{t=0} = \phi_p(D^{\alpha}y_0(t))|_{t=0} - \phi_p(D^{\alpha}y_1(t))|_{t=0} = 0. \end{cases}$$

Thus, in view of Lemma 2.6, we have that $\phi_p(D^{\alpha}x_1(t)) \ge \phi_p(D^{\alpha}x_0(t)),\quad \phi_p(D^{\alpha}y_0(t)) \ge \phi_p(D^{\alpha}y_1(t)),\ \ \forall\, t \in [0,1]$. Since $\Phi_p(x)$ is nondecreasing, we have $D^{\alpha}x_1(t) \ge D^{\alpha}x_0(t),\ D^{\alpha}y_0(t) \ge D^{\alpha}y_1(t),\ \ \forall\, t \in [0,1]$.

Let $\epsilon(t) = x_1(t) - x_0(t),\ \theta(t) = y_0(t) - y_1(t)$. It follows from (3.1) and (H_1), we have

$$\begin{cases} D^{\alpha}\epsilon(t) \ge 0, & t \in [0,1], \\ t^{1-\alpha}\epsilon(t)|_{t=0} \ge 0, \end{cases} \tag{3.3}$$

and

$$\begin{cases} D^{\alpha}\theta(t) \ge 0, & t \in [0,1], \\ t^{1-\alpha}\theta(t)|_{t=0} \ge 0, \end{cases} \tag{3.4}$$

By Lemma 2.5, we have $x_1(t) \ge x_0(t),\ y_0(t) \ge y_1(t),\ \ \forall\, t \in [0,1]$.

Now we put $w(t) = \phi_p(D^{\alpha}y_1(t)) - \phi_p(D^{\alpha}x_1(t))$. Applying (3.1) and (H_1), we obtain

$$\begin{aligned} D^{\beta}w(t) &= D^{\beta}(\phi_p(D^{\alpha}y_1(t))) - D^{\beta}(\phi_p(D^{\alpha}x_1(t))) \\ &= g(t, y_0(t), x_0(t), D^{\alpha}y_0(t), D^{\alpha}x_0(t)) + M\phi_p(D^{\alpha}y_0(t)) + N\phi_p(D^{\alpha}x_0(t)) - M\phi_p(D^{\alpha}y_1(t)) \\ &\quad -N\phi_p(D^{\alpha}x_1(t)) - f(t, x_0(t), y_0(t), D^{\alpha}x_0(t), D^{\alpha}y_0(t)) - M\phi_p(D^{\alpha}x_0(t)) - N\phi_p(D^{\alpha}y_0(t)) \\ &\quad +M\phi_p(D^{\alpha}x_1(t)) + N\phi_p(D^{\alpha}y_1(t)) \\ &\ge -M[\phi_p(D^{\alpha}y_0(t)) - \phi_p(D^{\alpha}x_0(t))] - N[\phi_p(D^{\alpha}x_0(t)) - \phi_p(D^{\alpha}y_0(t))] + M\phi_p(D^{\alpha}y_0(t)) \\ &\quad +N\phi_p(D^{\alpha}x_0(t)) - M\phi_p(D^{\alpha}y_1(t)) - N\phi_p(D^{\alpha}x_1(t)) - M\phi_p(D^{\alpha}x_0(t)) - N\phi_p(D^{\alpha}y_0(t)) \\ &\quad +M\phi_p(D^{\alpha}x_1(t)) + N\phi_p(D^{\alpha}y_1(t)) \\ &= -(M-N)w(t). \end{aligned}$$

Also, $w(t)|_{t=0} = \phi_p(D^{\alpha}y_1(t))|_{t=0} - \phi_p(D^{\alpha}x_1(t))|_{t=0} = 0$. In view of Lemma 2.4, we have that $w(t) \ge 0,\ \ \forall\, t \in J$. Thus we have the relation $\phi_p(D^{\alpha}x_1(t)) \le \phi_p(D^{\alpha}y_1(t))$. That is

$D^\alpha x_1(t) \leq D^\alpha y_1(t)$, since $\Phi_p(x)$ is nondecreasing. Therefore $D^\alpha x_0(t) \leq D^\alpha x_1(t) \leq D^\alpha y_1(t) \leq D^\alpha y_0(t), \quad \forall\, t \in J$ holds.

Let $\delta(t) = y_1(t) - x_1(t)$. It follows from (3.1) that

$$\begin{cases} D^\alpha \delta(t) = D^\alpha y_1(t) - D^\alpha x_1(t) \geq 0, \\ t^{1-\alpha}\delta(t)|_{t=0} = t^{1-\alpha} y_1(t)|_{t=0} - t^{1-\alpha} x_1(t)|_{t=0} = r_2 - r_1 \geq 0. \end{cases}$$

By Lemma 2.5, we obtain $y_1(t) \geq x_1(t), \quad \forall\, t \in (0,1]$. Hence, we have the relation $x_0(t) \leq x_1(t) \leq y_1(t) \leq y_0(t)$.

Now we assume that

$$x_{k-1} \leq x_k \leq y_k \leq y_{k-1}, \text{and } D^\alpha x_{k-1} \leq D^\alpha x_k \leq D^\alpha y_k \leq D^\alpha y_{k-1}, \quad \text{for some } k \geq 1,$$

we will prove that (3.2) is also true for $k+1$. Set

$u(t) = \phi_p(D^\alpha x_{k+1}(t)) - \phi_p(D^\alpha x_k(t))$, $v(t) = \phi_p(D^\alpha y_k(t)) - \phi_p(D^\alpha y_{k+1}(t))$, $w(t) = \phi_p(D^\alpha y_{k+1}(t)) - \phi_p(D^\alpha x_{k+1}(t))$, $\epsilon(t) = x_{k+1}(t) - x_k(t)$, $\theta(t) = y_k(t) - y_{k+1}(t)$, $\delta(t) = y_{k+1}(t) - x_{k+1}(t)$
By (H_2) and (3.1), we have that

$$\begin{cases} D^\beta u(t) \geq -Mu(t) + Nv(t), \\ D^\beta v(t) \geq -Mv(t) + Nu(t), \\ u(t)|_{t=0} = 0, \\ v(t)|_{t=0} = 0, \end{cases}$$

$$\begin{cases} D^\alpha \epsilon(t) \geq 0, \\ t^{1-\alpha}\epsilon(t)|_{t=0} \geq 0, \end{cases}$$

$$\begin{cases} D^\alpha \theta(t) \geq 0, \\ t^{1-\alpha}\theta(t)|_{t=0} \geq 0, \end{cases}$$

and

$$\begin{cases} D^\beta w(t) \geq -(M-N)w(t), \\ w(t)|_{t=0} = 0, \end{cases}$$

$$\begin{cases} D^\alpha \delta(t) \geq 0, \\ t^{1-\alpha}\delta(t)|_{t=0} \geq 0. \end{cases}$$

In view of Lemma 2.4, 2.5 and 2.6, we obtain

$$x_k \leq x_{k+1} \leq y_{k+1} \leq y_k \text{ and } D^\alpha x_k \leq D^\alpha x_{k+1} \leq D^\alpha y_{k+1} \leq D^\alpha y_k, \qquad \forall\, t \in [0,1]$$

.

From the above, by induction, it is not difficult to prove that

$$x_0 \leq x_1 \leq \cdots \leq x_n \leq \cdots \leq y_n \leq \cdots \leq y_1 \leq y_0.$$

and

$$D^\alpha x_0 \leq D^\alpha x_1 \leq \cdots \leq D^\alpha x_n \leq \cdots \leq D^\alpha y_n \leq \cdots \leq D^\alpha y_1 \leq D^\alpha y_0.$$

8

Applying the standard arguments, it is easy to show $\{x_n\}$ and $\{y_n\}$ are uniformly bounded and equi-continuous in $[x_0, y_0]$. By Arzela-Ascoli theorem, we have

$$\lim_{n\to\infty} x_n(t) = x^*(t), \quad \lim_{n\to\infty} y_n(t) = x^*(t), \qquad \forall\ t \in [0,1]$$

and

$$\lim_{n\to\infty} D^\alpha x_n(t) = D^\alpha x^*(t), \quad \lim_{n\to\infty} D^\alpha y_n(t) = D^\alpha x^*(t), \qquad \forall\ t \in [0,1]$$

and the limit function x^* and y^* satisfy (1.1). Moreover, $x^*, y^* \in [x_0, y_0]$. Taking the limits $n \to \infty$ in (3.1), we find that (x^*, y^*) is a solution of problem (1.1) in $[x_0, y_0] \times [x_0, y_0]$.

Finally, we show that (x^*, y^*) is an extremal solution of the system (1.1). Assume that $(x, y) \in [x_0, y_0] \times [x_0, y_0]$ is any solution for the problem (1.1), that is

$$\begin{cases} D^\beta(\phi_p(D^\alpha x(t))) = f(t, x(t), y(t), D^\alpha x(t), D^\alpha y(t)), & t \in [0,1], \\ D^\alpha x(t)|_{t=0} = 0, \qquad t^{1-\alpha}x(t)|_{t=0} = r_1, \\ D^\beta(\phi_p(D^\alpha y(t))) = g(t, y(t), x(t), D^\alpha y(t), D^\alpha x(t)), & t \in [0,1], \\ D^\alpha y(t)|_{t=0} = 0, \qquad t^{1-\alpha}y(t)|_{t=0} = r_2, \end{cases} \tag{3.5}$$

Applying (3.1), (3.5), (H_2), Lemma 2.5 and 2.6, we have

$$x_n \le x,\ y \le y_n, \quad D^\alpha x_n \le D^\alpha x,\ D^\alpha y \le D^\alpha y_n, \quad n = 1, 2, \ldots \tag{3.6}$$

Taking the limit $n \to \infty$ in (3.6), we have $x^* \le x,\ y \le y^*$. The proof is complete.

Example: Consider the following problem

$$\begin{cases} D^{\frac{2}{3}}(\phi_4(D^{\frac{1}{2}}x(t))) = \frac{1}{6\Gamma(1-\frac{2}{3})}x^{\frac{1}{3}}(t)[D^{\frac{1}{2}}x(t)]^{\frac{1}{3}} - y^3(t)\Big[D^{\frac{1}{2}}y(t) - \frac{2\Gamma(\frac{7}{4})}{\Gamma(\frac{5}{4})}t^{\frac{1}{4}}\Big]^3, & t \in (0,1], \\ D^{\frac{2}{3}}(\phi_4(D^{\frac{1}{2}}y(t))) = \frac{1}{6\Gamma(1-\frac{2}{3})}y^{\frac{1}{3}}(t)[D^{\frac{1}{2}}y(t)]^{\frac{1}{3}} - x^3(t)\Big[D^{\frac{1}{2}}x(t) - \frac{2\Gamma(\frac{7}{4})}{\Gamma(\frac{5}{4})}t^{\frac{1}{4}}\Big]^3, & t \in (0,1], \\ D^{\frac{1}{2}}x(t)|_{t=0} = 0, \quad t^{1-\frac{1}{2}}x(t)|_{t=0} = 0, \\ D^{\frac{1}{2}}y(t)|_{t=0} = 0, \quad t^{1-\frac{1}{2}}y(t)|_{t=0} = 0, \end{cases} \tag{3.7}$$

where $\alpha = \frac{1}{2},\ \beta = \frac{2}{3},\ p = 4$ and

$$\begin{cases} f(t, x(t), y(t), D^{\frac{1}{2}}x(t), D^{\frac{1}{2}}y(t)) = \frac{1}{6\Gamma(1-\frac{2}{3})}x^{\frac{1}{3}}[D^{\frac{1}{2}}x]^{\frac{1}{3}} - y^3\Big[D^{\frac{1}{2}}y - \frac{2\Gamma(\frac{7}{4})}{\Gamma(\frac{5}{4})}t^{\frac{1}{4}}\Big]^3 \\ g(t, y(t), x(t), D^{\frac{1}{2}}y(t), D^{\frac{1}{2}}x(t)) = \frac{1}{6\Gamma(1-\frac{2}{3})}y^{\frac{1}{3}}[D^{\frac{1}{2}}y]^{\frac{1}{3}} - x^3\Big[D^{\frac{1}{2}}x - \frac{2\Gamma(\frac{7}{4})}{\Gamma(\frac{5}{4})}t^{\frac{1}{4}}\Big]^3 \end{cases}$$

Take $x_0(t) = \frac{1}{2}t^{\frac{3}{2}},\ y_(t) = 2t^{\frac{3}{4}}$, then $D^{\frac{1}{2}}x_0(t) = \frac{1}{2}\Gamma(\frac{5}{2})t,\ D^{\frac{1}{2}}y_0(t) = \frac{2\Gamma(\frac{7}{4})}{\Gamma(\frac{5}{4})}t^{\frac{1}{4}}$. It is not difficult to show that (H_1) holds.

Since the function $\sqrt[3]{x} + x^3$ is monotone increasing for $x \in R$, we obtain

$$\begin{aligned} & f(t, \overline{x(t)}, \overline{y(t)}, D^{\frac{1}{2}}\overline{x(t)}, D^{\frac{1}{2}}\overline{y(t)}) - f(t, x(t), y(t), D^{\frac{1}{2}}x(t), D^{\frac{1}{2}}y(t)) \\ = \ & \frac{1}{6\Gamma(1-\frac{2}{3})}(\overline{x(t)})^{\frac{1}{3}}[D^{\frac{1}{2}}\overline{x(t)}]^{\frac{1}{3}} - (\overline{y(t)})^3\Big[D^{\frac{1}{2}}\overline{y(t)} - \frac{2\Gamma(\frac{7}{4})}{\Gamma(\frac{5}{4})}t^{\frac{1}{4}}\Big]^3 - \frac{1}{6\Gamma(1-\frac{2}{3})}x^{\frac{1}{3}}(t)[D^{\frac{1}{2}}x(t)]^{\frac{1}{3}} \\ + \ & y^3(t)\Big[D^{\frac{1}{2}}y(t) - \frac{2\Gamma(\frac{7}{4})}{\Gamma(\frac{5}{4})}t^{\frac{1}{4}}\Big]^3, \end{aligned}$$

9

$$
\begin{aligned}
&\leq \frac{1}{6\Gamma(1-\frac{2}{3})}\sqrt[3]{2}[(D^{\frac{1}{2}}\overline{x(t)})^{\frac{1}{3}} - (D^{\frac{1}{2}}x(t))^{\frac{1}{3}}] \\
&\leq \frac{1}{6\Gamma(1-\frac{2}{3})}\sqrt[3]{2}[(D^{\frac{1}{2}}x(t))^{3} - (D^{\frac{1}{2}}\overline{x(t)})^{3}] \\
&= \frac{1}{6\Gamma(1-\frac{2}{3})}\sqrt[3]{2}[\Phi_4(D^{\frac{1}{2}}x(t)) - \Phi_4(D^{\frac{1}{2}}\overline{x(t)})]
\end{aligned}
$$

where $x_0(t) \leq \overline{x(t)} \leq x(t) \leq y_0(t)$, and $x_0(t) \leq y(t) \leq \overline{y(t)} \leq y_0(t)$.
Note $M = \frac{\sqrt[3]{2}}{6\Gamma(1-\frac{2}{3})}, N = 0$ and

$$(M+N)\Gamma(1-\frac{2}{3}) = \frac{\sqrt[3]{2}}{6} < 1,$$

thus the condition (H_2) holds. Therefore, there exist monotone iterative sequence $\{x_n\}$ and $\{y_n\}$, which converge uniformly to solutions of fractional problem (3.7) in $[x_0, y_0]$ by Theorem 3.1.

References

[1] Z.wei,Q.Li,J.Che, Initial value problems for fractional differential equations involving Riemann-Liouville sequential fractional derivative,J.Math.Anal.Appl. 367(1) (2010)260-272

[2] F.Li,J.Sun,M.Jia,Monotone iterative method for second-order three-point boundary value problem with upper and lower solutions in the reversed order, Appl. Math. Comput.217 (2011)4840-4847.

[3] R.A.Khan,J.R.L.Webb, Existence of at least three solutions of a second-order three-point bounday value problem,Nonlinear Anal.64(2006)1356-1366

[4] J.Henderson, Existence of multiple solutions for second order boundary value problems,J.Differ.Equ. 166 (2000) 443-454.

[5] C.Yang,J.Yan,Positive solutions for third-order Sturm-Liouville boundary problems with p-Laplacian, Appl. Math. Comput. 59 (2010) 2059-2066.

[6] J.Wang,H.Xiang,Z.Liu,Positive solutions for three-point boundary value problems of nonlinear fractional differential equations with p-Laplacian,Far East J.Appl.math.37 (2009)33-47.

[7] J.Wang,H.Xiang,Z.Liu, Upper and lower solutions method for a class of singular fractional boundary value problems with p-Laplacian operator,Abstr. Appl. Anal.219 (2013) 4680-4691.

[8] X.Zhang,L.Liu,Y.Wu,Y.Lu The iterative solutions for nonlinear factional differential equation,Appl. Math. Comput. 219 (2013) 4680-4691 .

[9] X.Zhang,Y.Han,Existence and uniqueness of positive solutions for higher order nonlocal fractional differential equations, Appl.Math.Lett. 25 (2012)555-560

[10] Y.Li,S.Lin,Positive solutions for the nonlinear Hadamard type fractional differential equations with p-Laplacian, J.Funct.Spaces Appl.2013 (2013)(Art. 951643).

[11] G.Wang,R.Agarwal,A.Cabada, Existence results and the monotone iterative technique for systems of nonlinear fractional differential equations, Appl.Math.Lett. 25 (2012) 1019-1024.

[12] L.Zhang,B.Ahmad,G.Wang, The existence of an extremal solution to a nonlinear system with the right-handed Riemann-Liouville fractional derivative,Appl.Math.Lett. 31 (2014) 1-6.

The Growth and Zeros of Linear Differential Equations with Entire Coefficients of $[p, q] - \varphi(r)$ Order *†

Sheng Gui Liu[1], Jin Tu[2,*], Hong Zhang[3]

[1] School of Mechanics and Civil Engineering, China University of Mining and Technology(Beijing), Beijing, 100083, China
[2] College of Mathematics and Information Science, Jiangxi Normal University, Nanchang 330022, China
[3] School of the Tourism and Urban Management,Jiangxi University of Finance and Economics, Nanchang 330032, China

Abstract

In this paper, the authors investigate the growth and zeros of the solutions and the coefficients of higher order linear differential equations with entire coefficients of $[p,\ q] - \varphi$ order, where $p,\ q$ are positive integers and satisfy $p \geq q \geq 1$. The theorems that we obtain extend and improve many previous results.
Key words: linear differential equations; entire functions; $[p, q] - \varphi$ order;
AMS Subject Classification(2010): 30D35, 34M10

1. Introduction and Results

In this paper, we shall assume that readers are familiar with the fundamental results and the standard notations of the Nevanlinna's theory of meromorphic functions and the theory of complex linear differential equations (e.g. [9,14]). The theory of complex linear differential equations has been developed since 1960s. Many authors have investigated the complex linear differential equations

$$f^{(k)}(z) + A_{k-1}(z)f^{(k-1)}(z) + \cdots + A_0(z)f(z) = 0 \tag{1.1}$$

and

$$f^{(k)}(z) + A_{k-1}(z)f^{(k-1)}(z) + \cdots + A_0(z)f(z) = F(z) \tag{1.2}$$

and achieved many valuable results when the coefficients $A_0(z), \cdots,\ A_{k-1}(z),\ F(z)(k \geq 2)$ in (1.1) and (1.2) are entire functions of finite order or finite iterate order (e.g. [1-2, 4-7, 13, 14]). In 2010, J. Tu and his co-authors investigated the complex oscillation properties of solutions of (1.1) and (1.2) when the coefficients in (1.1) or (1.2) are entire functions of $[p, q]-$order (see [21]). In the following, we introduce some notations about $[p, q]-$order, where p, q are positive integers

*Corresponding author E-mail adress:tujin2008@sina.com
†This project is supported by the National Natural Science Foundation of China (Grant No.41472130, 11561031), the Natural Science Foundation of Jiangxi Province in China (Grant No.20161BAB201020, 20132BAB211002) and the Foundation of Education Bureau of Jiangxi Province in China (GJJ151331,GJJ14272).

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 27, NO.4, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC

and satisfy $p \geq q \geq 1$. For $r \in (0, +\infty)$, we define $\exp_1 r = e^r$ and $\exp_{i+1} r = \exp(\exp_i r),\ i \in \mathbb{N}$ and for all sufficiently large r, we define $\log_1 r = \log r$ and $\log_{i+1} r = \log(\log_i r),\ i \in \mathbb{N}$. Especially, we have $\exp_0 r = r = \log_0 r$ and $\exp_{-1} r = \log_1 r$. Secondly, we denote the linear measure and the logarithmic measure of a set $E \subset (1, +\infty)$ by $mE = \int_E dt$ and $m_l E = \int_E \frac{dt}{t}$.

Definition 1.1. ([12,15,16]) If $f(z)$ is a meromorphic function, the $[p, q]$-order of $f(z)$ is defined by

$$\sigma_{[p,q]}(f) = \overline{\lim_{r\to\infty}} \frac{\log_p T(r, f)}{\log_q r}, \tag{1.3}$$

especially if $f(z)$ is an entire function, the $[p, q]$-order of $f(z)$ is defined by

$$\sigma_{[p,q]}(f) = \overline{\lim_{r\to\infty}} \frac{\log_p T(r, f)}{\log_q r} = \overline{\lim_{r\to\infty}} \frac{\log_{p+1} M(r, f)}{\log_q r}. \tag{1.4}$$

Definition 1.2. ([15,16]) The $[p, q]$-exponent of convergence of the (distinct) zero-sequence of $f(z)$ are respectively defined by

$$\lambda_{[p,q]}(f) = \overline{\lim_{r\to\infty}} \frac{\log_p n(r, \frac{1}{f})}{\log_q r} = \overline{\lim_{r\to\infty}} \frac{\log_p N(r, \frac{1}{f})}{\log_q r}, \tag{1.5}$$

$$\overline{\lambda}_{[p,q]}(f) = \overline{\lim_{r\to\infty}} \frac{\log_p \overline{n}(r, \frac{1}{f})}{\log_q r} = \overline{\lim_{r\to\infty}} \frac{\log_p \overline{N}(r, \frac{1}{f})}{\log_q r}. \tag{1.6}$$

In recent years, many authors investigated the equations (1.1) and (1.2) with entire coefficients or meromorphic coefficients of $[p, q]-$order (e.g. [3, 11, 15, 16]) and obtain the following results.

Theorem A. ([15]) Let $A_j(z)$ $(j = 0, 1, \cdots, k-1)$ be entire functions satisfying $\max\{\sigma_{[p,q]}(A_j)| j \neq 0\} < \sigma_{[p,q]}(A_0) < \infty$. Then every nontrivial solution $f(z)$ of (1.1) satisfies $\sigma_{[p+1,q]}(f) = \sigma_{[p,q]}(A_0)$.

Theorem B. ([15]) Let $F(z) \not\equiv 0, A_j(z)$ $(j = 0, 1, \cdots, k-1)$ be entire functions satisfying $\max\{\sigma_{[p,q]}(A_j), \sigma_{[p+1,q]}\ (F)|j = 1, \cdots, k-1\} < \sigma_{[p,q]}(A_0)$. Then every solution $f(z)$ of (1.2) satisfies

$$\overline{\lambda}_{[p+1,q]}(f) = \lambda_{[p+1,q]}(f) = \sigma_{[p+1,q]}(f) = \sigma_{[p,q]}(A_0),$$

with at most one exceptional solution f_0 satisfying $\sigma_{[p+1,q]}(f_0) < \sigma_{[p,q]}(A_0)$.

Theorem C. ([15]) Let $F(z) \not\equiv 0,\ A_j(z)(j = 0, \cdots, k-1)$ be entire functions, and let $f(z)$ be a solution of (1.2) satisfying $\max\{\sigma_{[p,q]}(A_j), \sigma_{[p,q]}(F)|j = 0, 1, \cdots, k-1\} < \sigma_{[p,q]}(f)$. Then $\overline{\lambda}_{[p,q]}(f) = \lambda_{[p,q]}(f) = \sigma_{[p,q]}(f)$.

On the basic of Definitions 1.1 and 1.2, some researchers introduce the notations of $[p, q]-\varphi(r)$ order of entire functions or analytic functions in [17, 18, 19], where p, q are positive integers and satisfy $p \geq q \geq 1$.

Definition 1.3.([17,18]) Let $\varphi:[0,+\infty)\to(0,+\infty)$ be a non-decreasing unbounded continuous function, the $[p,q]-\varphi(r)$ order and $[p,q]-\varphi(r)$ lower order of an entire function $f(z)$ are respectively defined by

$$\sigma_{[p,q]}(f,\varphi)=\overline{\lim_{r\to\infty}}\frac{\log_p T(r,f)}{\log_q \varphi(r)};\quad \mu_{[p,q]}(f,\varphi)=\underline{\lim_{r\to\infty}}\frac{\log_p T(r,f)}{\log_q \varphi(r)}. \tag{1.7}$$

Similar with Definition 1.3, we can also define $[p,q]-\varphi(r)$ exponent of convergence of (distinct) zero-sequence of an entire function $f(z)$.

Definition 1.4. ([17]) The $[p,q]-\varphi(r)$ exponent of convergence of (distinct) zero-sequence of an entire function $f(z)$ are respectively defined by

$$\lambda_{[p,q]}(f,\varphi)=\overline{\lim_{r\to\infty}}\frac{\log_p n(r,\frac{1}{f})}{\log_q \varphi(r)};\quad \overline{\lambda}_{[p,q]}(f,\varphi)=\overline{\lim_{r\to\infty}}\frac{\log_p \overline{n}(r,\frac{1}{f})}{\log_q \varphi(r)}. \tag{1.8}$$

Remark 1.5. If $\varphi(r)=r$, the Definitions 1.1-1.2 are special cases of Definitions 1.3-1.4.

In order to get similar results with Theorems $A-C$ by replacing $[p,q]-$order with $[p,q]-\varphi(r)$ order, we suppose that $\varphi(r)$ has the following properties without special instructions:

Proposition 1.6. Suppose that $\varphi(r):[0,+\infty)\to(0,+\infty)$ is a non-decreasing unbounded continuous function and satisfies (i) $\lim_{r\to\infty}\frac{\log_{p+1} r}{\log_q \varphi(r)}=0$, (ii) $\lim_{r\to\infty}\frac{\log_q \varphi(\alpha r)}{\log_q \varphi(r)}=1$ for some $\alpha>1$.

Proposition 1.7. ([17]) If $\varphi(r)$ satisfies the above two conditions $(i)-(ii)$ in Proposition 1.6: (i) then for any entire function $f(z)$, we have

$$\sigma_{[p,q]}(f,\varphi)=\overline{\lim_{r\to\infty}}\frac{\log_p T(r,f)}{\log_q \varphi(r)}=\overline{\lim_{r\to\infty}}\frac{\log_{p+1} M(r,f)}{\log_q \varphi(r)};$$
$$\mu_{[p,q]}(f,\varphi)=\underline{\lim_{r\to\infty}}\frac{\log_p T(r,f)}{\log_q \varphi(r)}=\underline{\lim_{r\to\infty}}\frac{\log_{p+1} M(r,f)}{\log_q \varphi(r)}.$$

(ii) then for any meromorphic function $f(z)$, we have

$$\lambda_{[p,q]}(f,\varphi)=\overline{\lim_{r\to\infty}}\frac{\log_p n(r,\frac{1}{f})}{\log_q \varphi(r)}=\overline{\lim_{r\to\infty}}\frac{\log_p N(r,\frac{1}{f})}{\log_q \varphi(r)};$$
$$\overline{\lambda}_{[p,q]}(f,\varphi)=\overline{\lim_{r\to\infty}}\frac{\log_p \overline{n}(r,\frac{1}{f})}{\log_q \varphi(r)}=\overline{\lim_{r\to\infty}}\frac{\log_p \overline{N}(r,\frac{1}{f})}{\log_q \varphi(r)}.$$

In this paper, we investigate the growth and zeros of solutions of (1.1) and (1.2) with entire coefficients of $[p,q]-\varphi(r)$ order and obtain the following results.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 27, NO.4, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC

Theorem 1.8. Let $A_j(z)$ $(j = 0, 1, \cdots, k-1)$ be entire functions satisfying $\max\{\sigma_{[p,q]}(A_j, \varphi) | j = 1, 2, \cdots, k-1\} < \sigma_{[p,q]}(A_0, \varphi) < \infty$. Then every solution $f(z) \not\equiv 0$ of (1.1) satisfies $\sigma_{[p+1,q]}(f, \varphi) = \sigma_{[p,q]}(A_0, \varphi)$.

Theorem 1.9. Let $F(z) \not\equiv 0$, $A_j(z) (j = 0, \cdots, k-1)$ be entire functions, and let $f(z)$ be a solution of (1.2) satisfying $\max\{\sigma_{[p,q]}(A_j, \varphi), \sigma_{[p,q]}(F, \varphi) | j = 0, 1, \cdots, k-1\} < \sigma_{[p,q]}(f, \varphi)$. Then $\overline{\lambda}_{[p,q]}(f, \varphi) = \lambda_{[p,q]}(f, \varphi) = \sigma_{[p,q]}(f, \varphi)$.

Theorem 1.10. Let $F(z) \not\equiv 0$, $A_j(z)$ $(j = 0, 1, \cdots, k-1)$ be entire functions satisfying $\max\{\sigma_{[p,q]}(A_j, \varphi),\ \sigma_{[p+1,q]}(F, \varphi) | j = 1, \cdots, k-1\} < \sigma_{[p,q]}(A_0, \varphi)$. Then every solution $f(z)$ of (1.2) satisfies

$$\overline{\lambda}_{[p+1,q]}(f, \varphi) = \lambda_{[p+1,q]}(f, \varphi) = \sigma_{[p+1,q]}(f, \varphi) = \sigma_{[p,q]}(A_0, \varphi),$$

with at most one exceptional solution f_0 satisfying $\sigma_{[p+1,q]}(f_0, \varphi) < \sigma_{[p,q]}(A_0, \varphi)$.

Remark 1.11. The above Theorems 1.8-1.10 generalize and extend Theorems A-C and some previous results.

2. Preliminary Lemmas

Lemma 2.1. ([10, 14]) Let $f(z)$ be a transcendental entire function, and let z be a point with $|z| = r$ at which $|f(z)| = M(r, f)$. Then for all $|z| = r$ outside a set E_1 of r of finite logarithmic measure, we have

$$\frac{f^{(j)}(z)}{f(z)} = \left(\frac{v_f(r)}{z}\right)^j (1 + o(1)) \quad (j \in \mathbb{N}),$$

where $v_f(r)$ is the central index of $f(z)$, $E_1 \subset (1, +\infty)$ is a set of r of finite logarithmic measure or finite linear measure in this paper, not necessarily the same at each occurrence.

Lemma 2.2. ([7, 14]) Let $g : [0, +\infty) \longrightarrow R$ and $h : [0, +\infty) \longrightarrow R$ be monotone increasing functions such that $g(r) \leq h(r)$ outside of an exceptional set $E_1 \subset [1, +\infty)$ of finite logarithmic measure or finite linear measure. Then for any $d > 1$, there exists $r_0 > 0$ such that $g(r) \leq h(dr)$ for all $r > r_0$.

Lemma 2.3.([17]) Let $f(z)$ be an entire function satisfying $\sigma_{[p,q]}(f, \varphi) = \sigma_1$ and $\mu_{[p,q]}(f, \varphi) = \mu_1$. Then

$$\overline{\lim_{r\to\infty}} \frac{\log_p v_f(r)}{\log_q \varphi(r)} = \sigma_1, \quad \underline{\lim_{r\to\infty}} \frac{\log_p v_f(r)}{\log_q \varphi(r)} = \mu_1.$$

Lemma 2.4. Let $f(z)$ be an entire function of $[p, q] - \varphi(r)$ order satisfying $\sigma_{[p,q]}(f, \varphi) = \sigma_2$, where $\varphi(r)$ only satisfies $\lim\limits_{r\to\infty} \frac{\log_q \varphi(\alpha r)}{\log_q \varphi(r)} = 1$ for some $\alpha > 1$. Then there exists a set $E_2 \subset (1, +\infty)$ having infinite logarithmic measure such that for all $r \in E_2$, we have

$$\lim_{r\to\infty} \frac{\log_p T(r, f)}{\log_q \varphi(r)} = \sigma_2 \quad (r \in E_2).$$

Proof. By Definition 1.3, there exists an increasing sequence $\{r_n\}_{n=1}^{\infty}$ tending to ∞ satisfying $(1+\frac{1}{n})r_n < r_{n+1}$ and

$$\lim_{n\to\infty}\frac{\log_p T(r_n,f)}{\log_q \varphi(r_n)} = \sigma_{[p,q]}(f,\varphi)=\sigma_2,$$

there exists an $n_1(\in\mathbb{N})$ such that for $n\geq n_1$ and for any $r\in E_2=\cup_{n=n_1}^{\infty}[r_n,(1+\frac{1}{n})r_n]$, we have

$$\frac{\log_p T(r_n,f)}{\log_q \varphi((1+\frac{1}{n})r_n)} \leq \frac{\log_p T(r,f)}{\log_q \varphi(r)}. \tag{2.1}$$

By (2.1), for all $r\in E_2$, we have

$$\lim_{n\to\infty}\frac{\log_p T(r_n,f)}{\log_q \varphi(r_n)}\cdot\lim_{n\to\infty}\frac{\log_q \varphi(r_n)}{\log_q \varphi((1+\frac{1}{n})r_n)} \leq \lim_{r\to\infty}\frac{\log_p T(r,f)}{\log_q \varphi(r)}. \tag{2.2}$$

By (2.12) and $\lim\limits_{r\to\infty}\frac{\log_q \varphi(\alpha r)}{\log_q \varphi(r)}=1$ $(\alpha>1)$, for all $r\in E_2$, we have

$$\lim_{r\to\infty}\frac{\log_p T(r,f)}{\log_q \varphi(r)} \geq \sigma_2. \tag{2.3}$$

On the other hand, by Definition 1.3, for all $r\in E_2$, we have

$$\overline{\lim_{r\to\infty}}\frac{\log_p T(r,f)}{\log_q \varphi(r)} \leq \sigma_2. \tag{2.4}$$

By (2.3) and (2.4), for any $r\in E_2$, we have

$$\lim_{r\to\infty}\frac{\log_p T(r,f)}{\log_q \varphi(r)} = \sigma_2.$$

where $m_l E_2=\sum_{n=n_1}^{\infty}\int_{r_n}^{(1+\frac{1}{n})r_n}\frac{dt}{t}=\sum_{n=n_1}^{\infty}\log(1+\frac{1}{n})=\infty$.

By Lemma 2.4, it is easy to obtain the following Lemma 2.5.

Lemma 2.5. Let $f_1(z)$, $f_2(z)$ be entire functions of $[p,q]-\varphi(r)$ order satisfying $\sigma_{[p,q]}(f_1,\varphi)>\sigma_{[p,q]}(f_2)$. Then there exists a set $E_3\subset(1,+\infty)$ having infinite logarithmic measure such that for all $r\in E_3$, we have

$$\lim_{r\to\infty}\frac{T(r,f_2)}{T(r,f_1)}=0 \quad (r\in E_3).$$

Lemma 2.6. ([8]) Let $f(z)$ be a transcendental meromorphic function, and let $\beta>1$ be a given constant, for any given $\varepsilon>0$, there exist a set $E_1\subset(1,+\infty)$ that has finite logarithmic measure

and a constant $B > 0$ that depends only on β and (i, j) (i, j *are integers with* $0 \le i < j$) such that for all $|z| = r \notin [0, 1] \cup E_1$, we have

$$\left|\frac{f^{(j)}(z)}{f^{(i)}(z)}\right| \le B\left[\frac{T(\beta r, f)}{r}(\log^{\beta} r)\log T(\beta r, f)\right]^{j-i}.$$

3. Proofs of Theorems 1.8 - 1.10

Proof of Theorem 1.8. We divide the proof into two parts.

(i) Set $\sigma_{[p,q]}(A_0, \varphi) = \sigma_3$, first, we prove that every solution of (1.1) satisfies $\sigma_{[p+1,q]}(f, \varphi) \le \sigma_3$. It is easy to know that equation (1.1) has no polynomial solutions under the assumptions. If $f(z)$ is a transcendental solution of (1.1), by (1.1), we get

$$\left|\frac{f^{(k)}(z)}{f(z)}\right| \le |A_{k-1}|\left|\frac{f^{(k-1)}(z)}{f(z)}\right| + \cdots + |A_s|\left|\frac{f^{(s)}(z)}{f(z)}\right| + \cdots + |A_0|. \tag{3.1}$$

Since $\max\{\sigma_{[p,q]}(A_j, \varphi)|j = 0, 1, \cdots, k-1\} \le \sigma_3$, for any given $\varepsilon > 0$ and for sufficiently large r, we have

$$|A_j(z)| \le \exp_{p+1}\{(\sigma_3 + \varepsilon)\log_q \varphi(r)\} \quad (j = 0, 1, \cdots, k-1). \tag{3.2}$$

By Lemma 2.1, there exists a set $E_1 \subset (1, +\infty)$ having finite logarithmic measure such that for all z satisfying $|z| = r \notin [0, 1] \cup E_1$ and $|f(z)| = M(r, f)$, we have

$$\frac{f^{(j)}(z)}{f(z)} = \left(\frac{v_f(r)}{z}\right)^j (1 + o(1)) \quad (j = 1, \cdots, k-1). \tag{3.3}$$

By (3.1)-(3.3), for all z satisfying $|z| = r \notin [0, 1] \cup E_1$ and $|f(z)| = M(r, f)$, we get

$$\left(\frac{v_f(r)}{r}\right)^k (1 + o(1)) \le k\exp_{p+1}\{(\sigma_3 + \varepsilon)\log_q \varphi(r)\}\left(\frac{v_f(r)}{r}\right)^{k-1}(1 + o(1)), \tag{3.4}$$

by (3.4) and Lemma 2.2, there exists some α_1 $(1 < \alpha_1 < \alpha)$ and $r \ge r_0$, we have

$$v_f(r) \le k\alpha_1 r\exp_{p+1}\{(\sigma_3 + \varepsilon)\log_q \varphi(\alpha_1 r)\}. \tag{3.5}$$

By Lemma 2.3 and the Proposition 1.6, we have $\sigma_{[p+1,q]}(f, \varphi) \le \sigma_3$.

(ii) On the other hand, if $f \not\equiv 0$, (1.1) can be written

$$-A_0 = \frac{f^{(k)}(z)}{f(z)} + \cdots + A_j\frac{f^{(j)}(z)}{f(z)} + \cdots + A_1\frac{f'(z)}{f(z)}. \tag{3.6}$$

By (3.6), we get

$$m(r, A_0) \le \sum_{i=1}^{k-1} m(r, A_j) + \sum_{j=1}^{k} m\left(r, \frac{f^{(k)}}{f}\right) + \log k. \tag{3.7}$$

Since $\max\{\sigma_{[p,q]}(A_j,\varphi)|j=1,2,\cdots,k-1\}<\sigma_3$ and by Lemma 2.5, there exists a set $E_2\subset(1,+\infty)$ with infinite logarithmic measure such that for all z satisfying $|z|=r\in E_2$, we have

$$\lim_{r\to\infty}\frac{\log_p m(r,A_0)}{\log_q\varphi(r)}=\sigma_3,\quad \frac{m(r,A_j)}{m(r,A_0)}\longrightarrow 0\quad (r\in E_2, j=1,\cdots,k-1). \tag{3.8}$$

By the lemma of logarithmic derivative, we have

$$m\left(r,\frac{f^{(j)}}{f}\right)=O\{\log rT(r,f)\}\quad (r\notin E_1). \tag{3.9}$$

By (3.7)-(3.9), for all sufficiently large $r\in E_2\setminus E_1$, we have

$$\frac{1}{2}m(r,A_0)\le O\{\log rT(r,f)\}.$$

Hence by Proposition 1.6, we have $\sigma_{[p+1,q]}(f,\varphi)\ge\sigma_3$. Therefore, every solution $f(z)\not\equiv 0$ of (1.1) satisfies $\sigma_{[p+1,q]}(f,\varphi)=\sigma_{[p,q]}(A_0,\varphi)$.

Proof of Theorem 1.9. Proof. If $f(z)\not\equiv 0$, by (1.2), we get

$$\frac{1}{f}=\frac{1}{F}\left(\frac{f^{(k)}}{f}+A_{k-1}\frac{f^{(k-1)}}{f}+\cdots+A_0\right), \tag{3.10}$$

it is easy to see that if $f(z)$ has a zero at z_0 of order α ($\alpha>k$), and $A_0,\cdots,A_{k-1}$ are analytic at z_0, then F must have a zero at z_0 of order $\alpha-k$, hence

$$n\left(r,\frac{1}{f}\right)\le k\overline{n}\left(r,\frac{1}{f}\right)+n\left(r,\frac{1}{F}\right), \tag{3.11}$$

and

$$N\left(r,\frac{1}{f}\right)\le k\overline{N}\left(r,\frac{1}{f}\right)+N\left(r,\frac{1}{F}\right). \tag{3.12}$$

By the lemma of logarithmic derivative and (3.10), we have

$$m\left(r,\frac{1}{f}\right)\le m\left(r,\frac{1}{F}\right)+\sum_{j=0}^{k-1}m(r,A_j)+O\left(\log T(r,f)+\log r\right)\quad (r\notin E_1). \tag{3.13}$$

By (3.12), (3.13), we get

$$T(r,f)\le k\overline{N}\left(r,\frac{1}{f}\right)+T(r,F)+\sum_{j=0}^{k-1}T(r,A_j)+O\left\{\log\left(rT(r,f)\right)\right\}\ (r\notin E_1). \tag{3.14}$$

Since $\max\{\sigma_{[p,q]}(F,\varphi),\sigma_{[p,q]}(A_j,\varphi)|j=0,1,\cdots,k-1\}<\sigma_{[p,q]}(f,\varphi)$, by Lemma 2.5, there exists a set $E_3\subset(1,+\infty)$ having infinite logarithmic measure such that

$$\max\left\{\frac{T(r,F)}{T(r,f)},\ \frac{T(r,A_j)}{T(r,f)}\right\}\longrightarrow 0\quad (r\in E_3, j=0,\cdots,k-1). \tag{3.15}$$

Since for all sufficiently large r, we have

$$\log T(r,f)=o\{T(r,f)\}. \tag{3.16}$$

By (3.14)-(3.16), for all $|z|=r\in E_3\backslash E_1$, we have

$$(1-o(1))\,T(r,f)\le O\left\{\overline{N}\left(r,\frac{1}{f}\right)\right\}+O\{\log r\}. \tag{3.17}$$

By Definition 1.4 and Proposition 1.7 and (3.17), we get

$$\sigma_{[p,q]}(f,\varphi)\le\overline{\lambda}_{[p,q]}(f,\varphi). \tag{3.18}$$

Since $\sigma_{[p,q]}(f,\varphi)\ge\lambda_{[p,q]}(f,\varphi)\ge\overline{\lambda}_{[p,q]}(f,\varphi)$, and by (3.18), we have

$$\overline{\lambda}_{[p,q]}(f,\varphi)=\lambda_{[p,q]}(f,\varphi)=\sigma_{[p,q]}(f,\varphi).$$

Proof of Theorem 1.10. We assume that f is a solution of (1.2). By the elementary theory of differential equations, all the solutions of (1.2) are entire functions and have the form

$$f=f^*+C_1f_1+C_2f_2+\cdots+C_kf_k,$$

where $C_1,\cdots,C_k$ are complex constants, $\{f_1,\cdots,f_k\}$ is a solution base of (1.1), f^* is a solution of (1.2) and has the form

$$f^*=D_1f_1+D_2f_2+\cdots+D_kf_k, \tag{3.19}$$

where $D_1,\cdots,D_k$ are certain entire functions satisfying

$$D_j'=F\cdot G_j(f_1,\cdots,f_k)\cdot W(f_1,\cdots,f_k)^{-1}\quad (j=1,\cdots,k), \tag{3.20}$$

where $G_j(f_1,\cdots,f_k)$ are differential polynomials in $f_1,\cdots,f_k$ and their derivatives with constant coefficients, and $W(f_1,\cdots,f_k)$ is the Wronskian of $f_1,\cdots,f_k$. By Theorem 1.8, we have $\sigma_{[p+1,q]}(f_j,\varphi)=\sigma_{[p,q]}(A_0,\varphi)(j=1,2,\cdots,k)$, then by (3.19) and (3.20), we get

$$\sigma_{[p+1,q]}(f,\varphi)\le\max\{\sigma_{[p+1,q]}(f_j,\varphi),\sigma_{[p+1,q]}(F,\varphi)|j=1,\cdots,k\}\le\sigma_{[p,q]}(A_0,\varphi).$$

We affirm that (1.2) can only possess at most one exceptional solution f_0 satisfying $\sigma_{[p+1,q]}(f_0,\varphi)<\sigma_{[p,q]}(A_0,\varphi)$. In fact, if f_* is another solution satisfying $\sigma_{[p+1,q]}(f_*,\varphi)<\sigma_{[p,q]}(A_0,\varphi)$, then $\sigma_{[p+1,q]}(f_0-f_*,\varphi)<\sigma_{[p,q]}(A_0,\varphi)$. But f_0-f_* is a solution of (1.1), this contradicts Theorem 1.8. Then $\sigma_{[p+1,q]}(f,\varphi)=\sigma_{[p,q]}(A_0,\varphi)$ holds for all solutions of (1.2) with at most one exceptional solution f_0 satisfying $\sigma_{[p+1,q]}(f_0,\varphi)<\sigma_{[p,q]}(A_0,\varphi)$. By Theorem 1.9, we get that

$$\overline{\lambda}_{[p+1,q]}(f,\varphi)=\lambda_{[p+1,q]}(f,\varphi)=\sigma_{[p+1,q]}(f,\varphi)$$

holds for all solutions satisfying $\sigma_{[p+1,q]}(f,\varphi)=\sigma_{[p,q]}(A_0,\varphi)$ with at most one exceptional solution f_0 satisfying $\sigma_{[p+1,q]}(f_0,\varphi)<\sigma_{[p,q]}(A_0,\varphi)$.

References

[1] S. Bank and I. Laine, On the oscillation theory of $f'' + Af = 0$ where A is entire, Trans. Amer. Math. Soc. 273 (1982), 352-363.

[2] B. Belaïdi, On the iterated order and the fixed points of entire solutions of some complex linear differential equations, Electron. J. Qual. Theory Differ. Equ. No. 9, (2006), 1-11.

[3] B. Belaïdi, On the $[p, q]$-order of meromorphic solutions of linear differential equations. Acta Univ. M. Belii Ser. Math. 2015, 37-49

[4] L. G. Bernal, On growth k-order of solutions of a complex homogeneous linear differential equations, Proc. Amer. Math. Soc. 101 (1987), 317-322.

[5] Z. X. Chen and S. A. Gao, The complex oscillation theory of certain non-homogeneous linear differential equations with transcendental entire coefficients, J. Math. Anal. Appl. 179 (1993), 403-416.

[6] G. Frank and S. Hellerstein, On the meromorphic solutions of non-homogeneous linear differential equations with polynomial coefficients, Proc. London Math. Soc. 53 (3) (1986), 407-428.

[7] S. A. Gao, Z. X. Chen and T. W. Chen, The Complex Oscillation Theory of Linear Differential Equations, HuaZhong Univ. Sci. Tech. Press 1998 (in Chinese).

[8] G. Gundersen, Estimates for the logarithmic derivate of a meromorphic function, plus similar estimates, J. London Math. Soc. 37 (2) (1988), 88-104.

[9] W. Hayman, Meromorphic Functions, Clarendon Press, Oxford, 1964.

[10] W. Hayman, The local growth of power series: A survey of the Wiman-Valiron method, Canad. Math. Bull. 17 (1974), 317-358 .

[11] H. Hu and X. M. Zheng, Growth of solutions of linear differential equations with meromorphic coefficients of $[p, q]$-order, Math. Commun. 19 (1) (2014), 29-42.

[12] O. P. Juneja, G. P. Kapoor and S. K. Bajpai, On the (p, q)-order and lower (p,q)-order of an entire function, J. Reine Angew. Math. 282 (1976), 53-67.

[13] L. Kinnunen, Linear differential equations with solutions of finite iterated order, Southeast Asian Bull. Math. (4) 22 (1998), 385-405.

[14] I. Laine, Nevanlinna Theory and Complex Differential Equations, Walter de Gruyter, Berlin, 1993.

[15] J. Liu, J. Tu and L. Z. Shi, Linear differential equantials with entire cofficients of $[p, q]$-order in the complex plane, J. Math. Anal. Appl. 372 (2010), 55-67.

[16] L. M. Li and T. B. Cao, Solutions for linear differential equations with meromorphic coefficients of [p,q]-order in the plane, Electron. J. Diff. Equ. No. 195, (2012), 1-15.

[17] X. Shen, J. Tu and H. Y. Xu, Complex oscillation of a second-order linear differential equation with entire coefficients of $[p, q] - \varphi(r)$ order, Advances in Difference Equations No. 200, (2014), 1-14.

[18] J. Tu, C. Y. Liu and H, Y. Xu, Meromorphic Functions of Relative $[p, q]$ Order to $\varphi(r)$, J. Jiangxi Norm. Univ., Nat. Sci. 36 (1), (2012), 47-50.

[19] J. Tu, J. S. Wei, H. Y. Xu, The order and type of meromorphic functions and analytic functions of $[p, q] - \varphi(r)$ order in the unit disc, J. Jiangxi Norm. Univ., Nat. Sci. 39 (2), (2015), 207-211.

Some k-fractional integrals inequalities through generalized $\lambda_{\phi m}$-MT-preinvexity

Chunyan Luo[1] Tingsong Du[1,2*] Muhammad Adil Khan[3]
Artion Kashuri[4] Yanjun Shen[5]

[1]Department of Mathematics, College of Science, China Three Gorges University, Yichang 443002, China

[2]Three Gorges Mathematical Research Center, China Three Gorges University, Yichang 443002, China

E-mail: luochunyanctgu@gmail.com tingsongdu@ctgu.edu.cn

[3]Department of Mathematics, University of Peshawar, Pakistan

E-mail: adilswati@gmail.com

[4]Department of Mathematics, Faculty of Technical Science, University "Ismail Qemali", Vlora, Albania E-mail: artionkashuri@gmail.com

[5]Hubei Provincial Collaborative Innovation Center for New Energy Microgrid, China Three Gorges University, Yichang 443002, China E-mail: shenyj@ctgu.edu.cn

* Corresponding author Tingsong Du

Abstract

The authors introduce the concept of the generalized $\lambda_{\phi m}$-MT-preinvex functions and discover a new k-fractional integral identity concerning twice differentiable preinvex mappings defined on (ϕ, m)-invex set. By using this identity, we establish the right-sided new Hermite-Hadamard type inequalities for the generalized $\lambda_{\phi m}$-MT-preinvex mappings via k-fractional integrals. The new k-fractional integral inequalities are also applied to some special means.

2010 Mathematics Subject Classification: Primary 26A33; Secondary 26D07, 26D20, 41A55.

Key words and phrases: Hermite-Hadamard's inequality; $\lambda_{\phi m}$-MT-preinvex functions; k-Riemann-Liouville fractional integrals.

1 Introduction

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex mapping on the interval I of real numbers and $u, v \in I$ with $u < v$. Then the following well-know Hermite-Hadamard inequality holds

$$f\left(\frac{u+v}{2}\right) \le \frac{1}{v-u}\int_u^v f(x)\mathrm{d}x \le \frac{f(u)+f(v)}{2}. \tag{1.1}$$

This inequality is one of the famous results for convex functions.

Many researchers generalized and extended the inequalities (1.1) involving a variety of convex functions one can see [8, 9, 12, 15, 20–22, 40, 41] and the references mentioned in these papers.

In 2013, Sarikaya et al. [32] considered the following Hermite-Hadamard type inequalities via Riemann-Liouville fractional integrals.

Theorem 1.1 *Let $f : [u, v] \to \mathbb{R}$ be a positive function along with $0 \leq u < v$ and let $f \in L^1[u, v]$. Suppose f is a convex function on $[u, v]$, then the subsequent inequalities for fractional integrals hold:*

$$f\Big(\frac{u+v}{2}\Big) \leq \frac{\Gamma(\alpha+1)}{2(v-u)^\alpha}\big[J^\alpha_{u^+}f(v) + J^\alpha_{v^-}f(u)\big] \leq \frac{f(u)+f(v)}{2}, \tag{1.2}$$

where the symbols $J^\alpha_{u^+}f$ and $J^\alpha_{v^-}f$ denote respectively the left-sided and right-sided Riemann-Liouville fractional integrals of the order $\alpha \in \mathbb{R}^+$ defined by

$$J^\alpha_{u^+}f(x) = \frac{1}{\Gamma(\alpha)}\int_u^x (x-t)^{\alpha-1}f(t)\mathrm{d}t, \quad u < x$$

and

$$J^\alpha_{v^-}f(x) = \frac{1}{\Gamma(\alpha)}\int_x^v (t-x)^{\alpha-1}f(t)\mathrm{d}t, \quad x < v.$$

Here, $\Gamma(\alpha)$ is the gamma function and its definition is $\Gamma(\alpha) = \int_0^\infty e^{-\mu}\mu^{\alpha-1}\mathrm{d}\mu$.

Due to the wide applications of Riemann-Liouville fractional Hermite-Hadamard type inequalities in mathematical analysis, many researchers extended Hermite-Hadamard inequality for different classes of convex functions. For example, see for convex mappings [7, 10, 16, 17, 29], for m-convex mappings [37] and (s, m)-convex mappings [3], for h-preinvex mappings[13], for harmonically convex mappings [18], for preinvex mappings [25, 31] and the references mentioned in these papers.

Also in [4], Anastassiou presented a complete theory with respect to fractional differentiation inequalities.

In 2012, Mubeen and Habibullah [24] introduced a new fractional integral that generalizes the Riemann-Liouville fractional integrals.

Definition 1.1 *([24]) Let $f \in L^1[a, b]$, then k-Riemann-Liouville fractional integrals ${}_kJ^\mu_{a^+}f(x)$ and ${}_kJ^\mu_{b^-}f(x)$ of order $\mu > 0$ are defined by*

$${}_kJ^\mu_{a^+}f(x) = \frac{1}{k\Gamma_k(\mu)}\int_a^x (x-t)^{\frac{\mu}{k}-1}f(t)\mathrm{d}t, \quad (0 \leq a < x < b)$$

and

$${}_kJ^\mu_{b^-}f(x) = \frac{1}{k\Gamma_k(\mu)}\int_x^b (t-x)^{\frac{\mu}{k}-1}f(t)\mathrm{d}t, \quad (0 \leq a < x < b),$$

respectively, where $k > 0$ and $\Gamma_k(\mu)$ is the k-gamma function given as $\Gamma_k(\mu) = \int_0^\infty t^{\mu-1}e^{-\frac{t^k}{k}}\mathrm{d}t$. Note that $\Gamma_k(\mu+k) = \mu\Gamma_k(\mu)$ and ${}_kJ^0_{a^+}f(x) = {}_kJ^0_{b^-}f(x) = f(x)$.

The notion of k-Riemann-Liouville fractional integral is an significant extension of Riemann-Liouville fractional integrals. It is stressed that for $k \neq 1$ the properties of k-Riemann-Liouville fractional integrals are quite dissimilar from those of general Riemann-Liouville fractional integrals. For this, the k-Riemann-Liouville fractional integrals have aroused the interest of many researchers. Properties and integral inequalities concerning this operator can refer to [1, 2, 6, 33, 34, 38] and the references mentioned in these papers.

Let us evoke some basic definitions as follows.

Definition 1.2 *([5]) A set $K \subseteq \mathbb{R}^n$ is said to be invex set respecting the mapping $\eta : K \times K \to \mathbb{R}^n$ if $x + t\eta(y, x) \in K$ for any $x, y \in K$ and $t \in [0, 1]$.*

Definition 1.3 *([39]) A function f defined on the invex set $K \subseteq \mathbb{R}^n$ is said to be preinvex with respect to η, if*

$$f\big(x + t\eta(y, x)\big) \leq (1 - t)f(x) + tf(y), \quad \forall\ x, y \in K, t \in [0, 1].$$

Definition 1.4 *([27]) Let $x \in K \subseteq \mathbb{R}^n$ and let $\phi : K \to \mathbb{R}$ be a continuous function. Then the set K is said to be ϕ-convex at x respecting ϕ, if*

$$x + \lambda e^{i\phi}(y - x) \in K, \quad \forall\ x, y \in K, \lambda \in [0, 1].$$

Definition 1.5 *([26]) A set $K \subseteq \mathbb{R}^n$ is called ϕ-invex at x with respect to $\phi(\cdot)$, if there a continuous function $\phi(\cdot) : K \to \mathbb{R}$ and a bifunction $\eta(\cdot, \cdot) : K \times K \to \mathbb{R}^n$, such that*

$$x + te^{i\phi}\eta(y, x) \in K, \quad \forall\ x, y \in K, t \in [0, 1].$$

Definition 1.6 *([11]) A set $K \subseteq \mathbb{R}^n$ is said to be m-invex with respect to the mapping $\eta : K \times K \times (0, 1] \to \mathbb{R}^n$ for some fixed $m \in (0, 1]$, if $mx + t\eta(y, x, m) \in K$ holds for each $x, y \in K$ and any $t \in [0, 1]$.*

Definition 1.7 *([27]) The function f on the ϕ-convex set K is said to be ϕ-convex with respect to ϕ, if*

$$f\big(x + \lambda e^{i\phi}(y - x)\big) \leq (1 - \lambda)f(x) + \lambda f(y), \quad \forall\ x, y \in K, \lambda \in [0, 1]. \tag{1.3}$$

Definition 1.8 *([42]) The function f defined on the ϕ-invex set $K \subseteq \mathbb{R}^n$ is said to be ϕ-MT-preinvex, if it is nonnegative and for $\forall\ x, y \in K$ and $t \in (0, 1)$ satisfies the following inequality*

$$f\big(x + te^{i\phi}\eta(y, x)\big) \leq \frac{\sqrt{1 - t}}{2\sqrt{t}} f(x) + \frac{\sqrt{t}}{2\sqrt{1 - t}} f(y). \tag{1.4}$$

Definition 1.9 *([28]) A function: $I \subseteq \mathbb{R} \to \mathbb{R}$ is said to be m-MT-convex, if f is positive and for $\forall\ x, y{\in}I$, and $t \in (0, 1)$, with $m \in [0, 1]$, satisfies the following inequality*

$$f\big(tx + m(1 - t)y\big) \leq \frac{\sqrt{t}}{2\sqrt{1 - t}} f(x) + \frac{m\sqrt{1 - t}}{2\sqrt{t}} f(y). \tag{1.5}$$

Definition 1.10 *([14]) A function f: $I \subseteq \mathbb{R} \to \mathbb{R}$ is said to be λ-MT-convex function, if f is positve and $\forall\ x, y{\in}I$, $\lambda \in (0, \frac{1}{2}]$ and $t \in (0, 1)$, satisfies the following inequality*

$$f\big(tx + (1 - t)y\big) \leq \frac{\sqrt{t}}{2\sqrt{1 - t}} f(x) + \frac{(1 - \lambda)\sqrt{1 - t}}{2\lambda\sqrt{t}} f(y). \tag{1.6}$$

Clearly, when choosing $m = 1$ and $\lambda = \frac{1}{2}$ in Definition 1.9 and Definition 1.10, respectively, the function f reduces to MT-convex function in [35]. For some significant integral inequalities in association with MT-convex functions, one can see [19, 23, 30, 36] and the references therein.

The main purpose of this paper is to introduce the class of generalized $\lambda_{\phi m}$-MT-preinvex functions on (ϕ, m)-invex and to prove a k-fractional integral identity. By using this identity, we establish the right-sided new Hadamard-type inequalities for the generalized $\lambda_{\phi m}$-MT-preinvex functions via k-Riemann-Liouville fractional integrals. These inequalities can be viewed as generalization of recent results that appeared in Refs. [30] and [42].

2 New definitions and a lemma

As one can see, the definitions of the ϕ-invex and m-invex have similar configurations. This observation leads us to generalized these concepts. Firstly, the so-called '(ϕ, m)-invex' may be introduced as follows.

Definition 2.1 *A set $K_{\phi m} \subseteq \mathbb{R}^n$ is said to be (ϕ, m)-invex with respect to a continuous function $\phi(\cdot) : K_{\phi m} \to \mathbb{R}$ and the mapping $\eta : K_{\phi m} \times K_{\phi m} \times (0,1] \to \mathbb{R}^n$, for some fixed $m \in (0,1]$, if $mx + te^{i\phi}\eta(y,x,m) \in K_{\phi m}$ holds for any $x, y \in K_{\phi m}$ and $t \in (0,1)$.*

Let us note that:
- if $\phi = 0$, then we get the definition of an m-invex set,
- if the mapping $\eta(y,x,m)$ with $m = 1$ reduces to $\eta(y,x)$, then we obtain the definition of a ϕ-invex set,
- if $\phi = 0$ and $\eta(y,x,m) = y - mx$ with $m = 1$, then we obtain the definition of a convex set.

Now we define the concept of generalized $\lambda_{\phi m}$-MT-preinvex functions.

Definition 2.2 *Let $K_{\phi m} \subseteq \mathbb{R}$ is a (ϕ, m)-invex set with respect to η and ϕ. A function $f : K_{\phi m} \to \mathbb{R}_0$ is said to be generalized $\lambda_{\phi m}$-MT-preinvex, according to η and ϕ, and $\forall\, x, y \in K_{\phi m}$, $t \in (0,1)$ and $\lambda \in (0, \frac{1}{2}]$, along with some fixed $m \in (0,1]$ satisfies the coming inequality*

$$f\big(mx + te^{i\phi}\eta(y,x,m)\big) \leq \frac{m(1-\lambda)\sqrt{1-t}}{2\lambda\sqrt{t}} f(x) + \frac{\sqrt{t}}{2\sqrt{1-t}} f(y). \tag{2.1}$$

Let us note that:
- if the mapping $\eta(y,x,m)$ with $m = 1$ degenerates into $\eta(y,x)$, then we obtain the definition of λ_ϕ-MT-preinvex function,
- if the mapping $\eta(y,x,m)$ with $m = 1$ degenerates into $\eta(y,x)$ and $\lambda = \frac{1}{2}$, then we obtain the definition of ϕ-MT-preinvex function,
- if $\phi = 0$, the mapping $\eta(y,x,m) = y - mx$, and $\lambda = \frac{1}{2}$, then we obtain the definition of m-MT-convex function,
- if $\phi = 0$ and the mapping $\eta(y,x,m) = y - mx$ with $m = 1$, then we obtain the definition of λ-MT-convex function,
- if $\phi = 0$, the mapping $\eta(y,x,m) = y - mx$ with $m = 1$, and $\lambda = \frac{1}{2}$, then we obtain the definition of MT-convex function.

Before presenting our main results, we prove the following lemma.

Lemma 2.1 *Let $K_{\phi m} \subseteq \mathbb{R}$ be a (ϕ, m)-invex subset respecting $\phi(\cdot)$ and $\eta : K_{\phi m} \times K_{\phi m} \times (0,1] \to \mathbb{R}$, $a, b \in K_{\phi m}$ with $\eta(b,a,m) > 0$ and some fixed $m \in (0,1]$. Suppose that $f : K_{\phi m} \to \mathbb{R}$ is a twice differentiable mapping such that $f'' \in L[ma, ma + e^{i\phi}\eta(b,a,m)]$, we have the following identity via k-fractional integral with $k, \alpha > 0$ holds:*

$$R_f(\alpha, k; \phi, \eta, m, a, b) = \frac{\big(e^{i\phi}\eta(b,a,m)\big)^2}{2} \int_0^1 \frac{1 - t^{\frac{\alpha}{k}+1} - (1-t)^{\frac{\alpha}{k}+1}}{\frac{\alpha}{k}+1} f''\big(ma + te^{i\phi}\eta(b,a,m)\big)\mathrm{d}t, \tag{2.2}$$

where

$$\begin{aligned} R_f(\alpha, k; \phi, \eta, m, a, b) :=& \frac{f(ma) + f\big(ma + e^{i\phi}\eta(b,a,m)\big)}{2} - \frac{\Gamma_k(\alpha+k)}{2k\big(e^{i\phi}\eta(b,a,m)\big)^{\frac{\alpha}{k}}} \\ &\times \Big[{}_kJ^\alpha_{ma^+} f\big(ma + e^{i\phi}\eta(b,a,m)\big) + {}_kJ^\alpha_{(ma+e^{i\phi}\eta(b,a,m))^-} f(ma)\Big]. \end{aligned}$$

Proof. Set

$$I^* = \frac{\big(e^{i\phi}\eta(b,a,m)\big)^2}{2}\int_0^1 \frac{1-t^{\frac{\alpha}{k}+1}-(1-t)^{\frac{\alpha}{k}+1}}{\frac{\alpha}{k}+1} f''\big(ma+te^{i\phi}\eta(b,a,m)\big)\mathrm{d}t.$$

Since $a,b\in K_{\phi m}$ and $K_{\phi m}$ is a (ϕ,m)-invex subset respecting ϕ and η, for $\forall\, t\in(0,1)$, we have $ma+te^{i\phi}\eta(b,a,m)\in K_{\phi m}$. Integrating by part gives, we have

$$\begin{aligned}
I^* =&\frac{\big(e^{i\phi}\eta(b,a,m)\big)^2}{2}\bigg[\frac{1-t^{\frac{\alpha}{k}+1}-(1-t)^{\frac{\alpha}{k}+1}}{(\frac{\alpha}{k}+1)e^{i\phi}\eta(b,a,m)} f'\big(ma+te^{i\phi}\eta(b,a,m)\big)\Big|_0^1 \\
&-\int_0^1 \frac{-(\frac{\alpha}{k}+1)t^{\frac{\alpha}{k}}+(\frac{\alpha}{k}+1)(1-t)^{\frac{\alpha}{k}}}{(\frac{\alpha}{k}+1)e^{i\phi}\eta(b,a,m)} f'\big(ma+te^{i\phi}\eta(b,a,m)\big)\mathrm{d}t\bigg] \\
=&\frac{\big(e^{i\phi}\eta(b,a,m)\big)^2}{2}\bigg[\frac{t^{\frac{\alpha}{k}}-(1-t)^{\frac{\alpha}{k}}}{\big(e^{i\phi}\eta(b,a,m)\big)^2} f\big(ma+te^{i\phi}\eta(b,a,m)\big)\Big|_0^1 \\
&-\int_0^1 \frac{\frac{\alpha}{k}t^{\frac{\alpha}{k}-1}+\frac{\alpha}{k}(1-t)^{\frac{\alpha}{k}-1}}{\big(e^{i\phi}\eta(b,a,m)\big)^2} f\big(ma+te^{i\phi}\eta(b,a,m)\big)\mathrm{d}t\bigg] \\
=&\frac{f(ma)+f\big(ma+te^{i\phi}\eta(b,a,m)\big)}{2}-\frac{\alpha}{2k}\bigg[\int_0^1\Big(t^{\frac{\alpha}{k}-1}+(1-t)^{\frac{\alpha}{k}-1}\Big)f\big(ma+te^{i\phi}\eta(b,a,m)\big)\mathrm{d}t\bigg].
\end{aligned}$$

Using the reduction formula $\Gamma_k(\alpha+k)=\alpha\Gamma_k(\alpha)$ $(\alpha>0)$, we have

$$\frac{\alpha}{2k}\int_0^1 t^{\frac{\alpha}{k}-1} f\big(ma+te^{i\phi}\eta(b,a,m)\big)\mathrm{d}t = \frac{\Gamma_k(\alpha+k)}{2k\big(e^{i\phi}\eta(b,a,m)\big)^{\frac{\alpha}{k}}}\, {}_kJ^{\alpha}_{(ma+e^{i\phi}\eta(b,a,m))^-} f(ma)$$

and

$$\frac{\alpha}{2k}\int_0^1 (1-t)^{\frac{\alpha}{k}-1} f\big(ma+te^{i\phi}\eta(b,a,m)\big)\mathrm{d}t = \frac{\Gamma_k(\alpha+k)}{2k\big(e^{i\phi}\eta(b,a,m)\big)^{\frac{\alpha}{k}}}\, {}_kJ^{\alpha}_{ma^+} f\big(ma+e^{i\phi}\eta(b,a,m)\big).$$

Thus, we obtain conclusion (2.2).

Remark 2.1 *If we put $k=1$ in Lemma 2.1, then we have:*
(a) for the mapping $\eta(b,a,m)$ with $m=1$ reduces to $\eta(b,a)$, we obtain Lemma 3.1 in [14],
(b) for $\alpha=1=m$ with the mapping $\eta(b,a,m)$ reduces to $\eta(b,a)$, we obtain Lemma 2.3 in [42],
(c) for $\phi=0$, $\alpha=1=m$ with the mapping $\eta(b,a,m)=b-ma$, we obtain Lemma 1.3 in [37].

3 Main results

Using Lemma 2.1, we now state the following theorem.

Theorem 3.1 *Let $A_{\phi m}\subseteq\mathbb{R}_0$ be an open (ϕ,m)-invex subset respecting $\phi(\cdot)$ and $\eta: A_{\phi m}\times A_{\phi m}\times(0,1]\to\mathbb{R}_0$, $a,b\in A_{\phi m}$ with $\eta(b,a,m)>0$, $\lambda\in(0,\frac{1}{2}]$ and some fixed $m\in(0,1]$. If $f: A_{\phi m}\to\mathbb{R}$ is a twice differentiable mapping such that $f''\in L[ma,ma+e^{i\phi}\eta(b,a,m)]$ and $|f''|^q$ for $q\geq 1$ is generalized $\lambda_{\phi m}$-MT-preinvex on $A_{\phi m}$ and $x\in[ma,ma+e^{i\phi}\eta(b,a,m)]$, then we have the following inequality for k-fractional integrals with $k,\alpha>0$*

$$\begin{aligned}
&\Big|R_f(\alpha,k;\phi,\eta,m,a,b)\Big| \\
\leq&\frac{k\big(e^{i\phi}\eta(b,a,m)\big)^2}{2(\alpha+k)}\left[\frac{\pi}{4}-\frac{\sqrt{\pi}\Gamma\big(q(\frac{\alpha}{k}+1)+\frac{1}{2}\big)}{2\Gamma\big(q(\frac{\alpha}{k}+1)+1\big)}\right]^{\frac{1}{q}}\left\{\frac{m(1-\lambda)}{\lambda}\big|f''(a)|^q+\big|f''(b)\big|^q\right\}^{\frac{1}{q}}.
\end{aligned} \tag{3.1}$$

Proof. Using Lemma 2.1 and the power-mean integral inequality, we obtain

$$\begin{aligned}
&\Big|R_f(\alpha,k;\phi,\eta,m,a,b)\Big|\\
\le&\frac{\big(e^{i\phi}\eta(b,a,m)\big)^2}{2}\int_0^1\left|\frac{1-t^{\frac{\alpha}{k}+1}-(1-t)^{\frac{\alpha}{k}+1}}{\frac{\alpha}{k}+1}\right|\Big|f''\big(ma+te^{i\phi}\eta(b,a,m)\big)\Big|\mathrm{d}t\\
\le&\frac{\big(e^{i\phi}\eta(b,a,m)\big)^2}{2(\frac{\alpha}{k}+1)}\bigg(\int_0^1 1\mathrm{d}t\bigg)^{1-\frac{1}{q}}\\
&\times\bigg\{\int_0^1\Big(1-t^{\frac{\alpha}{k}+1}-(1-t)^{\frac{\alpha}{k}+1}\Big)^q\Big|f''\big(ma+te^{i\phi}\eta(b,a,m)\big)\Big|^q\mathrm{d}t\bigg\}^{\frac{1}{q}}\\
\le&\frac{k\big(e^{i\phi}\eta(b,a,m)\big)^2}{2(\alpha+k)}\bigg\{\int_0^1\Big(1-t^{q(\frac{\alpha}{k}+1)}-(1-t)^{q(\frac{\alpha}{k}+1)}\Big)\Big|f''\big(ma+te^{i\phi}\eta(b,a,m)\big)\Big|^q\mathrm{d}t\bigg\}^{\frac{1}{q}}.
\end{aligned}$$

To prove the third inequality above, we use the following inequality

$$\Big(1-(1-t)^{\frac{\alpha}{k}+1}-t^{\frac{\alpha}{k}+1}\Big)^q\le 1-(1-t)^{q(\frac{\alpha}{k}+1)}-t^{q(\frac{\alpha}{k}+1)},\tag{3.2}$$

for any $t\in(0,1)$, which follows from

$$(A-B)^q\le A^q-B^q,$$

for any $A>B\ge 0$ and $q\ge 1$.
Since $|f''|^q$ is generalized $\lambda_{\phi m}$-MT-preinvex on $A_{\phi m}$, it follows that

$$\begin{aligned}
&\int_0^1\Big(1-t^{q(\frac{\alpha}{k}+1)}-(1-t)^{q(\frac{\alpha}{k}+1)}\Big)\Big|f''\big(ma+te^{i\phi}\eta(b,a,m)\big)\Big|^q\mathrm{d}t\\
\le&\int_0^1\Big(1-t^{q(\frac{\alpha}{k}+1)}-(1-t)^{q(\frac{\alpha}{k}+1)}\Big)\bigg\{\frac{m(1-\lambda)\sqrt{1-t}}{2\lambda\sqrt{t}}|f''(a)|^q+\frac{\sqrt{t}}{2\sqrt{1-t}}|f''(b)|^q\bigg\}\mathrm{d}t\\
=&\frac{m(1-\lambda)}{\lambda}\bigg\{\frac{\pi}{4}-\frac{1}{2}\beta\Big(q\Big(\frac{\alpha}{k}+1\Big)+\frac{1}{2},\frac{3}{2}\Big)-\frac{1}{2}\beta\Big(\frac{1}{2},q\Big(\frac{\alpha}{k}+1\Big)+\frac{3}{2}\Big)\bigg\}\big|f''(a)\big|^q\\
&+\bigg\{\frac{\pi}{4}-\frac{1}{2}\beta\Big(q\Big(\frac{\alpha}{k}+1\Big)+\frac{3}{2},\frac{1}{2}\Big)-\frac{1}{2}\beta\Big(\frac{3}{2},q\Big(\frac{\alpha}{k}+1\Big)+\frac{1}{2}\Big)\bigg\}\big|f''(b)\big|^q\\
=&\left[\frac{\pi}{4}-\frac{\sqrt{\pi}\Gamma\big(q(\frac{\alpha}{k}+1)+\frac{1}{2}\big)}{2\Gamma\big(q(\frac{\alpha}{k}+1)+1\big)}\right]\bigg\{\frac{m(1-\lambda)}{\lambda}\big|f''(a)|^q+\big|f''(b)\big|^q\bigg\}.
\end{aligned}$$

Here, we utilize the following fact that

$$\int_0^1\frac{\sqrt{1-t}}{2\sqrt{t}}\mathrm{d}t=\int_0^1\frac{\sqrt{t}}{2\sqrt{1-t}}\mathrm{d}t=\frac{1}{2}\beta\Big(\frac{1}{2},\frac{3}{2}\Big)=\frac{\pi}{4},$$

$$\int_0^1 t^{q(\frac{\alpha}{k}+1)+\frac{1}{2}}(1-t)^{-\frac{1}{2}}\mathrm{d}t=\beta\Big(q\Big(\frac{\alpha}{k}+1\Big)+\frac{3}{2},\frac{1}{2}\Big)$$

and

$$\int_0^1(1-t)^{q(\frac{\alpha}{k}+1)-\frac{1}{2}}t^{\frac{1}{2}}\mathrm{d}t=\beta\Big(\frac{3}{2},q\Big(\frac{\alpha}{k}+1\Big)+\frac{1}{2}\Big)$$

where the beta function

$$\beta(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}\mathrm{d}t=\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)},\quad \forall\, x,y>0.$$

Hence, the proof is completed.

We now discuss some special cases of Theorem 3.1.

Corollary 3.1 *In Theorem 3.1, if $q=1$, then we have*

$$\Big|R_f(\alpha,k;\phi,\eta,m,a,b)\Big|\leq\frac{k\big(e^{i\phi}\eta(b,a,m)\big)^2}{2(\alpha+k)}\left[\frac{\pi}{4}-\frac{\sqrt{\pi}\Gamma\big(\frac{\alpha}{k}+\frac{3}{2}\big)}{2\Gamma\big(\frac{\alpha}{k}+2\big)}\right]\left\{\frac{m(1-\lambda)}{\lambda}\big|f''(a)\big|+\big|f''(b)\big|\right\}.$$

Corollary 3.2 *In Theorem 3.1, if we take $\lambda=\frac{1}{2}$, $q=1$ and the mapping $\eta(b,a,m)$ with $m=1$ degenerates into $\eta(b,a)$, then we have the following inequality for ϕ-MT-preinvex functions*

$$\begin{aligned}&\left|\frac{f(a)+f\big(a+e^{i\phi}\eta(b,a)\big)}{2}-\frac{\Gamma_k(\alpha+k)}{2k\big(e^{i\phi}\eta(b,a)\big)^{\frac{\alpha}{k}}}\left[{}_kJ^{\alpha}_{a^+}f\big(a+e^{i\phi}\eta(b,a)\big)+{}_kJ^{\alpha}_{(a+e^{i\phi}\eta(b,a))^-}f(a)\right]\right|\\&\leq\frac{k\big(e^{i\phi}\eta(b,a)\big)^2}{2(\alpha+k)}\left[\frac{\pi}{4}-\frac{\sqrt{\pi}\Gamma\big(\frac{\alpha}{k}+\frac{3}{2}\big)}{2\Gamma\big(\frac{\alpha}{k}+2\big)}\right]\left\{\big|f''(a)\big|+\big|f''(b)\big|\right\}.\end{aligned}$$

Remark 3.1 *In Corollary 3.2, if we put $\phi=0$ and $\eta(b,a)=b-a$, then we have the succeeding inequality for MT-preinvex functions*

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma_k(\alpha+k)}{2k(b-a)^{\frac{\alpha}{k}}}\left[{}_kJ^{\alpha}_{a^+}f(b)+{}_kJ^{\alpha}_{b^-}f(a)\right]\right|\\&\leq\frac{k(b-a)^2}{2(\alpha+k)}\left[\frac{\pi}{4}-\frac{\sqrt{\pi}\Gamma\big(\frac{\alpha}{k}+\frac{3}{2}\big)}{2\Gamma\big(\frac{\alpha}{k}+2\big)}\right]\left\{\big|f''(a)\big|+\big|f''(b)\big|\right\}.\end{aligned}$$

Especially if we take $k=1$ and $\alpha=1$, we have

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right|\leq\frac{\pi(b-a)^2}{64}\left\{\big|f''(a)\big|+\big|f''(b)\big|\right\}.$$

Corollary 3.3 *In Theorem 3.1, if $|f''(x)|\leq M$, $\lambda=\frac{1}{2}$ and $\eta(b,a,m)$ with $m=1$ degenerates into $\eta(b,a)$, then we have the forthcoming inequality for ϕ-MT-preinvex functions*

$$\begin{aligned}&\left|\frac{f(a)+f\big(a+e^{i\phi}\eta(b,a)\big)}{2}-\frac{\Gamma_k(\alpha+k)}{2k\big(e^{i\phi}\eta(b,a)\big)^{\frac{\alpha}{k}}}\left[{}_kJ^{\alpha}_{a^+}f\big(a+e^{i\phi}\eta(b,a)\big)+{}_kJ^{\alpha}_{(a+e^{i\phi}\eta(b,a))^-}f(a)\right]\right|\\&\leq\frac{kM\big(e^{i\phi}\eta(b,a)\big)^2}{2(\alpha+k)}\left[\frac{\pi}{2}-\frac{\sqrt{\pi}\Gamma\big(q(\frac{\alpha}{k}+1)+\frac{1}{2}\big)}{\Gamma\big(q(\frac{\alpha}{k}+1)+1\big)}\right]^{\frac{1}{q}}.\end{aligned}$$

Especially if we take $\alpha=1$, $q=1$ and $k=1$, we get

$$\Big|\frac{f(a)+f(a+e^{i\phi}\eta(b,a))}{2}-\frac{1}{e^{i\phi}\eta(b,a)}\int_a^{a+e^{i\phi}\eta(b,a)}f(x)\mathrm{d}x\Big|\leq\frac{M\pi\big(e^{i\phi}\eta(b,a)\big)^2}{32},\tag{3.3}$$

which is the result given in [42],Theorem 2.5. Obviously, if we choose $\phi=0$ and $\eta(b,a)=b-a$ in (3.3), then we obtain the result given in [30],Theorem 2.1.

Now, we are ready to prove our second theorem.

Theorem 3.2 *Suppose that all the assumptions of Theorem 3.1 are satisfied, then we have the following inequality*

$$\begin{aligned}&\Big|R_f(\alpha,k;\phi,\eta,m,a,b)\Big|\\&\le\frac{k\big(e^{i\phi}\eta(b,a,m)\big)^2}{2(\alpha+k)}\Big(\frac{\alpha}{\alpha+2k}\Big)^{1-\frac{1}{q}}\left[\frac{\pi}{4}-\frac{\sqrt{\pi}\Gamma(\frac{\alpha}{k}+\frac{3}{2})}{2\Gamma\big(\frac{\alpha}{k}+2\big)}\right]^{\frac{1}{q}}\left[\frac{m(1-\lambda)}{\lambda}|f''(a)|^q+\big|f''(b)\big|^q\right]^{\frac{1}{q}}.\end{aligned}\tag{3.4}$$

Proof. Using Lemma 2.1 and the Hölder's integral inequality for $q\ge 1$, we get

$$\begin{aligned}&\Big|R_f(\alpha,k;\phi,\eta,m,a,b)\Big|\\&\le\frac{\big(e^{i\phi}\eta(b,a,m)\big)^2}{2}\int_0^1\left|\frac{1-t^{\frac{\alpha}{k}+1}-(1-t)^{\frac{\alpha}{k}+1}}{\frac{\alpha}{k}+1}\right|\Big|f''\big(ma+te^{i\phi}\eta(b,a,m)\big)\Big|\mathrm{d}t\\&\le\frac{\big(e^{i\phi}\eta(b,a,m)\big)^2}{2(\frac{\alpha}{k}+1)}\left\{\int_0^1\Big(1-t^{\frac{\alpha}{k}+1}-(1-t)^{\frac{\alpha}{k}+1}\Big)\mathrm{d}t\right\}^{1-\frac{1}{q}}\\&\quad\times\left\{\int_0^1\Big(1-t^{\frac{\alpha}{k}+1}-(1-t)^{\frac{\alpha}{k}+1}\Big)\Big|f''\big(ma+te^{i\phi}\eta(b,a,m)\big)\Big|^q\mathrm{d}t\right\}^{\frac{1}{q}}\\&=\frac{k\big(e^{i\phi}\eta(b,a,m)\big)^2}{2(\alpha+k)}\left(\frac{\alpha}{\alpha+2k}\right)^{1-\frac{1}{q}}\left\{\int_0^1\Big(1-t^{\frac{\alpha}{k}+1}-(1-t)^{\frac{\alpha}{k}+1}\Big)\Big|f''\big(ma+te^{i\phi}\eta(b,a,m)\big)\Big|^q\mathrm{d}t\right\}^{\frac{1}{q}}.\end{aligned}$$

By the generalized $\lambda_{\phi m}$-MT-preinvexity of $|f''|^q$ on $A_{\phi m}$ for $q\ge 1$, we have

$$\begin{aligned}&\int_0^1\Big(1-t^{\frac{\alpha}{k}+1}-(1-t)^{\frac{\alpha}{k}+1}\Big)\Big|f''\big(ma+te^{i\phi}\eta(b,a,m)\big)\Big|^q\mathrm{d}t\\&\le\int_0^1\Big(1-t^{\frac{\alpha}{k}+1}-(1-t)^{\frac{\alpha}{k}+1}\Big)\Big(\frac{m(1-\lambda)\sqrt{1-t}}{2\lambda\sqrt{t}}|f''(a)|^q+\frac{\sqrt{t}}{2\sqrt{1-t}}|f''(b)|^q\Big)\mathrm{d}t\\&=\frac{m(1-\lambda)}{\lambda}\left[\frac{\pi}{4}-\frac{1}{2}\beta\Big(\frac{\alpha}{k}+\frac{3}{2},\frac{3}{2}\Big)-\frac{1}{2}\beta\Big(\frac{1}{2},\frac{\alpha}{k}+\frac{5}{2}\Big)\right]\big|f''(a)\big|^q\\&\quad+\left[\frac{\pi}{4}-\frac{1}{2}\beta\Big(\frac{\alpha}{k}+\frac{5}{2},\frac{1}{2}\Big)-\frac{1}{2}\beta\Big(\frac{3}{2},\frac{\alpha}{k}+\frac{3}{2}\Big)\right]\big|f''(b)\big|^q\\&=\left[\frac{\pi}{4}-\frac{\sqrt{\pi}\Gamma(\frac{\alpha}{k}+\frac{3}{2})}{2\Gamma\big(\frac{\alpha}{k}+2\big)}\right]\left[\frac{m(1-\lambda)}{\lambda}|f''(a)|^q+\big|f''(b)\big|^q\right].\end{aligned}$$

Hence, the proof is completed.

Let us discuss some special cases of Theorem 3.2.

Corollary 3.4 *In Theorem 3.2, if the mapping $\eta(b,a,m)$ with $m=1$ degenerates into $\eta(b,a)$, then we obtain the following inequality for λ_ϕ-MT-preinvex functions*

$$\begin{aligned}&\left|\frac{f(a)+f\big(a+e^{i\phi}\eta(b,a)\big)}{2}-\frac{\Gamma_k(\alpha+k)}{2k\big(e^{i\phi}\eta(b,a)\big)^{\frac{\alpha}{k}}}\left[{}_kJ^{\alpha}_{a^+}f\big(a+e^{i\phi}\eta(b,a)\big)+{}_kJ^{\alpha}_{(a+e^{i\phi}\eta(b,a))^-}f(a)\right]\right|\\&\le\frac{k\big(e^{i\phi}\eta(b,a)\big)^2}{2(\alpha+k)}\left(\frac{\alpha}{\alpha+2k}\right)^{1-\frac{1}{q}}\left[\frac{\pi}{4}-\frac{\sqrt{\pi}\Gamma(\frac{\alpha}{k}+\frac{3}{2})}{2\Gamma\big(\frac{\alpha}{k}+2\big)}\right]^{\frac{1}{q}}\left[\frac{(1-\lambda)}{\lambda}|f''(a)|^q+\big|f''(b)\big|^q\right]^{\frac{1}{q}}.\end{aligned}$$

Corollary 3.5 *In Theorem 3.2, if $\phi = 0$, $\lambda = \frac{1}{2}$ and $\eta(b, a, m) = b - ma$ with $m = 1$, then we have the following inequality for MT-convex functions*

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2} - \frac{\Gamma_k(\alpha+k)}{2k(b-a)^{\frac{\alpha}{k}}}\left[{}_kJ^{\alpha}_{a^+}f(b) + {}_kJ^{\alpha}_{b^-}f(a)\right]\right| \\ &\leq \frac{k(b-a)^2}{2(\alpha+k)}\Big(\frac{\alpha}{\alpha+2k}\Big)^{1-\frac{1}{q}}\left[\frac{\pi}{4} - \frac{\sqrt{\pi}\Gamma(\frac{\alpha}{k}+\frac{3}{2})}{2\Gamma\big(\frac{\alpha}{k}+2\big)}\right]^{\frac{1}{q}}\left[\left|f''(a)\right|^q + \left|f''(b)\right|^q\right]^{\frac{1}{q}}.\end{aligned}$$

Corollary 3.6 *In Theorem 3.2, if $|f''(x)| \leq M$, $\lambda = \frac{1}{2}$ and $\eta(b, a, m)$ with $m = 1$ degenerates into $\eta(b, a)$, then we have the following inequality for ϕ-MT-preinvex functions*

$$\begin{aligned}&\left|\frac{f(a)+f\big(a+e^{i\phi}\eta(b,a)\big)}{2} - \frac{\Gamma_k(\alpha+k)}{2k\big(e^{i\phi}\eta(b,a)\big)^{\frac{\alpha}{k}}}\left[{}_kJ^{\alpha}_{a^+}f\big(a+e^{i\phi}\eta(b,a)\big) + {}_kJ^{\alpha}_{(a+e^{i\phi}\eta(b,a))^-}f(a)\right]\right| \\ &\leq \frac{kM\big(e^{i\phi}\eta(b,a)\big)^2}{2(\alpha+k)}\Big(\frac{\alpha}{\alpha+2k}\Big)^{1-\frac{1}{q}}\left[\frac{\pi}{2} - \frac{\sqrt{\pi}\Gamma(\frac{\alpha}{k}+\frac{3}{2})}{\Gamma\big(\frac{\alpha}{k}+2\big)}\right]^{\frac{1}{q}}.\end{aligned}$$

Especially if we take $\alpha = 1 = k$, we get

$$\left|\frac{f(a)+f(a+e^{i\phi}\eta(b,a))}{2} - \frac{1}{e^{i\phi}\eta(b,a))}\int_a^{a+e^{i\phi}\eta(b,a)} f(x)\mathrm{d}x\right| \leq \frac{M(e^{i\phi}\eta(b,a))^2}{2}\Big(\frac{1}{6}\Big)^{1-\frac{1}{q}}\Big(\frac{\pi}{16}\Big)^{\frac{1}{q}}, \tag{3.5}$$

which is the result given in [42],Theorem 2.15. Clearly, if we put $\phi = 0$ and $\eta(b, a) = b - a$ in (3.5), we obtain the result given in [30], Theorem 2.4.

A different approach leads to the following results.

Theorem 3.3 *Let $A_{\phi m} \subseteq \mathbb{R}_0$ be an open (ϕ, m)-invex subset respecting $\phi(\cdot)$ and $\eta : A_{\phi m} \times A_{\phi m} \times (0, 1] \to \mathbb{R}_0$, $a, b \in A_{\phi m}$ with $\eta(b, a, m) > 0$, and let $f : A_{\phi m} \to \mathbb{R}$ be a twice differentiable mapping such that $f'' \in L[ma, ma + e^{i\phi}\eta(b, a, m)]$. If $|f''|^q$ is generalized $\lambda_{\phi m}$-MT-preinvex on $A_{\phi m}$, $\lambda \in (0, \frac{1}{2}]$, $m \in (0, 1]$, $q = \frac{p}{p-1}$, $q \neq p > 1$ and $x \in [ma, ma + e^{i\phi}\eta(b, a, m)]$, then we have the following inequality for k-fractional integrals with $k, \alpha > 0$*

$$\Big|R_f(\alpha,k;\phi,\eta,m,a,b)\Big| \leq \frac{k\big(e^{i\phi}\eta(b,a,m)\big)^2}{2(\alpha+k)}\Big(\frac{p(\alpha+k)-k}{p(\alpha+k)+k}\Big)^{\frac{1}{p}}\left[\frac{\pi}{4}\Big(\frac{m(1-\lambda)}{\lambda}\left|f''(a)\right|^q + \left|f''(b)\right|^q\Big)\right]^{\frac{1}{q}}. \tag{3.6}$$

Proof. Using Lemma 2.1 and Hölder's integral inequality leads to

$$\begin{aligned}&\Big|R_f(\alpha,k;\phi,\eta,m,a,b)\Big| \\ &\leq \frac{\big(e^{i\phi}\eta(b,a,m)\big)^2}{2}\int_0^1\left|\frac{1-t^{\frac{\alpha}{k}+1}-(1-t)^{\frac{\alpha}{k}+1}}{\frac{\alpha}{k}+1}\right|\Big|f''\big(ma+te^{i\phi}\eta(b,a,m)\big)\Big|\mathrm{d}t \\ &\leq \frac{k\big(e^{i\phi}\eta(b,a,m)\big)^2}{2(\alpha+k)}\left\{\int_0^1\Big(1-t^{\frac{\alpha}{k}+1}-(1-t)^{\frac{\alpha}{k}+1}\Big)^p\mathrm{d}t\right\}^{\frac{1}{p}}\left\{\int_0^1\Big|f''\big(ma+te^{i\phi}\eta(b,a,m)\big)\Big|^q\mathrm{d}t\right\}^{\frac{1}{q}}\end{aligned}$$

$$
\begin{aligned}
&\le \frac{k\big(e^{i\phi}\eta(b,a,m)\big)^2}{2(\alpha+k)}\left\{\int_0^1\left(1-t^{p(\frac{\alpha}{k}+1)}-(1-t)^{p(\frac{\alpha}{k}+1)}\right)\mathrm{d}t\right\}^{\frac{1}{p}}\\
&\quad\times\left\{\int_0^1\left(\frac{m(1-\lambda)\sqrt{1-t}}{2\lambda\sqrt{t}}|f''(a)|^q+\frac{\sqrt{t}}{2\sqrt{1-t}}|f''(b)|^q\right)\mathrm{d}t\right\}^{\frac{1}{q}}\\
&=\frac{k\big(e^{i\phi}\eta(b,a,m)\big)^2}{2(\alpha+k)}\left(\frac{p(\alpha+k)-k}{p(\alpha+k)+k}\right)^{\frac{1}{p}}\left[\frac{\pi}{4}\left(\frac{m(1-\lambda)}{\lambda}\big|f''(a)\big|^q+\big|f''(b)\big|^q\right)\right]^{\frac{1}{q}}.
\end{aligned}
$$

To prove the third inequality above, we use the same inequality (3.2) as Theorem 3.1, the generalized $\lambda_{\phi m}$-MT-preinvexity of $|f''|^q$ on $A_{\phi m}$ for $q>1$, and the following fact

$$
\int_0^1\left(1-(1-t)^{p(\frac{\alpha}{k}+1)}-t^{p(\frac{\alpha}{k}+1)}\right)\mathrm{d}t=\frac{p(\alpha+k)-k}{p(\alpha+k)+k}.
$$

This ends the proof of Theorem 3.3.

Let us point out some special cases of Theorem 3.3.

Corollary 3.7 *In Theorem 3.3, if the mapping $\eta(b,a,m)$ with $m=1$ degenerates into $\eta(b,a)$ and $\lambda=\frac{1}{2}$, then we have the following inequality for ϕ-MT-preinvex functions*

$$
\begin{aligned}
&\left|\frac{f(a)+f\big(a+e^{i\phi}\eta(b,a)\big)}{2}-\frac{\Gamma_k(\alpha+k)}{2k\big(e^{i\phi}\eta(b,a)\big)^{\frac{\alpha}{k}}}\left[{}_kJ^{\alpha}_{a^+}f\big(a+e^{i\phi}\eta(b,a)\big)+{}_kJ^{\alpha}_{(a+e^{i\phi}\eta(b,a))^-}f(a)\right]\right|\\
&\le\frac{k\big(e^{i\phi}\eta(b,a)\big)^2}{2(\alpha+k)}\left(\frac{p(\alpha+k)-k}{p(\alpha+k)+k}\right)^{\frac{1}{p}}\left[\frac{\pi}{4}\Big(|f''(a)|^q+|f''(b)|^q\Big)\right]^{\frac{1}{q}}.
\end{aligned}
$$

Corollary 3.8 *In Theorem 3.3, if we put $\phi=0$ and $\eta(b,a,m)=b-ma$ with $m=1$, then we have the following inequality for λ-MT-convex functions*

$$
\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma_k(\alpha+k)}{2k(b-a)^{\frac{\alpha}{k}}}\left[{}_kJ^{\alpha}_{a^+}f(b)+{}_kJ^{\alpha}_{b^-}f(a)\right]\right|\\
&\le\frac{k(b-a)^2}{2(\alpha+k)}\left(\frac{p(\alpha+k)-k}{p(\alpha+k)+k}\right)^{\frac{1}{p}}\left[\frac{\pi}{4}\left(\frac{(1-\lambda)}{\lambda}\big|f''(a)\big|^q+\big|f''(b)\big|^q\right)\right]^{\frac{1}{q}}.
\end{aligned}
$$

Especially if we take $k=1$ and $\lambda=\frac{1}{2}$, we have

$$
\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J^{\alpha}_{a^+}f(b)+J^{\alpha}_{b^-}f(a)\right]\right|\\
&\le\frac{(b-a)^2}{2(\alpha+1)}\left(\frac{p(\alpha+1)-1}{p(\alpha+1)+1}\right)^{\frac{1}{p}}\left[\frac{\pi}{4}\Big(\big|f''(a)\big|^q+\big|f''(b)\big|^q\Big)\right]^{\frac{1}{q}}.
\end{aligned}
$$

Corollary 3.9 *In Theorem 3.3, if $|f''(x)|\le M$, $\phi=0$, $\lambda=\frac{1}{2}$ and the mapping $\eta(b,a,m)=b-ma$ with $m=1$, then we have the following inequality for MT-convex functions*

$$
\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma_k(\alpha+k)}{2k(b-a)^{\frac{\alpha}{k}}}\left[{}_kJ^{\alpha}_{a^+}f(b)+{}_kJ^{\alpha}_{b^-}f(a)\right]\right|\le\frac{kM(b-a)^2}{2(\alpha+k)}\left(\frac{\pi}{2}\right)^{\frac{1}{q}}\left(\frac{p(\alpha+k)-k}{p(\alpha+k)+k}\right)^{\frac{1}{p}}.
$$

Especially if we take $k = 1$ and $\alpha = 1$, then we get

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right| \le \frac{M(b-a)^2}{4}\left(\frac{\pi}{2}\right)^{\frac{1}{q}}\left(\frac{2p-1}{2p+1}\right)^{\frac{1}{p}}.$$

Finally, we are in a position to present the following result.

Theorem 3.4 *Suppose that the assumptions of Theorem 3.3 are satisfied, then we have the following inequality*

$$\begin{aligned}
&\Big|R_f(\alpha,k;\phi,\eta,m,a,b)\Big| \\
&\le \frac{k\big(e^{i\phi}\eta(b,a,m)\big)^2}{2(\alpha+k)}\left[\frac{(q-p)\alpha-pk+k}{(q-p)\alpha+2qk-pk-k}\right]^{\frac{q-1}{q}} \\
&\quad\times\left[\frac{\pi}{4}-\frac{\sqrt{\pi}\Gamma\big(p(\frac{\alpha}{k}+1)+\frac{1}{2}\big)}{2\Gamma\big(p(\frac{\alpha}{k}+1)+1\big)}\right]^{\frac{1}{q}}\left[\frac{m(1-\lambda)}{\lambda}|f''(a)|^q+\big|f''(b)\big|^q\right]^{\frac{1}{q}}.
\end{aligned} \tag{3.7}$$

Proof. Using Lemma 2.1 and Hölder's inequality, we have

$$\begin{aligned}
&\Big|R_f(\alpha,k;\phi,\eta,m,a,b)\Big| \\
&\le \frac{\big(e^{i\phi}\eta(b,a,m)\big)^2}{2}\int_0^1\left|\frac{1-t^{\frac{\alpha}{k}+1}-(1-t)^{\frac{\alpha}{k}+1}}{\frac{\alpha}{k}+1}\right|\Big|f''\big(ma+te^{i\phi}\eta(b,a,m)\big)\Big|\mathrm{d}t \\
&\le \frac{\big(e^{i\phi}\eta(b,a,m)\big)^2}{2(\frac{\alpha}{k}+1)}\left[\int_0^1\Big(1-t^{\frac{\alpha}{k}+1}-(1-t)^{\frac{\alpha}{k}+1}\Big)^{\frac{q-p}{q-1}}\mathrm{d}t\right]^{\frac{q-1}{q}} \\
&\quad\times\left[\int_0^1\Big(1-t^{\frac{\alpha}{k}+1}-(1-t)^{\frac{\alpha}{k}+1}\Big)^{p}\Big|f''\big(ma+te^{i\phi}\eta(b,a,m)\big)\Big|^q\mathrm{d}t\right]^{\frac{1}{q}} \\
&\le \frac{k\big(e^{i\phi}\eta(b,a,m)\big)^2}{2(\alpha+k)}\left[\int_0^1\Big(1-t^{\frac{(\frac{\alpha}{k}+1)(q-p)}{q-1}}-(1-t)^{\frac{(\frac{\alpha}{k}+1)(q-p)}{q-1}}\Big)\mathrm{d}t\right]^{\frac{q-1}{q}} \\
&\quad\times\left[\int_0^1\Big(1-t^{p(\frac{\alpha}{k}+1)}-(1-t)^{p(\frac{\alpha}{k}+1)}\Big)\Big|f''\big(ma+te^{i\phi}\eta(b,a,m)\big)\Big|^q\mathrm{d}t\right]^{\frac{1}{q}}.
\end{aligned} \tag{3.8}$$

By the generalize $\lambda_{\phi m}$-MT-preinvexity of $|f''|^q$ on $A_{\phi m}$ for $q>1$, we have

$$\begin{aligned}
&\int_0^1\Big(1-t^{p(\frac{\alpha}{k}+1)}-(1-t)^{p(\frac{\alpha}{k}+1)}\Big)\Big|f''\big(ma+te^{i\phi}\eta(b,a,m)\big)\Big|^q\mathrm{d}t \\
\le&\int_0^1\Big(1-t^{p(\frac{\alpha}{k}+1)}-(1-t)^{p(\frac{\alpha}{k}+1)}\Big)\left(\frac{m(1-\lambda)\sqrt{1-t}}{2\lambda\sqrt{t}}|f''(a)|^q+\frac{\sqrt{t}}{2\sqrt{1-t}}|f''(b)|^q\right)\mathrm{d}t \\
=&\frac{m(1-\lambda)}{\lambda}\left[\frac{\pi}{4}-\frac{1}{2}\beta\Big(p\Big(\frac{\alpha}{k}+1\Big)+\frac{1}{2},\frac{3}{2}\Big)-\frac{1}{2}\beta\Big(\frac{1}{2},p\Big(\frac{\alpha}{k}+1\Big)+\frac{3}{2}\Big)\right]\big|f''(a)\big|^q \\
&+\left[\frac{\pi}{4}-\frac{1}{2}\beta\Big(p\Big(\frac{\alpha}{k}+1\Big)+\frac{3}{2},\frac{1}{2}\Big)-\frac{1}{2}\beta\Big(\frac{3}{2},p\Big(\frac{\alpha}{k}+1\Big)+\frac{1}{2}\Big)\right]\big|f''(b)\big|^q \\
=&\left[\frac{\pi}{4}-\frac{\sqrt{\pi}\Gamma\big(p(\frac{\alpha}{k}+1)+\frac{1}{2}\big)}{2\Gamma\big(p(\frac{\alpha}{k}+1)+1\big)}\right]\left[\frac{m(1-\lambda)}{\lambda}|f''(a)|^q+\big|f''(b)\big|^q\right].
\end{aligned} \tag{3.9}$$

11

Also

$$\int_0^1 \left(1 - t^{\frac{(\frac{\alpha}{k}+1)(q-p)}{q-1}} - (1-t)^{\frac{(\frac{\alpha}{k}+1)(q-p)}{q-1}}\right)\mathrm{d}t = \frac{(q-p)\alpha - pk + k}{(q-p)\alpha + 2qk - pk - k}. \tag{3.10}$$

Utilizing (3.9) and (3.10) in (3.8), we deduce the inequality (3.7). This completes the proof of Theorem 3.4 as well.

We next discuss some special cases of Theorem 3.4.

Corollary 3.10 *In Theorem 3.4, if the mapping $\eta(b,a,m)$ with $m = 1$ degenerates into $\eta(b,a)$, then we obtain the following inequality for λ_ϕ-MT-preinvex functions*

$$\begin{aligned}
&\left|\frac{f(a) + f\big(a + e^{i\phi}\eta(b,a)\big)}{2} - \frac{\Gamma_k(\alpha+k)}{2k\big(e^{i\phi}\eta(b,a)\big)^{\frac{\alpha}{k}}}\left[{}_kJ^{\alpha}_{a^+} f\big(a + e^{i\phi}\eta(b,a)\big) + {}_kJ^{\alpha}_{(a+e^{i\phi}\eta(b,a))^-} f(a)\right]\right| \\
&\le \frac{k\big(e^{i\phi}\eta(b,a)\big)^2}{2(\alpha+k)}\left[\frac{(q-p)\alpha - pk + k}{(q-p)\alpha + 2qk - pk - k}\right]^{\frac{q-1}{q}} \\
&\quad\times\left[\frac{\pi}{4} - \frac{\sqrt{\pi}\,\Gamma\big(p(\frac{\alpha}{k}+1) + \frac{1}{2}\big)}{2\Gamma\big(p(\frac{\alpha}{k}+1)+1\big)}\right]^{\frac{1}{q}}\left[\frac{(1-\lambda)}{\lambda}|f''(a)|^q + |f''(b)|^q\right]^{\frac{1}{q}}.
\end{aligned}$$

Corollary 3.11 *In Theorem 3.4, if we put $\phi = 0$ and $\eta(b,a,m) = b - ma$ with $m = 1$, then we obtain the following inequality for λ-MT-convex functions*

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2} - \frac{\Gamma_k(\alpha+k)}{2k(b-a)^{\frac{\alpha}{k}}}\left[{}_kJ^{\alpha}_{a^+} f(b) + {}_kJ^{\alpha}_{b^-} f(a)\right]\right| \\
&\le \frac{k(b-a)^2}{2(\alpha+k)}\left[\frac{(q-p)\alpha - pk + k}{(q-p)\alpha + 2qk - pk - k}\right]^{\frac{q-1}{q}} \\
&\quad\times\left[\frac{\pi}{4} - \frac{\sqrt{\pi}\,\Gamma\big(p(\frac{\alpha}{k}+1) + \frac{1}{2}\big)}{2\Gamma\big(p(\frac{\alpha}{k}+1)+1\big)}\right]^{\frac{1}{q}}\left[\frac{(1-\lambda)}{\lambda}|f''(a)|^q + |f''(b)|^q\right]^{\frac{1}{q}}.
\end{aligned}$$

Especially if we take $k = 1$ and $\lambda = \frac{1}{2}$, then we have the following inequality for MT-convex functions

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J^{\alpha}_{a^+} f(b) + J^{\alpha}_{b^-} f(a)\right]\right| \\
&\le \frac{(b-a)^2}{2(\alpha+1)}\left[\frac{(q-p)\alpha - p + 1}{(q-p)\alpha + 2q - p - 1}\right]^{\frac{q-1}{q}}\left[\frac{\pi}{4} - \frac{\sqrt{\pi}\,\Gamma\big(p(\alpha+1) + \frac{1}{2}\big)}{2\Gamma\big(p(\alpha+1)+1\big)}\right]^{\frac{1}{q}}\left[|f''(a)|^q + |f''(b)|^q\right]^{\frac{1}{q}}.
\end{aligned}$$

Corollary 3.12 *In Theorem 3.4, if $|f''(x)| \le M$, $\phi = 0$, $\lambda = \frac{1}{2}$ and the mapping $\eta(b,a,m) = b - ma$ with $m = 1$, then we have the following inequality for MT-convex functions*

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2} - \frac{\Gamma_k(\alpha+k)}{2k(b-a)^{\frac{\alpha}{k}}}\left[{}_kJ^{\alpha}_{a^+} f(b) + {}_kJ^{\alpha}_{b^-} f(a)\right]\right| \\
&\le \frac{kM(b-a)^2}{2(\alpha+k)}\left[\frac{(q-p)\alpha - pk + k}{(q-p)\alpha + 2qk - pk - k}\right]^{\frac{q-1}{q}}\left[\frac{\pi}{2} - \frac{\sqrt{\pi}\,\Gamma\big(p(\frac{\alpha}{k}+1) + \frac{1}{2}\big)}{\Gamma\big(p(\frac{\alpha}{k}+1)+1\big)}\right]^{\frac{1}{q}}.
\end{aligned}$$

4 Applications to special means

We begin this section by considering some particular means for two positive real numbers a, b and for this purpose we recall the following well-known means:

$$\text{Arithmetic mean: } A := A(a,b) = \tfrac{a+b}{2},$$

$$\text{Geometric mean: } G := G(a,b) = \sqrt{ab},$$

$$\text{Harmonic mean: } H := H(a,b) = \tfrac{2ab}{a+b},$$

$$\text{Power mean: } P_r := P_r(a,b) = \left(\tfrac{a^r+b^r}{2}\right)^{\frac{1}{r}}, r \geq 1,$$

$$\text{Identric mean: } = I(a,b) = \begin{cases} \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}}, & a \neq b, \\ a, & a = b, \end{cases}$$

$$\text{Logarthmic mean: } L(a,b) = \begin{cases} \frac{b-a}{\ln b - \ln a}, & a \neq b, \\ a, & a = b, \end{cases}$$

and

$$\text{Generalized mean: } L_p := L_p(a,b) = \begin{cases} \left[\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right]^{\frac{1}{p}}, & p \neq 0, -1, \text{ and } a \neq b, \\ L(a,b), & p = -1 \text{ and } a \neq b, \\ I(a,b), & p = 0 \text{ and } a \neq b, \\ a, & a = b. \end{cases}$$

Clearly, L_p is monotonic nondecreasing over $p \in \mathbb{R}$, with $L_{-1} := L$ and $L_0 := I$. In particular, we have $H \leq G \leq L \leq I \leq A$.

Let $0 < a < b$, $\lambda \in (0, \frac{1}{2}]$ and let $M := M(a,b) : [a + \eta(b,a)] \times [a, a + \eta(b,a)] \to \mathbb{R}_+$, which is one of the above mentioned means, one can obtain various inequalities for these means.

Now, if $\eta(b,a,m)$ with m=1 degenerates into $\eta(b,a)$ and $\eta(b,a) := M(b,a)$, for $\phi = 0$ in (3.1), (3.4), (3.6) and (3.7), we have the following interesting inequalities concerning the above means

$$\left|R_f(\alpha, k; 0, \eta, 1, a, b)\right| \leq \frac{kM^2}{2(\alpha+k)}\left[\frac{\pi}{4} - \frac{\sqrt{\pi}\Gamma\big(q(\frac{\alpha}{k}+1)+\frac{1}{2}\big)}{2\Gamma\big(q(\frac{\alpha}{k}+1)+1\big)}\right]^{\frac{1}{q}}\left\{\frac{(1-\lambda)}{\lambda}\left|f''(a)\right|^q + \left|f''(b)\right|^q\right\}^{\frac{1}{q}}, \tag{4.1}$$

$$\left|R_f(\alpha, k; 0, \eta, 1, a, b)\right| \leq \frac{kM^2}{2(\alpha+k)}\left(\frac{\alpha}{\alpha+2k}\right)^{1-\frac{1}{q}}\left[\frac{\pi}{4} - \frac{\sqrt{\pi}\Gamma\big(\frac{\alpha}{k}+\frac{3}{2}\big)}{2\Gamma\big(\frac{\alpha}{k}+2\big)}\right]^{\frac{1}{q}}\left\{\frac{(1-\lambda)}{\lambda}\left|f''(a)\right|^q + \left|f''(b)\right|^q\right\}^{\frac{1}{q}}, \tag{4.2}$$

$$\left|R_f(\alpha, k; 0, \eta, 1, a, b)\right| \leq \frac{kM^2}{2(\alpha+k)}\left(\frac{p(\alpha+k)-k}{p(\alpha+k)+k}\right)^{\frac{1}{p}}\left\{\frac{\pi}{4}\left(\frac{(1-\lambda)}{\lambda}\left|f''(a)\right|^q + \left|f''(b)\right|^q\right)\right\}^{\frac{1}{q}} \tag{4.3}$$

and

$$\begin{aligned}
&\Big|R_f(\alpha,k;0,\eta,1,a,b)\Big| \leq \frac{kM^2}{2(\alpha+k)}\left\{\frac{(q-p)\alpha-pk+k}{(q-p)\alpha+2qk-pk-k}\right\}^{\frac{q-1}{q}} \\
&\times\left[\frac{\pi}{4}-\frac{\sqrt{\pi}\Gamma\big(p(\frac{\alpha}{k}+1)+\frac{1}{2}\big)}{2\Gamma\big(p(\frac{\alpha}{k}+1)+1\big)}\right]^{\frac{1}{q}}\left\{\frac{(1-\lambda)}{\lambda}\big|f''(a)\big|^q+\big|f''(b)\big|^q\right\}^{\frac{1}{q}},
\end{aligned} \tag{4.4}$$

where

$$\begin{aligned}
\Big|R_f(\alpha,k;0,\eta,1,a,b)\Big| =&\frac{f(a)+f\big(a+M(a,b)\big)}{2}-\frac{\Gamma_k(\alpha+k)}{2kM^{\frac{\alpha}{k}}(a,b)} \\
&\times\Big[{}_kJ^{\alpha}_{a^+}f\big(a+M(b,a)\big)+{}_kJ^{\alpha}_{(a+M(b,a))^-}f(a)\Big].
\end{aligned}$$

Letting $M=A$, G, H, P_r, I, L, L_p in (4.1), (4.2), (4.3) and (4.4), we also get the required inequalities, and the more details are left to the reader to explore.

Acknowledgments

This work was partially supported by the National Natural Science Foundation of China under Grant No. 61374028.

References

[1] P. Agarwal, J. Tariboon, S. K. Ntouyas, Some generalized Riemann-Liouville k-fractional integral inequalities , *J. Inequal. Appl.*, **2016** (2016), Article ID 122, 13 pages.

[2] A. Ali, G. Gulshan, R. Hussain, A. Latif, M. Muddassar, Generalized inequalities of the type of Hermite-Hadamard-Fejer with quasi-convex functions by way of k-fractional derivatives, *J. Comput. Anal. Appl.*, **22** (7) (2017), 1208-1219.

[3] G. A. Anastassiou, Generalised fractional Hermite-Hadamard inequalities involving m-convexity and (s,m)-convexity, *Facta Univ. Ser. Math. Inform.*, **28** (2) (2013), 107-126.

[4] G. A. Anastassiou, Fractional Differentiation Inequalities, Research Monograph, Springer, New York, 2009.

[5] T. Antczak, Mean value in invexity analysis, *Nonlinear Anal.*, **60** (2005), 1473-1484.

[6] M. U. Awan, M. A. Noor, M. V. Mihai, K. I. Noor, On bounds involving k-Appell's hypergeometric functions, *J. Inequal. Appl.*, **2017** (2017), Article ID 118, 15 pages.

[7] F. X. Chen, Extensions of the Hermite-Hadamard inequality for convex functions via fractional integrals, *J. Math. Inequal.*, **10** (2016), 75–81.

[8] Y.-M. Chu, M. A. Khan, T. Ali, S. S. Dragomir, Inequalities for α-fractional differentiable functions, *J. Inequal. Appl.*, **2017** (2017), Article ID 93, 12 pages.

[9] Y.-M. Chu, M. A. Khan, T. U. Khan, T. Ali, Generalizations of Hermite-Hadamard type inequalities for MT-convex functions, *J. Nonlinear Sci. Appl.*, **9** (2016), 4305–4316.

[10] S. S. Dragomir, M. I. Bhatti, M. Iqbal, M. Muddassar, Some new Hermite-Hadamard's type fractional integral inequalities, *J. Comput. Anal. Appl.*, **18** (4) (2015), 655-661.

[11] T. S. Du, J. G. Liao, Y. J. Li, Properties and integral inequalities of Hadamard-Simpson type for the generalized (s, m)-preinvex functions, *J. Nonlinear Sci. Appl.*, **9** (5) (2016), 3112-3126.

[12] T. S. Du, Y. J. Li, Z. Q. Yang, A generalization of Simpson's inequality via differentiable mapping using extended (s, m)-convex functions, *Appl. Math. Comput.*, **293** (2017), 358-369.

[13] T. S. Du, S. H. Wu, S. J. Zhao, M. U. Awan, Riemann-Liouville fractional Hermite-Hadamard inequalities for h-preinvex functions, *J. Comput. Anal. Appl.*, **25** (2) (2018), 364-384.

[14] S. Ermeydan, H. Yildirim, Riemann-Liouville Fractional Hermite-Hadamard Inequalities for differentiable $\lambda\phi$-preinvex functions, *Malaya J. Mat.*, **4** (3) (2016), 430-437.

[15] S. Hussain, S. Qaisar, More results on Hermite-Hadamard type inequality through (α, m)-preinvexity, *J. Appl. Anal. Comput.*, **6** (2016), 293-305.

[16] S. R. Hwang, K. L. Tseng, K. C. Hsu, New inequalities for fractional integrals and their applications, *Turkish J. Math.*, **40** (2016), 471-486.

[17] M. Iqbal, M.I. Bhatti, K. Nazeer, Generalization of inequalities analogous to Hermite-Hadamard inequality via fractional integrals, *Bull Korean Math. Soc.*, **52** (3) (2015), 707-716.

[18] İ. İşcan, S. H. Wu, Hermite-Hadamard type inequalities for harmonically convex functions via fractional integrals, *Appl. Math. Comput.*, **238** (2014), 237-244.

[19] A. Kashuri, R. Liko, Generalizations of Hermite-Hadamard and Ostrowski type inequalities for MT_m-preinvex functions, *Proyecciones (Antofagasta)*, **36** (1) (2017), 45-80.

[20] M. A. Khan, T. Ali, S. S. Dragomir, Hermite-Hadamard type inequalities for conformable fractional integrals, *Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Math.*, (2017), DOI 10.1007/s13398-017-0408-5.

[21] M. A. Latif, S. S. Dragomir, Generalization of Hermite-Hadamard type inequalities for n-times differentiable functions which are s-preinvex in the second sense with applications, *Hacet. J. Math. Stat.*, **44** (2015), 839-853.

[22] Y. J. Li, T. S. Du, A generalization of Simpson type inequality via differentiable functions using extended $(s, m)_\phi$-preinvex functions, *J. Comput. Anal. Appl.*, **22** (4) (2017), 613-632.

[23] W. J. Liu, Ostrowski type fractional integral inequalities for MT-convex functions, *Miskolc Math. Notes*, **16** (2015), 249-256.

[24] S. Mubeen, G. M. Habibullah, k-fractional integrals and applications, *Int. J. Contemp. Math. Sciences*, **7** (2) (2012), 89-94.

[25] M. A. Noor, K. I. Noor, M. V. Mihai, M. U. Awana, Fractional Hermite-Hadamard inequalities for some classes of differentiable preinvex functions, *Politehn. Univ. Bucharest Sci. Bull. Ser. A Appl. Math. Phys.*, **78** (3) (2016), 163-174.

[26] M. A. Noor, K. I. Noor, M. U. Awan, S. Khan, Hermite-Hadamard type inequalities for differentiable h_φ-preinvex functions, *Arab. J. Math.*, **4** (2015), 63-76.

[27] M. A. Noor, Some new classes of nonconvex functions, *Nonlinear Funct. Anal. Appl.*, **12** (2006), 165-171.

[28] O. Omotoyinbo, A. Mogbodemu, Some new Hermite-Hadamard integral inequalities for convex functions, *Int. J. Sci. Innovation Tech.*, **1** (1) (2014), 001-012.

[29] M. E. Özdemir, S. S. Dragomir, Ç. Yildiz, The Hadamard inequality for convex function via fractional integrals, *Acta. Math. Sci. Ser. B Engl. Ed.*, **33B** (5) (2013), 1293-1299.

[30] J. Park, Hermite-Hadamard-like type inequalities for twice differentiable MT-convex functions, *Appl. Math. Sci.*, **9** (2015), 5235-5250.

[31] S. Qaisar, M. Iqbal, M. Muddassar, New Hermite-Hadamard's inequalities for preinvex functions via fractional integrals, *J. Comput. Anal. Appl.*, **20** (7) (2016), 1318-1328.

[32] M. Z. Sarikaya, E. Set, H. Yaldiz, N. Başak, Hermite-Hadamard's inequalities for fractional integrals and related fractional inequalities, *Math. Comput. Modelling*, **57** (2013), 2403-2407.

[33] M. Z. Sarikaya, A. Karaca, On the k-Riemann-Liouville fractional integral and applications, *Int. J. Stat. Math.*, **1** (3) (2014), 33-43.

[34] E. Set, M. Tomar, M. Z. Sarikaya, On generalized Grüss type inequalities for k-fractional integrals, *Appl. Math. Comput.*, **269** (2015), 29-34.

[35] M. Tunç, Y. Şubaş, I. Karabayir, On some Hadamard type inequalities for MT-convex functions, *Int. J. Open Problems Compt. Math.*, **6** (2) (2013), 102-113.

[36] M. Tunç, Ostrowski type inequalities for functions whose derivatives are MT-convex, *J. Comput. Anal. Appl.*, **17** (2014), 691-696.

[37] J. R. Wang, X. Z. Li, M. Fečkan, Y. Zhou, Hermite-Hadamard-type inequalities for Riemann-Liouville fractional integrals via two kinds of convexity, *Appl. Anal.*, **92** (11) (2013), 2241-2253.

[38] H. Wang, T. S. Du, Y. Zhang, k-fractional integral trapezium-like inequalities through (h,m)-convex and (α,m)-convex mappings, *J. Inequal. Appl.*, **2017** (2017), Article ID 311, 20 pages.

[39] T. Weir, B. Mond, Pre-invex functions in multiple objective optimization, *J. Math. Anal. Appl.*, **136** (1988), 29-38.

[40] S. H. Wu, B. Sroysang, J. S. Xie, Y. M. Chu, Parametrized inequality of Hermite-Hadamard type for functions whose third derivative absolute values are quasi-convex, *SpringerPlus*, **4** (2015), Article ID 831, 9 pages.

[41] Y. C. Zhang, T. S. Du, J. Pan, On new inequalities of Fejér-Hermite-Hadamard type for differentiable (α,m)-preinvex mappings, *ScienceAsia*, **43** (4) (2017), 258-266.

[42] S. Zheng, T. S. Du, S. S. Zhao, L. Z. Chen, New Hermite-Hadamard inequalities for twice differentiable ϕ-MT-preinvex functions, *J. Nonlinear Sci. Appl.*, **9** (10) (2016), 5648-5660.

Some generalizations of operator inequalities for positive linear map

Chaojun Yang and Fangyan Lu*

Abstract:

We generalize some inequalities for positive unital linear map as follows: Let A, B be positive operators on a Hilbert space with $0 < m \leq A \leq m' < M' \leq B \leq M$. Then for every positive unital linear map Φ, $\mu \in [0,1]$ and $p > 0$,

$$\Phi^p(A\nabla_\mu B + 2rMm(A^{-1}\nabla B^{-1} - A^{-1}\sharp B^{-1})) \leq \alpha^p \Phi^p(A\sharp_\mu B)$$

and

$$\Phi^p(A\nabla_\mu B + 2rMm(A^{-1}\nabla B^{-1} - A^{-1}\sharp B^{-1})) \leq \alpha^p (\Phi(A)\sharp_\mu \Phi(B))^p$$

where $r = \min\{\mu, 1-\mu\}$, $h' = \frac{M'}{m'}$ and $\alpha = \max\left\{\frac{(M+m)^2}{4MmK(\sqrt{h'},2)^R}, \frac{(M+m)^2}{4^{\frac{2}{p}}MmK(\sqrt{h'},2)^R}\right\}$. Furthermore, we give a squaring reversed Karcher mean inequality involving positive linear map.

1. Introduction

Through this paper, let m, m', M, M' be scalars. Other capital letters denote general elements of the C^*-algebra $B(\mathcal{H})$ of all bounded linear operators on a complex separable Hilbert space $(\mathcal{H}, \langle\cdot,\cdot\rangle)$. The Kantorovich constant is defined by $K(h,2) = \frac{(h+1)^2}{4h}$ for $h > 0$. We write $A \geq 0 (A > 0)$ to mean the self-adjoint operator A is positive(strictly positive). The partial order $A \leq B$ is defined as $B - A \geq 0$.

For each $\mu \in [0,1]$, the weighted arithmetic mean ∇_μ and weight geometric mean $\sharp_\mu$ for strictly positive operator A and B are defined below:

$$A\nabla_\mu B = (1-\mu)A + \mu B \quad and \quad A\sharp_\mu B = A^{\frac{1}{2}}(A^{\frac{-1}{2}}BA^{\frac{-1}{2}})^\mu A^{\frac{1}{2}}.$$

When $\mu = \frac{1}{2}$ we write $A\nabla B$ and $A\sharp B$ for brevity, respectively.

A linear map $\Phi : B(\mathcal{H}) \to B(\mathcal{H})$ is called positive (strictly positive) if $\Phi(A) \geq 0$ ($\Phi(A) > 0$) whenever $A \geq 0$ ($A > 0$), and Φ is said to be unital if $\Phi(I) = I$.

The arithmetic-geometric mean for positive operator A and B states

$$\frac{A+B}{2} \geq A\sharp B.$$

In [8], Lin give a reversed arithmetic-geometric mean inequality involving a positive linear map

$$\Phi(\frac{A+B}{2}) \leq \frac{(M+m)^2}{4Mm}\Phi(A\sharp B) \tag{1}$$

where $0 < m \leq A, B \leq M$ and Φ is a positive unital linear map.

---

* Corresponding author.

2010 Mathematics Subject Classification. Primary 47A63 ; Secondary 47B20

The research was supported by NNSFC (No. 11571247).

Keywords and phrases : positive linear map; operator inequality; Kantorovich constant; Karcher mean.

1

It is well known that t^α is operator monotone function on $[0,\infty)$ if and only if $\alpha\in[0,1]$. Since t^2 is not an operator monotone function, we can not obtain $A^2\geq B^2$ directly by $A\geq B\geq 0$. However Lin [8] gave a p-th powering$(0<p\leq 2)$ of inequality (1), namely the inequality

$$\Phi^p(\tfrac{A+B}{2})\leq\left(\frac{(M+m)^2}{4^{\frac{2}{p}}Mm}\right)^p\Phi^p(A\sharp B) \tag{2}$$

and

$$\Phi^p(\tfrac{A+B}{2})\leq\left(\frac{(M+m)^2}{4^{\frac{2}{p}}Mm}\right)^p(\Phi(A)\sharp\Phi(B))^p \tag{3}$$

where $0<m\leq A,B\leq M$ and Φ is a positive unital linear map.

In [6], the authors extend (2) and (3) to $p>2$, which states

$$\Phi^p(\tfrac{A+B}{2})\leq\left(\frac{(M+m)^2}{4^{\frac{2}{p}}Mm}\right)^p\Phi^p(A\sharp B) \tag{4}$$

and

$$\Phi^p(\tfrac{A+B}{2})\leq\left(\frac{(M+m)^2}{4^{\frac{2}{p}}Mm}\right)^p(\Phi(A)\sharp\Phi(B))^p \tag{5}$$

where $0<m\leq A,B\leq M$ and Φ is a positive unital linear map.

Recently the author in [4] gives inequalities that generalize the inequalities (2) to (5) and state as follows

$$\Phi^p(A\nabla_\mu B+2rMm(A^{-1}\nabla B^{-1}-A^{-1}\sharp B^{-1}))\leq\alpha^p\Phi^p(A\sharp_\mu B) \tag{6}$$

and

$$\Phi^p(A\nabla_\mu B+2rMm(A^{-1}\nabla B^{-1}-A^{-1}\sharp B^{-1}))\leq\alpha^p(\Phi(A)\sharp_\mu\Phi(B))^p \tag{7}$$

where $0<m\leq A\leq m'<M'\leq B\leq M$, Φ be a positive unital linear map on $B(\mathcal{H})$, $\mu\in[0,1]$, $p>0$, $r=\min\{\mu,1-\mu\}$ and $\alpha=\max\left\{\frac{(M+m)^2}{4Mm},\frac{(M+m)^2}{4^{\frac{2}{p}}Mm}\right\}$.

The $\omega-$weighted Karcher mean $\Lambda(\omega;A_1,\ldots,A_n)$(or$\Lambda(\omega;\mathbb{A})$) of $A_1,\cdots,A_n>0$ is defined to be the unique positive definite solution of equation

$$\sum_{i=1}^n\omega_i log(X^{-\frac{1}{2}}A_iX^{-\frac{1}{2}})=0,$$

where $\omega=(w_1,\ldots,w_n)$ is a probability vector. Next we cite some basic properties of the Karcher mean as follows, for more details about Karcher mean, see[7].

Proposition 1.1. [7] The Karcher mean satisfies the following properties:

(i) (consistency with scalars) $\Lambda(\omega;\mathbb{A})=A_1^{\omega_1}\cdots A_n^{\omega_n}$ *if the* A_i *is commute.*

(ii) (self duality) $\Lambda(\omega;A_1^{-1},\cdots,A_n^{-1})^{-1}=\Lambda(\omega;A_1,\cdots,A_n)$.

(iii) (AGH weighted mean inequalities) $(\sum_{i=1}^n\omega_iA_i^{-1})^{-1}\leq\Lambda(\omega;A_1,\cdots,A_n)\leq\sum_{i=1}^n\omega_iA_i$.

(iv) $\Phi(\Lambda(\omega;\mathbb{A}))\leq\Lambda(\omega;\Phi(\mathbb{A}))$ *for any positive unital linear map* Φ.

(v) (monotonicity) If $B_i\leq A_i$ *for all* $1\leq i\leq n$*, then* $\Lambda(\omega;\mathbb{B})\leq\Lambda(\omega;\mathbb{A})$.

As mentioned in the abstract, we shall give refinements of inequalities (6) and (7), along with presenting a reversed Karcher mean inequality related to *(iv)* in Proposition 1.1 and a squaring version thereafter.

2. Main Results

Lemma 2.1.(Choi inequality.) [5] Let Φ be a unital positive linear map, then
(i) when $A > 0$ and $-1 \leq p \leq 0$, then $\Phi(A)^p \leq \Phi(A^p)$, in particular,$\Phi(A)^{-1} \leq \Phi(A^{-1})$;
(ii) when $A \geq 0$ and $0 \leq p \leq 1$, then $\Phi(A)^p \geq \Phi(A^p)$;
(iii) when $A \geq 0$ and $1 \leq p \leq 2$, then $\Phi(A)^p \leq \Phi(A^p)$.

Lemma 2.2. [1] Let Φ be a unital positive linear map and A, B be positive operators. Then for $\alpha \in [0, 1]$

$$\Phi(A \sharp_\alpha B) \leq \Phi(A) \sharp_\alpha \Phi(B).$$

Lemma 2.3. [3] Let $A, B \geq 0$. Then the following norm inequality holds:

$$\|AB\| \leq \tfrac{1}{4}\|A + B\|^2.$$

Lemma 2.4. [2] Let $A, B \geq 0$. Then for $1 \leq r < +\infty$,

$$\|A^r + B^r\| \leq \|(A + B)^r\|.$$

Lemma 2.5. [5] (L-H inequality) If $0 \leq \alpha \leq 1$, $A, B \geq 0$ and $A \geq B$, then $A^\alpha \geq B^\alpha$.

Lemma 2.6. [9] For two operators $A, B \geq 0$ and $1 < h \leq A^{-\frac{1}{2}} B A^{-\frac{1}{2}} \leq h'$ or $0 < h' \leq A^{-\frac{1}{2}} B A^{-\frac{1}{2}} \leq h < 1$, we have

$$A \nabla_\mu B - 2r(A \nabla B - A \sharp B) \geq K(\sqrt{h}, 2)^R A \sharp_\mu B$$

for all $\mu \in [0, 1]$, where $r = \min\{\mu, 1 - \mu\}$ and $R = \min\{2r, 1 - 2r\}$.

Lemma 2.7. Let $0 < m \leq A \leq m' < M' \leq B \leq M$, then

$$A^{-1} \nabla_\mu B^{-1} - 2r(A^{-1} \nabla B^{-1} - A^{-1} \sharp B^{-1}) \geq K(\sqrt{h'}, 2)^R A^{-1} \sharp_\mu B^{-1}$$

for all $\mu \in [0, 1]$, where $r = \min\{\mu, 1 - \mu\}$, $h' = \frac{M'}{m'}$ and $R = \min\{2r, 1 - 2r\}$.

Proof. Since $0 < m \leq A \leq m' < M' \leq B \leq M$, we have $0 < \frac{m}{M} \leq (A^{-1})^{-\frac{1}{2}}(B^{-1})(A^{-1})^{-\frac{1}{2}} \leq \frac{m\prime}{M\prime} < 1$. Thus by Lemma 2.6 we have

$$A^{-1} \nabla_\mu B^{-1} - 2r(A^{-1} \nabla B^{-1} - A^{-1} \sharp B^{-1}) \geq K(\sqrt{h'}, 2)^R A^{-1} \sharp_\mu B^{-1}$$

where $K(\sqrt{h'}, 2) = K(\sqrt{\frac{1}{h'}}, 2)$.

Theorem 2.8. Let $0 < m \leq A \leq m' < M' \leq B \leq M$, Φ be a positive unital linear map on $B(\mathcal{H})$, $\mu \in [0, 1]$ and $p > 0$, we have

$$\Phi^p(A \nabla_\mu B + 2rMm(A^{-1} \nabla B^{-1} - A^{-1} \sharp B^{-1})) \leq \alpha^p \Phi^p(A \sharp_\mu B) \tag{8}$$

and

$$\Phi^p(A \nabla_\mu B + 2rMm(A^{-1} \nabla B^{-1} - A^{-1} \sharp B^{-1})) \leq \alpha^p (\Phi(A) \sharp_\mu \Phi(B))^p \tag{9}$$

where $r=\min\{\mu,1-\mu\}$, $h'=\frac{M'}{m'}$ and $\alpha=\max\left\{\frac{(M+m)^2}{4MmK(\sqrt{h'},2)^R},\frac{(M+m)^2}{4^{\frac{2}{p}}MmK(\sqrt{h'},2)^R}\right\}$.

Proof. By $< m\leq A\leq m'<M'\leq B\leq M$ we have

$$A+MmA^{-1}\leq M+m \quad and \quad B+MmB^{-1}\leq M+m.$$

Thus we have

$$(1-\mu)A+(1-\mu)MmA^{-1}\leq(1-\mu)(M+m) \quad and \quad \mu B+\mu MmB^{-1}\leq\mu(M+m). \tag{10}$$

By (10) we obtain

$$A\nabla_\mu B+MmA^{-1}\nabla_\mu B^{-1}\leq M+m. \tag{11}$$

By Lemma 2.7 we have

$$A^{-1}\nabla_\mu B^{-1}-2r(A^{-1}\nabla B^{-1}-A^{-1}\sharp B^{-1})\geq K(\sqrt{h'},2)^R A^{-1}\sharp_\mu B^{-1} \tag{12}$$

Compute

$$\begin{aligned}
&||\Phi(A\nabla_\mu B+2rMm(A^{-1}\nabla B^{-1}-A^{-1}\sharp B^{-1}))K(\sqrt{h'},2)^R Mm\Phi^{-1}(A\sharp_\mu B)||\\
&\leq\tfrac14||\Phi(A\nabla_\mu B+2rMm(A^{-1}\nabla B^{-1}-A^{-1}\sharp B^{-1}))+K(\sqrt{h'},2)^R Mm\Phi^{-1}(A\sharp_\mu B)||^2\\
&\leq\tfrac14||\Phi(A\nabla_\mu B+2rMm(A^{-1}\nabla B^{-1}-A^{-1}\sharp B^{-1}))+K(\sqrt{h'},2)^R\Phi((A\sharp_\mu B)^{-1})||^2\\
&=\tfrac14||\Phi(A\nabla_\mu B+2rMm(A^{-1}\nabla B^{-1}-A^{-1}\sharp B^{-1}))+K(\sqrt{h'},2)^R\Phi(A^{-1}\sharp_\mu B^{-1})||^2\\
&=\tfrac14||\Phi(A\nabla_\mu B)+Mm\Phi(2r(A^{-1}\nabla B^{-1}-A^{-1}\sharp B^{-1})+K(\sqrt{h'},2)^R(A^{-1}\sharp_\mu B^{-1}))||^2\\
&\leq\tfrac14||\Phi(A\nabla_\mu B)+Mm\Phi(A^{-1}\nabla_\mu B^{-1})||^2,\\
&\leq\tfrac14(M+m)^2
\end{aligned}$$

where the first inequality is derived by Lemma 2.3, the second one is derived by Lemma 2.1, the third one is derived by (12) and the last one is derived by (11).

Therefore

$$||\Phi(A\nabla_\mu B+2rMm(A^{-1}\nabla B^{-1}-A^{-1}\sharp B^{-1}))\Phi^{-1}(A\sharp_\mu B)||\leq\frac{(M+m)^2}{4MmK(\sqrt{h'},2)^R}.$$

Hence

$$\Phi^2(A\nabla_\mu B+2rMm(A^{-1}\nabla B^{-1}-A^{-1}\sharp B^{-1}))\leq(\frac{(M+m)^2}{4MmK(\sqrt{h'},2)^R})^2\Phi^2(A\sharp_\mu B).$$

If $0<p\leq2$, then $0<\frac{p}{2}\leq1$. Therefore by the L-H inequality we get

$$\Phi^p(A\nabla_\mu B+2rMm(A^{-1}\nabla B^{-1}-A^{-1}\sharp B^{-1}))\leq(\frac{(M+m)^2}{4MmK(\sqrt{h'},2)^R})^p\Phi^p(A\sharp_\mu B).$$

Now we obtain inequality (8) for $0<p\leq2$.

Next we prove (9) for $0<p\leq2$. Through

$$\begin{aligned}
&||\Phi(A\nabla_\mu B+2rMm(A^{-1}\nabla B^{-1}-A^{-1}\sharp B^{-1}))K(\sqrt{h'},2)^R Mm(\Phi(A)\sharp_\mu\Phi(B))^{-1}||\\
&\le \tfrac{1}{4}||\Phi(A\nabla_\mu B+2rMm(A^{-1}\nabla B^{-1}-A^{-1}\sharp B^{-1}))+K(\sqrt{h'},2)^R Mm(\Phi(A)\sharp_\mu\Phi(B))^{-1}||^2\\
&\le \tfrac{1}{4}||\Phi(A\nabla_\mu B+2rMm(A^{-1}\nabla B^{-1}-A^{-1}\sharp B^{-1}))+K(\sqrt{h'},2)^R \Phi(A\sharp_\mu B)^{-1}||^2\\
&\le \tfrac{1}{4}||\Phi(A\nabla_\mu B+2rMm(A^{-1}\nabla B^{-1}-A^{-1}\sharp B^{-1}))+K(\sqrt{h'},2)^R \Phi(A^{-1}\sharp_\mu B^{-1})||^2\\
&= \tfrac{1}{4}||\Phi(A\nabla_\mu B)+Mm\Phi(2r(A^{-1}\nabla B^{-1}-A^{-1}\sharp B^{-1})+K(\sqrt{h'},2)^R(A^{-1}\sharp_\mu B^{-1}))||^2\\
&\le \tfrac{1}{4}||\Phi(A\nabla_\mu B)+Mm\Phi(A^{-1}\nabla_\mu B^{-1})||^2,\\
&\le \tfrac{1}{4}(M+m)^2
\end{aligned}$$

where the second inequality is obtained by Lemma 2.2. Hence we get inequality (9) for $0<p\le 2$.

Next, put $p>2$. We can obtain

$$\begin{aligned}
&(K(\sqrt{h'},2)^R Mm)^{\frac{p}{2}}||\Phi^{\frac{p}{2}}(A\nabla_\mu B+2rMm(A^{-1}\nabla B^{-1}-A^{-1}\sharp B^{-1}))\Phi^{-\frac{p}{2}}(A\sharp_\mu B)||\\
&= ||\Phi^{\frac{p}{2}}(A\nabla_\mu B+2rMm(A^{-1}\nabla B^{-1}-A^{-1}\sharp B^{-1}))(K(\sqrt{h'},2)^R Mm)^{\frac{p}{2}}\Phi^{-\frac{p}{2}}(A\sharp_\mu B)||\\
&\le \tfrac{1}{4}||\Phi^{\frac{p}{2}}(A\nabla_\mu B+2rMm(A^{-1}\nabla B^{-1}-A^{-1}\sharp B^{-1}))+(K(\sqrt{h'},2)^R Mm)^{\frac{p}{2}}\Phi^{-\frac{p}{2}}(A\sharp_\mu B)||^2\\
&\le \tfrac{1}{4}||\Phi(A\nabla_\mu B+2rMm(A^{-1}\nabla B^{-1}-A^{-1}\sharp B^{-1}))+K(\sqrt{h'},2)^R Mm\Phi^{-1}(A\sharp_\mu B)||^p\\
&\le \tfrac{1}{4}||\Phi(A\nabla_\mu B+2rMm(A^{-1}\nabla B^{-1}-A^{-1}\sharp B^{-1}))+K(\sqrt{h'},2)^R Mm\Phi(A^{-1}\sharp_\mu B^{-1})||^p\\
&= \tfrac{1}{4}||\Phi(A\nabla_\mu B)+Mm\Phi(2r(A^{-1}\nabla B^{-1}-A^{-1}\sharp B^{-1})+K(\sqrt{h'},2)^R(A^{-1}\sharp_\mu B^{-1}))||^p\\
&\le \tfrac{1}{4}||\Phi(A\nabla_\mu B)+Mm\Phi(A^{-1}\nabla_\mu B^{-1})||^p,\\
&\le \tfrac{1}{4}(M+m)^p
\end{aligned}$$

where the second inequality is obtained by Lemma 2.4.

Therefore, we get inequality (8) for $p>2$. Likewise, we have

$$\begin{aligned}
&(K(\sqrt{h'},2)^R Mm)^{\frac{p}{2}}||\Phi^{\frac{p}{2}}(A\nabla_\mu B+2rMm(A^{-1}\nabla B^{-1}-A^{-1}\sharp B^{-1}))(\Phi(A)\sharp_\mu\Phi(B))^{-\frac{p}{2}}||\\
&= ||\Phi^{\frac{p}{2}}(A\nabla_\mu B+2rMm(A^{-1}\nabla B^{-1}-A^{-1}\sharp B^{-1}))(K(\sqrt{h'},2)^R Mm)^{\frac{p}{2}}(\Phi(A)\sharp_\mu\Phi(B))^{-\frac{p}{2}}||\\
&\le \tfrac{1}{4}||\Phi^{\frac{p}{2}}(A\nabla_\mu B+2rMm(A^{-1}\nabla B^{-1}-A^{-1}\sharp B^{-1}))+(K(\sqrt{h'},2)^R Mm)^{\frac{p}{2}}(\Phi(A)\sharp_\mu\Phi(B))^{-\frac{p}{2}}||^2\\
&\le \tfrac{1}{4}||\Phi(A\nabla_\mu B+2rMm(A^{-1}\nabla B^{-1}-A^{-1}\sharp B^{-1}))+K(\sqrt{h'},2)^R Mm(\Phi(A)\sharp_\mu\Phi(B))^{-1}||^p\\
&\le \tfrac{1}{4}||\Phi(A\nabla_\mu B+2rMm(A^{-1}\nabla B^{-1}-A^{-1}\sharp B^{-1}))+K(\sqrt{h'},2)^R Mm\Phi(A\sharp_\mu B)^{-1}||^p\\
&\le \tfrac{1}{4}||\Phi(A\nabla_\mu B+2rMm(A^{-1}\nabla B^{-1}-A^{-1}\sharp B^{-1}))+K(\sqrt{h'},2)^R Mm\Phi((A\sharp_\mu B)^{-1})||^p\\
&= \tfrac{1}{4}||\Phi(A\nabla_\mu B+2rMm(A^{-1}\nabla B^{-1}-A^{-1}\sharp B^{-1}))+K(\sqrt{h'},2)^R \Phi(A^{-1}\sharp_\mu B^{-1})||^p\\
&= \tfrac{1}{4}||\Phi(A\nabla_\mu B)+Mm\Phi(2r(A^{-1}\nabla B^{-1}-A^{-1}\sharp B^{-1})+K(\sqrt{h'},2)^R(A^{-1}\sharp_\mu B^{-1}))||^p\\
&\le \tfrac{1}{4}||\Phi(A\nabla_\mu B)+Mm\Phi(A^{-1}\nabla_\mu B^{-1})||^p,\\
&\le \tfrac{1}{4}(M+m)^p.
\end{aligned}$$

Remark 2.9. Since $\frac{(M+m)^2}{4MmK(\sqrt{h'},2)^R}\le\frac{(M+m)^2}{4Mm}$ and $\frac{(M+m)^2}{4^{\frac{2}{p}}MmK(\sqrt{h'},2)^R}\le\frac{(M+m)^2}{4^{\frac{2}{p}}Mm}$, so under a stronger condition as Theorem 2.8, we see (8) and (9) are refinements of (6) and (7), respectively.

Corollary 2.10. Let $0<m\le A\le m'<M'\le B\le M$, $\mu\in[0,1]$ and $p>0$, we have

$$(A\nabla_\mu B + 2rMm(A^{-1}\nabla B^{-1} - A^{-1}\sharp B^{-1}))^p \le \alpha^p (A\sharp_\mu B)^p$$

where $r = \min\{\mu, 1-\mu\}$, $h' = \frac{M'}{m'}$ and $\alpha = \max\left\{\frac{(M+m)^2}{4MmK(\sqrt{h'},2)^R}, \frac{(M+m)^2}{4^{\frac{2}{p}}MmK(\sqrt{h'},2)^R}\right\}$.

Proof. Put $\Phi(A) = A$ for all $A \in B(\mathcal{H})$ in Theorem 2.3, we then get the desired result.

Theorem 2.11. Let Φ be a strictly unital positive linear map, $0 < m \le A_i \le M$ for $i = 1, \cdots n$, $\omega = (\omega_1, \cdots, \omega_n)$ be a probability vector, $t \in [-1, 0)$. Then we have

$$\Lambda(\omega; \Phi(\mathbb{A})) \le \tfrac{(m+M)^2}{4mM}\Phi(\Lambda(\omega; \mathbb{A})). \tag{13}$$

Proof. By Proposition 1.1 and $0 < m \le A_i \le M$ we have

$$\sum_{i=1}^n \omega_i A_i + Mm(\sum_{i=1}^n \omega_i A_i^{-1}) \le M + m.$$

First we show

$$(\sum_{i=1}^n \omega_i A_i)^2 \le (\tfrac{(m+M)^2}{4mM})^2 (\sum_{i=1}^n \omega_i A_i^{-1})^{-2}.$$

This inequality equals to

$$\| \sum_{i=1}^n \omega_i A_i \sum_{i=1}^n \omega_i A_i^{-1} \| \le \frac{(M+m)^2}{4Mm}.$$

Note that

$$\begin{aligned} &\|(\sum_{i=1}^n \omega_i A_i) Mm (\sum_{i=1}^n \omega_i A_i^{-1})\| \\ &\le \tfrac{1}{4}\|\sum_{i=1}^n \omega_i A_i + Mm(\sum_{i=1}^n \omega_i A_i^{-1})\|^2 \\ &\le \tfrac{1}{4}(M+m)^2. \end{aligned}$$

Use Lemma 2.5 we get

$$\sum_{i=1}^n \omega_i A_i \le \tfrac{(m+M)^2}{4mM}(\sum_{i=1}^n \omega_i A_i^{-1})^{-1}. \tag{14}$$

Thus by Proposition 1.1 and (14) we get

$$\Lambda(\omega; \Phi(\mathbb{A})) \le \sum_{i=1}^n \omega_i \Phi(A_i) = \Phi(\sum_{i=1}^n \omega_i A_i) \le \frac{(m+M)^2}{4mM}\Phi((\sum_{i=1}^n \omega_i A_i^{-1})^{-1}) \le \frac{(m+M)^2}{4mM}\Phi(\Lambda(\omega; \mathbb{A})).$$

Remark 2.12. Since $\Phi(\Lambda(\omega; \mathbb{A})) \le \Lambda(\omega; \Phi(\mathbb{A}))$ for any positive unital linear map, we get a reversed version of this inequality by Theorem 2.11.

Next we give a squaring version of inequality (13).

Theorem 2.13. Suppose all the assumptions of Theorem 2.11 be satisfied. Then

$$(\Lambda(\omega; \Phi(\mathbb{A})))^2 \le \psi \Phi^2(\Lambda(\omega; \mathbb{A}))$$

where $\psi = \begin{cases} \frac{K(\frac{M}{m},2)^2(M+m)^2}{4Mm} & for\ m \le t_0 \\ \frac{K(\frac{M}{m},2)(M+m)-M}{m} & for\ m \ge t_0 \end{cases}$, $t_0 = \frac{2Mm}{K(\frac{M}{m},2)(M+m)}$ and $K(\frac{M}{m},2) = \frac{(M+m)^2}{4Mm}$.

Proof. According to the assumption one can see that

$$m \le \Phi(\Lambda(\omega;\mathbb{A})) \le M \tag{15}$$

and

$$m \le \Lambda(\omega;\Phi(\mathbb{A})) \le M \tag{16}$$

inequality (15) implies

$$\Phi^2(\Lambda(\omega;\mathbb{A})) \le (M+m)\Phi(\Lambda(\omega;\mathbb{A})) - Mm,$$

and inequality (16) give us

$$\Lambda^2(\omega;\Phi(\mathbb{A})) \le (M+m)\Lambda(\omega;\Phi(\mathbb{A})) - Mm.$$

Hence

$$\begin{aligned} &\Phi^{-1}(\Lambda(\omega;\mathbb{A}))\Lambda^2(\omega;\Phi(\mathbb{A}))\Phi^{-1}(\Lambda(\omega;\mathbb{A})) \\ &\le \Phi^{-1}(\Lambda(\omega;\mathbb{A}))((M+m)\Lambda(\omega;\Phi(\mathbb{A})) - Mm)\Phi^{-1}(\Lambda(\omega;\mathbb{A})) \\ &\le (K(\tfrac{M}{m},2)(M+m)\Phi(\Lambda(\omega;\mathbb{A})) - Mm)\Phi^{-2}(\Lambda(\omega;\mathbb{A})) \end{aligned} \tag{17}$$

where the second inequality is derived by Theorem 2.11.

Consider the real function $f(t)$ on $(0,\infty)$ defined as

$$f(t) = \frac{K(\frac{M}{m},2)(M+m)t - Mm}{t^2}.$$

As a matter of fact, the inequality (17) implies that

$$\Phi^{-1}(\Lambda(\omega;\mathbb{A}))\Lambda^2(\omega;\Phi(\mathbb{A}))\Phi^{-1}(\Lambda(\omega;\mathbb{A})) \le \max_{m\le t\le M} f(t).$$

Notice that

$$f(m) \ge f(M)$$

and

$$f'(t) = \frac{2Mm - K(\frac{M}{m},2)(M+m)t}{t^3}.$$

The function has an maximum point on

$$t_0 = \frac{2Mm}{K(\frac{M}{m},2)(M+m)}$$

with the maximum value

$$f(t_0) = \frac{K(\frac{M}{m},2)^2(M+m)^2}{4Mm}.$$

Whence

$$\max_{m\le t\le M} f(t) = \begin{cases} f(t_0) & for\ m \le t_0 \\ f(m) & for\ m \ge t_0. \end{cases}$$

Notice that

$$f(m) = \frac{K(\frac{M}{m},2)(M+m)-M}{m}.$$

This completes the proof.

References

[1] F. Kubo, T. Ando, *Means of positive linear operators*, Math. Ann., **246** (1980), 205-224.
[2] R. Bhatia, *Positive definite matrices*, Princeton (NJ): Princeton University Press, (2007).
[3] R. Bhatia, F. Kittaneh, *Notes on matrix arithmetic-geometric mean inequalities*, Linear Algebra Appl., **308** (2000), 203–211.
[4] M. Bakherad, *Refinements of a reversed AM-GM operator inequality*, Linear Multilinear Algebra, **64** (2016), 1687-1695.
[5] T. Furuta, J. Mićić Hot, J. Pečarić and Y. Seo, *Mond-Pečarić method in operator inequalities*, Monographs in inequalities 1, Element, Zágreb, (2005).
[6] X. Fu, C. He, *Some operator inequalities for positive linear maps*, Linear Multilinear Algebra, **63** (2015), 571-577.
[7] Y. Lim , M. Pálfia, *Matrix power means and the Karcher mean*, J.Funct.Anal., **262** (2012), 1498-1514.
[8] M. Lin, *Squaring a reverse AM-GM inequality*, Studia Math., **215** (2013), 187-194.
[9] X. Zhao, L. Li and H Zuo *Operator iteration on the Young inequality*, Journal of Inequalities and Applications, Doi: 10.1186/s13660-016-1249-z.

Chaojun Yang
Department of Mathematics, Soochow University, Suzhou 215006, P. R. China
E-mail address: cjyangmath@163.com

Fangyan Lu
Department of Mathematics, Soochow University, Suzhou 215006, P. R. China
E-mail address: fylu@suda.edu.cn

Locally and globally small Riemann sums and Henstock integral of fuzzy-number-valued functions in E^n

Muawya Elsheikh Hamid[a,b*], Luoshan Xu [a]
[a] School of Mathematical Science, Yangzhou University, Yangzhou 225002, P.R. China
[b] School of Management, Ahfad University for Women, Omdurman, Sudan

Abstract: In this paper, the notions of locally and globally small Riemann sums modifications with respect to a fuzzy-number-valued functions in E^n are introduced and studied. The basic properties and characterizations are presented. In particular, it is proved that a fuzzy-number-valued functions in E^n is Henstock (H) integrable on $[a, b]$ if and only if it has $(LSRS)$, and also it is proved that a fuzzy-number-valued functions in E^n is Henstock (H) integrable on $[a, b]$ if and only if it has $(GSRS)$.

Keywords: Fuzzy-number-valued functions in E^n; Henstock integral (H); Locally small Riemann sums $(LSRS)$; Globally small Riemann sums $(GSRS)$.

1 Introduction

Since the concept of fuzzy sets was firstly introduced by Zadeh in 1965 [12], it has been studied extensively from many different aspects of the theory and applications, such as fuzzy topology, fuzzy analysis, fuzzy decision making and fuzzy logic, information science and so on.

The locally and globally small Riemann sums have been introduced by many authors from different points of views including [2, 3, 5, 6, 8, 9]. In 1986, Schurle characterized the Lebesgue integral in $(LSRS)$ (locally small Riemann sums) property [8]. The $(LSRS)$ property has been used to characterized the Perron (P) integral on $[a, b]$ [9]. By considering the equivalency between the (P) integral and the Henstock-Kurzweil (HK) integral, the $(LSRS)$ property has been used to characterized the (HK) integral on $[a, b]$ [6].

The $(LSRS)$ property brought a research to have global characterization on the Riemann sums of an (HK) integrable function on $[a, b]$. This research has been done by considering the following fact: Every (HK) integrable function on $[a, b]$ is measurable, however, there is no guarantee the boundedness of the function. A measurable function f is (HK) integrable on $[a, b]$ depends on it behaves on the set of x in which $|f(x)|$ is large, i.e. $|f(x)| \geq N$ for some N. This fact has been characterized in $(GSRS)$ (globally small Riemann sums) property [6]. The $(GSRS)$ property involves one characteristic of the primitive of an (HK) integrable function. That is the primitive of the (HK) integral on $[a, b]$ is ACG* (generalized strongly absolutely continuous) on $[a, b]$. This is not a simple concept. In 2015, Indrati [5] introduced a countably Lipschitz condition of a function which is simpler than the ACG*, and proved that the (HK) integrable function or it's primitive could be characterized in countably Lipschitz condition. Also, by considering the characterization of the (HK) integral in the $(GSRS)$ property, it showed that the relationship between $(GSRS)$ property and countably Lipschitz condition of an (HK) integrable function on $[a, b]$.

In 2018, Hamid et al. [2] investigated locally and globally small Riemann sums for fuzzy-number-valued functions and proved two main theorems: (1) A fuzzy-number-valued functions $\tilde{f}(x)$ is Henstock integrable on $[a, b]$ if and only if $\tilde{f}(x)$ has $(LSRS)$. (2) A fuzzy-number-valued functions $\tilde{f}(x)$ is Henstock integrable on $[a, b]$ if and only if $\tilde{f}(x)$ has $(GSRS)$.

*Corresponding author. Tel.: +8613218977118. E-mail address: muawya.ebrahim@gmail.com, mowia-84@hotmail.com (M.E. Hamid).

In this paper, we generalize locally and globally small Riemann sums from fuzzy-valued functions to n-dimensional fuzzy-numbers by means of support function. The notions of locally small Riemann sums for n-dimensional fuzzy-number-valued functions are presented and discussed. Finally, we provide a characterizations of globally small Riemann sums in n-dimensional fuzzy-number-valued functions.

The rest of this paper is organized as follows, in Section 2 we shall review the relevant concepts and properties of fuzzy-number-valued functions in E^n and the definition of Henstock integrals for fuzzy-number-valued functions in E^n. Section 3 is devoted to discussing the support function characterizations of locally small Riemann sums and Henstock integral for fuzzy-number-valued functions in E^n. In section 4 we shall investigate the support function characterizations of globally small Riemann sums and Henstock integral for fuzzy-number-valued functions in E^n. The last section provides the Conclusions.

2 Preliminaries

In this paper the close interval $[a, b]$ denotes a compact interval on R. The set of intervals-point $\{([a_1, b_1], \xi_1), ([a_2, b_2], \xi_2), \cdots, ([a_k, b_k], \xi_k)\}$ is called a division of $[a, b]$ that is $\xi_1, \xi_2, \cdots, \xi_k \in [a, b]$, intervals $[a_1, b_1], [a_2, b_2], \cdots, [a_k, b_k]$ are non-intersect and $\bigcup\limits_{i=1}^{k} [a_i, b_i] = [a, b]$. Marking the division of $[a, b]$ as $P = \{([a_1, b_1], \xi_1), ([a_2, b_2], \xi_2), \cdots, ([a_k, b_k], \xi_k)\}$, shortening as $P = \{[u, v]; \xi\}$ [7].

Definition 2.1 [4, 6] Let $\delta : [a, b] \to \mathbb{R}^+$ be a positive real-valued function. $P = \{[x_{i-1}, x_i]; \xi_i\}$ is said to be a δ-fine division, if the following conditions are satisfied:

(1) $a = x_0 < x_1 < x_2 < ... < x_n = b$;

(2) $\xi_i \in [x_{i-1}, x_i] \subset (\xi_i - \delta(\xi_i), \xi_i + \delta(\xi_i))(i = 1, 2, \cdots, n)$.

For brevity, we write $P = \{[u, v]; \xi\}$, where $[u, v]$ denotes a typical interval in P and ξ is the associated point of $[u, v]$.

Definition 2.2 [11] E^n is said to be a fuzzy number space if $E^n = \{u :\ R^n \to [0, 1] : u$ satisfies (1)-(4) below$\}$:

(1) u is normal, i.e., there exists a $x_0 \in R^n$ such that $u(x_0) = 1$;

(2) u is a convex fuzzy set, i.e., $u(rx + (1 - r)y) \geqslant \min(u(x), u(y)),\ x, y \in R^n,\ r \in [0, 1]$;

(3) u is upper semi-continuous;

(4) $[u]^0 = \overline{\{x \in R^n : u(x) > 0\}}$ is compact, for $0 < r \leq 1$, denote $[u]^r = \{x : x \in R^n \ and\ u(x) \geqslant r\}$, $[u]^0 = \overline{\bigcup_{r\in(0,1]} [u]^r}$.

Form (1)-(4), it follows that for any $u \in E^n$ and $r \in [0, 1]$ the $r-$level set $[u]^r$ is a compact convex set. For any $u, v \in E^n$

$$D(u, v) = \sup_{r\in[0,1]} d([u]^r, [v]^r), \tag{2.1}$$

where d is Hausdorff metric. It is well known that (E^n, d) is an metric space [11]. The norm of fuzzy number $u \in E^n$ is defined by

$$\|u\| = D(u, \tilde{0}) = \sup_{\alpha\in[u]^0} |\alpha|, \tag{2.2}$$

where the $\| \cdot \|$ is norm on E^n, $\tilde{0}$ is fuzzy number on E^n and $\tilde{0} = \chi_{\{0\}}$.

Definition 2.3 [11] For $A \in P_k(R^n),\ x \in S^{n-1}$, define the support function of A as $\sigma(x, A) = \sup\limits_{y\in A}\langle y, x\rangle$, where$S^{n-1}$ is the unit sphere of R^n, i.e., $S^{n-1} = \{x \in R^n : \|x\| = 1\}$, $\langle\cdot, \cdot\rangle$ is the inner product in R^n.

Definition 2.4 [10] A fuzzy-number-valued function $\tilde{f} : [a, b] \to E^n$ is said to be Henstock integrable to $\tilde{A} \in E^n$ if for every $\varepsilon > 0$, there is a function $\delta(t) > 0$ such that for any δ-fine division $P = \{[u, v]; \xi\}$ of $[a, b]$, we have

$$D\big(\sum \tilde{f}(\xi)(v - u), \tilde{A}\big) < \varepsilon, \tag{2.3}$$

where the sum $\sum$ is understood to be over P and we write $(FH) \int\limits_a^b \tilde{f}(t)dt = \tilde{A}$, and $\tilde{f}(t) \in FH[a, b]$.

Lemma 2.1 [11] If $u, v \in E^n,\ k \in R,$ for any $r \in [0,1]$, we have

$$[u+v]^r = [u]^r + [v]^r,\ [ku]^r = k[u]^r. \tag{2.4}$$

Lemma 2.2 [11] Suppose $u \in E^n$, then
(1) $u^*(r, x+y) \le u^*(r,x) + u^*(r,y)$,
(2) if $u, v \in E^n$, $r \in [0,1]$, then

$$d([u]^r, [v]^r) = \sup_{x \in S^{n-1}} |u^*(r,x) - v^*(r,x)|, \tag{2.5}$$

(3) $(u+v)^*(r,x) = u^*(r,x) + v^*(r,x)$,
(4) $(ku)^*(r,x) = ku^*(r,x),\ k \ge 0$.

Lemma 2.3 [1, 11] Given $u, v \in E^n$ the distance $D : E^n \times E^n \to [0, +\infty)$ between u and v is defined by the equation $D(u,v) = \sup_{r \in [0,1]} d([u]^r, [v]^r)$, then
(1) (E^n, D) is a complete metric space,
(2) $D(u+w, v+w) = D(u,v)$,
(3) $D(u+v, w+e) \leqslant D(u,w) + D(v,e)$,
(4) $D(ku, kv) = |k| D(u,v), k \in R$,
(5) $D(u+v, \tilde{0}) \leqslant D(u, \tilde{0}) + D(v, \tilde{0})$,
(6) $D(u+v, w) \leqslant D(u,w) + D(v, \tilde{0})$.
Where $u, v, w, e, \widetilde{0} \in E^n,\ \widetilde{0} = \mathcal{X}_{(\{0\})}$.

Lemma 2.4 [1] If $\tilde{f} : [a,b] \to E^n$, then the following statements are equivalent:
(1) $\tilde{f}$ is (FH) integrable.
(2) $f^*(\xi)(r,x)$ is (RH) integrable for any $r \in [0,1]$ uniformly, i.e., for every $\varepsilon > 0$ there is a $\delta(\xi) > 0$ which is independent of $r \in [0,1]$, such that for any δ-fine division $P = \{[u,v]; \xi\}$ and $r \in [0,1]$ we have

$$|\sum f^*(\xi)(r,x)(v-u) - A^*(r,x)| < \varepsilon. \tag{2.6}$$

3 Support function characterizations of locally small Riemann sums and Henstock integral for fuzzy-number-valued functions in E^n

In this section, we define the locally small Riemann sums for fuzzy-number-valued functions in n-dimensional and investigate their properties. We star with the following definition.

Definition 3.1 A fuzzy-number-valued function $\tilde{f} : [a,b] \to E^n$ is said to be have locally small Riemann sums or $(LSRS)$ if for every $\varepsilon > 0$ there is a $\delta(\xi) > 0$ such that for every $t \in [a,b]$, we have

$$||\sum \tilde{f}(\xi)(v-u)||_{E^n} < \varepsilon, \tag{3.1}$$

whenever $P = \{[u,v]; \xi\}$ is a δ-fine division of an interval $C \subset (t - \delta(t), t + \delta(t))$, $t \in C$ and Σ sums over P. (Where $C = [y, z]$).

The following Theorem 3.1 shows that $\tilde{f}$ has $(LSRS)$ is equal to the type of it's support functions.

Theorem 3.1 Let $\tilde{f} : [a,b] \to E^n$ be a fuzzy-number-valued function, the support-function-wise $f^*(\xi)(r,x)$ of $\tilde{f}$ has locally small Riemann sums or $(LSRS)$ if and only if for every $\varepsilon > 0$, there is a $\delta(\xi) > 0$ such that for every $t \in [a,b]$, we have

$$|\sum f^*(\xi)(r,x)(v-u)| < \varepsilon, \tag{3.2}$$

uniformly for any $r \in [0,1]$ and $x \in S^{n-1}$, whenever $P = \{[u,v]; \xi\}$ is a δ-fine division of an interval $C \subset (t - \delta(t), t + \delta(t))$, $t \in C$ and Σ sums over P.

Proof Let $\tilde{0} \in E^n$ denote the (FH) integral of $\tilde{f}$ on $[a,b]$. Given $\varepsilon > 0$ there is a $\delta(\xi) > 0$ such that for any δ-fine division $P = \{[u,v];\xi\}$ of $[a,b]$, we have

$$D(\sum \tilde{f}(\xi)(v-u), \tilde{0}) < \varepsilon. \tag{3.3}$$

That is

$$\sup_{r\in[0,1]} d([\sum \tilde{f}(\xi)(v-u)]^r, [\tilde{0}]^r) < \varepsilon. \tag{3.4}$$

By Lemma 2.2 we have

$$\sup_{r\in[0,1]} \sup_{x\in S^{n-1}} |(\sum \tilde{f}(\xi)(v-u))^*(r,x) - \sigma(x,0)| < \varepsilon. \tag{3.5}$$

Furthermore, by $\sigma(x,A) = \sup\limits_{y\in A}\langle y,x\rangle$, we have

$$\sup_{r\in[0,1]} \sup_{x\in S^{n-1}} |\sum f^*(\xi)(r,x)(v-u) - \sigma(x,0)| < \varepsilon. \tag{3.6}$$

Hence, for any $r \in [0,1]$, $x \in S^{n-1}$ and for any δ-fine division P we have

$$|\sum f^*(\xi)(r,x)(v-u)| < \varepsilon. \tag{3.7}$$

Where $\sigma(x,0) = 0$.
This completes the proof. □

Lemma 3.1 (Henstock Lemma). Let $\tilde{f} : [a,b] \to E^n$ be a fuzzy-number-valued function and Henstock integrable to $\tilde{A}$. Then, the support-function-wise $f^*(\xi)(r,x)$ of $\tilde{f}$ on $[a,b]$ is Henstock integrable to $A^*(r,x)$ uniformly for any $r \in [0,1]$, $x \in S^{n-1}$ and $\tilde{A} \in E^n$, i.e., for every $\varepsilon > 0$ there is a positive function $\delta(\xi) > 0$, for δ-fine division $P = \{[u,v];\xi\}$ of $[a,b]$ and for any $x \in S^{n-1}$, we have

$$|\sum f^*(\xi)(r,x)(v-u) - A^*(r,x)| < \varepsilon. \tag{3.8}$$

Furthermore, for any sum of parts $\sum\limits_1$ from $\sum$ we have

$$|\sum_1 f^*(\xi)(r,x)(v-u) - A^*(r,x)| < \varepsilon. \tag{3.9}$$

Proof Let $\tilde{A} \in E^n$ denote the (FH) integral of $\tilde{f}$ on $[a,b]$. Given $\varepsilon > 0$ there is a $\delta(\xi) > 0$ such that for any δ-fine division $P = \{[u,v];\xi\}$ of $[a,b]$, we have

$$D(\sum \tilde{f}(\xi)(v-u), \tilde{A}) < \varepsilon. \tag{3.10}$$

That is

$$\sup_{r\in[0,1]} d([\sum \tilde{f}(\xi)(v-u)]^r, [\tilde{A}]^r) < \varepsilon. \tag{3.11}$$

By Lemma 2.2 we have

$$\sup_{r\in[0,1]} \sup_{x\in S^{n-1}} |(\sum \tilde{f}(\xi)(v-u))^*(r,x) - A^*(r,x)| < \varepsilon. \tag{3.12}$$

Furthermore, by $A^*(r,x) = \sup\limits_{y\in[A]^r}\langle y,x\rangle$, we have

$$\sup_{r\in[0,1]} \sup_{x\in S^{n-1}} |\sum f^*(\xi)(r,x)(v-u) - A^*(r,x)| < \varepsilon. \tag{3.13}$$

Hence, for any $r \in [0,1]$, $x \in S^{n-1}$ and for any δ-fine division P we have

$$|\sum f^*(\xi)(r,x)(v-u) - A^*(r,x)| < \varepsilon.$$

For proof

$$|\sum_1 f^*(\xi)(r,x)(v-u) - A^*(r,x)| < \varepsilon, \tag{3.14}$$

the proof is similar to the Theorem 3.7 in [6].
This completes the proof. □

Hamid et al. [2] showed that if a fuzzy-number-valued functions $\tilde{f}(x)$ is Henstock integrable on $[a,b]$ then $\tilde{f}(x)$ has $LSRS$. In next Theorem, we prove the above result to n-dimensional fuzzy-number-valued functions, which is an extension of the above result of Muawya et al. [2].

Theorem 3.2 Let $\tilde{f} : [a, b] \to E^n$ be a fuzzy-number-valued function. If $\tilde{f}$ is Henstock integrable to $\tilde{F}([a, b])$, then $\tilde{f}$ has $LSRS$.

Proof Since $\tilde{f}$ is Henstock integrable to $\tilde{F}([a, b])$, by Theorem 3.1 the support-function-wise $f^*(\xi)(r, x)$ of $\tilde{f}$ on $[a, b]$ is Henstock integrable to $F^*([a, b])(r, x)$ uniformly for any $r \in [0, 1]$, $x \in S^{n-1}$, i.e., for every $\varepsilon > 0$ there is a positive function $\delta(\xi) > 0$, for δ-fine division $P = \{[u, v]; \xi\}$ of $[a, b]$ and for any $x \in S^{n-1}$, we have

$$|\sum f^*(\xi)(r, x)(v - u) - F^*([a, b])(r, x)| < \frac{\varepsilon}{2}. \tag{3.15}$$

For each $t \in [a, b]$, there is a closed interval $C = [y, z] \subset (t - \delta(t), t + \delta(t))$ such that

$$|F^*([y, z])(r, x)| < \frac{\varepsilon}{2}. \tag{3.16}$$

According to Henstock lemma, for each $t \in [a, b]$ and δ-fine division $P = \{[u, v]; \xi\}$ of $C \subset (t - \delta(t), t + \delta(t))$, we have

$$\begin{aligned} |\sum f^*(\xi)(r, x)(v - u)| &\leq |\sum f^*(\xi)(r, x)(v - u) - F^*([a, b])(r, x)| + |F^*([y, z])(r, x)| \\ &< \varepsilon. \end{aligned}$$

Applies Theorem 3.1 again $\tilde{f}$ has $LSRS$.
This completes the proof. □

Lemma 3.2 Let $\tilde{f} : [a, b] \to E^n$ be a fuzzy-number-valued function. If $\tilde{f}$ is (FH) integrable with the $\tilde{F}$ as primitive then for each number $\varepsilon > 0$ there is a positive function $\delta(\xi) > 0$, such that for any $[u, v] \subset [a, b]$ with $v - u < \delta(\xi)$, we have

$$\|\tilde{F}([u, v])\|_{E^n} = \|(FH)\int_{[u,v]} \tilde{f}\mathrm{d}x\|_{E^n} < \varepsilon. \tag{3.17}$$

Proof The continuity follows from Lemma 3.1 and the following inequality:

$$\begin{aligned} \|\tilde{F}(t) - \tilde{F}(\xi)\|_{E^n} &\leq \|\tilde{F}(t) - \tilde{F}(\xi) - \tilde{f}(\xi)(t - \xi)\|_{E^n} + \|\tilde{f}(\xi)(t - \xi)\|_{E^n} \\ &< \varepsilon. \end{aligned}$$

We only need set $\delta(\xi) < \frac{\varepsilon}{2(\|\tilde{f}(\xi)\|_{E^n}+1)}$.
This completes the proof. □

Theorem 3.3 Let a fuzzy-number-valued function $\tilde{f} : [a, b] \to E^n$ has $LSRS$, then $\tilde{f}$ is (FH) integrable on $[a, b]$.

Proof Given any $\varepsilon > 0$ and $P = \{([a, b], \xi)\} = \{([a_1, b_1],\ \xi_1),\ ([a_2, b_2], \xi_2), \cdots, ([a_n, b_n], \xi_n)\}$ is a δ-fine partition of $[a, b]$. For each $i (i = 1, 2, \cdots, n)$ there is a positive function δ_i with $P_i = \{([u_i, v_i], \xi_i)\}$ is a δ_i-fine partition of $[a_i, b_i]$. Since $\tilde{f}$ has $LSRS$ on $[a_i, b_i]$, then we have

$$\|\sum_{P_i} \tilde{f}(\xi)(v - u)\|_{E^n} < \frac{\varepsilon}{2n}. \tag{3.18}$$

Taken $\eta = \max\{\delta(\xi),\ \xi \in [a, b]\}$, according to the Lemma 3.2 we have

$$\|\tilde{F}([a_i, b_i])\|_{E^n} = \|(FH)\int_{[a_i,b_i]} \tilde{f}\mathrm{d}x\|_{E^n} < \frac{\varepsilon}{2n}. \tag{3.19}$$

Therefore, for any δ_i-fine partition $P_i = \{([u_i, v_i], \xi_i)\}$ of $[a_i, b_i]$, we have

$$\begin{aligned} D(\sum_{P_i} \tilde{f}(\xi)(v - u), \tilde{F}([a_i, b_i])) &\leq \|\sum_{P_i} \tilde{f}(\xi)(v - u)\|_{E^n} + \|\tilde{F}([a_i, b_i])\|_{E^n} \\ &< \frac{\varepsilon}{2n} + \frac{\varepsilon}{2n} = \frac{\varepsilon}{n}, \end{aligned}$$

for each i.

Subsequently taken $\delta^*(\xi) = \min\{\delta(\xi), \delta_i(\xi)\}$, then $P = \bigcup_{i=1}^{n} P_i$ denote δ^*-fine partition of $[a, b]$.

Therefore we have

$$\begin{aligned} D(\sum_{P} \tilde{f}(\xi)(v - u), \tilde{F}([a, b])) &= \sum_{i=1}^{n} D(\sum_{P_i} \tilde{f}(\xi)(v - u), \tilde{F}([a_i, b_i])) \\ &< n \cdot \frac{\varepsilon}{n} = \varepsilon. \end{aligned}$$

Then $\tilde{f}$ is FH integral on $[a, b]$.
This completes the proof. □

4 Support function characterizations of globally small Riemann sums and Henstock integral for fuzzy-number-valued functions in E^n

The main purpose in this part is to introduce the concept of globally small Riemann sums for fuzzy-number-valued functions in n-dimensional and discuss their properties. We begin with the following definition.

Definition 4.1 A fuzzy-number-valued function $\tilde{f} : [a,b] \to E^n$ is said to be have globally small Riemann sums or $(GSRS)$ if for every $\varepsilon > 0$ there exists a positive integer N such that for every $n \geqslant N$ there is a $\delta_n(\xi) > 0$ and for every δ_n-fine division $P = \{[u,v];\xi\}$ of $[a,b]$, we have

$$\| \sum_{\|\tilde{f}(\xi)\|_{E^n} > n} \tilde{f}(\xi)(v-u)\|_{E^n} < \varepsilon, \tag{4.1}$$

where the $\sum$ is taken over P and for which $\|\tilde{f}(\xi)\|_{E^n} >_n$.

The following Theorem 4.1 shows that $\tilde{f}$ has $(GSRS)$ is equal to the type of it's support functions.

Theorem 4.1 Let $\tilde{f} : [a,b] \to E^n$ be a fuzzy-number-valued function, the support-function-wise $f^*(\xi)(r,x)$ of $\tilde{f}$ has globally small Riemann sums or $(GSRS)$ if and only if for every $\varepsilon > 0$, there exists a positive integer N such that for every $n \geqslant N$ there is a $\delta_n(\xi) > 0$ and for every δ_n-fine division $P = \{[u,v];\xi\}$ of $[a,b]$, we have

$$\Big| \sum_{|f^*(\xi)(r,x)| > n} f^*(\xi)(r,x)(v-u)\Big| < \varepsilon, \tag{4.2}$$

uniformly for any $r \in [0,1]$ and $x \in S^{n-1}$, where the $\sum$ is taken over P and for which $|f^*(\xi)(r,x)| > n$.

Proof First, we can prove the following statements are equivalent:

(1) $\|\tilde{f}(\xi)\|_{E^n} > n$.

(2) $|f^*(\xi)(r,x)| > n$.

In fact

$$\begin{aligned} \|\tilde{f}(\xi)\|_{E^n} > n &= \sup_{r\in[0,1]} d([\tilde{f}(\xi)]^r, [\tilde{0}]^r) \\ &= \sup_{r\in[0,1]} \sup_{x \in S^{n-1}} |f^*(\xi)(r,x)|. \end{aligned}$$

Second, let $\tilde{0} \in E^n$ denote the (FH) integral of $\tilde{f}$ on $[a,b]$. Given $\varepsilon > 0$ there exists a positive integer N such that for every $n \geqslant N$ there is a $\delta_n(\xi) > 0$ and for every δ_n-fine division $P = \{[u,v];\xi\}$ of $[a,b]$, we have

$$D(\sum_{\|\tilde{f}(\xi)\|_{E^n} > n} \tilde{f}(\xi)(v-u), \tilde{0}) < \varepsilon. \tag{4.3}$$

That is

$$\sup_{r\in[0,1]} d([\sum_{\|\tilde{f}_r(\xi)\|_{E^n} > n} \tilde{f}(\xi)(v-u)]^r, [\tilde{0}]^r) < \varepsilon. \tag{4.4}$$

By Lemma 2.2 we have

$$\sup_{r\in[0,1]} \sup_{x\in S^{n-1}} |(\sum_{|f^*(\xi)(r,x)| > n} f(\xi)(v-u))^*(r,x) - \sigma(x,0)| < \varepsilon. \tag{4.5}$$

Furthermore, by $\sigma(x,A) = \sup_{y\in A} \langle y, x\rangle$, we have

$$\sup_{r\in[0,1]} \sup_{x\in S^{n-1}} | \sum_{|f^*(\xi)(r,x)| > n} f^*(\xi)(r,x)(v-u)) - \sigma(x,0)| < \varepsilon. \tag{4.6}$$

Hence, for any $r \in [0,1]$, $x \in S^{n-1}$ and for any δ-fine division P we have

$$| \sum_{|f^*(\xi)(r,x)| > n} f^*(\xi)(r,x)(v-u)| < \varepsilon. \tag{4.7}$$

Where $\sigma(x,0)=0$.
This completes the proof. □

Hamid et al. [2] investigated that a fuzzy-number-valued functions $\tilde{f}(x)$ is Henstock integrable on $[a,b]$ if and only if $\tilde{f}(x)$ has $GSRS$. In next Theorem 4.3, we extend this result to n-dimensional fuzzy-number-valued functions. To prove this result, we need to prove the following Theorem.

Theorem 4.2 Let $\tilde{f}:[a,b]\to E^n$ be a fuzzy-number-valued function. If $\tilde{f}$ has $GSRS$ then $\tilde{f}$ is Henstock integrable on $[a,b]$.

Proof Because $\tilde{f}$ has $GSRS$, then by Theorem 4.1 for every $\varepsilon>0$, there exists a positive integer N such that for every $n\geqslant N$ there is a $\delta_n(\xi)>0$ and for every δ_n-fine division $P=\{[u,v];\xi\}$ of $[a,b]$, we have

$$|\sum_{|f^*(\xi)(r,x)|>n} f^*(\xi)(r,x)(v-u)|<\varepsilon. \tag{4.8}$$

uniformly for any $r\in[0,1]$ and $x\in S^{n-1}$, where the $\sum$ is taken over P and for which $|f^*(\xi)(r,x)|>n$.

For each two δ-fine divisions $P_1=\{[u_1,v_1];\xi_1\}$, $P_2=\{[u_2,v_2];\xi_2\}$ of $[a,b]$, we have

$$\begin{aligned}
& |\sum f^*(\xi_1)(r,x)(v_1-u_1)-\sum f^*(\xi_2)(r,x)(v_2-u_2)| \\
\le\ & |\sum f^*(\xi_1)(r,x)(v_1-u_1)|+|\sum f^*(\xi_2)(r,x)(v_2-u_2)| \\
\le\ & |\sum_{|f^*(\xi_1)(r,x)|>n} f^*(\xi_1)(r,x)(v_1-u_1)|+|\sum_{|f^*(\xi_1)(r,x)|\le n} f^*(\xi_1)(r,x)(v_1-u_1)| \\
+\ & |\sum_{|f^*(\xi_2)(r,x)|>n} f^*(\xi_2)(r,x)(v_2-u_2)|+|\sum_{|f^*(\xi_2)(r,x)|\le n} f^*(\xi_2)(r,x)(v_2-u_2)| \\
<\ & 4\varepsilon.
\end{aligned}$$

According to the properties of Cauchy, $\tilde{f}$ is Henstock integrable on $[a,b]$.
This completes the proof. □

Theorem 4.3 Given a fuzzy-number-valued function $\tilde{f}:[a,b]\to E^n$, for each $r\in[0,1]$ and $x\in S^{n-1}$ defined the support function $f_n^*(\xi)(r,x)$ of $\tilde{f}_n$ by the formula:

$$f_n^*(\xi)(r,x)=\begin{cases} f^*(\xi)(r,x),\xi\in[a,b] & \text{if } |f^*(\xi)(r,x)|\le n, \\ 0, & \text{others.}\end{cases}$$

A fuzzy-number-valued function $\tilde{f}$ is Henstock integrable if and only if $\tilde{f}$ has $GSRS$ and $\tilde{F}_n([a,b])\to\tilde{F}([a,b])$ as $n\to\infty$. (Where $\tilde{F}([a,b])$ and $\tilde{F}_n([a,b])$ the integral of $\tilde{f}$ and $\tilde{f}_n$ respectively).

Proof First we shall prove the necessity. Because a fuzzy-number-valued function $\tilde{f}$ is Henstock integrable on $[a,b]$ uniformly for any $r\in[0,1]$ and $x\in S^{n-1}$, i.e., for every $\varepsilon>0$ there is a positive function δ^*, for δ^*-fine division $P=\{[u,v];\xi\}$ of $[a,b]$, we have

$$|\sum f^*(\xi)(r,x)(v-u)-F^*([a,b])(r,x)|<\frac{\varepsilon}{3}. \tag{4.9}$$

For each $n\in\mathbb{N}$, there is a positive function δ_n, for δ_n-fine division $P=\{[u,v];\xi\}$ of $[a,b]$, we have

$$|\sum f_n^*(\xi)(r,x)(v-u)-F_n^*([a,b])(r,x)|<\frac{\varepsilon}{3}, \tag{4.10}$$

for each $r\in[0,1]$ and $x\in S^{n-1}$.

Because $\{F_n^*([a,b])(r,x)\}$ converge to $F^*([a,b])(r,x)$ of $[a,b]$ then there is a positive number N so if $n\ge N$ we have

$$|F_n^*([a,b])(r,x)-F^*([a,b])(r,x)|<\frac{\varepsilon}{3}. \tag{4.11}$$

For $n\ge N$, defined a positive function δ on $[a,b]$ by the formula:

$$\delta(\xi)=\min\{\delta^*(\xi),\delta_n(\xi)\}. \tag{4.12}$$

Therefor, for each δ-fine division $P=\{[u,v];\xi\}$ of $[a,b]$, we have

$$\begin{aligned}
& |\sum_{|f^*(\xi)(r,x)|>n} f^*(\xi)(r,x)(v-u)| \\
= & |\sum f^*(\xi)(r,x)(v-u) - \sum f_n^*(\xi)(r,x)(v-u)| \\
\leq & |\sum f^*(\xi)(r,x)(v-u) - F^*([a,b])(r,x)| + |F_n^*([a,b])(r,x) - F^*([a,b])(r,x)| \\
+ & |F^*([a,b])(r,x) - \sum f_n^*(\xi)(r,x)(v-u)| \\
< & \frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\frac{\varepsilon}{3}=\varepsilon.
\end{aligned}$$

Then $\tilde{f}$ has $GSRS$.

Second we shall prove the sufficiency. Because $\tilde{f}$ has $GSRS$, then by Theorem 4.1 for every $\varepsilon>0$, there exists a positive integer N such that for every $n \geqslant N$ there is a $\delta_n(\xi)>0$ and for every δ_n-fine division $P=\{[u,v];\xi\}$ of $[a,b]$, we have

$$|\sum_{|f^*(\xi)(r,x)|>n} f^*(\xi)(r,x)(v-u)\big| < \varepsilon, \tag{4.13}$$

uniformly for any $r\in[0,1]$ and $x \in S^{n-1}$, where the $\sum$ is taken over P and for which $|f^*(\xi)(r,x)|>n$.

Note that $\tilde{f}_n$, is Henstock integrable on $[a,b]$ for all n. Choose N so that whenever $n,m \geqslant N$ we have

$$|F_n^*([a,b])(r,x) - F_m^*([a,b])(r,x)| < \varepsilon. \tag{4.14}$$

Then for $n,m \geqslant N$ and a suitably chosen δ-fine division $P=\{[u,v];\xi\}$, we have

$$\begin{aligned}
& |F_n^*([a,b])(r,x) - F_m^*([a,b])(r,x)| \\
\leq & |F_n^*([a,b])(r,x) - \sum_{|f^*(\xi)(r,x)|\leq n} f^*(\xi)(r,x)(v-u)| + |\sum_{|f^*(\xi)(r,x)|>n} f^*(\xi)(r,x)(v-u)| \\
+ & |\sum_{|f^*(\xi)(r,x)|\leq m} f^*(\xi)(r,x)(v-u) - F_m^*([a,b])(r,x)| + |\sum_{|f^*(\xi)(r,x)|>m} f^*(\xi)(r,x)(v-u)| \\
< & 4\varepsilon.
\end{aligned}$$

That is, $\{F_n^*([a,b])(r,x)\}$ converge to $F^*([a,b])(r,x)$, as $n\to\infty$. Again, for suitably chosen N and $\delta(\xi)$ and for every δ-fine division $P=\{[u,v];\xi\}$, we have

$$\begin{aligned}
& |\sum f^*(\xi)(r,x)(v-u) - F^*([a,b])(r,x)| \\
\leq & |\sum f^*(\xi)(r,x)(v-u) - F_N^*([a,b])(r,x)| + |F_N^*([a,b])(r,x) - F^*([a,b])(r,x)| \\
\leq & |\sum_{|f^*(\xi)(r,x)|\leq N} f^*(\xi)(r,x)(v-u) - F_N^*([a,b])(r,x)| + |\sum_{|f^*(\xi)(r,x)|>N} f^*(\xi)(r,x)(v-u)| \\
+ & |F_N^*([a,b])(r,x) - F^*([a,b])(r,x)| \\
< & 3\varepsilon.
\end{aligned}$$

That is, $\tilde{f}$ is Henstock integrable on $[a,b]$.
This completes the proof. □

5 conclusions

This paper introduces, first of all, the generalization of locally and globally small Riemann sums from fuzzy-valued functions to n-dimensional fuzzy-numbers by means of support function. In addition, the concept of locally small Riemann sums for n-dimensional fuzzy-number-valued functions is presented and discussed. Finally, an important result of this paper is a characterizations of globally small Riemann sums for n-dimensional fuzzy-number-valued functions.

References

[1] S.X. Hai, Z.T. Gong, On Henstock integral of fuzzy-number-valued functions in R^n, International Journal of Pure and Applied Mathematics, 7(1), 111-121(2003).

[2] M.E. Hamid, L.S. Xu, Z.T. Gong, Locally and globally small Riemann sums and Henstock integral of fuzzy-number-valued functions, Journal of Computational analysis and applications, 25(1), 11-18(2018).

[3] M.E. Hamid, L.S. Xu, Z.T. Gong, Locally and globally small Riemann sums and Henstock-Stieltjes integral of fuzzy- number-valued functions, Journal of Computational analysis and applications, 25(6), 1107-1115(2018).

[4] R. Henstock, Theory of Integration, Butterworth, London (1963).

[5] C.R. Indrati, Some Characteristics of the Henstock-Kurzweil in Countably Lipschitz Condition, The 7th SEAMS-UGM Conference (2015).

[6] P.Y. Lee, Lanzhou Lectures on Henstock Integration, World Scientific, Singapore (1989).

[7] P.Y. Lee, R. Vyborny, The Integral: An Easy Approach after Kurzweil and Henstock, Cambridge University Press (2000).

[8] A.W. Schurle, A new property equivalent to Lebesgue integrability, Proceedings of the American Mathematical Society, 96(1), 103-106(1986).

[9] A.W. Schurle, A function is Perron integrable if it has locally small Riemann sums, Journal of the Australian Mathematical Society (Series A), 41(2), 224-232(1986).

[10] C.X. Wu, Zengtai Gong, On Henstock integral of fuzzy-number-valued functions (I), Fuzzy Sets and Systems, 120, 523-532(2001).

[11] C.X. Wu, M. Ma, J.X. Fang, Structure Theory of Fuzzy Analysis, Guizhou Scientific Publication (1994), In Chinese.

[12] L.A. Zadeh, Fuzzy sets, Information Control, 8, 338-353(1965).

On systems of fractional Langevin equations of Riemann-Liouville type with generalized nonlocal fractional integral boundary conditions

Chatthai Thaiprayoon[a,*], Sotiris K. Ntouyas[b,c] and Jessada Tariboon[d,e]

[a] Department of Mathematics, Faculty of Science, Burapha University, Chonburi, 20131, Thailand
E-mail: chatthai@buu.ac.th

[b] Department of Mathematics, University of Ioannina, 451 10 Ioannina, Greece
E-mail: sntouyas@uoi.gr

[c] Nonlinear Analysis and Applied Mathematics (NAAM)-Research Group, Department of Mathematics, Faculty of Science, King Abdulaziz University, P.O. Box 80203, Jeddah 21589, Saudi Arabia

[d] Nonlinear Dynamic Analysis Research Center,
Department of Mathematics, Faculty of Applied Science,
King Mongkut's University of Technology North Bangkok,
Bangkok 10800, Thailand

[e] Centre of Excellence in Mathematics,
CHE, Sri Ayutthaya Rd., Bangkok 10400, Thailand
E-mail: jessada.t@sci.kmutnb.ac.th

Abstract

By applying Krasnoselskii's and O'Regan's fixed point theorems, in this paper, we study the existence of solutions for a coupled system consisting from Langevin fractional differential equations of Riemann-Liouville type subject to the generalized nonlocal integral boundary conditions. Examples illustrating our results are also presented.

Key words and phrases: Fractional differential equations, Krasnoselskii's fixed point theorem, O'Regan's fixed point theorem, generalized fractional integral.
AMS (MOS) Subject Classifications: 26A33; 34A08.

1 Introduction

In this paper we concentrate on the study of existence of solutions for a coupled system of Langevin fractional differential equations of Riemann-Liouville type subject to the generalized nonlocal integral boundary conditions of the form

$$\begin{cases} D^{p_1}\left(D^{p_2}+\lambda_1\right)x(t) = f(t,x(t),y(t)), \quad 0<t<T, \\ D^{q_1}\left(D^{q_2}+\lambda_2\right)y(t) = g(t,x(t),y(t)), \quad 0<t<T, \\ x(0)=0, \qquad x(\eta)=\displaystyle\sum_{i=1}^{n}\alpha_i \, {}^{\mu_i}I^{\gamma_i}x(\xi_i), \\ y(0)=0, \qquad y(\kappa)=\displaystyle\sum_{j=1}^{m}\beta_j \, {}^{\delta_j}I^{\phi_j}y(\zeta_j), \end{cases} \tag{1}$$

where D^{χ} is the Riemann-Liouville fractional derivative of order $\chi \in \{p_1,p_2,q_1,q_2\}$, ${}^{\mu_i}I^{\gamma_i}$, ${}^{\delta_j}I^{\phi_j}$ are the Katugampola fractional integrals of orders $\gamma_i,\phi_j>0$, respectively, $\xi_i,\zeta_j\in(0,T)$ and $\alpha_i,\beta_j\in\mathbb{R}$ for

*Corresponding author

all $i = 1, 2, \ldots, n, j = 1, 2, \ldots, m,\ f, g : [0, T] \times \mathbb{R}^2 \to \mathbb{R}$ are continuous functions and $\lambda_1,\ \lambda_2$ are given constants.

Fractional differential equations arise in many engineering and scientific disciplines as the mathematical modeling of systems and processes in the fields of physics, chemistry, control theory, biology, economics, etc. A comprehensive study of fractional calculus and its applications is introduced in several books (see [1]-[3]). Initial and boundary value problems of nonlinear fractional differential equations and inclusions have been addressed by several researchers. For some recent results on fractional differential equations we refer in a series of papers ([4]-[12]).

In fractional calculus, the fractional derivatives are defined via fractional integrals. There are several known forms of the fractional integrals which have been studied extensively for their applications. Two of the most known fractional integrals are the Riemann-Liouville and the Hadamard fractional integral. A new fractional integral, called *generalized Riemann-Liouville fractional integral*, which generalizes the Riemann-Liouville and the Hadamard integrals into a single form, was introduced in [13]. The corresponding fractional derivatives were introduced in [14]. This integral is now known as "Katugampola fractional integral" see for example [15, pp 15, 123]. For some recent work with this new operator, for example, see [16]-[17] and the references cited therein.

The Langevin equation (first formulated by Langevin in 1908) is found to be an effective tool to describe the evolution of physical phenomena in fluctuating environments [18]. For some new developments on the fractional Langevin equation in physics, see, for example, [19]-[23]. For recent results on Langevin equations with different kinds of boundary conditions we refer to [24]-[28] and the references therein.

Recently in [16], we have studied the existence and the uniqueness of solutions of a class of boundary value problems for fractional Langevin equations of Riemann-Liouville type with generalized nonlocal integral boundary conditions. Here we extend the results of [16], to a coupled system of Langevin fractional differential equations of Riemann-Liouville type subject to the generalized nonlocal integral boundary conditions. Usually in the literature the Banach's contraction mapping principle is used to prove he existence and the uniqueness of solutions, and he existence of solutions is proved via Leray-Schauder alternative. Here we apply Krasnoselskii's and O'Regan's fixed point theorems. To the best of our knowledge this is the first paper using Krasnoselskii's and O'Regan's fixed point theorems to prove the existence of solutions for coupled systems.

The paper is organized as follows: In Section 2 we will present some useful preliminaries and some auxiliary lemmas. In Section 3, we establish the main existence results by using Krasnoselskii's and O'Regan's fixed point theorems. Examples illustrating our results are presented in the final Section 4.

2 Preliminaries

In this section, we introduce some notations and definitions of fractional calculus [1, 2] and present preliminary results needed in our proofs later.

Definition 2.1 *[2] The Riemann-Liouville fractional integral of order $p > 0$ of a continuous function $f : (0, \infty) \to \mathbb{R}$ is defined by*

$$J^p f(t) = \frac{1}{\Gamma(p)} \int_0^t (t-s)^{p-1} f(s) ds,$$

provided the right-hand side is point-wise defined on $(0, \infty)$, where Γ is the gamma function defined by $\Gamma(p) = \int_0^\infty e^{-s} s^{p-1} ds$.

Definition 2.2 *[2] The Riemann-Liouville fractional derivative of order $p > 0$ of a continuous function $f : (0, \infty) \to \mathbb{R}$ is defined by*

$$D^p f(t) = \frac{1}{\Gamma(n-p)} \left(\frac{d}{dt}\right)^n \int_0^t (t-s)^{n-p-1} f(s) ds, \quad n-1 \leq p < n,$$

where $n = [p]+1$, $[p]$ denotes the integer part of a real number p, provided the right-hand side is point-wise defined on $(0, \infty)$.

Lemma 2.3 *[2] Let $p>0$ and $x\in C(0,T)\cap L(0,T)$. Then the fractional differential equation $D^p x(t)=0$ has a unique solution $x(t)=\sum_{i=1}^n c_i t^{p-i}$, and the following formula holds: $J^pD^px(t)=x(t)+\sum_{i=1}^n c_i t^{p-i}$, where $c_i\in\mathbb{R}$, $i=1,2,\ldots,n$, and $n-1\le p<n$.*

Lemma 2.4 *([2], page 71) Let $\alpha>0$, $\beta>0$ and $a\ge 0$. Then the following properties hold:*

$$J^\alpha(x-a)^{\beta-1}(t) \;=\; \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)}(t-a)^{\beta+\alpha-1}$$

Definition 2.5 *[14] The generalized (Katugampola) fractional integral of order $q>0$ and $\rho>0$, of a function f, for all $0<t<\infty$, is defined as*

$${}^\rho I^q f(t)=\frac{\rho^{1-q}}{\Gamma(q)}\int_0^t \frac{s^{\rho-1}f(s)}{(t^\rho-s^\rho)^{1-q}}\,ds,$$

provided the right-hand side is point-wise defined on $(0,\infty)$.

Lemma 2.6 *[16] Let constants $\rho,q>0$ and $p>0$. Then the following formula holds*

$${}^\rho I^q t^p=\frac{\Gamma\left(\frac{p+\rho}{\rho}\right)}{\Gamma\left(\frac{p+\rho q+\rho}{\rho}\right)}\frac{t^{p+\rho q}}{\rho^q}. \tag{2}$$

For convenience to prove our results, we set constants

$$\Omega_1=\frac{\Gamma(p_1)}{\Gamma(p_1+p_2)}\eta^{p_1+p_2-1}, \tag{3}$$

$$\Omega_2=\sum_{i=1}^n\frac{\alpha_i\Gamma(p_1)}{\Gamma(p_1+p_2)}\frac{\Gamma\left(\frac{p_1+p_2+\mu_i-1}{\mu_i}\right)}{\Gamma\left(\frac{p_1+p_2+\mu_i\gamma_i+\mu_i-1}{\mu_i}\right)}\frac{\xi_i^{p_1+p_2+\mu_i\gamma_i-1}}{\mu_i^{\gamma_i}}, \tag{4}$$

$$\Omega=\Omega_2-\Omega_1\neq 0, \tag{5}$$

and

$$\Psi_1=\frac{\Gamma(q_1)}{\Gamma(q_1+q_2)}\kappa^{q_1+q_2-1}, \tag{6}$$

$$\Psi_2=\sum_{j=1}^m\frac{\beta_j\Gamma(q_1)}{\Gamma(q_1+q_2)}\frac{\Gamma\left(\frac{q_1+q_2+\delta_j-1}{\delta_j}\right)}{\Gamma\left(\frac{q_1+q_2+\delta_j\phi_j+\delta_j-1}{\delta_j}\right)}\frac{\zeta_j^{q_1+q_2+\delta_j\phi_j-1}}{\delta_j^{\phi_j}}, \tag{7}$$

$$\Psi=\Psi_2-\Psi_1\neq 0. \tag{8}$$

Lemma 2.7 *Let $\Omega,\Psi\neq 0$, $0<p_1,p_2,q_1,q_2\le 1$, $\mu_i,\gamma_i>0,\delta_j,\phi_j>0$, η, κ, $\xi_i,\zeta_j\in(0,T)$, $\alpha_i,\beta_j\in\mathbb{R}$ for all $i=1,2,\ldots,n, j=1,2,\ldots,m$ and $h,g\in C([0,T],\mathbb{R})$. Then the problem*

$$D^{p_1}(D^{p_2}+\lambda_1)x(t)=h(t), \qquad 0<t<T, \tag{9}$$

$$D^{q_1}(D^{q_2}+\lambda_2)y(t)=g(t), \qquad 0<t<T, \tag{10}$$

$$x(0)=0, \qquad x(\eta)=\sum_{i=1}^n\alpha_i\ {}^{\mu_i}I^{\gamma_i}x(\xi_i), \tag{11}$$

$$y(0)=0, \qquad y(\kappa)=\sum_{j=1}^m\beta_j\ {}^{\delta_j}I^{\phi_j}y(\zeta_j), \tag{12}$$

has a unique solution given by

$$x(t) \;=\; \frac{\Gamma(p_1)}{\Gamma(p_1+p_2)}\frac{t^{p_1+p_2-1}}{\Omega}\Bigg[J^{p_1+p_2}h(\eta)-\lambda_1\ J^{p_2}x(\eta)$$

$$-\sum_{i=1}^{n}\alpha_i\ {}^{\mu_i}I^{\gamma_i}\left(J^{p_1+p_2}h(s)-\lambda_1\ J^{p_2}x(s)\right)(\xi_i)\Bigg]+J^{p_1+p_2}h(t)-\lambda_1\ J^{p_2}x(t),$$

and

$$\begin{aligned}y(t) &= \frac{\Gamma(q_1)}{\Gamma(q_1+q_2)}\frac{t^{q_1+q_2-1}}{\Psi}\Bigg[J^{q_1+q_2}g(\kappa)-\lambda_2\ J^{q_2}y(\kappa)\\ &\quad -\sum_{j=1}^{m}\beta_j\ {}^{\delta_j}I^{\phi_j}\left(J^{q_1+q_2}g(s)-\lambda_2\ J^{q_2}y(s)\right)(\zeta_j)\Bigg]+J^{q_1+q_2}g(t)-\lambda_2\ J^{q_2}y(t).\end{aligned}$$

Proof. Applying Lemma 2.3 to the equations (9) and (10), we obtain

$$(D^{p_2}+\lambda_1)x(t)=J^{p_1}h(t)+c_1t^{p_1-1},\quad \text{and}\quad (D^{q_2}+\lambda_2)y(t)=J^{q_1}g(t)+d_1t^{q_1-1},$$

which give

$$x(t)=J^{p_1+p_2}h(t)-\lambda_1J^{p_2}x(t)+c_1\frac{\Gamma(p_1)}{\Gamma(p_1+p_2)}t^{p_1+p_2-1}+c_2t^{p_2-1},$$

$$y(t)=J^{q_1+q_2}g(t)-\lambda_2J^{q_2}y(t)+d_1\frac{\Gamma(q_1)}{\Gamma(q_1+q_2)}t^{q_1+q_2-1}+d_2t^{q_2-1},$$

for $c_1,c_2,d_1,d_2\in\mathbb{R}$. It is easy to see that the conditions $x(0)=0, y(0)=0$ imply that $c_2=0, d_2=0$. Thus

$$x(t)=J^{p_1+p_2}h(t)-\lambda_1J^{p_2}x(t)+c_1\frac{\Gamma(p_1)}{\Gamma(p_1+p_2)}t^{p_1+p_2-1}, \tag{13}$$

$$y(t)=J^{q_1+q_2}g(t)-\lambda_2J^{q_2}y(t)+d_1\frac{\Gamma(q_1)}{\Gamma(q_1+q_2)}t^{q_1+q_2-1}. \tag{14}$$

Taking the generalized fractional integral of order $\mu_i>0,\gamma_i>0$, to (13) and $\phi_j>0,\delta_j>0$ to (14) , we have

$${}^{\mu_i}I^{\gamma_i}x(t)={}^{\mu_i}I^{\gamma_i}\left(J^{p_1+p_2}h(s)-\lambda_1J^{p_2}x(s)\right)(t)+c_1\frac{\Gamma(p_1)}{\Gamma(p_1+p_2)}\frac{\Gamma(\frac{p_1+p_2+\mu_i-1}{\mu_i})}{\Gamma(\frac{p_1+p_2+\mu_i\gamma_i+\mu_i-1}{\mu_i})}\frac{t^{p_1+p_2+\mu_i\gamma_i-1}}{\mu_i^{\gamma_i}}, \tag{15}$$

and

$${}^{\delta_j}I^{\phi_j}y(t)={}^{\delta_j}I^{\phi_j}\left(J^{q_1+q_2}g(s)-\lambda_2J^{q_2}y(s)\right)(t)+d_1\frac{\Gamma(q_1)}{\Gamma(q_1+q_2)}\frac{\Gamma(\frac{q_1+q_2+\delta_j-1}{\delta_j})}{\Gamma(\frac{q_1+q_2+\delta_j\phi_j+\delta_j-1}{\delta_j})}\frac{t^{q_1+q_2+\delta_j\phi_j-1}}{\delta_j^{\phi_j}}. \tag{16}$$

Using the second condition of $(11),(12)$ to $(15),(16)$ respectively, we get

$$J^{p_1+p_2}h(\eta)-\lambda_1J^{p_2}x(\eta)+c_1\Omega_1=\sum_{i=1}^{n}\alpha_i\ {}^{\mu_i}I^{\gamma_i}\left(J^{p_1+p_2}h(s)-\lambda_1J^{p_2}x(s)\right)(\xi_i)+c_1\Omega_2,$$

and

$$\begin{aligned}&J^{q_1+q_2}g(\kappa)-\lambda_2J^{p_2}y(\kappa)+d_1\Psi_1\\ &=\sum_{j=1}^{m}\beta_j\ {}^{\delta_j}I^{\phi_j}\left(J^{q_1+q_2}g(s)-\lambda_2J^{q_2}y(s)\right)(\zeta_j)+d_1\Psi_2.\end{aligned}$$

Solving the above equations for finding constants c_1,d_1, we obtain

$$c_1=\frac{1}{\Omega}\left[J^{p_1+p_2}h(\eta)-\lambda_1J^{p_2}x(\eta)-\sum_{i=1}^{n}\alpha_i\,{}^{\mu_i}I^{\gamma_i}\left(J^{p_1+p_2}h(s)-\lambda_1J^{p_2}x(s)\right)(\xi_i)\right],$$

and

$$d_1=\frac{1}{\Psi}\left[J^{q_1+q_2}g(\kappa)-\lambda_2J^{q_2}y(\kappa)-\sum_{j=1}^{m}\beta_j\,{}^{\delta_j}I^{\phi_j}\left(J^{q_1+q_2}g(s)-\lambda_2J^{q_2}y(s)\right)(\zeta_j)\right].$$

Substituting the constants c_1,d_1 into $(13),(14)$, we obtain (13) and (13). The proof is completed. □

3 Main results

Let $\mathcal{C} = C([0,T],\mathbb{R})$ denotes the Banach space of all continuous functions from $[0,T]$ to $\mathbb{R}$. Let us introduce the space $X = \{x(t)|x(t) \in C([0,T])\}$ endowed with the norm $\|x\| = \sup\{|x(t)|, t \in [0,T]\}$. Obviously $(X, \|\cdot\|)$ is a Banach space. Also let $Y = \{y(t)|y(t) \in C([0,T])\}$ be endowed with the norm $\|y\| = \sup\{|y(t)|, t \in [0,T]\}$. Obviously the product space $(X\times Y, \|(x,y)\|)$ is a Banach space with norm $\|(x,y)\| = \|x\| + \|y\|$.

Throughout this paper, for convenience, we use the following expressions

$$J^z h(s,x(s),y(s))(\tau) = \frac{1}{\Gamma(z)}\int_0^\tau (\tau-s)^{z-1} f(s,x(s),y(s))ds,$$

and

$${}^\rho I^z h(s,x(s),y(s))(\tau) = \frac{\rho^{1-z}}{\Gamma(z)}\int_0^\tau \frac{s^{\rho-1}f(s,x(s),y(s))}{(\tau^\rho - s^\rho)^{1-z}}ds,$$

where $\rho, z > 0$ and $\tau \in [0,T]$.

In view of Lemma 2.7, we define an operator $\mathcal{F} : X\times Y \to X \times Y$ by

$$\mathcal{F}(x,y)(t) = \begin{pmatrix} \mathcal{P}(x,y)(t) \\ \mathcal{Q}(x,y)(t) \end{pmatrix}, \tag{17}$$

where

$$\begin{aligned}\mathcal{P}(x,y)(t) = \;& \frac{\Gamma(p_1)}{\Gamma(p_1+p_2)}\frac{t^{p_1+p_2-1}}{\Omega}\Bigg[J^{p_1+p_2} f(s,x(s),y(s))(\eta) - \lambda_1\, J^{p_2}x(s)(\eta) \\ & -\sum_{i=1}^n \alpha_i\, {}^{\mu_i}I^{\gamma_i}\Big(J^{p_1+p_2}f(s,x(s),y(s))(\tau) - \lambda_1\, J^{p_2}x(s)(\tau)\Big)(\xi_i)\Bigg] \\ & + J^{p_1+p_2}f(s,x(s),y(s))(t) - \lambda_1\, J^{p_2}x(s)(t),\end{aligned} \tag{18}$$

and

$$\begin{aligned}\mathcal{Q}(x,y)(t) = \;& \frac{\Gamma(q_1)}{\Gamma(q_1+q_2)}\frac{t^{q_1+q_2-1}}{\Psi}\Bigg[J^{q_1+q_2} g(s,x(s),y(s))(\kappa) - \lambda_2\, J^{q_2}y(s)(\kappa) \\ & -\sum_{j=1}^m \beta_j\, {}^{\delta_j}I^{\phi_j}\left(J^{q_1+q_2}g(s,x(s),y(s))(s) - \lambda_2\, J^{q_2}x(s)\right)(\zeta_j)\Bigg] \\ & + J^{q_1+q_2}g(t) - \lambda_2\, J^{q_2}y(t).\end{aligned} \tag{19}$$

To simplify the notations, we use in the following constants

$$\begin{aligned}\Phi(a) \;=\;& \frac{T^{a+p_2}}{\Gamma(1+a+p_2)} + \frac{\Gamma(p_1)}{\Gamma(p_1+p_2)}\frac{T^{p_1+p_2-1}}{|\Omega|}\Bigg(\frac{\eta^{a+p_2}}{\Gamma(1+a+p_2)} \\ & + \sum_{i=1}^n |\alpha_i|\Bigg[\frac{1}{\Gamma(1+a+p_2)}\frac{\xi_i^{a+p_2+\mu_i\gamma_i}}{\mu_i^{\gamma_i}}\frac{\Gamma\left(\frac{a+p_2+\mu_i}{\mu_i}\right)}{\Gamma\left(\frac{a+p_2+\mu_i\gamma_i+\mu_i}{\mu_i}\right)}\Bigg]\Bigg),\end{aligned} \tag{20}$$

and

$$\begin{aligned}\Lambda(b) \;=\;& \frac{T^{b+q_2}}{\Gamma(1+b+q_2)} + \frac{\Gamma(q_1)}{\Gamma(q_1+q_2)}\frac{T^{q_1+q_2-1}}{|\Psi|}\Bigg(\frac{\kappa^{b+q_2}}{\Gamma(1+b+q_2)} \\ & + \sum_{j=1}^m |\beta_j|\Bigg[\frac{1}{\Gamma(1+b+q_2)}\frac{\zeta_j^{b+q_2+\delta_j\phi_j}}{\delta_j^{\phi_j}}\frac{\Gamma\left(\frac{b+q_2+\delta_j}{\delta_j}\right)}{\Gamma\left(\frac{b+q_2+\delta_j\phi_j+\delta_j}{\delta_j}\right)}\Bigg]\Bigg),\end{aligned} \tag{21}$$

where $a \in \{p_1, 0\}$ and $b \in \{q_1, 0\}$.

3.1 Existence result via Krasnoselskii's fixed point theorem

The next result is based on the following fixed point theorem.

Lemma 3.1 *(Krasnoselskii's fixed point theorem) [29]. Let M be a closed, bounded, convex and nonempty subset of a Banach space X. Let A, B be the operators such that (a) $Ax + By \in M$ whenever $x, y \in M$; (b) A is compact and continuous; (c) B is a contraction mapping. Then there exists $z \in M$ such that $z = Az + Bz$.*

Theorem 3.2 *Suppose that the folloeing conditions hold:*

(H_1) $|f(t,u,v)| \leq \psi(t), \quad \forall (t,u,v) \in [0,T] \times \mathbb{R}^2, \quad$ *and* $\psi \in C([0,T], \mathbb{R}^+)$;

(H_2) $|g(t,u,v)| \leq \omega(t), \quad \forall (t,u,v) \in [0,T] \times \mathbb{R}^2, \quad$ *and* $\omega \in C([0,T], \mathbb{R}^+)$;

If

$$\Upsilon = \max\{|\lambda_1|\Phi(0), |\lambda_2|\Lambda(0)\} < 1, \tag{22}$$

where $\Phi(0)$ and $\Lambda(0)$ are defined by (20) and (21) with $a = b = 0$, respectively. Then the problem (1) has at least one solution on $[0,T]$.

Proof. To prove our result, we set $\sup_{t\in[0,T]} |\psi(t)| = \|\psi\|, \sup_{t\in[0,T]} |\omega(t)| = \|\omega\|$ and choose

$$R \geq \frac{\|\psi\|\Phi(p_1) + \|\omega\|\Lambda(q_1)}{1 - \Upsilon}, \tag{23}$$

where $\Phi(p_1)$ and $\Lambda(q_1)$ are defined by (20) and (21) with $a = p_1$ and $b = q_1$, respectively. Let $B_R = \{(x,y) \in X \times Y : \|(x,y)\| \leq R\}$. We define four operators by

$$\begin{aligned}
\mathcal{P}_1(x,y)(t) &= J^{p_1+p_2} f(s,x(s),y(s))(t) + \frac{\Gamma(p_1)}{\Gamma(p_1+p_2)} \frac{t^{p_1+p_2-1}}{\Omega} \Bigg[J^{p_1+p_2} f(s,x(s),y(s))(\eta) \\
&\quad - \sum_{i=1}^{n} \alpha_i \, {}^{\mu_i}I^{\gamma_i} \Big(J^{p_1+p_2} f(s,x(s),y(s))(\tau) \Big)(\xi_i) \Bigg],
\end{aligned}$$

$$\begin{aligned}
\mathcal{P}_2(x)(t) &= -\lambda_1 \, J^{p_2} x(s)(t) - \lambda_1 \frac{\Gamma(p_1)}{\Gamma(p_1+p_2)} \frac{t^{p_1+p_2-1}}{\Omega} \Bigg[J^{p_2} x(s)(\eta) \\
&\quad - \sum_{i=1}^{n} \alpha_i \, {}^{\mu_i}I^{\gamma_i} \Big(J^{p_2} x(s)(\tau) \Big)(\xi_i) \Bigg],
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{Q}_1(x,y)(t) &= J^{q_1+q_2} g(s,x(s),y(s))(t) + \frac{\Gamma(q_1)}{\Gamma(q_1+q_2)} \frac{t^{q_1+q_2-1}}{\Psi} \Bigg[J^{q_1+q_2} g(s,x(s),y(s))(\kappa) \\
&\quad - \sum_{j=1}^{m} \beta_j \, {}^{\delta_j}I^{\phi_j} (J^{q_1+q_2} g(s,x(s),y(s))(s))(\zeta_j) \Bigg],
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{Q}_2(y)(t) &= -\lambda_2 \, J^{q_2} y(s)(t) - \lambda_2 \frac{\Gamma(q_1)}{\Gamma(q_1+q_2)} \frac{t^{q_1+q_2-1}}{\Psi} \Bigg[J^{q_2} y(s)(\kappa) \\
&\quad - \sum_{j=1}^{m} \beta_j \, {}^{\delta_j}I^{\phi_j} (J^{q_2} x(s)(\tau))(\zeta_j) \Bigg],
\end{aligned}$$

and

$$\mathcal{F}_1(x,y)(t) = \begin{pmatrix} \mathcal{P}_1(x,y)(t) \\ \mathcal{Q}_1(x,y)(t) \end{pmatrix}, \quad \mathcal{F}_2(x,y)(t) = \begin{pmatrix} \mathcal{P}_2(x)(t) \\ \mathcal{Q}_2(y)(t) \end{pmatrix}. \tag{24}$$

Observe that $\mathcal{P} = \mathcal{P}_1 + \mathcal{P}_2$, $\mathcal{Q} = \mathcal{Q}_1 + \mathcal{Q}_2$ and $\mathcal{F} = \mathcal{F}_1 + \mathcal{F}_2$. For any $(x_1, y_1), (x_2, y_2) \in B_R$ we have

$$\begin{aligned}
& |\mathcal{P}_1(x_1,y_1)(t) + \mathcal{P}_2(x_2)(t)| \\
= & \Bigg| J^{p_1+p_2} f(s, x_1(s), y_1(s))(t) + \frac{\Gamma(p_1)}{\Gamma(p_1+p_2)} \frac{t^{p_1+p_2-1}}{\Omega} \Bigg[J^{p_1+p_2} f(s, x_1(s), y_1(s))(\eta) \\
& - \sum_{i=1}^{n} \alpha_i \ {}^{\mu_i}I^{\gamma_i} \Big(J^{p_1+p_2} f(s, x_1(s), y_1(s))(\tau) \Big)(\xi_i) \Bigg] - \lambda_1 \ J^{p_2} x_2(s)(t) \\
& - \lambda_1 \frac{\Gamma(p_1)}{\Gamma(p_1+p_2)} \frac{t^{p_1+p_2-1}}{\Omega} \Bigg[J^{p_2} x_2(s)(\eta) - \sum_{i=1}^{n} \alpha_i \ {}^{\mu_i}I^{\gamma_i} \Big(J^{p_2} x_2(s)(\tau) \Big)(\xi_i) \Bigg] \Bigg| \\
\leq & \|\psi\| \Bigg(J^{p_1+p_2}(T) + \frac{\Gamma(p_1)}{\Gamma(p_1+p_2)} \frac{T^{p_1+p_2-1}}{|\Omega|} \Bigg[J^{p_1+p_2}(\eta) + \sum_{i=1}^{n} |\alpha_i| \ {}^{\mu_i}I^{\gamma_i} \Big(J^{p_1+p_2}(\tau) \Big)(\xi_i) \Bigg] \Bigg) \\
& + |\lambda_1| \|x_2\| \Bigg(J^{p_2}(T) + \frac{\Gamma(p_1)}{\Gamma(p_1+p_2)} \frac{T^{p_1+p_2-1}}{|\Omega|} \Bigg[J^{p_2}(\eta) + \sum_{i=1}^{n} |\alpha_i| \ {}^{\mu_i}I^{\gamma_i} (J^{p_2}(\tau))(\xi_i) \Bigg] \Bigg) \\
\leq & \|\psi\| \Phi(p_1) + |\lambda_1| \|x_2\| \Phi(0).
\end{aligned}$$

In a similar way, we get

$$\begin{aligned}
& |\mathcal{Q}_1(x_1,y_1)(t) + \mathcal{Q}_2(y_2)(t)| \\
= & \Bigg| J^{q_1+q_2} g(s, x_1(s), y_1(s))(t) + \frac{\Gamma(q_1)}{\Gamma(q_1+q_2)} \frac{t^{q_1+q_2-1}}{\Psi} \Bigg[J^{q_1+q_2} g(s, x_1(s), y_1(s))(\kappa) \\
& - \sum_{j=1}^{m} \beta_j \ {}^{\delta_j}I^{\phi_j} (J^{q_1+q_2} g(s, x_1(s), y_1(s))(s))(\zeta_j) \Bigg] - \lambda_2 \ J^{q_2} y_2(s)(t) \\
& - \lambda_2 \frac{\Gamma(q_1)}{\Gamma(q_1+q_2)} \frac{t^{q_1+q_2-1}}{\Psi} \Bigg[J^{q_2} y_2(s)(\kappa) - \sum_{j=1}^{m} \beta_j \ {}^{\delta_j}I^{\phi_j} (J^{q_2} y_2(s)(\tau))(\zeta_j) \Bigg] \Bigg| \\
\leq & \|\omega\| \Bigg(J^{q_1+q_2}(T) + \frac{\Gamma(q_1)}{\Gamma(q_1+q_2)} \frac{T^{q_1+q_2-1}}{|\Psi|} \Bigg[J^{q_1+q_2}(\kappa) + \sum_{j=1}^{m} |\beta_j| \ {}^{\delta_j}I^{\phi_j} (J^{q_1+q_2}(\tau))(\zeta_j) \Bigg] \Bigg) \\
& + |\lambda_2| \|y_2\| \Bigg(J^{q_2}(T) + \frac{\Gamma(q_1)}{\Gamma(q_1+q_2)} \frac{T^{q_1+q_2-1}}{|\Psi|} \Bigg[J^{q_2}(\kappa) + \sum_{j=1}^{m} |\beta_j| \ {}^{\delta_j}I^{\phi_j} (J^{q_2}(\tau))(\zeta_j) \Bigg] \Bigg) \\
\leq & \|\omega\| \Lambda(q_1) + |\lambda_2| \|y_2\| \Lambda(0),
\end{aligned}$$

which imply that $\|\mathcal{F}_1(x,y) + \mathcal{F}_2(x,y)\| \leq R$. This shows that $\mathcal{F}_1(x,y) + \mathcal{F}_2(x,y) \in B_R$.

For $(x_1, y_1), (x_2, y_2) \in X \times Y$ and for each $t \in [0, T]$ we have

$$\|\mathcal{P}_2(x_1) - \mathcal{P}_2(x_2)\| \leq |\lambda_1| \Phi(0) \|x_1 - x_2\|,$$

and

$$\|\mathcal{Q}_2(y_1) - \mathcal{Q}_2(y_2)\| \leq |\lambda_2| \Lambda(0) \|y_1 - y_2\|.$$

Thus

$$\|\mathcal{F}_2(x_1,y_1) - \mathcal{F}_2(x_2,y_2)\| \leq \Upsilon \|x_1 - x_2\| + \Upsilon \|y_1 - y_2\| = \Upsilon \|(x_1 - x_2, y_1 - y_2)\|,$$

which implies that $\mathcal{F}_2$ is a contraction mapping by (22). The continuity of f implies that the operator $\mathcal{F}_1$ is continuous. Also, $\mathcal{F}_1$ is uniformly bounded on B_R as

$$\|\mathcal{P}_1(x,y)\| \leq \|\psi\| \Phi(p_1), \quad \text{and} \quad \|\mathcal{Q}_1(x,y)\| \leq \|\omega\| \Lambda(q_1).$$

Thus

$$\|\mathcal{F}_1(x,y)\| \leq \|\psi\|\Phi(p_1) + \|\omega\|\Lambda(q_1).$$

Next we will prove the compactness of the operator $\mathcal{F}_1$. Let $t_1, t_2 \in [0,T]$ with $t_1 < t_2$. Then we have

$$\begin{aligned}
& |\mathcal{P}_1(x,y)(t_2) - \mathcal{P}_1(x,y)(t_1)| \\
\leq\ & \Big| J^{p_1+p_2} f(s,x(s),y(s))(t_2) - J^{p_1+p_2} f(s,x(s),y(s))(t_1) \\
& + \frac{\Gamma(p_1)(t_2^{p_1+p_2-1} - t_1^{p_1+p_2-1})}{\Omega\Gamma(p_1+p_2)} \Big[J^{p_1+p_2} f(s,x(s),y(s))(\eta) \\
& - \sum_{i=1}^{n} \alpha_i \ {}^{\mu_i}I^{\gamma_i} \Big(J^{p_1+p_2} f(s,x(s),y(s))(\tau) \Big)(\xi_i) \Big] \Big| \\
\leq\ & \|\psi\| \frac{(t_2^{p_1+p_2} - t_1^{p_1+p_2}) + 2(t_2-t_1)^{p_1+p_2}}{\Gamma(p_1+p_2+1)} + \|\psi\| \frac{\Gamma(p_1)(t_2^{p_1+p_2-1} - t_1^{p_1+p_2-1})}{|\Omega|\Gamma(p_1+p_2)} J^{p_1+p_2}(\eta) \\
& + \sum_{i=1}^{n} |\alpha_i| \ {}^{\mu_i}I^{\gamma_i} \Big(J^{p_1+p_2}(\tau) \Big)(\xi_i)
\end{aligned}$$

and

$$\begin{aligned}
& |\mathcal{Q}_1(x,y)(t_2) - \mathcal{Q}_1(x,y)(t_1)| \\
\leq\ & \Big| J^{q_1+q_2} g(s,x(s),y(s))(t_2) - J^{q_1+q_2} g(s,x(s),y(s))(t_1) \\
& + \frac{\Gamma(q_1)(t_2^{q_1+q_2-1} - t_1^{q_1+q_2-1})}{\Psi\Gamma(q_1+q_2)} \Big[J^{q_1+q_2} g(s,x(s),y(s))(\kappa) \\
& - \sum_{j=1}^{m} \beta_j \ {}^{\delta_j}I^{\phi_j} \Big(J^{q_1+q_2} g(s,x(s),y(s))(\tau) \Big)(\zeta_j) \Big] \Big| \\
\leq\ & \|\omega\| \frac{(t_2^{q_1+q_2} - t_1^{q_1+q_2}) + 2(t_2-t_1)^{q_1+q_2}}{\Gamma(q_1+q_2+1)} + \|\omega\| \frac{\Gamma(q_1)(t_2^{q_1+q_2-1} - t_1^{q_1+q_2-1})}{|\Psi|\Gamma(q_1+q_2)} J^{q_1+q_2}(\kappa) \\
& + \sum_{j=1}^{m} |\beta_j| \ {}^{\delta_j}I^{\phi_j} \Big(J^{q_1+q_2}(\tau) \Big)(\zeta_j),
\end{aligned}$$

which is independent of (x,y) and tends to zero as $t_2 - t_1 \to 0$. Thus, $\mathcal{F}_1$ is equicontinuous. So $\mathcal{F}_1$ is relatively compact on B_R. Hence, by the Arzelá-Ascoli theorem, $\mathcal{F}_1$ is compact on B_R. Thus all the assumptions of Lemma 3.1 are satisfied. So the conclusion of Lemma 3.1 implies that the problem (1) has at least one solution on $[0,T]$. This completes the proof. □

3.2 Existence result via O'Regan's fixed point theorem

Our next existence result relies on a fixed point theorem due to O'Regan in [30].

Lemma 3.3 *Denote by U an open set in a closed, convex set C of a Banach space E. Assume $0 \in U$. Also assume that $F(\bar{U})$ is bounded and that $F : \bar{U} \to C$ is given by $F = F_1 + F_2$, in which $F_1 : \bar{U} \to E$ is continuous and completely continuous and $F_2 : \bar{U} \to E$ is a nonlinear contraction (i.e., there exists a nonnegative nondecreasing function $\phi : [0,\infty) \to [0,\infty)$ satisfying $\phi(z) < z$ for $z > 0$, such that $\|F_2(x) - F_2(y)\| \leq \phi(\|x-y\|)$ for all $x, y \in \bar{U}$). Then, either*

(C1) F has a fixed point $u \in \bar{U}$; or

(C2) there exist a point $u \in \partial U$ and $\lambda \in (0,1)$ with $u = \lambda F(u)$, where $\bar{U}$ and ∂U, respectively, represent the closure and boundary of U.

In the sequel, we will use Lemma 3.3 by taking C to be E. For more details of such fixed point theorems, we refer a paper [31] by Petryshyn. Let

$$K_r = \{(x,y) \in X \times Y : \|(x,y)\| \leq R\}.$$

Theorem 3.4 *Let $f, g : [0,T] \times \mathbb{R} \to \mathbb{R}$ be continuous functions. Suppose that (22) holds. In addition we assume that:*

(H_3) *there exist a nonnegative function $z_1 \in C([0,T],\mathbb{R})$ and nondecreasing functions $\psi_1, \psi_2 : [0,\infty) \to [0,\infty)$ such that*

$$|f(t,u,v)| \leq z_1(t)[\psi_1(\|u\|) + \psi_2(\|v\|)] \quad \textit{for all} \quad (t,u,v) \in [0,T] \times \mathbb{R}^2;$$

(H_4) *there exist a nonnegative function $z_2 \in C([0,T],\mathbb{R})$ and nondecreasing functions $\omega_1, \omega_2 : [0,\infty) \to [0,\infty)$ such that*

$$|g(t,u,v)| \leq z_2(t)[\omega_1(\|u\|) + \omega_2(\|v\|)] \quad \textit{for all} \quad (t,u,v) \in [0,T] \times \mathbb{R}^2;$$

(H_5) $\displaystyle\sup_{r\in(0,\infty)} \frac{r}{\|z_1\|[\psi_1(r)+\psi_2(r)]\Phi(p_1) + \|z_2\|[\omega_1(r)+\omega_2(r)]\Lambda(q_1)} > \frac{1}{1-\Upsilon}$, *where $\Phi(p_1), \Lambda(q_1)$ and Υ are defined in (20), (21)and (22) respectively.*

Then the he problem (1) has at least one solution on $[0,T]$.

Proof. Consider the operator $\mathcal{F} : X \times Y \to X \times Y$ as that defined in (18). We decompose $\mathcal{F}$ into a sum of two operators

$$\mathcal{F}(x,y)(t) = \mathcal{F}_1(x,y)(t) + \mathcal{F}_2(x,y)(t)$$

where $\mathcal{F}_1(x,y), \mathcal{F}_2(x,y)$ defined in (24). From (H_5) there exists a number $r_0 > 0$ such that

$$\frac{r_0}{\|z_1\|[\psi_1(r_0)+\psi_2(r_0)]\Phi(p_1) + \|z_2\|[\omega_1(r_0)+\omega_2(r_0)]\Lambda(q_1)} > \frac{1}{1-\Upsilon}. \tag{25}$$

We shall prove that the operators $\mathcal{F}_1$ and $\mathcal{F}_2$ satisfy all the conditions of Lemma 3.3.

Step 1. *The set $\mathcal{F}(K_{r_0})$ is bounded.* We first show that $\mathcal{F}_1(K_{r_0})$ is bounded. For any $(x,y) \in \bar{K}_{r_0}$ we have

$$\|\mathcal{P}_1(x,y)\| \leq \|z_1\|[\psi_1(r_0)+\psi_2(r_0)]\Phi(p_1),$$

and

$$\|\mathcal{Q}_1(x,y)\| \leq \|z_2\|[\omega_1(r_0)+\omega_2(r_0)]\Lambda(q_1).$$

Thus

$$\mathcal{F}_1(x,y)\| \leq \|z_1\|[\psi_1(r_0)+\psi_2(r_0)]\Phi(p_1) + \|z_2\|[\omega_1(r_0)+\omega_2(r_0)]\Lambda(q_1).$$

This proves that $\mathcal{F}_1(\bar{K}_{r_0})$ is uniformly bounded. In a similar way we have

$$\|\mathcal{P}_2(x)\| \leq |\lambda_1|\Phi(0)\|x\|, \quad \text{and} \quad \|\mathcal{Q}_2(y)\| \leq |\lambda_2|\Lambda(0)\|y\|,$$

and thus

$$\|\mathcal{F}_2(x,y)\| \leq \Upsilon r_0.$$

Step 2. *The operator $\mathcal{F}_1$ is continuous and completely continuous.*

By Step 1, $\mathcal{F}_1(\bar{K}_{r_0})$ is uniformly bounded. In addition for any $t_1, t_2 \in [0,T]$, we have:

$$\begin{aligned}
& |\mathcal{P}_1(x,y)(t_2) - \mathcal{P}_1(x,y)(t_1)| \\
\leq\ & \|z_1\|[\psi_1(r_0)+\psi_2(r_0)]\Bigg[\frac{1}{\Gamma(p_1+p_2+1)}\Big(t_2^{p_1+p_2} - t_1^{p_1+p_2} + 2(t_2-t_1)^{p_1+p_2}\Big) \\
& + \frac{\Gamma(p_1)(t_2^{p_1+p_2-1} - t_1^{p_1+p_2-1})}{|\Omega|\Gamma(p_1+p_2)} J^{p_1+p_2}(\eta) + \sum_{i=1}^{n} |\alpha_i|\ {}^{\mu_i}I^{\gamma_i}\Big(J^{p_1+p_2}(\tau)\Big)(\xi_i)\Bigg],
\end{aligned}$$

and

$$\begin{aligned}
&|\mathcal{Q}_1(x,y)(t_2)-\mathcal{Q}_1(x,y)(t_1)|\\
\leq\ & \|z_2\|[\omega_1(r_0)+\omega_2(r_0)]\Bigg[\frac{1}{\Gamma(q_1+q_2+1)}\Big(t_2^{q_1+q_2}-t_1^{q_1+q_2}+2(t_2-t_1)^{q_1+q_2}\Big)\\
&+\frac{\Gamma(q_1)(t_2^{q_1+q_2-1}-t_1^{q_1+q_2-1})}{|\Psi|\Gamma(q_1+q_2)}J^{q_1+q_2}(\kappa)+\sum_{j=1}^{m}|\beta_j|\ {}^{\delta_j}I^{\phi_j}\Big(J^{q_1+q_2}(\tau)\Big)(\zeta_j)\Bigg],
\end{aligned}$$

which are independent of (x,y) and tends to zero as $t_2-t_1\to 0$. Thus, $\mathcal{F}_1$ is equicontinuous. Hence, by the Arzelá-Ascoli Theorem, $\mathcal{F}_1(\bar{K}_{r_0})$ is a relatively compact set. Now, let $(x_n,y_n)\subset\bar{K}_{r_0}$ with $\|(x_n,y_n)-(x,y)\|\to 0$. Then the limit $\|(x_n,y_n)(t)-(x,y)(t)\|\to 0$ is uniformly valid on $[0,T]$. From the uniform continuity of $f(t,x,y)$ and $g(t,x,y)$ on the compact set $[0,T]\times[-r_0,r_0]\times[-r_0,r_0]$, it follows that $\|f(t,x_n(t),y_n(t))-f(t,x(t),y(t))\|\to 0$ and $\|g(t,x_n(t),y_n(t))-g(t,x(t),y(t))\|\to 0$ are uniformly valid on $[0,T]$. Hence $\|\mathcal{F}_1(x_n,y_n)-\mathcal{F}_1(x,y)\|\to 0$ as $n\to\infty$ which proves the continuity of $\mathcal{F}_1$. Therefore the operator $\mathcal{F}_1$ is continuous and completely continuous

Step 3. *The operator $\mathcal{F}_2$ is contractive.* This was proved in Theorem 3.2.

Step 4. *Finally, it will be shown that the case (C2) in Lemma 3.3 does not hold.* On the contrary, we suppose that (C2) holds. Then, we have that there exist $\theta\in(0,1)$ and $(x,y)\in\partial\bar{K}_{r_0}$ such that $(x,y)=\theta\mathcal{F}(x,y)$. So, we have $\|(x,y)\|=r_0$ and

$$\|x\|\leq\|z_1\|[\psi_1(r_0)+\psi_2(r_0)]\Phi(p_1)+|\lambda_1|\Phi(0)\|x\|,$$

and

$$\|y\|\leq\|z_2\|[\omega_1(r_0)+\omega_2(r_0)]\Lambda(q_1)+|\lambda_2|\Lambda(0)\|y\|,$$

from which we get

$$\|x\|+\|y\|\leq\|z_1\|[\psi_1(r_0)+\psi_2(r_0)]\Phi(p_1)+\|z_2\|[\omega_1(r_0)+\omega_2(r_0)]\Lambda(q_1)+\Upsilon r_0,$$

or

$$\frac{r_0}{\|z_1\|[\psi_1(r_0)+\psi_2(r_0)]\Phi(p_1)+\|z_2\|[\omega_1(r_0)+\omega_2(r_0)]\Lambda(q_1)}\leq\frac{1}{1-\Upsilon},$$

which contradicts to (25). Consequently, we have proved that the operators $\mathcal{F}_1$ and $\mathcal{F}_2$ satisfy all the conditions in Lemma 3.3. Hence, the operator $\mathcal{F}$ has at least one fixed point $(x,y)\in\bar{K}_{r_0}$, which is the solution of the he problem (1). The proof is completed. □

Theorem 3.5 *Let $f,g:[0,T]\times\mathbb{R}\to\mathbb{R}$ be continuous functions. Suppose that (22) holds. In addition we assume that:*

(H_6) *there exist a nonnegative function $z_1\in C([0,T],\mathbb{R})$ and a nondecreasing function $\psi:[0,\infty)\to[0,\infty)$ such that*

$$|f(t,u,v)|\leq z_1(t)\psi(\|u\|+\|v\|)\ \ \textit{for all}\ \ (t,u,v)\in[0,T]\times\mathbb{R}^2;$$

(H_7) *there exist a nonnegative function $z_2\in C([0,T],\mathbb{R})$ and a nondecreasing function $\omega:[0,\infty)\to[0,\infty)$ such that*

$$|g(t,u,v)|\leq z_2(t)\omega(\|u\|+\|v\|)\ \ \textit{for all}\ \ (t,u,v)\in[0,T]\times\mathbb{R}^2;$$

(H_8) $\sup\limits_{r\in(0,\infty)}\dfrac{r}{\|z_1\|\psi(r)\Phi(q_1)+\|z_2\|\omega(r)\Lambda(q_1)}>\dfrac{1}{1-\Upsilon}$, *where $\Phi(p_1),\Lambda(q_1)$ and Υ are defined in (20), (21)and (22) respectively.*

Then the he problem (1) has at least one solution on $[0,T]$.

Proof. The proof is similar to that of Theorem 3.4 and it is omitted. □

To establish some special cases, we set constants

$$R_1 = \sum_{i=1}^{n} \frac{\alpha_i \Gamma(p_1)}{\Gamma(p_1+p_2)} \frac{\Gamma(p_1+p_2)}{\Gamma(p_1+p_2+\gamma_i)} \xi_i^{p_1+p_2+\gamma_i-1},\ R = R_1 - \Omega_1 \neq 0,$$

and

$$L_1 = \sum_{j=1}^{m} \frac{\beta_j \Gamma(q_1)}{\Gamma(q_1+q_2)} \frac{\Gamma(q_1+q_2)}{\Gamma(q_1+q_2+\phi_j)} \zeta_j^{q_1+q_2+\phi_j-1},\ L = L_1 - \Psi_1 \neq 0,$$

$$\begin{aligned} \chi(a) \quad &= \quad \frac{T^{a+p_2}}{\Gamma(1+a+p_2)} + \frac{\Gamma(p_1)}{\Gamma(p_1+p_2)} \frac{T^{p_1+p_2-1}}{|R|} \Bigg(\frac{\eta^{a+p_2}}{\Gamma(1+a+p_2)} \\ &\quad + \sum_{i=1}^{n} |\alpha_i| \Bigg[\frac{\xi_i^{a+p_2+\gamma_i}}{\Gamma(1+a+p_2)} \frac{\Gamma(a+p_2+1)}{\Gamma(a+p_2+\gamma_i+1)} \Bigg] \Bigg), \end{aligned} \tag{26}$$

and

$$\begin{aligned} \Theta(b) \quad &= \quad \frac{T^{b+q_2}}{\Gamma(1+b+q_2)} + \frac{\Gamma(q_1)}{\Gamma(q_1+q_2)} \frac{T^{q_1+q_2-1}}{|L|} \Bigg(\frac{\kappa^{b+q_2}}{\Gamma(1+b+q_2)} \\ &\quad + \sum_{j=1}^{m} |\beta_j| \Bigg[\frac{\zeta_j^{b+q_2+\phi_j}}{\Gamma(1+b+q_2)} \frac{\Gamma(b+q_2+1)}{\Gamma(b+q_2+\phi_j+1)} \Bigg] \Bigg), \end{aligned} \tag{27}$$

where $a = \{p_1, 0\}$ and $b = \{q_1, 0\}$

By setting $\mu_i = 1$ and $\delta_j = 1$, we have a boundary value problem with nonlocal Riemann-Liouville fractional integral conditions

$$\begin{cases} D^{p_1}(D^{p_2} + \lambda_1)x(t) = f(t, x(t), y(t)), \quad 0 < t < T, \\ D^{q_1}(D^{q_2} + \lambda_2)y(t) = g(t, x(t), y(t)), \quad 0 < t < T, \\ x(0) = 0, \qquad x(\eta) = \sum_{i=1}^{n} \alpha_i\ J^{\gamma_i} x(\xi_i), \\ y(0) = 0, \qquad y(\kappa) = \sum_{j=1}^{m} \beta_j\ J^{\phi_j} y(\zeta_j). \end{cases} \tag{28}$$

Using the above constants, we have the following corollaries.

Corollary 3.6 *Suppose that (H1) and (H2) holds. If*

$$M = \max\{|\lambda_1|\chi(0), |\lambda_2|\Theta(0)\} < 1, \tag{29}$$

then the problem (28) has at least one solution on $[0, T]$.

Corollary 3.7 *Let* $f, g : [0, T] \times \mathbb{R} \to \mathbb{R}$ *be continuous functions. Suppose that (29),* (H_3) *and* (H_4) *holds. In addition we assume that:*

(H_9) $\sup_{r\in(0,\infty)} \frac{r}{\|z_1\|[\psi_1(r)+\psi_2(r)]\chi(p_1) + \|z_2\|[\omega_1(r)+\omega_2(r)]\Theta(q_1)} > \frac{1}{1-M}$, *where* $\chi(p_1), \Theta(q_1)$ *and* M *are defined in (26), (27)and (29) respectively.*

Then the he problem (28) has at least one solution on $[0, T]$.

Corollary 3.8 *Let* $f, g : [0, T] \times \mathbb{R} \to \mathbb{R}$ *be continuous functions. Suppose that (29),* (H_6) *and* (H_7) *holds. In addition we assume that:*

(H_{10}) $\sup_{r\in(0,\infty)} \frac{r}{\|z_1\|\psi(r)\chi(q_1) + \|z_2\|\omega(r)\Theta(q_1)} > \frac{1}{1-M}$, *where* $\chi(p_1), \Theta(q_1)$ *and* M *are defined in (26), (27)and (29) respectively.*

Then the he problem (28) has at least one solution on $[0, T]$.

4 Examples

In this section we present examples to illustrate our results.

Example 4.1 *Consider the following system of fractional Langevin equation subject to the nonlocal Katugampola fractional integral conditions*

$$\begin{cases} D^{1/2}\left(D^{3/5}+0.2\right)x(t)=\dfrac{t\sin 3t}{t+1}\dfrac{\arctan x(t)}{3|x(t)|+2}+\dfrac{2\cos t}{3t^2+2}\dfrac{\sin y(t)}{2|y(t)|+3}, \quad 0<t<1,\\ D^{2/5}\left(D^{4/5}+0.25\right)y(t)=\dfrac{3t^2}{4t+3}\dfrac{3x(t)}{5|x(t)|+1}+\dfrac{2y(t)+3}{3|y(t)|+4}, \quad 0<t<1,\\ x(0)=0, \qquad x(0.6)=0.2\ {}^{1/2}I^{7/10}x(0.3)+0.3\ {}^{2/5}I^{3/5}x(0.6),\\ y(0)=0, \qquad y(0.2)=0.2\ {}^{3/10}I^{4/5}y(0.3)+0.3\ {}^{3/5}I^{2/5}y(0.7)+0.4\ {}^{2/5}I^{9/10}y(0.9), \end{cases} \tag{30}$$

Here $p_1=1/2$, $p_2=3/5$, $q_1=2/5$, $q_2=4/5$, $\lambda_1=0.2$, $\lambda_2=0.25$, $\eta=0.6$, $\kappa=0.2$, $\alpha_1=0.2$, $\alpha_2=0.3$, $\beta_1=0.2$, $\beta_2=0.3$, $\beta_3=0.4$, $\mu_1=1/2$, $\mu_2=2/5$, $\gamma_1=7/10$, $\gamma_2=3/5$, $\delta_1=3/10$, $\delta_2=3/5$, $\delta_3=2/5$, $\phi_1=4/5$, $\phi_2=2/5$, $\phi_3=9/10$, $\xi_1=0.3$, $\xi_2=0.6$, $\zeta_1=0.3$, $\zeta_2=0.7$, $\zeta_3=0.9$, $T=1$, $f(t,x,y)=(t\sin 3t\arctan x(t))/((t+1)(3|x(t)|+2))+(2\cos t\sin y(t))/((3t^2+2)(2|y(t)|+3))$ and $g(t,x,y)=(9t^2x(t))/((4t+3)(5|x(t)|+1))+(2y(t)+3)/(3|y(t)|+4)$. Since $f(t,x,y)\le(t\sin 3t)/(3t+3)+(2\cos t)/(6t^2+4)$, $g(t,x,y\le(9t^2)/(20t+15)+(2/3)$ and by using the Maple program, we can find

$$\begin{aligned}\Phi(0) &= \frac{T^{p_2}}{\Gamma(1+p_2)}\\ &\quad+\frac{\Gamma(p_1)}{\Gamma(p_1+p_2)}\frac{T^{p_1+p_2-1}}{|\Omega|}\left(\frac{\eta^{p_2}}{\Gamma(1+p_2)}+\sum_{i=1}^{2}|\alpha_i|\left[\frac{1}{\Gamma(1+p_2)}\frac{\xi_i^{p_2+\mu_i\gamma_i}}{\mu_i^{\gamma_i}}\frac{\Gamma\left(\frac{p_2+\mu_i}{\mu_i}\right)}{\Gamma\left(\frac{p_2+\mu_i\gamma_i+\mu_i}{\mu_i}\right)}\right]\right)\\ &\approx 4.318646369,\end{aligned}$$

and

$$\begin{aligned}\Lambda(0) &= \frac{T^{q_2}}{\Gamma(1+q_2)}\\ &\quad+\frac{\Gamma(q_1)}{\Gamma(q_1+q_2)}\frac{T^{q_1+q_2-1}}{|\Psi|}\left(\frac{\kappa^{q_2}}{\Gamma(1+q_2)}+\sum_{j=1}^{3}|\beta_j|\left[\frac{1}{\Gamma(1+q_2)}\frac{\zeta_i^{q_2+\delta_j\phi_j}}{\delta_j^{\phi_j}}\frac{\Gamma\left(\frac{q_2+\delta_j}{\delta_j}\right)}{\Gamma\left(\frac{q_2+\delta_j\phi_j+\delta_j}{\delta_j}\right)}\right]\right)\\ &\approx 3.234126953.\end{aligned}$$

Thus $\Upsilon\approx 0.8637292738<1$. Hence, by Theorem 3.2, the system (30) has at least one solution on $[0,1]$.

Example 4.2 *Consider the following system of fractional Langevin equation subject to the nonlocal Katugampola fractional integral conditions*

$$\begin{cases} D^{3/10}\left(D^{4/5}+0.25\right)x(t)=\dfrac{t}{15}\left(\dfrac{|x|^2+2|x|}{|x|+4}+\dfrac{|y|^2+2|y|+2}{3|y|+4}\right), \quad 0<t<1,\\ D^{2/5}\left(D^{9/10}+0.2\right)y(t)=\dfrac{t}{5}\left(\dfrac{|x|^2+|x|+1}{2|x|+5}+\dfrac{|y|^2+1}{|y|+5}\right), \quad 0<t<1,\\ x(0)=0, \qquad x(0.1)=1.5\ {}^{7/10}I^{1/2}x(0.6)+2\ {}^{3/10}I^{1/5}x(0.8)+2.5\ {}^{3/5}I^{3/10}x(0.9),\\ y(0)=0, \qquad y(0.8)=3\ {}^{7/10}I^{4/5}y(0.7)+2.5\ {}^{3/10}I^{9/10}y(0.8), \end{cases} \tag{31}$$

Here $p_1=3/10$, $p_2=4/5$, $q_1=2/5$, $q_2=9/10$, $\lambda_1=0.25$, $\lambda_2=0.2$, $\eta=0.1$, $\kappa=0.8$, $\alpha_1=1.5$, $\alpha_2=2$, $\alpha_3=2.5$, $\beta_1=3$, $\beta_2=2.5$, $\mu_1=7/10$, $\mu_2=3/10$, $\mu_3=3/5$, $\gamma_1=1/2$, $\gamma_2=1/5$, $\gamma_3=3/10$, $\delta_1=7/10$, $\delta_2=3/10$, $\phi_1=4/5$, $\phi_2=9/10$, $\xi_1=0.6$, $\xi_2=0.8$, $\xi_3=0.9$, $\zeta_1=0.7$, $\zeta_2=0.8$, $T=1$, $f(t,x,y)=(t/15)[((|x|^2+2|x|)/(|x|+4))+((|y|^2+2|y|+2)/(3|y|+4))]$ and $g(t,x,y)=(t/5)[((|x|^2+|x|+1)/(2|x|+5))+((|y|^2+1)/(|y|+5))]$. By using the Maple program, we can find

$$\Phi(0) = \frac{T^{p_2}}{\Gamma(1+p_2)}$$

$$
\begin{aligned}
&+\frac{\Gamma(p_1)}{\Gamma(p_1+p_2)}\frac{T^{p_1+p_2-1}}{|\Omega|}\left(\frac{\eta^{p_2}}{\Gamma(1+p_2)}+\sum_{i=1}^{3}|\alpha_i|\left[\frac{1}{\Gamma(1+p_2)}\frac{\xi_i^{p_2+\mu_i\gamma_i}}{\mu_i^{\gamma_i}}\frac{\Gamma\left(\frac{p_2+\mu_i}{\mu_i}\right)}{\Gamma\left(\frac{p_2+\mu_i\gamma_i+\mu_i}{\mu_i}\right)}\right]\right)\\
\approx\ & 1.892763483,
\end{aligned}
$$

and

$$
\begin{aligned}
\Lambda(0) &= \frac{T^{q_2}}{\Gamma(1+q_2)}\\
&+\frac{\Gamma(q_1)}{\Gamma(q_1+q_2)}\frac{T^{q_1+q_2-1}}{|\Psi|}\left(\frac{\kappa^{q_2}}{\Gamma(1+q_2)}+\sum_{j=1}^{2}|\beta_j|\left[\frac{1}{\Gamma(1+q_2)}\frac{\zeta_i^{q_2+\delta_j\phi_j}}{\delta_j^{\phi_j}}\frac{\Gamma\left(\frac{q_2+\delta_j}{\delta_j}\right)}{\Gamma\left(\frac{q_2+\delta_j\phi_j+\delta_j}{\delta_j}\right)}\right]\right)\\
&\approx 1.824427804.
\end{aligned}
$$

Thus $\Upsilon\approx 0.4731908708<1$. Since $|f(t,x,y)|\le (t/15)[(|x|^2+2|x|)/4+(|y|^2+2|y|+2)/4]$, $|g(t,x,y)|\le (t/5)[(|x|^2+|x|+1)/5+(|y|^2+1)/5]$, we choose $z_1(t)=t/15$, $\psi_1(x)=(|x|^2+2|x|)/4$, $\psi_2(y)=(|y|^2+2|y|+2)/4$, $z_2(t)=t/5$, $\omega_1(x)=(|x|^2+|x|+1)/5$, $\omega_2(y)=(|y|^2+1)/5$. We can show that

$$
\begin{aligned}
&\sup_{r\in(0,\infty)}\frac{r}{\|z_1\|[\psi_1(r)+\psi_2(r)]\Phi(p_1)+\|z_2\|[\omega_1(r)+\omega_2(r)]\Lambda(q_1)}\\
\approx\ & 2.080080186>1.898220711\approx\frac{1}{1-\Upsilon}.
\end{aligned}
$$

Hence, by Theorem 3.4, the system (31) has at least one solution on $[0,1]$.

Example 4.3 *Consider the following system of fractional Langevin equation subject to the nonlocal Katugampola fractional integral conditions*

$$
\begin{cases}
D^{4/5}\left(D^{9/10}+0.3\right)x(t)=\dfrac{t}{5}\left(\dfrac{2(|x+y|)^3+2|x|+|y|}{3|x|+4}\right), \quad 0<t<\dfrac{2}{3},\\
D^{3/10}\left(D^{9/10}+0.35\right)y(t)=\dfrac{t}{3}\left(\dfrac{(|x+y|)^2+1}{|x|+2|y|+3}\right), \quad 0<t<\dfrac{2}{3},\\
x(0)=0, \qquad x(0.6)=0.4\ {}^{2/5}I^{7/10}x(0.2)+0.4\ {}^{4/5}I^{2/5}x(0.6),\\
y(0)=0, \qquad y(0.3)=0.8\ {}^{4/5}I^{4/5}y(0.2)+0.7\ {}^{1/5}I^{9/10}y(0.5)+0.8\ {}^{7/10}I^{7/10}y(0.6),
\end{cases}
\tag{32}
$$

Here $p_1=4/5$, $p_2=9/10$, $q_1=3/10$, $q_2=9/10$, $\lambda_1=0.3$, $\lambda_2=0.35$, $\eta=0.6$, $\kappa=0.3$, $\alpha_1=0.4$, $\alpha_2=0.4$, $\beta_1=0.8$, $\beta_2=0.7$, $\beta_3=0.8$, $\mu_1=2/5$, $\mu_2=4/5$, $\gamma_1=7/10$, $\gamma_2=2/5$, $\delta_1=4/5$, $\delta_2=1/5$, $\delta_3=7/10$, $\phi_1=4/5$, $\phi_2=9/10$, $\phi_3=7/10$, $\xi_1=0.2$, $\xi_2=0.6$, $\zeta_1=0.2$, $\zeta_2=0.5$, $\zeta_3=0.6$, $T=\frac{2}{3}$, $f(t,x,y)=(t/5)\left[(2(|x+y|)^3+2|x|+|y|)/(3|x|+4)\right]$ and $g(t,x,y)=(t/3)\left[((|x+y|)^2+1)/(|x|+2|y|+3)\right]$. By using the Maple program, we can find

$$
\begin{aligned}
\Phi(0) &= \frac{T^{p_2}}{\Gamma(1+p_2)}+\frac{\Gamma(p_1)}{\Gamma(p_1+p_2)}\frac{T^{p_1+p_2-1}}{|\Omega|}\left(\frac{\eta^{p_2}}{\Gamma(1+p_2)}\right.\\
&\left.+\sum_{i=1}^{2}|\alpha_i|\left[\frac{1}{\Gamma(1+p_2)}\frac{\xi_i^{p_2+\mu_i\gamma_i}}{\mu_i^{\gamma_i}}\frac{\Gamma\left(\frac{p_2+\mu_i}{\mu_i}\right)}{\Gamma\left(\frac{p_2+\mu_i\gamma_i+\mu_i}{\mu_i}\right)}\right]\right)\\
&\approx 2.401980728,
\end{aligned}
\tag{33}
$$

and

$$
\begin{aligned}
\Lambda(0) &= \frac{T^{q_2}}{\Gamma(1+q_2)}+\frac{\Gamma(q_1)}{\Gamma(q_1+q_2)}\frac{T^{q_1+q_2-1}}{|\Psi|}\left(\frac{\kappa^{q_2}}{\Gamma(1+q_2)}\right.\\
&\left.+\sum_{j=1}^{3}|\beta_j|\left[\frac{1}{\Gamma(1+q_2)}\frac{\zeta_i^{q_2+\delta_j\phi_j}}{\delta_j^{\phi_j}}\frac{\Gamma\left(\frac{q_2+\delta_j}{\delta_j}\right)}{\Gamma\left(\frac{q_2+\delta_j\phi_j+\delta_j}{\delta_j}\right)}\right]\right)
\end{aligned}
$$

$$\approx \quad 1.427481620. \tag{34}$$

Thus $\Upsilon \approx 0.7205942184 < 1$.
Since $|f(t,x,y)| \leq (t/5)\left[((|x+y|)^3+|x|+|y|)/2\right]$, $|g(t,x,y)| \leq (t/3)\left[((|x+y|)^2+1)/3\right]$, we choose $z_1(t)=t/10$, $\psi(x+y)=|x+y|)^3+|x|+|y|$, $z_2(t)=t/9$, $\omega(x+y)=(|x+y|)^2+1)$. We can show that

$$\sup_{r\in(0,\infty)} \frac{r}{\|z_1\|[\psi_1(r)+\psi_2(r)]\Phi(p_1)+\|z_2\|[\omega_1(r)+\omega_2(r)]\Lambda(q_1)}$$
$$\approx \quad 3.980031158 > 3.579024007 \approx \frac{1}{1-\Upsilon}.$$

Hence, by Theorem 3.5, the system (32) has at least one solution on $\left[0, \frac{2}{3}\right]$.

Acknowledgement:

This work was financially supported by the Research Grant of Burapha University through National Research Council of Thailand (Grant no. 144/2560).

References

[1] I. Podlubny, *Fractional Differential Equations*, Academic Press, San Diego, 1999.

[2] A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, *Theory and Applications of Fractional Differential Equations*, North-Holland Mathematics Studies, 204. Elsevier Science B.V., Amsterdam, 2006.

[3] K. Diethelm, *The Analysis of Fractional Differential Equations*, Lecture Notes in Mathematics, Springer-verlag Berlin Heidelberg, 2010.

[4] R.P. Agarwal, Y. Zhou, Y. He, Existence of fractional neutral functional differential equations, *Comput. Math. Appl.* **59** (2010), 1095-1100.

[5] B. Ahmad, S.K. Ntouyas, J. Tariboon, Existence results for mixed Hadamard and Riemann-Liouville fractional integro-differential equations, *Adv. Difference Equ.* (2015) **2015:293**.

[6] B. Ahmad, J.J. Nieto, Riemann-Liouville fractional integro-differential equations with fractional nonlocal integral boundary conditions, *Bound. Value Probl.* (2011) **2011:36**.

[7] B. Ahmad, S.K. Ntouyas, A. Alsaedi, New existence results for nonlinear fractional differential equations with three-point integral boundary conditions, *Adv. Difference Equ.* (2011) Art. ID 107384, 11pp.

[8] J. Tariboon, S.K. Ntouyas, P. Thiramanus, Riemann-Liouville fractional differential equations with Hadamard fractional integral conditions, *Inter. J. Appl. Math. Stat.* **54** (2016), 119-134.

[9] B. Ahmad, S.K. Ntouyas, A. Alsaedi, A study of nonlinear fractional differential equations of arbitrary order with Riemann-Liouville type multistrip boundary conditions, *Math. Probl. Eng.* (2013), Art. ID 320415, 9 pp.

[10] B. Ahmad, J.J. Nieto, Boundary value problems for a class of sequential integrodifferential equations of fractional order, *J. Funct. Spaces Appl.* 2013, Art. ID 149659, 8 pp.

[11] L. Zhang, B. Ahmad, G. Wang, R.P. Agarwal, Nonlinear fractional integro-differential equations on unbounded domains in a Banach space, *J. Comput. Appl. Math.* **249** (2013), 51–56.

[12] X. Liu, M. Jia, W. Ge, Multiple solutions of a p-Laplacian model involving a fractional derivative, *Adv. Difference Equ.* (2013), **2013:126**.

[13] U.N. Katugampola, New Approach to a generalized fractional integral, *Appl. Math. Comput.* **218** (2015), 860-865.

[14] U.N. Katugampola, A new approach to generalized fractional derivatives, *Bull. Math. Anal. Appl.* **6** (2014), 1-15.

[15] A.B. Malinowska, T. Odzijewicz, D.F.M. Torres, *Advanced Methods in the Fractional Calculus of Variations*, Springer, 2015.

[16] Ch. Thaiprayoon, S.K. Ntouyas, J. Tariboon, On the nonlocal Katugampola fractional integral conditions for fractional Langevin equation, *Adv. Difference Equ.* (2015) **2015:374**.

[17] S.K. Ntouyas, J. Tariboon, Langevin fractional differential inclusions with nonlocal Katugampola fractional integral boundary conditions, *J. Comput. Appl. Anal.*, to appear.

[18] W.T. Coffey, Yu.P. Kalmykov, J.T. Waldron, *The Langevin Equation*, second ed., World Scientific, Singapore, 2004.

[19] S.C. Lim, M. Li, L.P. Teo, Langevin equation with two fractional orders, *Phys. Lett. A* **372** (2008), 6309-6320.

[20] S.C. Lim, L.P. Teo, The fractional oscillator process with two indices, *J. Physics A: Math. Theor.* **42** (2009) 34. Art. ID 065208.

[21] M. Uranagase, T. Munakata, Generalized Langevin equation revisited: mechanical random force and self-consistent structure, *J. Phys. A: Math. Theor.* **43** (2010) 11. Art. ID 455003.

[22] S.I. Denisov, H. Kantz, P. Hänggi, Langevin equation with super-heavy-tailed noise, *J. Phys. A: Math. Theor.* **43** (2010) 10. Art. ID 285004.

[23] A. Lozinski, R.G. Owens, T.N. Phillips, *The Langevin and Fokker-Planck Equations in Polymer Rheology*, 2011, Handbook of Numerical Analysis 16 (C), pp. 211-303.

[24] J. Tariboon, S.K. Ntouyas, C. Thaiprayoon, Nonlinear Langevin equation of Hadamard-Caputo type fractional derivatives with nonlocal fractional integral conditions, *Adv. Math. Phys.* Volume 2014 (2014), Article ID 372749, 15 pages.

[25] A. Alsaedi, S.K. Ntouyas, B. Ahmad, Existence results for Langevin fractional differential inclusions involving two fractional orders with four-point multi-term fractional integral boundary conditions, *Abstr. Appl. Anal.* Volume 2013 (2013), Article ID 869837, 17 pages

[26] J. Tariboon, S.K. Ntouyas, Ch. Thaiprayoon, Nonlinear Langevin equation of Hadamard-Caputo type fractional derivatives with nonlocal fractional integral conditions, *Adv. Math. Phys.* Volume 2014 (2014), Article ID 372749, 15 pages.

[27] W. Yukunthorn, S.K. Ntouyas, J. Tariboon, Nonlinear fractional Caputo-Langevin equation with nonlocal Riemann-Liouville fractional integral conditions, *Adv. Difference Equ.* (2014), **2014:315**.

[28] W. Sudsutad, S.K. Ntouyas, J. Tariboon, Systems of fractional Langevin equation via Riemann-Liouville and Hadamard types and their fractional integral conditions, *Adv. Difference Equ.* (2015) **2015:235**.

[29] M.A. Krasnoselskii, Two remarks on the method of successive approximations, *Uspekhi Mat. Nauk* **10** (1955), 123-127.

[30] D. O'Regan, Fixed-point theory for the sum of two operators, *Appl. Math. Lett.* **9** (1996), 1-8.

[31] W.V. Petryshyn, P. M. Fitzpatric, A degree theory, fixed point theorems, and mapping theorems for multivalued noncompact maps, *Trans. Amer. Math. Soc.*, **194** (1974), 1-25.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 27, NO.4, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC

SUBORDINATION RESULTS FOR CERTAIN CLASS OF ANALYTIC FUNCTIONS ASSOCIATED WITH MITTAG-LEFFLER FUNCTION

MANSOUR F. YASSEN

Abstract. In this paper, we introduce a new class of analytic functions associated with Mittag-Leffler fuction in the open unit disk. Several properties of functions belonging to this class are derived.

1. Introduction

Let $\mathbf{U}$ be the open unit disc $\mathbf{U} = \{z : |z| < 1\}$. Also, Let $\mathcal{A}(p)$ the class of functions $f(z)$ of the form

$$f(z) = z^p + \sum_{n=2}^{\infty} a_n z^{n+p-1}, \tag{1.1}$$

which are analytic in $\mathbf{U}$, where $p \in \mathbb{N} = \{1, 2, 3, ...\}$. Also $f_i(z) \in \mathcal{A}(p), (i = 1, 2)$ defined by

$$f_i(z) = z^p + \sum_{n=2}^{\infty} a_{n,i} z^{n+p-1}, \qquad (i = 1, 2) \tag{1.2}$$

the convolution (or Hadamard product) of $f_1(z)$ and $f_2(z)$ given by:

$$(f_1 * f_2)(z) := (f_2 * f_1)(z) := z^p + \sum_{n=2}^{\infty} a_{n,1} a_{n,2} z^{n+p-1}. \tag{1.3}$$

The Mittag-Leffler function ([11],[12]) is defined by:

$$E_\alpha(z) = 1 + \frac{z}{\alpha!} + \frac{z^2}{(2\alpha)!} + \frac{z^3}{(3\alpha)!} + ... = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)}, \quad (\alpha \in \mathbb{C}; Re(\alpha) > 0). \tag{1.4}$$

Some interesting properties and general of Mittag-Leffler function can be found *e.g.* in [2], [3], [4], [5], [6], [9], [13], [14], [15], [16], [18], [21], [22] and [23]. The function $E^{\eta,k}_{\alpha,\beta}(z)(z \in \mathbb{C})$ introduced by Srivastava and Tomovski [20] in the form:

$$E^{\eta,k}_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{(\eta)_{nk} z^n}{\Gamma(\alpha n + \beta) n!}, \quad (\alpha, \beta, \eta \in \mathbb{C}; Re(\alpha) > max\{0, Re(k)-1\}; Re(k) > 0), \tag{1.5}$$

where

$$(\eta)_n = \frac{\Gamma(\eta + n)}{\Gamma(\eta)} = \begin{cases} 1, & n = 0, \\ \eta(\eta+1)(\eta+2)\dots(\eta+n-1), & n \in \mathbb{N}. \end{cases} \tag{1.6}$$

2010 *Mathematics Subject Classification.* 30C45.

Key words and phrases. Analytic functions, Hadamard product,starlike functions, prestar-like functions, Differential subordination, Mittag-Leffler function.

1

The function $E^{\eta,k}_{\alpha,\beta}(z)$ proved by Srivastava and Tomovski [20] is an entire function in the complex z-plane. Attiya [1] defined the function $Q^{\eta,k}_{\alpha,\beta}(z)$ by

$$Q^{\eta,k}_{\alpha,\beta}(z) = \frac{\Gamma(\alpha+\beta)}{(\eta)_k}\left(E^{\eta,k}_{\alpha,\beta}(z) - \frac{1}{\Gamma(\beta)}\right), \quad (z \in \mathbf{U}), \tag{1.7}$$

very recently, Attiya [1] introduce the operator

$$\mathcal{H}^{\eta,k}_{\alpha,\beta}(f(z)) : \mathcal{A}(1) \to \mathcal{A}(1),$$

defined, in terms of convolution by

$$\begin{aligned}\mathcal{H}^{\eta,k}_{\alpha,\beta}(f(z)) &= Q^{\eta,k}_{\alpha,\beta}(z) * f(z) \\ &= z + \sum_{n=2}^{\infty} \frac{\Gamma(\eta+nk)\Gamma(\alpha+\beta)}{\Gamma(\eta+k)\Gamma(n\alpha+\beta)} a_n z^n \quad (z \in \mathbf{U}).\end{aligned} \tag{1.8}$$

Analogous to $\mathcal{H}^{\eta,k}_{\alpha,\beta}(f(z))$, we introduce the operator $\mathcal{H}^{\eta,k}_{\alpha,\beta,p}(f(z))$ as follows

$$\mathcal{H}^{\eta,k}_{\alpha,\beta,p}(f(z)) : \mathcal{A}(p) \to \mathcal{A}(p), \tag{1.9}$$

where

$$\mathcal{H}^{\eta,k}_{\alpha,\beta,p}(f(z)) = Q^{\eta,k}_{\alpha,\beta,p}(z) * f(z), \quad (z \in \mathbf{U}). \tag{1.10}$$

and

$$Q^{\eta,k}_{\alpha,\beta,p}(z) = \frac{z^{p-1}\Gamma(\alpha+\beta)}{(\eta)_k}\left(E^{\eta,k}_{\alpha,\beta}(z) - \frac{1}{\Gamma(\beta)}\right), \quad (z \in \mathbf{U}), \tag{1.11}$$

from equations (1.9), (1.10) and (1.11) we not that

$$\begin{aligned}\mathcal{H}^{\eta,k}_{\alpha,\beta,p}(f(z)) &= Q^{\eta,k}_{\alpha,\beta,p}(z) * f(z) \\ &= z^p + \sum_{n=2}^{\infty} \frac{\Gamma(\eta+nk)\Gamma(\alpha+\beta)}{\Gamma(\eta+k)\Gamma(n\alpha+\beta)} a_n z^{n+p-1} \quad (z \in \mathbf{U}),\end{aligned} \tag{1.12}$$

when $p = 1$, the operator $\mathcal{H}^{\eta,k}_{\alpha,\beta,1}(f(z))$ is the Attiya operator $\mathcal{H}^{\eta,k}_{\alpha,\beta}(f(z))$ [1].

A function $f(z) \in \mathcal{A}(1)$ is said to be in the class $\mathcal{S}^*(\sigma)$ [7] and [19] or (star-like of order σ in $\mathbf{U}$) if:

$$\mathcal{S}^*(\sigma) := \left\{ f(z) : Re\left(\frac{zf'(z)}{f(z)}\right) > \sigma, 0 \le \sigma < 1, z \in \mathbf{U} \right\}. \tag{1.13}$$

A function $f(z) \in \mathcal{A}(1)$ is said to be in the class $\Re(\sigma)$ [7] and [17] or (pre-starlike of order σ in $\mathbf{U}$) if:

$$\Re(\sigma) := \left\{ f(z) : \frac{z}{(1-z)^{2(1-\sigma)}} * f(z) \in \mathcal{S}^*(\sigma), \sigma < 1, z \in \mathbf{U} \right\}. \tag{1.14}$$

The function $g(z)$ is called subordinate to $G(z)$, if there exist a Schwarz function $h(z)$, analytic in $\mathbf{U}$, with $h(0) = 0$ and $|h(z)| \le 1$, such that $g(z) = G(h(z))$ for all $z \in \mathbf{U}$.

This subordination is denoted by $g(z) \prec G(z)$. If the function $G(z)$ is univalent in $\mathbf{U}$, then $g(z) \prec G(z)$ if and only if $g(0) = G(0)$ and $g(\mathbf{U}) \subset G(\mathbf{U})$.

Let $\mathfrak{T}$ be the class of function $w(z)$ with $w(0) = 1$, which are analytic and univalent in $\mathbf{U}$.

Definition 1. Let $f(z) \in \mathcal{A}(p)$, then $f(z)$ is said to be in the class $\mathfrak{T}^{\eta,k}_{\alpha,\beta,p}(\delta; w)$ if it satisfies the following condition

$$\frac{(1-\delta)}{p} z^{-p+1} \left(\mathcal{H}^{\eta,k}_{\alpha,\beta,p}(f(z))\right)' + \frac{\delta}{p(p-1)} z^{-p+2} \left(\mathcal{H}^{\eta,k}_{\alpha,\beta,p}(f(z))\right)'' \prec w(z), \quad (1.15)$$

where $\delta \in \mathbb{C}$), $p \in \mathbb{N} \setminus \{1\}$ and $w(z) \in \mathfrak{T}$.

The main object of our paper is to investigate and introduce some subordination results of the class $\mathfrak{T}^{\eta,k}_{\alpha,\beta,p}(\delta; w)$.

2. Some Lemmas

In our paper, we use the following lemmas:

Lemma 1.1 [10]. Let $G(z)$ be analytic function in $\mathbf{U}$ and $w(z)$ be analytic and convex univalent in $\mathbf{U}$ with $G(0) = w(0)$, if

$$G(z) + \frac{1}{\vartheta} zG'(z) \prec w(z), \quad (2.1)$$

where $Re(\vartheta) \geq 0$ and $\vartheta \neq 0$, then $G(z) \prec w(z)$.

Lemma 1.2 [17]. Let $\sigma < 1$, $f(z) \in \mathcal{S}^*(\sigma)$, and $G(z) \in \Re(\sigma)$, then, for analytic function $F(z)$ in $\mathbf{U}$,

$$\frac{G * (fF)}{G * f}(\mathbf{U}) \subset \overline{co}\left(F(\mathbf{U})\right), \quad (2.2)$$

where $\overline{co}\left(F(\mathbf{U})\right)$ denote the closed convex hull of $F(\mathbf{U})$.

Lemma 1.3 [8]. Let $G(z) = 1 + \sum_{n=k}^{\infty} d_n z^n, (k \in \mathbb{N})$ be analytic function and convex univalent function in $\mathbf{U}$. If $Re\{G(z)\} > 0, (z \in \mathbf{U})$, then

$$Re\{G(z)\} \geq \frac{1-|z|^k}{1+|z|^k} \quad (k \in \mathbb{N}; z \in \mathbf{U}). \quad (2.3)$$

3. Properties of the class $\mathfrak{T}^{\eta,k}_{\alpha,\beta,p}(\delta; w)$

In this section, we let $p \in \mathbb{N}$ and $p > 1$.

Theorem 3.1. *let* $0 \leq \delta_1 < \delta_2$. *Then* $\mathfrak{T}^{\eta,k}_{\alpha,\beta,p}(\delta_2; w) \subset \mathfrak{T}^{\eta,k}_{\alpha,\beta,p}(\delta_1; w)$.

Proof. Let $f(z) \in \mathfrak{T}^{\eta,k}_{\alpha,\beta,p}(\delta_2; w)$ and $0 \leq \delta_1 < \delta_2$. Suppose that

$$\phi(z) = \frac{z^{-p+1}}{p} \left(\mathcal{H}^{\eta,k}_{\alpha,\beta,p}(f(z))\right)'. \quad (3.1)$$

Therefore, the function $\phi(z)$ in the above equation is analytic in $\mathbf{U}$ with $\phi(0) = 1$. Differentiating the both sides of the above equation w.r.t. z, we have

$$\phi'(z) = \frac{(1-p)z^{-p}}{p} \left(\mathcal{H}^{\eta,k}_{\alpha,\beta,p}(f(z))\right)' + \frac{z^{-p+1}}{p} \left(\mathcal{H}^{\eta,k}_{\alpha,\beta,p}(f(z))\right)''. \quad (3.2)$$

By using Equation (1.15), we have

$$\begin{aligned} \frac{(1-\delta_2)z^{-p+1}}{p} \left(\mathcal{H}^{\eta,k}_{\alpha,\beta,p}(f(z))\right)' + \frac{\delta_2 z^{-p+2}}{p(p-1)} \left(\mathcal{H}^{\eta,k}_{\alpha,\beta,p}(f(z))\right)'' &= \phi(z) + \frac{\delta_2 z\phi'(z)}{(p-1)} \\ &\prec w(z). \quad (3.3) \end{aligned}$$

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 27, NO.4, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC

Using Lemma 1.1, we have

$$\phi(z) \prec w(z). \tag{3.4}$$

Since $0 \leq \delta_1/\delta_2 < 1$ and $w(z)$ is univalent in $\mathbf{U}$, using equations (3.1) and (3.4), we given that

$$\frac{(1-\delta_1)z^{-p+1}}{p}\left(\mathcal{H}^{\eta,k}_{\alpha,\beta,p}(f(z))\right)' + \frac{\delta_1 z^{-p+2}}{p(p-1)}\left(\mathcal{H}^{\eta,k}_{\alpha,\beta,p}(f(z))\right)'' = \left(1-\frac{\delta_1}{\delta_2}\right)\phi(z)$$

$$+\frac{\delta_1}{\delta_2}\left(\frac{(1-\delta_2)z^{-p+1}}{p}\left(\mathcal{H}^{\eta,k}_{\alpha,\beta,p}(f(z))\right)' + \frac{\delta_2 z^{-p+2}}{p(p-1)}\left(\mathcal{H}^{\eta,k}_{\alpha,\beta,p}(f(z))\right)''\right) \prec w(z). \tag{3.5}$$

Therefore $f(z) \in \mathfrak{T}^{\eta,k}_{\alpha,\beta,p}(\delta_1; w)$, and the proof of Theorem 1.1 is completed. ∎

Theorem 3.2. *Let $\delta > 0, \rho > 0$, and $f(z) \in \mathfrak{T}^{\eta,k}_{\alpha,\beta,p}(\delta; \rho w + 1 - \rho)$. If $\rho \leq \rho_0$, where*

$$\rho_0 = \frac{1}{2}\left(1 - \frac{(p-1)}{\delta}\int_0^1 \frac{t^{((p-1)/\delta)-1}}{1+t}dt\right)^{-1}. \tag{3.6}$$

Then $f(z) \in \mathfrak{T}^{\eta,k}_{\alpha,\beta,p}(0; w)$. The bound ρ_0 is sharp in the case $w(z) = 1/(1-z)$.

Proof. Let $f(z) \in \mathfrak{T}^{\eta,k}_{\alpha,\beta,p}(\delta; \rho w + 1 - \rho)$ with $\delta > 0, \rho > 0$. Suppose that

$$\phi(z) = \frac{z^{-p+1}}{p}\left(\mathcal{H}^{\eta,k}_{\alpha,\beta,p}(f(z))\right)'. \tag{3.7}$$

Then we have

$$\begin{aligned}\phi(z) + \frac{\delta z\phi'(z)}{(p-1)} &= \frac{(1-\delta)z^{-p+1}}{p}\left(\mathcal{H}^{\eta,k}_{\alpha,\beta,p}(f(z))\right)' + \frac{\delta z^{-p+2}}{p(p-1)}\left(\mathcal{H}^{\eta,k}_{\alpha,\beta,p}(f(z))\right)'' \\ &\prec \rho w(z) + 1 - \rho. \end{aligned}\tag{3.8}$$

Using Lemma 1.1, we have

$$\phi(z) \prec \frac{\rho(p-1)z^{(-(p-1)/\delta)}}{\delta}\int_0^z u^{((p-1)/\delta)}w(u)du + 1 - \rho = (w * \varphi)(z), \tag{3.9}$$

where

$$\varphi(z) = \frac{\rho(p-1)z^{(-(p-1)/\delta)}}{\delta}\int_0^z \frac{u^{((p-1)/\delta)-1}}{1-u}du + 1 - \rho. \tag{3.10}$$

If $0 < \rho \leq \rho_0$ where $\rho_0(>1)$ is given by (3.6), then it follows from (3.10) that

$$\begin{aligned}Re\{\varphi(z)\} &= \frac{\rho(p-1)}{\delta}\int_0^1 t^{((p-1)/\delta)-1}Re\left\{\frac{1}{1-tz}\right\}dt + 1 - \rho \\ &> \frac{\rho(p-1)}{\delta}\int_0^1 \frac{t^{((p-1)/\delta)-1}}{1+t}dt + 1 - \rho \\ &\geq \frac{1}{2} \qquad (z \in \mathbf{U}).\end{aligned}\tag{3.11}$$

Using the Herglotz representation for $\varphi(z)$. Also, from Equations (3.7) and (3.9) we obtain

$$\frac{z^{-p+1}}{p}\left(\mathcal{H}^{\eta,k}_{\alpha,\beta,p}(f(z))\right)' \prec (w * \varphi)(z) \prec w(z), \tag{3.12}$$

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 27, NO.4, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC

since $w(z)$ is convex univalent in $\mathbf{U}$. Therefore $f(z) \in \mathfrak{T}^{\eta,k}_{\alpha,\beta,p}(0;w)$. If $w(z) = 1/(1-z)$ and $f(z) \in \mathcal{A}(p)$ defined by:

$$\frac{z^{-p+1}}{p}\left(\mathcal{H}^{\eta,k}_{\alpha,\beta,p}(f(z))\right)' = \frac{\rho(p-1)z^{-(p-1)/\delta}}{\delta}\int_0^z \frac{u^{((p-1)/\delta)-1}}{1-u}du + 1 - \rho, \quad (3.13)$$

we can see that

$$\frac{(1-\delta)z^{-p+1}}{p}\left(\mathcal{H}^{\eta,k}_{\alpha,\beta,p}(f(z))\right)' + \frac{\delta z^{-p+2}}{p(p-1)}\left(\mathcal{H}^{\eta,k}_{\alpha,\beta,p}(f(z))\right)'' = \rho w(z) + 1 - \rho. \quad (3.14)$$

Thus $f(z) \in \mathfrak{T}^{\eta,k}_{\alpha,\beta,p}(\delta;\rho h + 1 - \rho)$. Also, for $\rho > \rho_0$, we have (at $z \to -1$)

$$Re\left\{\frac{z^{-p+1}}{p}\left(\mathcal{H}^{\eta,k}_{\alpha,\beta,p}(f(z))\right)'\right\} \longrightarrow \frac{\rho(p-1)}{\delta}\int_0^1 \frac{t^{((p-1)/\delta)-1}}{1+t}dt + 1 - \rho < \frac{1}{2}, \quad (3.15)$$

which obtains $f(z) \notin \mathfrak{T}^{\eta,k}_{\alpha,\beta,p}(0;w)$. Therefore, the value ρ_0 cannot be increased when $w(z) = 1/(1-z)$. ∎

Theorem 3.3. *Let* $f(z) \in \mathfrak{T}^{\eta,k}_{\alpha,\beta,p}(\delta;w), \phi(z) \in \mathcal{A}(p)$, *and*

$$Re\left\{z^{-p}\phi(z)\right\} > \frac{1}{2} \qquad (z \in \mathbf{U}). \quad (3.16)$$

Then $(f * \phi)(z) \in \mathfrak{T}^{\eta,k}_{\alpha,\beta,p}(\delta;w)$.

Proof. For $f(z) \in \mathfrak{T}^{\eta,k}_{\alpha,\beta,p}(\delta;w)$ and $\phi(z) \in \mathcal{A}(p)$, we have

$$\begin{aligned}
&\frac{(1-\delta)z^{-p+1}}{p}\left(\mathcal{H}^{\eta,k}_{\alpha,\beta,p}((f*\phi)(z))\right)' + \frac{\delta z^{-p+2}}{p(p-1)}\left(\mathcal{H}^{\eta,k}_{\alpha,\beta,p}((f*\phi)(z))\right)'' \\
=\ & \frac{(1-\delta)}{p}\left(z^{-p}\phi(z)\right) * \left(z^{-p+1}\left(\mathcal{H}^{\eta,k}_{\alpha,\beta,p}(f)(z))\right)'\right) \\
&+\frac{\delta}{p(p-1)}\left(z^{-p}\phi(z)\right) * \left(z^{-p+2}\left(\mathcal{H}^{\eta,k}_{\alpha,\beta,p}(f)(z))\right)''\right) \\
=\ & (z^{-p}\phi(z)) * \varphi(z), \qquad (3.17)
\end{aligned}$$

where

$$\varphi(z) = \frac{(1-\delta)z^{-p+1}}{p}\left(\mathcal{H}^{\eta,k}_{\alpha,\beta,p}(f(z))\right)' + \frac{\delta z^{-p+2}}{p(p-1)}\left(\mathcal{H}^{\eta,k}_{\alpha,\beta,p}(f(z))\right)''. \quad (3.18)$$

Using (3.16), the function $z^{-p}\phi(z)$ has the Herglotz Representation

$$z^{-p}\phi(z) = \int_{|y|=1}\frac{d\mu(y)}{(1-yz)} \qquad (z \in \mathbf{U}), \quad (3.19)$$

where $\mu(y)$ is a probability measure defined on the circle $|y| = 1$ and

$$\int_{|y|=1} d\mu(y) = 1.$$

Since $w(z)$ is convex univalent in $\mathbf{U}$, it follows from (3.17) to (3.19) that

$$\begin{aligned}
\frac{(1-\delta)z^{-p+1}}{p}\left(\mathcal{H}^{\eta,k}_{\alpha,\beta,p}((f*\phi)(z))\right)' &+ \frac{\delta z^{-p+2}}{p(p-1)}\left(\mathcal{H}^{\eta,k}_{\alpha,\beta,p}((f*\phi)(z))\right)'' \\
&= \int_{|y|=1}\varphi(yz)d\mu(y) \prec w(z). \qquad (3.20)
\end{aligned}$$

This shows that $(f*\phi)(z) \in \mathfrak{T}^{\eta,k}_{\alpha,\beta,p}(\delta;w)$ and the theorem is proved. ∎

Theorem 3.4. *Let $f(z) \in \mathfrak{T}^{\eta,k}_{\alpha,\beta,p}(\delta;w), \phi(z) \in \mathcal{A}(p)$, and*

$$z^{-p+1}\phi(z) \in \Re(\sigma) \qquad (\sigma < 1, z \in \mathbf{U}). \tag{3.21}$$

Then $(f*\phi)(z) \in \mathfrak{T}^{\eta,k}_{\alpha,\beta,p}(\delta;w)$.

Proof. For $f(z) \in \mathfrak{T}^{\eta,k}_{\alpha,\beta,p}(\delta;w)$ and $\phi(z) \in \mathcal{A}(p)$ from (3.17), we have

$$\begin{aligned}&\frac{(1-\delta)z^{-p+1}}{p}\left(\mathcal{H}^{\eta,k}_{\alpha,\beta,p}((f*\phi)(z))\right)' + \frac{\delta z^{-p+2}}{p(p-1)}\left(\mathcal{H}^{\eta,k}_{\alpha,\beta,p}((f*\phi)(z))\right)''\\ =\;& \frac{(z^{-p+1}\phi(z))*(z\varphi(z))}{(z^{-p+1}\phi(z))*z} \qquad (z\in\mathbf{U}),\end{aligned} \tag{3.22}$$

where $\varphi(z)$ is defined as in (3.18). Since $w(z)$ is convex univalent in $\mathbf{U}$,

$$\varphi(z) \prec w(z), \quad z^{-p+1}\phi(z) \in \Re(\sigma) \quad \text{and} \quad z \in \mathcal{S}^*(\sigma), \quad (\sigma < 1).$$

It follows from (3.22) and Lemma 1.2 the desired result. ∎

Theorem 3.5. *Let $\delta \geq 0$ and*

$$f_i(z) = z^p + \sum_{n=2}^{\infty} a_{n,i}z^{n+p-1} \in \mathfrak{T}^{\eta,k}_{\alpha,\beta,p}(\delta;w), \qquad (i=1,2) \tag{3.23}$$

where

$$w_i(z) = \gamma_i + (1-\gamma_i)\frac{1+z}{1-z} \qquad \text{and} \qquad \gamma_i < 1, (i=1,2). \tag{3.24}$$

If $f(z) \in \mathcal{A}(p)$ is defined by

$$\mathcal{H}^{\eta,k}_{\alpha,\beta,p}(f(z)) = \int_0^z \left(\mathcal{H}^{\eta,k}_{\alpha,\beta,p}(f_1(u))\right)' * \left(\mathcal{H}^{\eta,k}_{\alpha,\beta,p}(f_2(u))\right)' du. \tag{3.25}$$

Then $f(z) \in \mathfrak{T}^{\eta,k}_{\alpha,\beta,p}(\delta;w)$, where

$$w(z) = \gamma + (1-\gamma)\frac{1+z}{1-z}, \tag{3.26}$$

where γ is given by

$$\gamma = \begin{cases} p - 4p(1-\gamma_1)(1-\gamma_2)\left(1 - \frac{p-1}{\delta}\int_0^1 \frac{t^{((p-1)/\delta)-1}}{1+t}dt\right); & (\delta > 0)\\ p - 2p(1-\gamma_1)(1-\gamma_2); & (\delta = 0),\end{cases} \tag{3.27}$$

the value of γ is the best possible.

Proof. For the case when $\delta > 0$. by putting

$$G_i(z) = \frac{(1-\delta)z^{-p+1}}{p}\left(\mathcal{H}^{\eta,k}_{\alpha,\beta,p}(f_i(z))\right)' + \frac{\delta z^{-p+2}}{p(p-1)}\left(\mathcal{H}^{\eta,k}_{\alpha,\beta,p}(f_i(z))\right)'' \quad (i=1,2), \tag{3.28}$$

for $f_i(z), (i=1,2)$ given by (3.23), we find that

$$G_i(z) = 1 + \sum_{n=2}^{\infty} b_{n,i}z^{n-1} \prec \gamma_i + (1-\gamma_i)\frac{1+z}{1-z} \qquad (i=1,2),$$

and

$$\left(\mathcal{H}^{\eta,k}_{\alpha,\beta,p}(f_i(z))\right)' = \frac{p(p-1)z^{-(p-1)(1-\delta)/\delta}}{\delta}\int_0^z u^{((p-1)/\delta)-1}G_i(u)du \quad (i=1,2).$$

Now, if $f(z) \in \mathcal{A}(p)$ is defined by (3.25), we find from (3.30) that

$$\begin{aligned}
\left(\mathcal{H}^{\eta,k}_{\alpha,\beta,p}(f(z))\right)' &= \left(\mathcal{H}^{\eta,k}_{\alpha,\beta,p}(f_1(z))\right)' * \left(\mathcal{H}^{\eta,k}_{\alpha,\beta,p}(f_2(z))\right)' \\
&= \left(\frac{p(p-1)z^{p-1}}{\delta}\int_0^1 t^{((p-1)/\delta)-1}G_1(tz)dt\right) \\
&\quad * \left(\frac{p(p-1)z^{p-1}}{\delta}\int_1^z t^{((p-1)/\delta)-1}G_2(tz)dt\right) \\
&= \left(\frac{p(p-1)z^{p-1}}{\delta}\int_1^z t^{((p-1)/\delta)-1}G(tz)dt\right) \qquad (3.29)
\end{aligned}$$

where

$$G(z) = \frac{p(p-1)}{\delta}\int_0^1 u^{((p-1)/\delta)-1}(G_1 * G_2)(tz)dt. \qquad (3.30)$$

Also, by using (3.29) and the Herglotz theorem,we see that

$$Re\left\{\left(\frac{G_1(z)-\gamma_1}{1-\gamma_1}\right) * \left(\frac{1}{2} + \frac{G_2(z)-\gamma_2}{2(1-\gamma_2)}\right)\right\} > 0 \qquad (z \in \mathbf{U}), \qquad (3.31)$$

which gives

$$Re\left\{(G_1 * G_2)(z)\right\} > \gamma_0 = 1 - 2(1-\gamma_1)(1-\gamma_2) \qquad (z \in \mathbf{U}). \qquad (3.32)$$

According to Lemma 1.3, we have

$$Re\left\{(G_1 * G_2)(z)\right\} \geq \gamma_0 + (1-\gamma_0)\left(\frac{1-|z|}{1+|z|}\right) \qquad (z \in \mathbf{U}). \qquad (3.33)$$

Now it follows from (3.31) to (3.35) that

$$\begin{aligned}
Re &\quad \left\{\frac{(1-\delta)z^{-p+1}}{p}\left(\mathcal{H}^{\eta,k}_{\alpha,\beta,p}(f(z))\right)' + \frac{\delta z^{-p+2}}{p(p-1)}\left(\mathcal{H}^{\eta,k}_{\alpha,\beta,p}(f(z))\right)''\right\} = Re\{G(z)\} \\
&= \frac{p(p-1)}{\delta}\int_0^1 t^{((p-1)/\delta)-1} Re\{(G_1 * G_2)(tz)\}dt \\
&\geq \frac{p(p-1)}{\delta}\int_0^1 t^{((p-1)/\delta)-1}\left(\beta_0 + (1-\beta_0)\frac{1-t|z|}{1+t|z|}\right)dt \\
&> p\gamma_0 + \frac{p(p-1)(1-\gamma_0)}{\delta}\int_0^1 t^{((p-1)/\delta)-1}\frac{1-t}{1+t}dt \\
&= p - 4p(1-\gamma_1)(1-\gamma_2)\left(1 - \frac{p-1}{\delta}\int_0^1 \frac{t^{((p-1)/\delta)-1}}{1+t}dt\right) \\
&= \gamma \qquad (z \in \mathbf{U}). \qquad (3.34)
\end{aligned}$$

which proves that $f(z) \in \mathfrak{T}^{\eta,k}_{\alpha,\beta,p}(\delta; w)$ for the function $w(z)$ given by (3.26).In order to show that the bound γ is Sharp, we take the functions $f_i(z) \in \mathcal{A}(p)$ $(i = 1, 2)$ defined by

$$\begin{aligned}
\left(\mathcal{H}^{\eta,k}_{\alpha,\beta,p}(f_i(z))\right)' &= \frac{p(p-1)z^{-(p-1)(1-\delta)/\delta}}{\delta} \\
&\quad \times \int_0^z u^{((p-1)/\delta)-1}\left(\gamma_i + (1-\gamma_i)\frac{1+u}{1-u}\right)du, \qquad (3.35)
\end{aligned}$$

for $i = 1, 2$ and, we have

$$\begin{aligned} G_i(z) &= \frac{(1-\delta)z^{-p+1}}{p}\left(\mathcal{H}^{\eta,k}_{\alpha,\beta,p}(f_i(z))\right)' + \frac{\delta z^{-p+2}}{p(p-1)}\left(\mathcal{H}^{\eta,k}_{\alpha,\beta,p}(f_i(z))\right)'' \\ &= \gamma_i + (1-\gamma_i)\frac{1+z}{1-z} \quad (i = 1, 2), \end{aligned} \tag{3.36}$$

and

$$(G_1 * G_2)(z) = 1 + 4(1-\gamma_1)(1-\gamma_2)\frac{z}{1-z}. \tag{3.37}$$

Hence, for $f(z) \in \mathcal{A}(p)$ given by (3.25), we obtain

$$\begin{aligned} &\frac{(1-\delta)z^{-p+1}}{p}\left(\mathcal{H}^{\eta,k}_{\alpha,\beta,p}(f(z))\right)' + \frac{\delta z^{-p+2}}{p(p-1)}\left(\mathcal{H}^{\eta,k}_{\alpha,\beta,p}(f(z))\right)'' \\ = &\frac{p(p-1)}{\delta}\int_0^1 t^{((p-1)/\delta)-1}\left(1 + 4(1-\gamma_1)(1-\gamma_2)\frac{tz}{1-tz}\right)dt \\ &\longrightarrow \gamma \quad (as \quad z \longrightarrow -1). \end{aligned} \tag{3.38}$$

The proof is simple in the case of $\delta = 0$, therefore, we omit the details involved. ∎

Conclusions

we introduced the class $\mathfrak{T}^{\eta,k}_{\alpha,\beta,p}(\delta; w)$ of analytic functions associated with Mittag-Leffler function. Conclusion property of the class $\mathfrak{T}^{\eta,k}_{\alpha,\beta,p}(\delta; w)$ is obtained, sufficient condition of the class $\mathfrak{T}^{\eta,k}_{\alpha,\beta,p}(\delta; w)$ is also derived. Furthermore, several properties of functions belonging to this class are derived.

References

[1] A. A. Attiya, Some Applications of Mittag-Leffler Function in the Unit Disk, Filomat,30,(7)(2016), 2075-2081.

[2] M. Garg, P. Manoha and S.L. Kalla, A Mittag-Leffler-type function of two variables, Integral Transforms Spec. Funct.24,(11)(2013), 934-944.

[3] A.A. Kilbas, H.M. Srivastava, and J.J. Trujillo, Theory and Applications of Fractional Differential Equations, North-Holland Mathematics Studies, 204. Elsevier Science B.V., Amsterdam,(2006).

[4] V. Kiryakova, Generalized fractional calculus and applications, Pitman Research Notes in Mathematics Series,301. Longman Scientific Technical, Harlow; copublished in the United States with John Wiley Sons, Inc., New York,1994.

[5] V. S. Kiryakova, Multiple (multiindex) Mittag-Leffler functions and relations to generalized fractional calculus.Higher transcendental functions and their applications, J. Comput. Appl. Math., 118,(1-2)(2000), 241-259.

[6] V. Kiryakova, The multi-index Mittag-Leffler functions as an important class of special functions of fractional calculus, Comput. Math. Appl. 59,(5)(2010), 1885-1895.

[7] J. Liu, subordinations for certain multivalent analytic funcations associated with the generalized Srivastava-Attiya operator, Integral Trans. and Special Functions, 19,(12)(2008), 893-901.

[8] T. H. MacGregor, Functions whose derivative has a postitive real part, Trans. Am. Math. Soc., 104,(1962), 532-537.

[9] F. Mainardia and R. Gorenflo, On Mittag-Leffler-type functions in fractional evolution processes. Higher transcendental functions and their applications, J. Comput. Appl. Math. 118,(1-2) (2000), 283299.

[10] S. S. Miller and P. T. Mocanu, Differential subordinations and univalent functions, Michigan Math. J., 28,(1981), 157-171.

[11] G. M. Mittag-leffler, Sur la nouvelle function, C.R. Acad. Sci.,Paris, 137,(1903), 554-558.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 27, NO.4, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC

[12] G. M. Mittag-leffler, Sur la representation analytique d'une function monogene (cinquieme note), Acta Math., 29,(1905), 101-181.
[13] M. A. Ozarslan and B. Ylmaz, The extended Mittag-Leffler function and its properties, J. Inequal. Appl.No.1, Vol. 2014,(2014).
[14] I. Podlubny, Fractional Differential Equations. An Introduction to Fractional Derivatives, Fractional Differential Equations, to Methods of Their Solution and Some of Their Applications. Mathematics in Science and Engineering, 198. Academic Press, Inc.,San Diego, CA,(1999).
[15] T. R. Prabhakar, A singular integral equation with a generalized Mittag-Leffeer function in the Kernal, Yokohoma Math. J., 19,(1971), 7-15.
[16] J. C. Prajapati, R.K.Jana, R. K. Saxena and A. K. Shukla, Some results on the generalized Mittag-Leffler function operator, J. Inequal.Appl., 6,(2013),1-6.
[17] S. Ruscheweyh, Convolutions in geometric function theory, Gaetan Morin Editeur Ltee,83,(1982).
[18] A. K. Shukla and J.C. Prajapati, On a generalization of MittagLeffler function and its properties, J. Math. Anal. Appl.,336, (2007),797-811.
[19] R. Singh, On a class of star-like functions, Compositio Mathematica,19,(1)(1968), 78-82.
[20] H. M. Srivastava and Z. Tomovski, Fractional calculus with an itegral operator containing a generalized Mittag-Leffler function in the kernal, Appl. Math. Comp., 211,(2009), 198-210.
[21] Z. Tomovski, R. Hilfer and H.M. Srivastava, Fractional and operational calculus with generalized fractional derivative operators and Mittag-Leffler type functions, Integral Transforms Spec. Funct., 21,(11) (2010), 797-814.
[22] Z. Tomovski, Generalized Cauchy type problems for nonlinear fractional differential equations with composite fractional derivative operator, Nonlinear Anal., 75,(7) (2012), 3364-3384.
[23] A. Wiman, Uber den Fundamental Salz in der Theorie der Funktionen, Acta. Math.,29, (1905),191-201.

Department of Mathematics, Faculty of Science, Damietta University, New Damietta 34517, Egypt.

Department of Mathematics, College of Science and Humanities in Al-Aflaj, Prince Sattam Bin Abdulaziz University, Kingdom of Saudi Arabia
E-mail address: mansouralieg@yahoo.com

TABLE OF CONTENTS, JOURNAL OF COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 27, NO. 4, 2019

TABLE OF CONTENTS, JOURNAL OF COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 27, NO. 4, 2019

(continued)

Volume 27, Number 5 October 30, 2019
ISSN:1521-1398 PRINT,1572-9206 ONLINE

Journal of Computational Analysis and Applications

EUDOXUS PRESS,LLC

Journal of Computational Analysis and Applications
ISSNno.'s:1521-1398 PRINT,1572-9206 ONLINE

SCOPE OF THE JOURNAL

An international publication of Eudoxus Press, LLC
(fifteen times annually)

Editor in Chief: George Anastassiou
Department of Mathematical Sciences,
University of Memphis, Memphis, TN 38152-3240, U.S.A
ganastss@memphis.edu
http://www.msci.memphis.edu/~ganastss/jocaaa

The main purpose of "J.Computational Analysis and Applications" is to publish high quality research articles from all subareas of Computational Mathematical Analysis and its many potential applications and connections to other areas of Mathematical Sciences. Any paper whose approach and proofs are computational,using methods from Mathematical Analysis in the broadest sense is suitable and welcome for consideration in our journal, except from Applied Numerical Analysis articles. Also plain word articles without formulas and proofs are excluded. The list of possibly connected mathematical areas with this publication includes, but is not restricted to: Applied Analysis, Applied Functional Analysis, Approximation Theory, Asymptotic Analysis, Difference Equations, Differential Equations, Partial Differential Equations, Fourier Analysis, Fractals, Fuzzy Sets, Harmonic Analysis, Inequalities, Integral Equations, Measure Theory, Moment Theory, Neural Networks, Numerical Functional Analysis, Potential Theory, Probability Theory, Real and Complex Analysis, Signal Analysis, Special Functions, Splines, Stochastic Analysis, Stochastic Processes, Summability, Tomography, Wavelets, any combination of the above, e.t.c.

"J.Computational Analysis and Applications" is a peer-reviewed Journal. See the instructions for preparation and submission of articles to JoCAAA. Assistant to the Editor: Dr.Razvan Mezei,mezei_razvan@yahoo.com, Madison,WI,USA.

Journal of Computational Analysis and Applications(JoCAAA) is published by **EUDOXUS PRESS,LLC**,1424 Beaver Trail Drive,Cordova,TN38016,USA,anastassioug@yahoo.com http://www.eudoxuspress.com. **Annual Subscription Prices**:For USA and Canada,Institutional:Print $800, Electronic OPEN ACCESS. Individual:Print $400. For any other part of the world add $160 more(handling and postages) to the above prices for Print. No credit card payments.

JoCAAA is reviewed and abstracted by AMS Mathematical Reviews,MATHSCI,and Zentralblaat MATH.

Dartmouth College
8000 Cummings Hall,
Hanover, NH 03755-8000
603-646-3843 (X 3546 Secr.)
e-mail:george.cybenko@dartmouth.edu
Approximation Theory and Neural Networks

Sever S. Dragomir
School of Computer Science and Mathematics, Victoria University,
PO Box 14428,
Melbourne City,
MC 8001, AUSTRALIA
Tel. +61 3 9688 4437
Fax +61 3 9688 4050
sever.dragomir@vu.edu.au
Inequalities, Functional Analysis, Numerical Analysis, Approximations, Information Theory, Stochastics.

Oktay Duman
TOBB University of Economics and Technology,
Department of Mathematics, TR-06530,
Ankara, Turkey,
oduman@etu.edu.tr
Classical Approximation Theory, Summability Theory, Statistical Convergence and its Applications

Saber N. Elaydi
Department Of Mathematics
Trinity University
715 Stadium Dr.
San Antonio, TX 78212-7200
210-736-8246
e-mail: selaydi@trinity.edu
Ordinary Differential Equations, Difference Equations

J .A. Goldstein
Department of Mathematical Sciences
The University of Memphis
Memphis, TN 38152
901-678-3130
jgoldste@memphis.edu
Partial Differential Equations, Semigroups of Operators

H. H. Gonska
Department of Mathematics
University of Duisburg
Duisburg, D-47048
Germany
011-49-203-379-3542
e-mail: heiner.gonska@uni-due.de
Approximation Theory, Computer Aided Geometric Design

John R. Graef
Department of Mathematics
University of Tennessee at Chattanooga
Chattanooga, TN 37304 USA
John-Graef@utc.edu
Ordinary and functional differential equations, difference equations, impulsive systems, differential inclusions, dynamic equations on time scales, control theory and their applications

Weimin Han
Department of Mathematics
University of Iowa
Iowa City, IA 52242-1419
319-335-0770
e-mail: whan@math.uiowa.edu
Numerical analysis, Finite element method, Numerical PDE, Variational inequalities, Computational mechanics

Tian-Xiao He
Department of Mathematics and Computer Science
P.O. Box 2900, Illinois Wesleyan University
Bloomington, IL 61702-2900, USA
Tel (309)556-3089
Fax (309)556-3864
the@iwu.edu
Approximations, Wavelet, Integration Theory, Numerical Analysis, Analytic Combinatorics

Margareta Heilmann
Faculty of Mathematics and Natural Sciences, University of Wuppertal
Gaußstraße 20
D-42119 Wuppertal, Germany,
heilmann@math.uni-wuppertal.de
Approximation Theory (Positive Linear Operators)

Xing-Biao Hu
Institute of Computational Mathematics
AMSS, Chinese Academy of Sciences
Beijing, 100190, CHINA
hxb@lsec.cc.ac.cn

Instructions to Contributors
Journal of Computational Analysis and Applications

An international publication of Eudoxus Press, LLC, of TN.

Editor in Chief: George Anastassiou

Department of Mathematical Sciences
University of Memphis
Memphis, TN 38152-3240, U.S.A.

1. Manuscripts files in Latex and PDF and in English, should be submitted via email to the Editor-in-Chief:

Prof.George A. Anastassiou
Department of Mathematical Sciences
The University of Memphis
Memphis,TN 38152, USA.
Tel. 901.678.3144
e-mail: ganastss@memphis.edu

Authors may want to recommend an associate editor the most related to the submission to possibly handle it.

Also authors may want to submit a list of six possible referees, to be used in case we cannot find related referees by ourselves.

2. Manuscripts should be typed using any of TEX,LaTEX,AMS-TEX,or AMS-LaTEX and according to EUDOXUS PRESS, LLC. LATEX STYLE FILE. (Click HERE to save a copy of the style file.)They should be carefully prepared in all respects. Submitted articles should be brightly typed (not dot-matrix), double spaced, in ten point type size and in 8(1/2)x11 inch area per page. Manuscripts should have generous margins on all sides and should not exceed 24 pages.

3. Submission is a representation that the manuscript has not been published previously in this or any other similar form and is not currently under consideration for publication elsewhere. A statement transferring from the authors(or their employers,if they hold the copyright) to Eudoxus Press, LLC, will be required before the manuscript can be accepted for publication.The Editor-in-Chief will supply the necessary forms for this transfer.Such a written transfer of copyright,which previously was assumed to be implicit in the act of submitting a manuscript,is necessary under the U.S.Copyright Law in order for the publisher to carry through the dissemination of research results and reviews as widely and effective as possible.

4. The paper starts with the title of the article, author's name(s) (no titles or degrees), author's affiliation(s) and e-mail addresses. The affiliation should comprise the department, institution (usually university or company), city, state (and/or nation) and mail code.

The following items, 5 and 6, should be on page no. 1 of the paper.

5. An abstract is to be provided, preferably no longer than 150 words.

6. A list of 5 key words is to be provided directly below the abstract. Key words should express the precise content of the manuscript, as they are used for indexing purposes.

The main body of the paper should begin on page no. 1, if possible.

7. All sections should be numbered with Arabic numerals (such as: 1. INTRODUCTION) .
Subsections should be identified with section and subsection numbers (such as 6.1. Second-Value Subheading).
If applicable, an independent single-number system (one for each category) should be used to label all theorems, lemmas, propositions, corollaries, definitions, remarks, examples, etc. The label (such as Lemma 7) should be typed with paragraph indentation, followed by a period and the lemma itself.

8. Mathematical notation must be typeset. Equations should be numbered consecutively with Arabic numerals in parentheses placed flush right, and should be thusly referred to in the text [such as Eqs.(2) and (5)]. The running title must be placed at the top of even numbered pages and the first author's name, et al., must be placed at the top of the odd numbed pages.

9. Illustrations (photographs, drawings, diagrams, and charts) are to be numbered in one consecutive series of Arabic numerals. The captions for illustrations should be typed double space. All illustrations, charts, tables, etc., must be embedded in the body of the manuscript in proper, final, print position. In particular, manuscript, source, and PDF file version must be at camera ready stage for publication or they cannot be considered.

Tables are to be numbered (with Roman numerals) and referred to by number in the text. Center the title above the table, and type explanatory footnotes (indicated by superscript lowercase letters) below the table.

10. List references alphabetically at the end of the paper and number them consecutively. Each must be cited in the text by the appropriate Arabic numeral in square brackets on the baseline.
References should include (in the following order):
initials of first and middle name, last name of author(s)
title of article,

name of publication, volume number, inclusive pages, and year of publication.

Authors should follow these examples:

Journal Article

1. H.H.Gonska,Degree of simultaneous approximation of bivariate functions by Gordon operators, (journal name in italics) *J. Approx. Theory*, 62,170-191(1990).

Book

2. G.G.Lorentz, (title of book in italics) *Bernstein Polynomials* (2nd ed.), Chelsea,New York,1986.

Contribution to a Book

3. M.K.Khan, Approximation properties of beta operators,in(title of book in italics) *Progress in Approximation Theory* (P.Nevai and A.Pinkus,eds.), Academic Press, New York,1991,pp.483-495.

11. All acknowledgements (including those for a grant and financial support) should occur in one paragraph that directly precedes the References section.

12. Footnotes should be avoided. When their use is absolutely necessary, footnotes should be numbered consecutively using Arabic numerals and should be typed at the bottom of the page to which they refer. Place a line above the footnote, so that it is set off from the text. Use the appropriate superscript numeral for citation in the text.

13. After each revision is made please again submit via email Latex and PDF files of the revised manuscript, including the final one.

14. Effective 1 Nov. 2009 for current journal page charges, contact the Editor in Chief. Upon acceptance of the paper an invoice will be sent to the contact author. The fee payment will be due one month from the invoice date. The article will proceed to publication only after the fee is paid. The charges are to be sent, by money order or certified check, in US dollars, payable to Eudoxus Press, LLC, to the address shown on the Eudoxus homepage.

No galleys will be sent and the contact author will receive one (1) electronic copy of the journal issue in which the article appears.

15. This journal will consider for publication only papers that contain proofs for their listed results.

Fixed point theorems for F-contractions on closed ball in partial metric spaces

Muhammad Nazam[1], Choonkil Park[2], Aftab Hussain[3], Muhammad Arshad[1] and Jung-Rye Lee[4*]

[1]Department of Mathematics and Statistics, International Islamic University, H-10, Islamabad, Pakistan
e-mail: nazim254.butt@gmail.com, marshadzia@iiu.edu.pk

[2]Research Institute for Natural Sciences, Hanyang University, Seoul 04763, Korea
e-mail: baak@hanyang.ac.kr

[3]Depatment of Mathematics, Khawaja Farid University, Rahim Yar Khan, Punjab, Pakistan
e-mail: aftabshh@gmail.com

[4]Depatment of Mathematics, Daejin University,Kyunggi 11159
e-mail: jrlee@daejin.ac.kr

Abstract. In this paper, we present new fixed point theorems for Kannan type F_p-contraction and Kannan type (α, η, GF_p)-contraction on a closed ball contained in a complete partial metric space. Some comparative examples are constructed to illustrate the significance of these results. Our results provide substantial generalizations and improvements of several well known results existing in the comparable literature.

1. Introduction and preliminaries

The recent study in Fixed Point Theory is due to a Polish mathematician Stefan Banach who, in 1922, presented a revolutionary contraction principle known as Banach's Contraction Principle. He proved that every contraction T in a complete metric space X has a unique fixed point ($T(x) = x$; $x \in X$). After the appearance of this remarkable result many generalizations of this result have appeared in literature (see for example [1–3, 6–11, 13, 14, 16, 19, 20, 22, 24, 25, 29]). One of these generalizations is known as F-contraction presented by Wardowski [30]. Wardowski [30] evinced that every F-contraction defined on a complete metric space has a unique fixed point. The concept of F-contraction proved another milestone in fixed point theory and numerous research papers on F-contraction have been published (see [21, 23, 28, 31]). Hussain *et al.* [12] introduced an α-GF-contraction with respect to a general family of functions G and established Wardowski type fixed point results in ordered metric spaces. Batra *et al.* [4, 5] extended the concept of F-contraction on graphs and altered distances and proved some fixed point and coincidence point results.

Motivated by Kannan [15], Wardowski [30], Matthews [18] and Kryeyszig [17], in this paper, we introduce Kannan type F-contraction and Kannan type (α, η, GF)-contraction on a closed ball contained in a complete partial metric space and present related fixed point theorems. We construct examples to illustrate these results. F-contraction on partial metric spaces is more general than F-contraction defined on metric spaces.

The notion of a partial metric space (PMS) was introduced in 1992 by Matthews [18] to model computation over a metric space. The PMS is a generalization of the usual metric space in which the self-distance is no longer necessarily zero.

Definition 1. [18] Let X be a nonempty set and $p : X \times X \to \mathbb{R}_0^+$ satisfy the following properties: for all $x, y, z \in X$,

(p_1) $x = y \Leftrightarrow p(x, x) = p(x, y) = p(y, y)$,
(p_2) $p(x, x) \leq p(x, y)$,
(p_3) $p(x, y) = p(y, x)$,
(p_4) $p(x, z) + p(y, y) \leq p(x, y) + p(y, z)$.

[0]2010 Mathematics Subject Classification: 47H09; 47H10; 54H25
[0]Keywords: partial metric space; fixed point; F-contraction; closed ball.
*Corresponding author.

Then (X,p) is called PMS. We present some new nontrivial examples of PMS.

Example 1. Let the set of rational numbers be $\mathbb{Q}=\{x_1,x_2,\cdots\}$. We define $p:\mathbb{R}\times\mathbb{R}\to\mathbb{R}^+$ by

$$p(x,y)=\begin{cases} 1 & \text{if } x=y\in\mathbb{R}-\mathbb{Q};\\ \frac{3}{2} & \text{if } x\neq y\in\mathbb{R}-\mathbb{Q};\\ \frac{1}{3} & \text{if } x=y\in\mathbb{Q};\\ 1+\frac{1}{m}+\frac{1}{n} & \text{if } x=x_m, y=x_n \text{ and } m\neq n;\\ 1+\frac{1}{n} & \text{if } \{x,y\}\cap\mathbb{Q}=\{x_n\} \text{ and } \{x,y\}-\mathbb{Q}\neq\phi. \end{cases}$$

Clearly p satisfies $(p_1)-(p_3)$. To prove p_4, let $x,y,z\in\mathbb{R}-\mathbb{Q}$ and $m\neq n$. Then

$$\begin{aligned} p(x,y)+p(z,z) &\leq p(x,z)+p(y,z);\\ p(x_n,y)+p(z,z) &= 2+\frac{1}{n}\leq p(x_n,z)+p(y,z);\\ p(x_n,x_n)+p(z,z) &= \frac{4}{3}<p(x_n,z)+p(x_n,z);\\ p(x_m,x_n)+p(z,z) &= 2+\frac{1}{m}+\frac{1}{n}=p(x_m,z)+p(x_n,z);\\ p(x,y)+p(x_k,x_k) &< 2<p(x,x_k)+p(y,x_k);\\ p(x_n,y)+p(x_k,x_k) &= \frac{4}{3}+\frac{1}{n}<2+\frac{1}{n}+\frac{2}{k}=p(x_n,x_k)+p(y,x_k);\\ p(x_n,x_n)+p(x_k,x_k) &= \frac{2}{3}\leq p(x_n,x_k)+p(x_n,x_k);\\ p(x_m,x_n)+p(x_k,x_k) &= \frac{4}{3}+\frac{1}{m}+\frac{1}{n}\leq p(x_m,x_k)+p(x_n,x_k). \end{aligned}$$

Example 2. Let X be uncountable, $a\in X$ and $\mathcal{T}=\{A\subset X:\ a\in A\}$ be a topology on X. It is easy to show that $(X,\mathcal{T})$ is a PMS with the partial metric p defined by

$$\begin{cases} p(a,a)=0, & \\ p(a,x)=p(x,x)=1 & \text{if } x\neq a,\\ p(x,y)=2 & \text{if } x=y \text{ and } x,y\in X-\{a\}. \end{cases}$$

Example 3. Let $X=\{x_i:i\in\mathbb{N}\}$ be a countably infinite set. Define $p:X\times X\to[0,\infty)$ by

$$p(x,y)=\begin{cases} 0 & \text{if } x=y=x_0,\\ \sum_{k=1}^{n}\frac{1}{2^k} & \text{if } (x,y)\in\{(x_m,x_n);0\leq m\leq n \text{ and } n\geq 1. \end{cases}$$

Then (X,p) is a PMS.

Example 4. Let $A=\{a_i:i\in\mathbb{N}\}$ and $B=\{b_i:i\in\mathbb{N}\}$ be two disjoint infinitely countable sets, and let $X=A\cup B$. Define $p:X\times X\to[0,\infty)$ by

$$p(x,y)=\begin{cases} 1 & \text{if } x=y\in A,\\ 0 & \text{if } x=y\in B,\\ 1+\frac{1}{i}+\frac{1}{j} & \text{if } x=y \text{ and } \{x,y\}\in\{\{a_i,a_j\},\{a_i,b_j\},\{b_i,b_j\}\}. \end{cases}$$

Then (X,p) is a PMS.

In [18], Matthews proved that every partial metric p on M induces a metric $d_p:M\times M\to\mathbb{R}_0^+$ defined by

$$d_p(r_1,r_2)=2p(r_1,r_2)-p(r_1,r_1)-p(r_2,r_2)$$

for all $r_1, r_2 \in M$. Matthews described that if $p(r_1, r_2) = \max\{r_1, r_2\}$, then $d_p(r_1, r_2) = |r_1 - r_2|$ a usual metric on M. Notice that every metric d on a set M is a partial metric p such that $p(r, r) = 0$ for all $r \in M$ and $p(r_1, r_2) = 0$ implies $r_1 = r_2$ (using (p_1) and (p_2)) but not conversely. The notions such as convergence, completeness, Cauchy sequence in the setting of partial metric spaces, can be found in [18] and references there in.

Definition 2. [18] Let (M, p) be a partial metric space.

(1) A sequence $\{r_n\}_{n\in\mathbb{N}}$ in (M, p) is called a Cauchy sequence if $\lim_{n,m\to\infty} p(r_n, r_m)$ exists and is finite.
(2) A partial metric space (M, p) is said to be complete if every Cauchy sequence $\{r_n\}_{n\in\mathbb{N}}$ in M converges, with respect to $\tau(p)$, to a point $r \in X$ such that $p(r, r) = \lim_{n,m\to\infty} p(r_n, r_m)$.

The following lemma will be helpful in the sequel.

Lemma 1. [18]

(1) A sequence r_n is a Cauchy sequence in a partial metric space (M, p) if and only if it is a Cauchy sequence in metric space (M, d_p)
(2) A partial metric space (M, p) is complete if and only if the metric space (M, d_p) is complete.
(3) A sequence $\{r_n\}_{n\in\mathbb{N}}$ in M converges to a point $r \in M$, with respect to $\tau(d_p)$ if and only if $\lim_{n\to\infty} p(r, r_n) = p(r, r) = \lim_{n,m\to\infty} p(r_n, r_m)$.
(4) If $\lim_{n\to\infty} r_n = \upsilon$ such that $p(\upsilon, \upsilon) = 0$ then $\lim_{n\to\infty} p(r_n, r) = p(\upsilon, r)$ for all $r \in M$.

Remark 1. *Since* $\left(\overline{B_p(x_0, r)}, p\right) \subseteq (X, p)$, *Lemma 1 holds for* $\left(\overline{B_p(x_0, r)}, p\right)$.

Let F_d denote F-contraction on metric spaces and F_p denote F-contraction on partial metric spaces. Wardowski [30] investigated a nonlinear function $F : \mathbb{R}^+ \to \mathbb{R}$ complying with the following axioms:

(F_1) F is strictly increasing;
(F_2) For each sequence $\{r_n\}$ of positive numbers $\lim_{n\to\infty} r_n = 0$ if and only if $\lim_{n\to\infty} F(r_n) = -\infty$;
(F_3) There exists $\theta \in (0, 1)$ such that $\lim_{\xi\to 0^+} (\xi)^\theta F(\xi) = 0$.

We denote by Δ_F the set of all functions satisfying the conditions $(F_1) - (F_3)$.

Example 5. Let $F : \mathbb{R}^+ \to \mathbb{R}$ be defined by

(a) $F(r) = \ln(r)$,
(b) $F(r) = r + \ln(r)$,
(c) $F(r) = \ln(r^2 + r)$,
(d) $F(r) = -\dfrac{1}{\sqrt{r}}$.

It is easy to check that (a),(b),(c) and (d) are members of Δ_F.

Wardowski utilized function F in an excellent manner and gave the following remarkable result.

Theorem 1. [30] Let (M, d) be a complete metric space and $T : M \to M$ be a mapping satisfying

$$(d(T(r_1), T(r_2)) > 0 \Rightarrow \tau + F(d(T(r_1), T(r_2)) \leq F(d(r_1, r_2)))$$

for all $r_1, r_2 \in M$ and some $\tau > 0$. Then T has a unique fixed point $\upsilon \in M$ and for every $r_0 \in M$ the sequence $\{T^n(r_0)\}$ for all $n \in \mathbb{N}$ is convergent to υ.

Remark 2. [30, Remark 2.1] In metric spaces a mapping giving fulfillment to F-contraction, is always a Banach contraction and hence a continuous map.

Example 6 explains that F_p-contraction is more general than F_d-contraction.

Example 6. Let $M = [0,1]$ and define partial metric by $p(r_1, r_2) = \max\{r_1, r_2\}$ for all $r_1, r_2 \in M$. The metric d induced by partial metric p is given by $d(r_1, r_2) = |r_1 - r_2|$ for all $r_1, r_2 \in M$. Define $F : \mathbb{R}^+ \to \mathbb{R}$ by $F(r) = \ln(r)$ and T by

$$T(r) = \begin{cases} \dfrac{r}{5} & \text{if } r \in [0,1), \\ 0 & \text{if } r = 1. \end{cases}$$

Note that for all $r_1, r_2 \in M$ with $r_1 \leq r_2$ or $r_2 \leq r_1$

$$\begin{aligned} \tau + F\left(p(T(r_1), T(r_2))\right) &\leq F\left(p(r_1, r_2)\right) \text{ implies} \\ \tau + F\left(\frac{r_1}{5}\right) &\leq F(r_1) \text{ or } \tau + F\left(\frac{r_2}{5}\right) \leq F(r_2). \end{aligned}$$

But T is neither continuous and nor satisfies F-contraction in a metric space (M, d). Indeed, for $r_1 = 1$ and $r_2 = \frac{5}{6}$, $d(T(r_1), T(r_2)) > 0$ and we have

$$\begin{aligned} \tau + F\left(d(T(r_1), T(r_2))\right) &\leq F\left(d(r_1, r_2)\right), \\ \tau + F\left(d(T(1), T(\frac{5}{6}))\right) &\leq F\left(d(1, \frac{5}{6})\right), \\ \tau + F\left(d(0, \frac{1}{6})\right) &\leq F\left(\frac{1}{6}\right), \\ \frac{1}{6} &< \frac{1}{6}, \end{aligned}$$

which is a contradiction for all possible values of τ.

The following result plays a vital role regarding the existence of the fixed point of the mapping satisfying a contractive condition on the closed ball.

Theorem 2. [17, Theorem 5.1.4] Let (X, d) be a complete metric space, $T : X \to X$ be a mapping, $r > 0$ and x_0 be an arbitrary point in X. Suppose there exists $k \in [0, 1)$ with

$$d(T(x), T(y)) \leq kd(x, y), \text{ for all } x, y \in Y = \overline{B(x_0, r)}$$

and $d(x_0, T(x_0)) < (1 - k)r$. Then there exists a unique point x^* in $\overline{B(x_0, r)}$ such that $x^* = T(x^*)$.

Definition 3. [15] Let (X, p) be a partial metric space. A mapping $T : X \to X$ is said to be a Kannan contraction if it satisfies the following condition:

$$p\left(T(x), T(y)\right) \leq \frac{k}{2}\left[p\left(x, T(x)\right) + p\left(y, T(y)\right)\right]$$

for all $x, y \in X$ and some $k \in [0, 1[$.

2. Kannan type F_p-contraction on closed ball

Definition 4. Let (X, p) be a partial metric space, $r > 0$ and x_0 be an arbitrary point in X. The mapping $T : X \to X$ is called Kannan type F_p-contraction on closed ball if for all $x, y \in \overline{B_p(x_0, r)} \subseteq X$ we have

$$\tau + F(p(T(x), T(y))) \leq F\left(\frac{k}{2}\left[p(x, T(x)) + p(y, T(y))\right]\right), \tag{2.1}$$

where $0 \leq k < 1$, $F \in \Delta_F$ and $\tau > 0$.

Remark 3. *(1) F_p-contraction and Kannan type F_p-contraction are independent.*
(2) Let F be a Kannan type F_p-contraction. From (2.1), for all $x, y \in \overline{B_p(x_0, r)}$ with $T(x) \neq T(y)$, we have

$$F(p(T(x), T(y))) \leq \tau + F(p(T(x), T(y))) \leq F\left(\frac{k}{2}\left[p(x, T(x)) + p(y, T(y))\right]\right).$$

Due to (F_1)*, we obtain*

$$p(T(x),T(y)) < \frac{k}{2}\left[p(x,T(x))+p(y,T(y))\right] \text{ for all } x,y\in X, T(x)\neq T(y).$$

Theorem 3. *Let* (X,p) *be a complete partial metric space,* $r>0$ *and* x_0 *be an arbitrary point in* X*. Assume that* $T:X\to X$ *is a Kannan type* F_p*-contraction on closed ball* $\overline{B_p(x_0,r)}\subseteq X$ *with*

$$p(x_0,T(x_0))\leq (1-\lambda)[r+p(x_0,x_0)], \lambda=\frac{k}{2-k}. \tag{2.2}$$

If T *or* F *is continuous, then there exists a point* x^* *in* $\overline{B_p(x_0,r)}$ *such that* $T(x^*)=x^*$ *with* $p(x^*,x^*)=0$*.*

Proof. Let x_0 be an initial point in X such that $x_1=T(x_0)$, $x_2=T(x_1)=T^2(x_0)$. Continuing in this way we can construct an iterative sequence $\{x_n\}$ such that $x_{n+1}=T(x_n)=T^n(x_0)$, for all $n\geq 0$. We show that $x_n\in\overline{B_p(x_0,r)}$ for all $n\in\mathbb{N}$. From (2.2), we have

$$p(x_0,x_1)=p(x_0,T(x_0))\leq(1-\lambda)[r+p(x_0,x_0)]<r+p(x_0,x_0),$$

which shows that $x_1\in\overline{B_p(x_0,r)}$. From (2.1) and (F_1), we get

$$F\left(p(x_1,x_2)\right)=F(p(T(x_0),T(x_1)))\leq F\left(\frac{k}{2}\left[p(x_0,x_1)+p(x_1,x_2)\right]\right)-\tau,$$

which implies

$$p(x_1,x_2)<\frac{k}{2}\left[p(x_0,x_1)+p(x_1,x_2)\right]<\lambda p(x_0,x_1)\leq\lambda[r+p(x_0,x_0)]$$

$$p(x_0,x_2)\leq p(x_0,x_1)+p(x_1,x_2)-p(x_1,x_1)<(1-\lambda)[r+p(x_0,x_0)]+\lambda[r+p(x_0,x_0)]=r+p(x_0,x_0).$$

This shows that $x_2\in\overline{B_p(x_0,r)}$. Inductively, we obtain that $x_n\in\overline{B_p(x_0,r)}$, for all $n\in\mathbb{N}$ and hence from the contractive condition (2.1), we have

$$\begin{aligned} F\left(p(x_n,x_{n+1})\right) &\leq F\left(\frac{k}{2}\left[p(x_{n-1},x_n)+p(x_n,x_{n+1})\right]\right)-\tau \\ &\leq F\left(\frac{k}{2}\left[p(x_{n-1},x_n)+\frac{k}{2-k}p(x_{n-1},x_n)\right]\right)-\tau \\ &\leq F\left(\frac{k}{2-k}p(x_{n-1},x_n)\right)-\tau \end{aligned} \tag{2.3}$$

and also

$$F\left(p(x_{n-1},x_n)\right)\leq F\left(\lambda p(x_{n-2},x_{n-1})\right)-\tau.$$

From (2.3), we obtain

$$F\left(p(x_n,x_{n+1})\right)\leq F\left(\lambda p(x_{n-2},x_{n-1})\right)-2\tau.$$

Repeating these steps, we get

$$F\left(p(x_n,x_{n+1})\right)\leq F\left(p(x_0,x_1)\right)-n\tau. \tag{2.4}$$

From (2.4), we obtain $\lim_{n\to\infty}F\left(p(x_n,x_{n+1})\right)=-\infty$. Since $F\in\Delta_F$,

$$\lim_{n\to\infty}p(x_n,x_{n+1})=0. \tag{2.5}$$

From the property (F_3) of F-contraction, there exists $\kappa\in(0,1)$ such that

$$\lim_{n\to\infty}\left((p(x_n,x_{n+1}))^{\kappa}F\left(p(x_n,x_{n+1})\right)\right)=0. \tag{2.6}$$

Following (2.4), for all $n\in\mathbb{N}$, we obtain

$$(p(x_n,x_{n+1}))^{\kappa}\left(F\left(p(x_n,x_{n+1})\right)-F\left(p(x_0,x_1)\right)\right)\leq-\left(p(x_n,x_{n+1})\right)^{\kappa}n\tau\leq 0. \tag{2.7}$$

Considering (2.5), (2.6) and letting $n\to\infty$ in (2.7), we have

$$\lim_{n\to\infty}\left(n\left(p(x_n,x_{n+1})\right)^{\kappa}\right)=0. \tag{2.8}$$

Since (2.8) holds, there exists $n_1 \in \mathbb{N}$ such that $n\left(p(x_n, x_{n+1})\right)^{\kappa} \leq 1$ for all $n \geq n_1$ or

$$p(x_n, x_{n+1}) \leq \frac{1}{n^{\frac{1}{\kappa}}} \text{ for all } n \geq n_1. \tag{2.9}$$

Using (2.9), we get for $m > n \geq n_1$,

$$\begin{aligned} p(x_n, x_m) &\leq p(x_n, x_{n+1}) + p(x_{n+1}, x_{n+2}) + p(x_{n+2}, x_{n+3}) + \cdots + p(x_{m-1}, x_m) - \sum_{j=n+1}^{m-1} p(x_j, x_j) \\ &\leq p(x_n, x_{n+1}) + p(x_{n+1}, x_{n+2}) + p(x_{n+2}, x_{n+3}) + \cdots + p(x_{m-1}, x_m) \\ &= \sum_{i=n}^{m-1} p(x_i, x_{i+1}) \leq \sum_{i=n}^{\infty} p(x_i, x_{i+1}) \leq \sum_{i=n}^{\infty} \frac{1}{i^{\frac{1}{k}}}. \end{aligned}$$

The convergence of the series $\sum_{i=n}^{\infty} \frac{1}{i^{\frac{1}{\kappa}}}$ entails that $\lim_{n,m\to\infty} p(x_n, x_m) = 0$. Hence $\{x_n\}$ is a Cauchy sequence in $\left(\overline{B_p(x_0, r)}, p\right)$. By Lemma 1, $\{x_n\}$ is a Cauchy sequence in $\left(\overline{B(x_0, r)}, d_p\right)$. Moreover, since $\left(\overline{B_p(x_0, r)}, p\right)$ is a complete partial metric space, by Lemma 1, $\left(\overline{B(x_0, r)}, d_p\right)$ is also a complete metric space. Thus there exists $x^* \in (\overline{B(x_0, r)}, d_p)$ such that $x_n \to x^*$ as $n \to \infty$ and using Lemma 1, we have

$$\lim_{n\to\infty} p(x^*, x_n) = p(x^*, x^*) = \lim_{n,m\to\infty} p(x_n, x_m). \tag{2.10}$$

Due to $\lim_{n,m\to\infty} p(x_n, x_m) = 0$, we infer from (2.10) that $p(x^*, x^*) = 0$ and $\{x_n\}$ converges to x^* with respect to $\mathcal{T}_p$. In order to show that x^* is a fixed point of T, we have two cases.

Case (1). T is *continuous.* We have

$$x^* = \lim_{n\to\infty} x_n = \lim_{n\to\infty} T^n(x_0) = \lim_{n\to\infty} T^{n+1}(x_0) = T(\lim_{n\to\infty} T^n(x_0)) = T(x^*).$$

Hence $x^* = T(x^*)$, that is, x^* is a fixed point of T.

Case (2). F is *continuous.* We complete this case in two steps. First, if for each $n \in \mathbb{N}$ there exists $b_n \in \mathbb{N}$ such that $x_{b_n+1} = T(x^*)$ and $b_n > b_{n-1}$ with $b_0 = 1$. Then we have

$$x^* = \lim_{n\to\infty} x_{b_n+1} = \lim_{n\to\infty} T(x^*) = T(x^*).$$

This shows that x^* is a fixed point of T. Second, there exists $n_0 \in \mathbb{N}$ such that $x_{n+1} \neq T(x^*)$ for all $n \geq n_0$. Using contractive condition (2.1), we obtain

$$F\left(p(T(x_n), T(x^*))\right) \leq F\left(\frac{k}{2}\left[p\left(x_n, x_{n+1}\right) + p\left(x^*, T(x^*)\right)\right]\right) - \tau.$$

On taking limit as $n \to \infty$ and using the continuity of F and the fact that $\lim_{n\to\infty} p(x_n, x_{n+1}) = 0$, we have

$$F(p(x^*, T(x^*))) < F(\frac{k}{2} p\left(x^*, T(x^*)\right)).$$

Since F is strictly increasing, the above inequality leads us to conclude that $p(x^*, T(x^*)) = 0$. Thus, by using the properties (p_1) and (p_2), we obtain $x^* = T(x^*)$, which completes the proof.

To prove the uniqueness of x^*, assume on contrary, that $y^* \in \overline{B_p(x_0, r)}$ is another fixed point of T, that is, $y^* = T(y^*)$. From (2.1), we have

$$\tau + F\left(p(T(x^*), T(y^*))\right) \leq F\left(\frac{k}{2}\left[p(x^*, T(x^*)) + p(y^*, T(y^*))\right]\right) \leq F\left(\frac{k}{2} \times 2p(x^*, y^*)\right). \tag{2.11}$$

The inequality (2.11) leads to a contradiction. Hence $p(x^*, y^*) = 0$. Thus, due to (p_1) and (p_2), we obtain $x^* = y^*$. $\square$

The following example explains the significance of Theorem 3.

Example 7. Let $X = \mathbb{R}^+$. Define $p : X^2 \to [0, \infty)$ by $p\left(x, y\right) = \max\left\{x, y\right\}$ for all $(x, y) \in X^2$. Then (X, p) is a complete partial metric space. Define the mapping $T : X \to X$ by

$$T\left(x\right) = \begin{cases} \frac{x}{14} & \text{if } x \in [0, 1], \\ x - \frac{1}{2} & \text{if } x \in (1, \infty). \end{cases}$$

Set $k=\frac{2}{5}$, $x_0=\frac{1}{2}$, $r=\frac{1}{2}$ and $p(x_0,x_0)=\frac{1}{2}$. Then $\overline{B_p(x_0,r)}=[0,1]$ and

$$p(x_0,T(x_0))=\max\left\{\frac{1}{2},\frac{1}{28}\right\}=\frac{1}{2}<(1-\lambda)[r+p(x_0,x_0)].$$

For all $x,y\in\overline{B_p(x_0,r)}$, we note that

$$\begin{aligned}p(T(x),T(y)) &= \max\left\{\frac{x}{14},\frac{y}{14}\right\}=\frac{1}{14}\max\{x,y\}\\ &< \frac{1}{5}[x+y]=\frac{1}{5}\left[\max\left\{x,\frac{x}{14}\right\}+\max\left\{y,\frac{y}{14}\right\}\right]\\ &= \frac{k}{2}\left[p(x,T(x))+p(y,T(y))\right]\end{aligned}$$

Thus

$$\tau+\ln\left(p(T(x),T(y))\right)\leq\ln\left(\frac{k}{2}\left[p(x,T(x))+p(y,T(y))\right]\right).$$

If $F(\alpha)=\ln(\alpha)$, $\alpha>0$ and $\tau>0$, then

$$\tau+F\left(p(T(x),T(y))\right)\leq F\left(\frac{k}{2}\left[p(x,T(x))+p(y,T(y))\right]\right).$$

However, for $x=100,y=10\in(1,\infty)$,

$$\begin{aligned}p(T(x),T(y)) &= \max\left\{x-\frac{1}{2},y-\frac{1}{2}\right\}\\ &\geq \frac{1}{5}[x+y]=\frac{k}{2}\left[p(x,T(x))+p(y,T(y))\right].\end{aligned}$$

Consequently, the contractive condition (2.1) does not hold on X. Hence, all the hypotheses of Theorem 3 are satisfied on closed ball and so $x=0$ is a fixed point of T.

3. Kannan type (α,η,GF_p)-contraction on closed ball

Definition 5. [27]. Let $T:X\to X$ and $\alpha:X\times X\to[0,+\infty)$ be two functions. We say that T is an α-admissible if for all $x,y\in X$, $\alpha(x,y)\geq 1$ implies $\alpha(T(x),T(y))\geq 1$.

Example 8. Let $X=\mathbb{R}$. Define $\alpha:X\times X\to[0,\infty)$ and $f:X\to X$ by

$$\alpha(x,y)=\begin{cases}e^{x+y} & \text{if } x,y\in[0,1],\\ 0 & \text{otherwise .}\end{cases}\quad f(x)=\begin{cases}\frac{x^2}{7} & \text{if } x\in[0,1],\\ \ln(x) & \text{if } x\in(1,\infty).\end{cases}$$

Apparently, $\alpha(x,y)\geq 1$ implies $\alpha(fx,fy)\geq 1$.

Definition 6. [26]. Let $T:X\to X$ and $\alpha,\eta:X\times X\to[0,+\infty)$ be two functions. We say that T is an α-admissible mapping with respect to η if for all $x,y\in X$, $\alpha(x,y)\geq\eta(x,y)$ implies $\alpha(T(x),T(y))\geq\eta(T(x),T(y))$.

Example 9. Let $X=\mathbb{R}$. Define $\alpha,\eta:X\times X\to[0,\infty)$ and $f:X\to X$ by

$$\alpha(x,y)=\begin{cases}\pi^{x+y} & \text{if } x,y\in[0,1],\\ 0 & \text{otherwise ,}\end{cases}\quad \eta(x,y)=\begin{cases}e^{x+y} & \text{if } x,y\in[0,1],\\ 0 & \text{otherwise .}\end{cases}$$

$$f(x)=\begin{cases}\frac{x^2}{7} & \text{if } x\in[0,1],\\ \ln(x) & \text{if } x\in(1,\infty).\end{cases}$$

Apparently, $\alpha(x,y)\geq\eta(x,y)$ implies $\alpha(fx,fy)\geq\eta(fx,fy)$.

If $\eta(x,y)=1$, then the above definition reduces to Definition 5.

We begin by introducing the following family of new functions.

Let Δ_G denote the set of all functions $G:(\mathbb{R}^+)^4\to\mathbb{R}^+$ which satisfy the property (G): for all $p_1,p_2,p_3,p_4\in\mathbb{R}^+$, if

$$\frac{p_1+p_2+p_3+p_4}{4}\leq\frac{p_i+p_{i+1}}{2},\ i=1,2,3,4,$$

then there exists $\tau>0$ such that $G(p_1,p_2,p_3,p_4)=\tau$.

Definition 7. Let (X,p) be a partial metric space and T be a self mapping on X. Suppose that $\alpha,\eta : X\times X \to [0,+\infty)$ are two functions. The mapping T is said to be an (α,η,GF_p)-contraction if for all $x,y\in X$, with $\eta(x,y)\le\alpha(x,y)$ and $d(T(x),T(y))>0$, we have

$$G(p(x,T(x)),p(y,T(y)),p(x,T(y)),p(y,T(x)))+F\left(p(T(x),T(y))\right)\le F\left(p(x,y)\right),$$

where $G\in\Delta_G$ and $F\in\Delta_F$.

Definition 8. Let (X,p) be a partial metric space and $T:X\to X$ and $\alpha,\eta : X\times X\to[0,+\infty)$ be two functions. T is said to be an (α,η)-continuous mapping on (X,p) if for a given $x\in X$, and the sequence $\{x_n\}_{n\in\mathbb{N}}$ converging to x

$$\alpha(x_n,x_{n+1})\ge\eta(x_n,x_{n+1}) \text{ implies } T(x_n)\to T(x).$$

Example 10. Let $X=[0,\infty)$ and $p:X\times X\to[0,\infty)$ be defined by $p(r_1,r_2)=\max\{r_1,r_2\}$ for all $r_1,r_2\in X$. Define

$$T(r)=\begin{cases}\sin(\pi r) & \text{if } r\in[0,1],\\ \cos(\pi r)+2 & \text{if } r\in(1,\infty),\end{cases}\quad \alpha(r_1,r_2)=\begin{cases} r_1^3+r_2^3+1 & \text{if } r_1,r_2\in[0,1],\\ 0 & \text{otherwise},\end{cases}$$

$$\eta(r_1,r_2)=\begin{cases} r_1^3+r_2^3 & \text{if } r_1,r_2\in[0,1],\\ 0 & \text{otherwise}.\end{cases}$$

Then apparently, T is not continuous on X, however T is an (α,η)-continuous.

Definition 9. Let (X,p) be a partial metric space and $\alpha,\eta : X\times X\to[0,+\infty)$ are two functions, $r>0$ and x_0 be an arbitrary point in X. The mapping $T:X\to X$ is said to be a Kannan type (α,η,GF_p)-contraction on closed ball if for all $x,y\in\overline{B_p(x_0,r)}\subseteq X$ with $\eta(x,y)\le\alpha(x,y)$ and $p(T(x),T(y))>0$, we have

$$\tau(G)+F\left(p(T(x),T(y))\right)\le F\left(\frac{k}{2}\left[p(x,T(x))+p(y,T(y))\right]\right), \tag{3.1}$$

where $\tau(G)=G(p(x,T(x)),p(y,T(y)),p(x,T(y)),p(y,T(x)))$, $0\le k<1, G\in\Delta_G$ and $F\in\Delta_F$.

Theorem 4. *Let (X,p) be a complete metric space and $T:X\to X$ be a Kannan type (α,η,GF_p)-contraction mapping on a closed ball $\overline{B_p(x_0,r)}$ satisfying the following assertions*

(1) *T is an α-admissible mapping with respect to η,*
(2) *there exists $x_0\in X$ such that $\alpha(x_0,T(x_0))\ge\eta(x_0,T(x_0))$,*
(3) *there exist $r>0$ and $x_0\in X$ such that $p(x_0,T(x_0))\le(1-\lambda)[r+p(x_0,x_0)]$, where $\lambda=\frac{k}{2-k}$.*

Then there exists a point x^* in $\overline{B_p(x_0,r)}$ such that $T(x^*)=x^*$ with $p(x^*,x^*)=0$.

Proof. Suppose that x_0 is an initial point of X, we can construct a sequence $\{x_n\}_{n=1}^{\infty}$ such that $x_{n+1}=T(x_n)=T^{n+1}(x_0)$ for all $n\in\mathbb{N}$. By assumption (2) there exists $x_0\in X$ such that $\alpha(x_0,T(x_0))\ge\eta(x_0,T(x_0))$. Since T is an α-admissible mapping with respect to η,

$$\alpha(x_0,T(x_0))\ge\eta(x_0,T(x_0)) \text{ implies } \alpha(x_1,x_2)\ge\eta(x_1,x_2), \text{ which implies } \alpha(x_2,x_3)\ge\eta(x_2,x_3).$$

In general, we have

$$\eta(x_{n-1},x_n)\le\alpha(x_{n-1},x_n), \text{ for all } n\in\mathbb{N}.$$

If there exists $n_0\in\mathbb{N}$ such that $p(x_{n_0},T(x_{n_0}))=0$, then x_{n_0} is a fixed point of T. We assume that $p(x_n,T(x_n))>0$ for all $n\in\mathbb{N}$. We show that $x_n\in\overline{B_p(x_0,r)}$ for all $n\in N$. Assumption (3) implies

$$p(x_0,x_1)=p(x_0,T(x_0))\le(1-\lambda)[r+p(x_0,x_0)]<[r+p(x_0,x_0)]$$

and thus $x_1\in\overline{B_p(x_0,r)}$. Note that $\tau(G)=\tau$. Indeed, $\tau(G)=G(p(x_0,x_1),p(x_1,x_2),p(x_0,x_2),p(x_1,x_1))$ satisfies

$$\frac{p(x_0,x_1)+p(x_1,x_2)+p(x_0,x_2)+p(x_1,x_1)}{4}\le\frac{p(x_0,x_1)+p(x_1,x_2)}{2}.$$

By the property (G), there exists $\tau>0$ such that

$$G(p(x_0,x_1),p(x_1,x_2),p(x_0,x_2),p(x_1,x_1))=\tau.$$

Due to (3.1) and (F_1), we have

$$\begin{aligned} F\left(p(x_1,x_2)\right) &= F(p(T(x_0),T(x_1))) \leq F\left(\frac{k}{2}\left[p(x_0,x_1)+p(x_1,x_2)\right]\right) - \tau(G) \\ &\leq F\left(\frac{k}{2}\left[p(x_0,x_1)+p(x_1,x_2)\right]\right) - \tau. \\ \text{This implies } p(x_1,x_2) &< \frac{k}{2}\left[p(x_0,x_1)+p(x_1,x_2)\right] < \lambda p(x_0,x_1) \leq \lambda[r+p(x_0,x_0)], \end{aligned}$$

$$\begin{aligned} p(x_0,x_2) &\leq p(x_0,x_1)+p(x_1,x_2)-p(x_1,x_1) \\ &< (1-\lambda)[r+p(x_0,x_0)]+\lambda[r+p(x_0,x_0)] = r+p(x_0,x_0). \end{aligned}$$

This shows that $x_2 \in \overline{B_p(x_0,r)}$. Inductively, we obtain that $x_n \in \overline{B_p(x_0,r)}$ for all $n \in \mathbb{N}$ and hence from the contractive condition (3.1), we have

$$F\left(p(x_n,x_{n+1})\right) \leq F\left(\frac{k}{2}\left[p(x_{n-1},x_n)+p(x_n,x_{n+1})\right]\right) - \tau(G). \tag{3.2}$$

Note that $\tau(G) = G(p(x_{n-1},x_n), p(x_n,x_{n+1}), p(x_{n-1},x_{n+1}), p(x_n,x_n))$ satisfies

$$\frac{p(x_{n-1},x_n)+p(x_n,x_{n+1})+p(x_{n-1},x_{n+1})+p(x_n,x_n)}{4} \leq \frac{p(x_{n-1},x_n)+p(x_n,x_{n+1})}{2},$$

and so by the property (G), there exists $\tau > 0$ such that

$$G(p(x_{n-1},x_n), p(x_n,x_{n+1}), p(x_{n-1},x_{n+1}), p(x_n,x_n)) = \tau.$$

Thus, from (3.2), we get

$$\begin{aligned} F\left(p(x_n,x_{n+1})\right) &\leq F\left(\frac{k}{2}\left[p(x_{n-1},x_n)+\frac{k}{2-k}p(x_{n-1},x_n)\right]\right) - \tau \\ &\leq F\left(\frac{k}{2-k}p(x_{n-1},x_n)\right) - \tau = F(\lambda p(x_{n-1},x_n)) - \tau \end{aligned} \tag{3.3}$$

but

$$F\left(p(x_{n-1},x_n)\right) \leq F\left(\lambda p(x_{n-2},x_{n-1})\right) - \tau.$$

From (3.3), we obtain

$$F\left(p(x_n,x_{n+1})\right) \leq F\left(\lambda p(x_{n-2},x_{n-1})\right) - 2\tau.$$

Continuing in the same way we obtain

$$F\left(p(x_n,x_{n+1})\right) \leq F\left(p(x_0,x_1)\right) - n\tau.$$

By the same reasoning as in the proof of Theorem 3, there exists $x^* \in \overline{B_p(x_0,r)}$ such that $p(x^*,x^*) = 0$ and $\{x_n\}$ converges to x^* with respect to $\mathcal{T}_p$. We show that x^* is a fixed point of T. We have two cases.

Case (1). T is an (α,η)-continuous.
Since $x_n \to x^*$ as $n \to \infty$ and $\eta(x_{n-1},x_n) \leq \alpha(x_{n-1},x_n)$ for all $n \in \mathbb{N}$, (α,η)-continuity of T implies $x_{n+1} = T(x_n) \to T(x^*)$ as $n \to \infty$. That is, $x^* = T(x^*)$. Hence x^* is a fixed point of T.

Case (2). F is *continuous*. We complete this case in two steps. First, if for each $n \in \mathbb{N}$ there exists $b_n \in \mathbb{N}$ such that $x_{b_n+1} = T(x^*)$ and $b_n > b_{n-1}$ with $b_0 = 1$, then we have

$$x^* = \lim_{n\to\infty} x_{b_n+1} = \lim_{n\to\infty} T(x^*) = T(x^*).$$

This shows that x^* is a fixed point of T. Second, there exists $n_0 \in \mathbb{N}$ such that $x_{n+1} \neq T(x^*)$ for all $n \geq n_0$. Using the contractive condition (3.1), we obtain

$$F\left(p(x_n,T(x^*))\right) \leq F\left(\frac{k}{2}\left[p\left(x_{n-1},x_n\right)+p\left(x^*,T(x^*)\right)\right]\right) - \tau(G),$$

where $\tau(G) = G(p(x_{n-1}, x_n), p(x^*, T(x^*)), p(x_{n-1}, T(x^*)), p(x^*, x_n))$. Using the continuity of F and the property (G), we have

$$F\left(\lim_{n\to\infty} p(x_n, T(x^*))\right) \leq F\left(\frac{k}{2}\left[\lim_{n\to\infty} p\left(x_{n-1}, x_n\right) + \lim_{n\to\infty} p\left(x^*, T(x^*)\right)\right]\right) - \tau.$$

Since F is strictly increasing, the above inequality leads us to conclude that $p(x^*, T(x^*)) = 0$. Thus, by using properties (p_1) and (p_2), we obtain $x^* = T(x^*)$, which completes the proof. □

Example 11. Let $X = \mathbb{R}^+$. Define $p : X^2 \to [0, \infty)$ by $p(x, y) = \max\{x, y\}$. Then (X, p) is a complete partial metric space. Define $T : X \to X$, $\alpha : X \times X \to [0, +\infty)$, $\eta : X \times X \to \mathbb{R}^+$, $G : (\mathbb{R}^+)^4 \to \mathbb{R}^+$ and $F : \mathbb{R}^+ \to \mathbb{R}$ by

$$T(x) = \begin{cases} \frac{5x}{19} & \text{if } x \in [0, 1], \\ x - \frac{1}{3} & \text{if } x \in (1, \infty), \end{cases} \quad \alpha(x, y) = \begin{cases} e^{x+y} & \text{if } x \in [0, 1], \\ \frac{1}{3} & \text{otherwise,} \end{cases}$$

$\eta(x, y) = \frac{1}{2}$ for all $x, y \in X$, $G(t_1, t_2, t_3, t_4) = \tau > 0$ and $F(t) = \ln(t)$ with $t > 0$. Set $k = \frac{4}{5}$ $x_0 = \frac{1}{2}$, $r = \frac{1}{2}$ and $p(x_0, x_0) = \frac{1}{2}$. Then $\overline{B(x_0, r)} = [0, 1]$, $\alpha(0, T(0)) \geq \eta(0, T(0))$ and

$$p\left(\frac{1}{2}, T\left(\frac{1}{2}\right)\right) = \max\left\{\frac{1}{2}, \frac{5}{38}\right\} < (1 - \lambda)[r + p(x_0, x_0)].$$

For if $x, y \in \overline{B(x_0, r)}$, then $\alpha(x, y) = e^{x+y} \geq \frac{1}{2} = \eta(x, y)$. On the other hand, $T(x) \in [0, 1]$ for all $x \in [0, 1]$ and so $\alpha(T(x), T(y)) \geq \eta(T(x), T(y))$ for $x \neq y$, $p(T(x), T(y)) = \left\{\frac{5x}{19}, \frac{5y}{19}\right\} > 0$. For all $x, y \in \overline{B_p(x_0, r)}$, we have

$$p(T(x), T(y)) = \left\{\frac{5x}{19}, \frac{5y}{19}\right\} = \frac{5}{19}\max\{x, y\},$$

$$\frac{5}{19}\max\{x, y\} < \frac{k}{2}\left[\max\left\{x, \frac{5x}{19}\right\} + \max\left\{y, \frac{5y}{19}\right\}\right] = \frac{14k}{38}[x + y].$$

Thus

$$p(T(x), T(y)) < \frac{k}{2}\left[p(x, T(x)) + p(y, T(y))\right].$$

Consequently,

$$\tau + \ln\left(p(T(x), T(y))\right) \leq \ln\left(\frac{k}{2}\left[p(x, T(x)) + p(y, T(y))\right]\right)$$

leads to

$$\tau + F\left(p(T(x), T(y))\right) \leq F\left(\frac{k}{2}\left[p(x, T(x)) + p(y, T(y))\right]\right).$$

If $x \notin \overline{B_p(x_0, r)}$ or $y \notin \overline{B_p(x_0, r)}$, then $\alpha(x, y) = \frac{1}{3} \ngeq \frac{1}{2} = \eta(x, y)$. Moreover, if $x = 100, y = 10 \in (1, \infty)$, then

$$p(T(x), T(y)) = \max\left\{x - \frac{1}{3}, y - \frac{1}{3}\right\} \geq \frac{k}{2}\left[p(x, T(x)) + p(y, T(y))\right].$$

Hence, all the hypotheses of Theorem 4 are satisfied on closed ball and $x = 0$ is a unique fixed point of T.

4. Conclusion

In this paper, the main aim of our paper is to present new concepts of F-contraction on closed ball which are different from those given in [12, 23, 30]. Existence and uniqueness of a fixed point of such type of F-contractions on closed ball in complete partial metric space are discussed. The study of such results is very useful in the sense that it requires the F-contraction mapping defined only on the closed ball instead the whole space. These new concepts shall lead the readers for further investigations and applications. It will also be interesting to apply these concepts in different metric spaces.

References

[1] Ya. I. Alber, S. Guerre-Delabriere, *Principles of weakly contractive maps in Hilbert space*, New Results in Operator Theory. Advances and Applications **98** (1977), pp. 7-22.

[2] G.A. Anastassiou, I.K. Argyros, *Approximating fixed points with applications in fractional calculus*, J. Comput. Anal. Appl. **21** (2016), 1225–1242.

[3] A. Batool, T. Kamran, S. Jang, C. Park, *Generalized φ-weak contractive fuzzy mappings and related fixed point results on complete metric space*, J. Comput. Anal. Appl. **21** (2016), 729–737.

[4] R. Batra, S. Vashistha, *Fixed points of an F-contraction on metric spaces with a graph*, Int. J. Comput. Math. **91** (2014), 1–8.

[5] R. Batra, S. Vashistha, R. Kumar, *A coincidence point theorem for F-contractions on metric spaces equipped with an altered distance*, J. Math. Comput. Sci. **4** (2014), 826–833.

[6] D. W. Boyd, J. S. W. Wong, *Nonlinear contractions*, Proc. Am. Math. Soc. **20** (1969), 458–464.

[7] J. Caristi, *Fixed point theorem for mapping satisfying inwardness conditions*, Trans. Am. Math. Soc. **215** (1976), 241–251.

[8] L. B. Círíc, *A generalization of Banachs contraction principle,* Proc. Am. Math. Soc. **45** (1974), 267–273.

[9] P. N. Dutta, B. S. Choudhury, *A generalisation of contraction principle in metric spaces*, Fixed Point Theory Appl. **2008** (2008), Art. ID 406368.

[10] M. Geraghty, *On contraction mappings*, Proc. Am. Math. Soc. **40** (1973), 604–608.

[11] N. Hussain, E. Karapínar, P. Salimi, F. Akbar, *α-Admissible mappings and related fixed point theorems*, J. Inequal. Appl. **2013**, 2013:114.

[12] N. Hussain, P. Salimi, *Suzuki-Wardowski type fixed point theorems for α-GF-contractions*, Taiwanese J. Math. **18** (2014), 1879–1895.

[13] J. R. Jachymski, *Equivalence of some contractivity properties over metrical structure*, Proc. Am. Math. Soc. **125** (1997), 2327–2335.

[14] O. Kada, T. Susuki, W. Takahashi, *Nonconvex minimization theorems and fixed point theorems in complete metric spaces*, Math. Jpn. **44** (1996), 381–391.

[15] R. Kannan, *Some results on fixed points*, Bull. Calcutta Math. Soc. **60** (1968), 71–76.

[16] W. A. Kirk, *Fixed points of asymptotic contractions*, J. Math. Anal. Appl. **277** (2003), 645–650.

[17] E. Kryeyszig., *Introductory Functional Analysis with Applications*, John Wiley Sons, New York, 1989.

[18] S. G. Matthews, *Partial metric topology*, Papers on general topology and application (Flushing, NY, 1992), pp. 183–197, Ann. New York Acad. Sci. **728**, New York Acad. Sci., New York, 1994.

[19] J. Matkowski, *Nonlinear contractions in metrically convex spaces*, Publ. Math. Debrecen **45** (1994), 103–114.

[20] A. Meir, E. Keeler, *A theorem on contraction mappings*, J. Math. Anal. Appl. **28** (1969), 326–329.

[21] G. Mínak, A. Halvaci, I. Altun, *Círíc type generalized F-contractions on complete metric spaces and fixed point results*, Filomat **28** (2014), 1143–1151.

[22] S. B. Nadler, *Multivalued contraction mappings*, Pacific J. Math. **30** (1969), 475–488.

[23] H. Piri, P. Kumam, *Some fixed point theorems concerning F-contraction in complete metric spaces*, Fixed Point Theory Appl. **2014**, 2014:210.

[24] E. Rakotch, *A note on contractive mappings*, Proc. Am. Math. Soc. **13** (1962), 459–465.

[25] B. E. Rhoades, *Some theorems on weakly contractive maps*, Nonlinear Anal. **47** (2001), 2683–2693.

[26] P. Salimi, A. Latif, N. Hussain, *Modified α-ψ-contractive mappings with applications*, Fixed Point Theory Appl. **2013**, 2013:151.

[27] B. Samet, C. Vetro, P. Vetro, *Fixed point theorems for α-ψ-contractive type mappings*, Nonlinear Anal. **75** (2012), 2154–2165.

[28] M. Sgroi, C. Vetro, *Multivalued F-contractions and the solution of certain functional and integral equations*, Filomat **27** (2013), 1259–1268.

[29] T. Suzuki, *A generalized Banach contraction principle that characterizes metric completeness*, Proc. Am. Math. Soc. **136** (2008), 1861–1869.

[30] D. Wardowski, *Fixed point theory of a new type of contractive mappings in complete metric spaces*, Fixed Point Theory Appl. **2012**, 2012:94.

[31] D. Wardowski, N. V. Dung, *Fixed points of F-weak contractions on complete metric space*, Demonstr. Math. **XLVII** (2014), 146–155.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 27, NO.5, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC

Lacunary sequence spaces defined by Euler transform and Orlicz functions

Abdullah Alotaibi[1], **Kuldip Raj**[2], **Ali H. Alkhaldi**[3] and **S. A. Mohiuddine**[4]

[1,4]Operator Theory and Applications Research Group, Department of Mathematics, Faculty of Science, King Abdulaziz University, P.O. Box 80203, Jeddah 21589, Saudi Arabia
[2]School of Mathematics, Shri Mata Vaishno Devi University, Katra 182320, J&K India
[3]Department of Mathematics, College of Science, King Khalid University, P.O. Box 9004, Abha, Saudi Arabia
Emails: [1]mathker11@hotmail.com; [2]kuldipraj68@gmail.com; [3]ahalkhaldi@kku.edu.sa; [4]mohiuddine@gmail.com

Abstract

We introduce lacunary sequence spaces over n-normed space defined by Euler transform and Musielak-Orlicz functions with the help of an infinite matrix. We also make an effort to study some topological properties and prove some inclusion relations. Finally, we study the notion of statistical convergence over mentioned sequence space.

Keywords and phrases: Musielak-Orlicz function; matrix transformation; Euler transformation; statistical convergence.
AMS subject classification (2010): 40A05; 40C05; 46A45.

1 Introduction and preliminaries

Euler transform is used for improving the convergence of certain series which is widely used in numerical analysis. These techniques are useful in computer science especially in making graphics. We may find the application of the results to physical chemistry and crystallography. Further, we may use these results in the accelerated convergence techniques to find eigenvalues and eigenvectors of the dynamical systems.

Let $\sum_{k=0}^{\infty} a_k$ be an infinite series with sequence of partial sums (s_k) and $q > 0$ be any real number. The Euler transform (E, q) of the sequence $S = (s_n)$ is defined by $E_n^q(S) = \frac{1}{(1+q)^n} \sum_{v=0}^{n} \begin{pmatrix} n \\ v \end{pmatrix} q^{n-v} s_v$. A series $\sum_{n=0}^{\infty} a_n$ is said to be summable (E, q) to the number s if $E_n^q(S) = \frac{1}{(1+q)^n} \sum_{v=0}^{n} \begin{pmatrix} n \\ v \end{pmatrix} q^{n-v} s_v \to s$ as $n \to \infty$, and is said to be absolutely summable (E, q) or summable $|E, q|$, if $\sum_k |E_k^q(S) - E_{k-1}^q(S)| < \infty$. Let $x = (x_n)$ be a sequence of scalars, for $k \geq 1$ we will denote by $N_n(x)$ the difference $E_n^q(x) - E_{n-1}^q(x)$, where E_n^q is defined as above. Using Abel's transform we have

$$N_n(x) = -\frac{1}{(1+q)^{n-1}} \sum_{k=0}^{n-2} x_{k+1} A_k + \frac{s_{n-1} A_{n-1}}{(1+q)^{n-1}} + \frac{s_n}{(1+q)^n} - \frac{q^{n-1}}{(1+q)^n} s_0,$$

where

$$A_k = \sum_{i=0}^{k} \Big[\frac{q}{1+q}\begin{pmatrix} n \\ i \end{pmatrix} - \begin{pmatrix} n-1 \\ i \end{pmatrix}\Big] q^{n-i-1}.$$

Clearly, for any sequence $x = (x_n)$, $y = (y_n)$ and scalar λ, we have: $N_n(x+y) = N_n(x) + N_n(y)$ and $N_n(\lambda x) = \lambda N_n(x)$.

Let

$$\mathfrak{M} = [m_{ij}] = \begin{bmatrix} p_1 & w_1^{(1)} & w_1^{(2)} & \dots \\ w_1^{(-1)} & p_2 & w_2^{(1)} & \dots \\ w_1^{(-2)} & w_2^{(-1)} & p_3 & \dots \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix},$$

where $p = (p_i)$ and $w^{(t)} = (w_i)^{(t)}$ are some fixed numerical sequences $t \in \mathbb{Z}\backslash\{0\}$. For a fixed $k_f \in \mathbb{N}$, we define a finite sequence t_n with k_f terms as $t_n = \begin{cases} \frac{n+1}{2}, & n \text{ is odd} \\ \frac{-n}{2}, & n \text{ is even} \end{cases}$. We construct a matrix $\mathfrak{M}_{(p,w^t,k_f)} = \mathfrak{M}$, $w^{t_i} = 0 \ \forall\, i > k_f$ and for $i = 1, 2, \dots, k_f$ we have some fixed sequences w^{t_i} and p.

Example 1.1. *For $k_f = 2$ we have $t_1 = 1, t_2 = -1$, we define $p_i = -1 \ \forall\, i$ and*

$$w_i^{(t)} = \begin{cases} 1, & for\, t = 1, -1 \\ 0, & \forall\, t \in \mathbb{Z}\backslash\{0, 1, -1\} \end{cases},$$

then we have

$$\mathfrak{M}_{(p,w^t,2)}x = \left\langle \sum_{j=1}^{\infty} m_{ij}\xi_j \right\rangle_n = < -\xi_1 + \xi_2, \xi_1 - \xi_2 + \xi_3, \xi_2 - \xi_3 + \xi_4, \xi_3 - \xi_4 + \xi_5 \cdots > .$$

An Orlicz function is a function $M : [0, \infty) \to [0, \infty)$ which is continuous, non-decreasing and convex with $M(0) = 0$, $M(x) > 0$ as $x > 0$ and $M(x) \to \infty$ $(x \to \infty)$. Clerly, if M is a convex function and $M(0) = 0$, then $M(\lambda x) \le \lambda M(x)$ for all $\lambda \in (0, 1)$. Using the idea of Orlicz function, Lindenstrauss and Tzafriri [15] constructed the sequence space

$$\ell_M = \left\{(x_k) \in w : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty, \text{ for some } \rho > 0\right\},$$

is called Orlicz sequence space and showed that ℓ_M is a Banach space with the following norm:

$$||x|| = \inf\left\{\rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \le 1\right\}.$$

The space ℓ_M is closely related to the space ℓ_p which is an Orlicz sequence space with $M(t) = |t|^p$ for $1 \le p < \infty$.

A sequence $\mathcal{M} = (M_k)$ of Orlicz functions is said to be a Musielak-Orlicz function [22]. A sequence $\mathcal{V} = (V_k)$ is defined by $V_k(v) = \sup\{|v|u - (M_k) : u \ge 0\}$ $(k = 1, 2, \cdots)$ is said to

complementary function of $\mathcal{M}$. For a given $\mathcal{M}$, the Musielak-Orlicz sequence space $t_{\mathcal{M}}$ and its subspace $h_{\mathcal{M}}$ are defined by

$$t_{\mathcal{M}} = \Big\{x \in w : I_{\mathcal{M}}(cx) < \infty \ \text{ for some } \ c > 0\Big\},$$

$$h_{\mathcal{M}} = \Big\{x \in w : I_{\mathcal{M}}(cx) < \infty \ \text{ for all } \ c > 0\Big\},$$

where $I_{\mathcal{M}}$ denotes the convex modular and is defined by

$$I_{\mathcal{M}}(x) = \sum_{k=1}^{\infty}(M_k)(x_k) \quad (x = (x_k) \in t_{\mathcal{M}}).$$

It is noted that $t_{\mathcal{M}}$ equipped with the Luxemburg norm or equipped with the Orlicz norm, where Luxemburg and Orlicz norms are given by

$$||x|| = \inf\Big\{k > 0 : I_{\mathcal{M}}\Big(\frac{x}{k}\Big) \leq 1\Big\} \ \text{ and } \ ||x||^0 = \inf\Big\{\frac{1}{k}\Big(1 + I_{\mathcal{M}}(kx)\Big) : k > 0\Big\},$$

respectively.

Kızmaz [14] was the first who introduced the idea of difference sequence spaces and studied $Z(\Delta) = \{x = (x_k) \in w : \Delta x \in Z\}$ $(Z = l_\infty, c, c_0)$, where $\Delta x = x_k - x_{k+1}$ for all $k \in \mathbb{N}$ ($\mathbb{N}$ and w denote the set of natural numbers and the set of all real and complex sequences) and the standard notations l_∞, c and c_0 denote bounded, convergent and null sequences respectively. Et and Çolak [7] presented a generalization of these difference sequence spaces and introduced the space $Z(\Delta^n)$ $(n \in \mathbb{N})$, in this case, $\Delta^n x$ is given by $\Delta^n x = \Delta(\Delta^{n-1}x) = \Delta^{n-1}x_k - \Delta^{n-1}x_{k+1}$ for $n \geq 2$, which is equivalent to the following binomial representation

$$\Delta^n x_k = \sum_{v=0}^{n}(-1)^v \binom{n}{v} x_{k+v}.$$

We remark that if we take $n = 1$, then difference sequence space $Z(\Delta^n)$ is reduced to $Z(\Delta)$.

Gähler [12] extended the usual notion of normed spaces into 2-normed spaces while the notion was again extended to n-normed spaces by Misiak [16]. Assume that X is a linear space over the field $\mathbb{K}$ of real or complex numbers of dimension $d \geq n \geq 2$, $n \in \mathbb{N}$ ($\mathbb{N}$ denotes the set of natural numbers). A real valued function $||\cdot, \cdots, \cdot||$ on X^n satisfying the conditions:

(N1) $||x_1, x_2, \cdots, x_n|| = 0$ if and only if $x_1, \cdots, x_n$ are linearly dependent in X;

(N2) $||x_1, x_2, \cdots, x_n||$ is invariant under permutation;

(N3) $||\alpha x_1, x_2, \cdots, x_n|| = |\alpha| \ ||x_1, x_2, \cdots, x_n||$ for any $\alpha \in \mathbb{K}$;

(N4) $||x_1 + x_1', x_2, \cdots, x_n|| \leq ||x_1, x_2, \cdots, x_n|| + ||x_1', x_2, \cdots, x_n||$

is called a n-norm on X, and the pair $(X, ||\cdot, \cdots, \cdot||)$ is called a n-normed space over $\mathbb{K}$.

For more details about these notions we refer to [3–5, 13, 18, 19, 21, 23] and references therein.

We used the standard notation $\theta = (k_r)$ to denotes the lacunary sequence, where θ is a sequence of positive integers such that $k_0 = 0$, $0 < k_r < k_{r+1}$ and $h_r = k_r - k_{r-1} \to \infty$ $(r \to \infty)$. The intervals determined by θ will be denoted by $I_r = (k_{r-1}, k_r]$ and the ratio $\frac{k_r}{k_r - 1}$ by q_r (see [9]).

2 Main results

Let $\mathcal{M} = (M_k)$ be a Musielak-Orlicz function, $p = (p_k)$ be a bounded sequence of positive real numbers and $u = (u_k)$ a sequence of strictly positive numbers. We define the following sequence space in the present paper:

$$E_n^q(\mathcal{M}, u, p, s, \mathfrak{M}_{(p,w^t,k_f)}, \Delta^r, \|\cdot, \cdots, \cdot\|) = \Big\{ x = (x_k) : \lim_r \frac{1}{h_r} \sum_{k=1}^{\infty} k^{-s}$$

$$\times \left[M_k \left(\left\| \frac{u_k N_k(\mathfrak{M}_{(p,w^t,k_f)} \Delta^r x)}{\rho}, z_1, \cdots, z_{n-1} \right\| \right) \right]^{p_k} < \infty, \quad s \geq 0, \text{ for some } \rho > 0 \Big\}.$$

We will use the following inequality to prove our results. If $0 \leq p_k \leq \sup p_k = H$, $K = \max(1, 2^{H-1})$ then

$$|a_k + b_k|^{p_k} \leq K\{|a_k|^{p_k} + |b_k|^{p_k}\}$$

for all k and $a_k, b_k \in \mathbb{C}$. Also $|a|^{p_k} \leq \max(1, |a|^H)$ for all $a \in \mathbb{C}$.

Theorem 2.1. *Let $\mathcal{M} = (M_k)$ be a Musielak-Orlicz function, $p = (p_k)$ be a bounded sequence of positive real numbers and $u = (u_k)$ be a sequence of strictly positive real numbers. Then the space $E_n^q(\mathcal{M}, u, p, s, \mathfrak{M}_{(p,w^t,k_f)}, \Delta^r, \|\cdot, \cdots, \cdot\|)$ is linear over the field $\mathbb{R}$ of real numbers.*

Proof. Let $x = (x_k), y = (y_k) \in E_n^q(\mathcal{M}, u, p, s, \mathfrak{M}_{(p,w^t,k_f)}, \Delta^r, \|\cdot, \cdots, \cdot\|)$ and $\alpha, \beta \in \mathbb{R}$. Then there exist positive integers ρ_1 and ρ_2 such that

$$\lim_r \frac{1}{h_r} \sum_{k=1}^{\infty} k^{-s} \left[M_k \left(\left\| \frac{u_k N_k(\mathfrak{M}_{(p,w^t,k_f)} \Delta^r x)}{\rho_1}, z_1, \cdots, z_{n-1} \right\| \right) \right]^{p_k} < \infty$$

and

$$\lim_r \frac{1}{h_r} \sum_{k=1}^{\infty} k^{-s} \left[M_k \left(\left\| \frac{u_k N_k(\mathfrak{M}_{(p,w^t,k_f)} \Delta^r y)}{\rho_2}, z_1, \cdots, z_{n-1} \right\| \right) \right]^{p_k} < \infty.$$

Let $\rho_3 = \max(2|\alpha|\rho_1, 2|\beta|\rho_2)$. Since (M_k) is nondecreasing and convex function so we have

$$\lim_r \frac{1}{h_r} \sum_{k=1}^{\infty} k^{-s} \left[M_k \left(\left\| \frac{u_k N_k(\mathfrak{M}_{(p,w^t,k_f)} (\alpha \Delta^r x + \beta \Delta^r y))}{\rho_3}, z_1, \cdots, z_{n-1} \right\| \right) \right]^{p_k}$$

$$\leq \lim_r \frac{1}{h_r} \sum_{k=1}^{\infty} k^{-s} \left[M_k \left(\left\| \frac{u_k N_k(\mathfrak{M}_{(p,w^t,k_f)} \alpha \Delta^r x)}{\rho_3}, z_1, \cdots, z_{n-1} \right\| \right) \right.$$

$$\left. + \left(\left\| \frac{u_k N_k(\mathfrak{M}_{(p,w^t,k_f)} \beta \Delta^r y)}{\rho_3}, z_1, \cdots, z_{n-1} \right\| \right) \right]^{p_k}$$

$$\leq K \lim_r \frac{1}{h_r} \sum_{k=1}^{\infty} k^{-s} \left[M_k \left(\left\| \frac{u_k N_k(\mathfrak{M}_{(p,w^t,k_f)} \Delta^r x)}{\rho_1}, z_1, \cdots, z_{n-1} \right\| \right) \right]^{p_k}$$

$$+ K \lim_r \frac{1}{h_r} \sum_{k=1}^{\infty} k^{-s} \left[M_k \left(\left\| \frac{u_k N_k(\mathfrak{M}_{(p,w^t,k_f)} \Delta^r y)}{\rho_2}, z_1, \cdots, z_{n-1} \right\| \right) \right]^{p_k}$$

$$< \infty.$$

Therefore, $\alpha x + \beta y \in E_n^q(\mathcal{M}, u, p, s, \mathfrak{M}_{(p,w^t,k_f)}, \Delta^r, ||\cdot, \cdots, \cdot||)$. This proves that $E_n^q(\mathcal{M}, u, p, s, \mathfrak{M}_{(p,w^t,k_f)}, \Delta^r, ||\cdot, \cdots, \cdot||)$ is a linear space. □

Theorem 2.2. *Let $\mathcal{M} = (M_k)$ be a Musielak-Orlicz function, $p = (p_k)$ be a bounded sequence of positive real numbers and $u = (u_k)$ be a sequence of strictly positive real numbers. Then, the space $E_n^q(\mathcal{M}, u, p, s, \mathfrak{M}_{(p,w^t,k_f)}, \Delta^r, \|\cdot, \cdots, \cdot\|)$ is paranormed space with the paranorm defined by*

$$g(x) = \inf\Bigg\{\rho^{p_n/H} : \Bigg(\lim_r \frac{1}{h_r}\sum_{k=1}^{\infty} k^{-s}\Bigg[M_k\Bigg(\Bigg\|\frac{u_k N_k(\mathfrak{M}_{(p,w^t,k_f)}\Delta^r x)}{\rho}, z_1, \cdots, z_{n-1}\Bigg\|\Bigg)\Bigg]^{p_k}\Bigg)^{\frac{1}{H}} \leq 1\Bigg\},$$

where $H = \max(1, \sup_k p_k)$.

Proof. Clearly $g(x) = g(-x)$ and $g(x+y) \leq g(x)+g(y)$. Since $M_k(0) = 0$, we get $\inf\{\rho^{p_n/H}\} = 0$ for $x = 0$. Finally, we prove that multiplication is continuous. Let λ be any number then,

$$g(\lambda x) = \inf\Bigg\{\rho^{p_n/H} : \lim_r \frac{1}{h_r}\sum_{k=1}^{\infty} k^{-s}\Bigg[M_k\Bigg(\Bigg\|\frac{\lambda u_k N_k(\mathfrak{M}_{(p,w^t,k_f)}\Delta^r x)}{\rho}, z_1, \cdots, z_{n-1}\Bigg\|\Bigg)\Bigg]^{p_k} \leq 1\Bigg\}$$

implies that

$$g(\lambda x) = \inf\Bigg\{(\lambda s)^{p_n/H} : \lim_r \frac{1}{h_r}\sum_{k=1}^{\infty} k^{-s}\Bigg[M_k\Bigg(\Bigg\|\frac{u_k N_k(\mathfrak{M}_{(p,w^t,k_f)}\Delta^r x)}{s}, z_1, \cdots, z_{n-1}\Bigg\|\Bigg)\Bigg]^{p_k} \leq 1\Bigg\},$$

where $s = \frac{\rho}{|\lambda|}$. Since $|\lambda|^{p_k} \leq \max(1, |\lambda|^H)$, then $|\lambda|^{p_k/H} \leq \Big(\max\big(1, |\lambda|^H\big)\Big)^{\frac{1}{H}}$. Hence

$$g(\lambda x) \leq \Big(\max\big(1, |\lambda|^H\big)\Big)^{\frac{1}{H}} \inf\Bigg\{(s)^{p_n/H} : \Bigg(\lim_r \frac{1}{h_r} \times \sum_{k=1}^{\infty} k^{-s}\Bigg[M_k\Bigg(\Bigg\|\frac{u_k N_k(\mathfrak{M}_{(p,w^t,k_f)}\Delta^r x)}{\rho}, z_1, \cdots, z_{n-1}\Bigg\|\Bigg)\Bigg]^{p_k}\Bigg)^{\frac{1}{H}} \leq 1\Bigg\}$$

and therefore, $g(x)$ converges to zero when $g(x)$ converges to zero in $E_n^q(\mathcal{M}, u, p, s, \mathfrak{M}_{(p,w^t,k_f)}, \Delta^r, \|\cdot, \cdots, \cdot\|)$. Now suppose that $\lambda_n \to 0$ as $n \to \infty$ and $x \in E_n^q(\mathcal{M}, u, p, s, \mathfrak{M}_{(p,w^t,k_f)}, \Delta^r, \|\cdot, \cdots, \cdot\|)$. For arbitrary $\epsilon > 0$, let n_0 be a positive integer such that

$$\lim_r \frac{1}{h_r}\sum_{k=n_0+1}^{\infty} k^{-s}\Bigg[M_k\Bigg(\Bigg\|\frac{u_k N_k(\mathfrak{M}_{(p,w^t,k_f)}\Delta^r x)}{\rho}, z_1, \cdots, z_{n-1}\Bigg\|\Bigg)\Bigg]^{p_k} < \frac{\epsilon}{2}$$

for some $\rho > 0$. This implies that

$$\Bigg(\lim_r \frac{1}{h_r}\sum_{k=n_0+1}^{\infty} k^{-s}\Bigg[M_k\Bigg(\Bigg\|\frac{u_k N_k(\mathfrak{M}_{(p,w^t,k_f)}\Delta^r x)}{\rho}, z_1, \cdots, z_{n-1}\Bigg\|\Bigg)\Bigg]^{p_k}\Bigg)^{\frac{1}{H}} \leq \frac{\epsilon}{2}.$$

Let $0 < |\lambda| < 1$, then using convexity of (M_k), we get

$$\lim_r \frac{1}{h_r} \sum_{k=n_0+1}^{\infty} k^{-s} \Big[M_k\Big(\Big\|\frac{\lambda u_k N_k(\mathfrak{M}_{(p,w^t,k_f)}\Delta^r x)}{\rho}, z_1, \cdots, z_{n-1}\Big\|\Big)\Big]^{p_k}$$
$$< |\lambda| \lim_r \frac{1}{h_r} \sum_{k=n_0+1}^{\infty} k^{-s} \Big[M_k\Big(\Big\|\frac{u_k N_k(\mathfrak{M}_{(p,w^t,k_f)}\Delta^r x)}{\rho}, z_1, \cdots, z_{n-1}\Big\|\Big)\Big]^{p_k} < \Big(\frac{\epsilon}{2}\Big)^H.$$

Since (M_k) is continuous everywhere in $[0, \infty)$, so

$$h(t) = \lim_r \frac{1}{h_r} \sum_{k=1}^{n_0} k^{-s} \Big[M_k\Big(\Big\|\frac{t u_k N_k(\mathfrak{M}_{(p,w^t,k_f)}\Delta^r x)}{\rho}, z_1, \cdots, z_{n-1}\Big\|\Big)\Big]^{p_k}$$

is continuous at 0. Hence, there is $0 < \delta < 1$ such that $|h(t)| < \epsilon/2$ for $0 < t < \delta$. Let K be such that $|\lambda_n| < \delta$ for $n > K$ we have

$$\Big(\lim_r \frac{1}{h_r} \sum_{k=1}^{n_0} k^{-s} \Big[M_k\Big(\Big\|\frac{\lambda_n u_k N_k(\mathfrak{M}_{(p,w^t,k_f)}\Delta^r x)}{\rho}, z_1, \cdots, z_{n-1}\Big\|\Big)\Big]^{p_k}\Big)^{\frac{1}{H}} < \frac{\epsilon}{2}.$$

Thus, for $n > K$,

$$\Big(\lim_r \frac{1}{h_r} \sum_{k=1}^{\infty} k^{-s} \Big[M_k\Big(\Big\|\frac{\lambda_n u_k N_k(\mathfrak{M}_{(p,w^t,k_f)}\Delta^r x)}{\rho}, z_1, \cdots, z_{n-1}\Big\|\Big)\Big]^{p_k}\Big)^{\frac{1}{H}} < \epsilon.$$

Hence, $g(\lambda x) \to 0$ as $\lambda \to 0$. This completes the proof of the theorem. □

Theorem 2.3. *If $\mathcal{M}' = (M'_k)$ and $\mathcal{M}'' = (M''_k)$ are two Musielak-Orlicz functions and s, s_1, s_2 be non-negative real numbers, then we have*

(i) $E^q_n(\mathcal{M}', u, p, s, \mathfrak{M}_{(p,w^t,k_f)}, \Delta^r, \|\cdot, \cdots, \cdot\|) \cap E^q_n(\mathcal{M}'', u, p, s, \mathfrak{M}_{(p,w^t,k_f)}, \Delta^r, \|\cdot, \cdots, \cdot\|) \subseteq E^q_n(\mathcal{M}' + \mathcal{M}'', u, p, s, \mathfrak{M}_{(p,w^t,k_f)}, \Delta^r, \|\cdot, \cdots, \cdot\|)$.

(ii) If the inequality $s_1 \leq s_2$ holds, then $E^q_n(\mathcal{M}', u, p, s_1, \mathfrak{M}_{(p,w^t,k_f)}, \Delta^r, \|\cdot, \cdots, \cdot\|) \subseteq E^q_n(\mathcal{M}', u, p, s_2, \mathfrak{M}_{(p,w^t,k_f)}, \Delta^r, \|\cdot, \cdots, \cdot\|)$.

(iii) If $\mathcal{M}'$ and $\mathcal{M}''$ are equivalent, then $E^q_n(\mathcal{M}', u, p, s, \mathfrak{M}_{(p,w^t,k_f)}, \Delta^r, \|\cdot, \cdots, \cdot\|) = E^q_n(\mathcal{M}'', u, p, s, \mathfrak{M}_{(p,w^t,k_f)}, \Delta^r, \|\cdot, \cdots, \cdot\|)$.

Proof. It is obvious so we omit the details. □

Theorem 2.4. *Suppose that $0 < r_k \leq p_k < \infty$ for each k. Then*

$$E^q_n(\mathcal{M}, u, r, s, \mathfrak{M}_{(p,w^t,k_f)}, \Delta^r, \|\cdot, \cdots, \cdot\|) \subseteq E^q_n(\mathcal{M}, u, p, s, \mathfrak{M}_{(p,w^t,k_f)}, \Delta^r, \|\cdot, \cdots, \cdot\|).$$

Proof. Let $x \in E^q_n(\mathcal{M}, u, r, s, \mathfrak{M}_{(p,w^t,k_f)}, \Delta^r, \|\cdot, \cdots, \cdot\|)$. Then there exists some $\rho > 0$ such that

$$\lim_r \frac{1}{h_r} \sum_{k=1}^{\infty} k^{-s} \Big[M_k\Big(\Big\|\frac{u_k N_k(\mathfrak{M}_{(p,w^t,k_f)}\Delta^r x)}{\rho}, z_1, \cdots, z_{n-1}\Big\|\Big)\Big]^{r_k} < \infty.$$

This implies that

$$M_k\left(\left\|\frac{u_kN_k(\mathfrak{M}_{(p,w^t,k_f)}\Delta^r x)}{\rho}, z_1, \cdots, z_{n-1}\right\|\right) \leq 1$$

for sufficiently large value of k, say $k \geq k_0$ for some fixed $k_0 \in \mathbb{N}$. Since (M_k) is nondecreasing, we have

$$\lim_r \frac{1}{h_r}\sum_{k\geq k_0}^{\infty} k^{-s}\left[M_k\left(\left\|\frac{u_kN_k(\mathfrak{M}_{(p,w^t,k_f)}\Delta^r x)}{\rho}, z_1, \cdots, z_{n-1}\right\|\right)\right]^{p_k}$$
$$\leq \lim_r \frac{1}{h_r}\sum_{k\geq k_0}^{\infty} k^{-s}\left[M_k\left(\left\|\frac{u_kN_k(\mathfrak{M}_{(p,w^t,k_f)}\Delta^r x)}{\rho}, z_1, \cdots, z_{n-1}\right\|\right)\right]^{r_k} < \infty.$$

Hence, $x \in E_n^q(\mathcal{M}, u, p, s, \mathfrak{M}_{(p,w^t,k_f)}, \Delta^r, \|\cdot, \cdots, \cdot\|)$. □

Theorem 2.5. *(i) If* $0 < p_k \leq 1$ *for each* k, *then* $E_n^q(\mathcal{M}, u, p, s, \mathfrak{M}_{(p,w^t,k_f)}, \Delta^r, \|\cdot, \cdots, \cdot\|) \subseteq E_n^q(\mathcal{M}, u, s, \mathfrak{M}_{(p,w^t,k_f)}, \Delta^r, \|\cdot, \cdots, \cdot\|)$.
(ii) If $p_k \geq 1$ *for all* k, *then* $E_n^q(\mathcal{M}, u, s, \mathfrak{M}_{(p,w^t,k_f)}, \Delta^r, \|\cdot, \cdots, \cdot\|) \subseteq E_n^q(\mathcal{M}, u, p, s, \mathfrak{M}_{(p,w^t,k_f)}, \Delta^r, \|\cdot, \cdots, \cdot\|)$.

Proof. It is easy to prove by using above so we omit the details. □

3 Applications to statistical convergence

Fast [8] extended the notion of usual convergence of a sequence of real or complex numbers and called it statistical convergence. This notion turned out to be one of the most active areas of research in summability theory after the works of Fridy [10] and Šalát [24]. Fridy and Orhan [11] defined and studied the notion of lacunary statistical convergence. Some recent related work and applications we refer to [1, 2, 6, 17, 20]. We are now ready to define following notions:

Definition 3.1. *Let* $\theta = (k_r)$ *be a lacunary sequence. Then, the sequence* $x = (x_k)$ *is* $N_k(u)$*-lacunary statistically convergent to the number* l *provided that for every* $\epsilon > 0$,

$$\lim_r \frac{1}{h_r}\left|\left\{k \in I_r : \left\|\frac{u_kN_k(\mathfrak{M}_{(p,w^t,k_f)}\Delta^r x) - l}{\rho}, z_1 \cdots, z_{n-1}\right\| \geq \epsilon\right\}\right| = 0.$$

In symbols, we shall write $[N_k, u, \mathfrak{M}_{(p,w^t,k_f)}, S, \Delta^r]_\theta\text{-}\lim x = l$ *or* $x_k \to l([N_k, u, \mathfrak{M}_{(p,w^t,k_f)}, S, \Delta^r]_\theta)$. *If we take* $\theta = (2^r)$, *then we shall write* $[N_k, u, \mathfrak{M}_{(p,w^t,k_f)}, S, \Delta^r]$ *instead of* $[N_k, u, \mathfrak{M}_{(p,w^t,k_f)}, S, \Delta^r]_\theta$.

Definition 3.2. *Let* $\theta = (k_r)$ *be a lacunary sequence,* $\mathcal{M} = (M_k)$ *be a Musielak-Orlicz function,* $u = (u_k)$ *be a sequence of strictly positive real numbers and* $p = (p_k)$ *be a bounded sequence of positive real numbers. We say that* $x = (x_k)$ *is strongly* $N_k(u, \mathfrak{M}_{(p,w^t,k_f)}, \Delta^r)$*-lacunary convergent to* l *with respect to* $\mathcal{M}$ *provided that*

$$\lim_r \frac{1}{h_r}\sum_{k\in I_r}\left[M_k\left(\left\|\frac{u_kN_k(\mathfrak{M}_{(p,w^t,k_f)}\Delta^r x) - l}{\rho}, z_1, \cdots, z_{n-1}\right\|\right)\right]^{p_k} = 0.$$

The set of all strongly $N_k(u, \mathfrak{M}_{(p,w^t,k_f)}, \Delta^r)$-lacunary convergent sequences to l with respect to $\mathcal{M}$ is denoted by $[N_k, u, \mathcal{M}, p, \mathfrak{M}_{(p,w^t,k_f)}, \Delta^r, \|\cdot, \cdots, \cdot\|]_\theta$. In symbols, we shall write $x_k \to l([N_k, u, \mathcal{M}, p, \mathfrak{M}_{(p,w^t,k_f)}, \Delta^r, \|\cdot, \cdots, \cdot\|]_\theta)$.

Note that, in the special case, $\mathcal{M}(x) = x$, $p_k = p_0$ for all $k \in \mathbb{N}$, we shall write $[N_k, u, \mathfrak{M}_{(p,w^t,k_f)}, \Delta^r, \|\cdot, \cdots, \cdot\|]_\theta$ instead of $[N_k, u, \mathcal{M}, p, \mathfrak{M}_{(p,w^t,k_f)}, \Delta^r, \|\cdot, \cdots, \cdot\|]_\theta$.

Theorem 3.3. *Let $\theta = (k_r)$ be a lacunary sequence.*

(i) If a sequence $x = (x_k)$ is strongly $N_k(u, \mathfrak{M}_{(p,w^t,k_f)}, \Delta^r)$-lacunary convergent to l, then it is $N_k(u, \mathfrak{M}_{(p,w^t,k_f)}, \Delta^r)$-lacunary statistically convergent to l.

(ii) If a bounded sequence $x = (x_k)$ is $N_k(u, \mathfrak{M}_{(p,w^t,k_f)}, \Delta^r)$-lacunary statistically convergent to l, then it is strongly $N_k(u, \mathfrak{M}_{(p,w^t,k_f)}, \Delta^r)$-lacunary convergent to l.

Proof. (i) Let $\epsilon > 0$ and $x_k \to l([N_k, u, \mathfrak{M}_{(p,w^t,k_f)}, \Delta^r, \|\cdot, \cdots, \cdot\|]_\theta)$. Then, we have

$$\sum_{k \in I_r} \left\| \frac{u_k N_k(\mathfrak{M}_{(p,w^t,k_f)} \Delta^r x) - l}{\rho}, z_1, \cdots, z_{n-1} \right\|^{p_0}$$

$$\geq \sum_{\substack{k \in I_r \\ \left\| \frac{u_k N_k(\mathfrak{M}_{(p,w^t,k_f)} \Delta^r x) - l}{\rho}, z_1, \cdots, z_{n-1} \right\| \geq \epsilon}} \left\| \frac{u_k N_k(\mathfrak{M}_{(p,w^t,k_f)} \Delta^r x) - l}{\rho}, z_1, \cdots, z_{n-1} \right\|^{p_0}$$

$$\geq \epsilon^{p_0} \left| \left\{ k \in I_r : \left\| \frac{u_k N_k(\mathfrak{M}_{(p,w^t,k_f)} \Delta^r x) - l}{\rho}, z_1, \cdots, z_{n-1} \right\| \geq \epsilon \right\} \right|.$$

Hence, $x_k \to l([N_k, u, \mathfrak{M}_{(p,w^t,k_f)}, S, \Delta^r]_\theta)$.

(ii) Suppose that $x_k \to l([N_k, u, \mathfrak{M}_{(p,w^t,k_f)}, S, \Delta^r]_\theta)$ and let $x \in l_\infty$. Let $\epsilon > 0$ be given and take N_ϵ such that

$$\lim_r \frac{1}{h_r} \left| \left\{ k \in I_r : \left\| \frac{u_k N_k(\mathfrak{M}_{(p,w^t,k_f)} \Delta^r x) - l}{\rho}, z_1, \cdots, z_{n-1} \right\| \geq \left(\frac{\epsilon}{2} \right)^{\frac{1}{p_0}} \right\} \right| \leq \frac{\epsilon}{2K^{p_0}}$$

for all $r > N_\epsilon$ and set

$$T_r = \left\{ k \in I_r : \left\| \frac{u_k N_k(\mathfrak{M}_{(p,w^t,k_f)} \Delta^r x) - l}{\rho}, z_1, \cdots, z_{n-1} \right\| \geq \left(\frac{\epsilon}{2} \right)^{\frac{1}{p_0}} \right\},$$

where $K = \sup_k |x_k| < \infty$. Now for all $r > N_\epsilon$ we have

$$\lim_r \frac{1}{h_r} \sum_{k \in I_r} \left\| \frac{u_k N_k(\mathfrak{M}_{(p,w^t,k_f)} \Delta^r x) - l}{\rho}, z_1, \cdots, z_{n-1} \right\|^{p_0}$$

$$= \lim_r \frac{1}{h_r} \sum_{\substack{k \in I_r \\ k \in T_r}} \left\| \frac{u_k N_k(\mathfrak{M}_{(p,w^t,k_f)} \Delta^r x) - l}{\rho}, z_1, \cdots, z_{n-1} \right\|^{p_0}$$

$$+\lim_r \frac{1}{h_r}\sum_{\substack{k\in I_r\\ k\notin T_r}}\Big\|\frac{u_kN_k(\mathfrak{M}_{(p,w^t,k_f)}\Delta^r x)-l}{\rho},z_1,\cdots,z_{n-1}\Big\|^{p_0}$$

$$\leq \lim_r \frac{1}{h_r}\Big(\frac{h_r\epsilon}{2K^{p_0}}\Big)K^{p_0}+\frac{\epsilon}{2h_r}h_r=\epsilon.$$

Thus, $(x_k)\in[N_k,u,\mathfrak{M}_{(p,w^t,k_f)},\Delta^r,\|\cdot,\cdots,\cdot\|]_\theta$. □

Theorem 3.4. *For any lacunary sequence θ, if $\lim_{r\to\infty}\inf q_r>1$ then*

$$[N_k,u,\mathfrak{M}_{(p,w^t,k_f)},S,\Delta^r]\subset[N_k,u,\mathfrak{M}_{(p,w^t,k_f)},S,\Delta^r]_\theta.$$

Proof. If $\lim_{r\to\infty}\inf q_r>1$, then there exists a $\delta>0$ such that $1+\delta\leq q_r$ for sufficiently large r. Since $h_r=k_r-k_{r-1}$, we have $\frac{k_r}{h_r}\leq\frac{1+\delta}{\delta}$. Let $x_k\to l([N_k,u,\mathfrak{M}_{(p,w^t,k_f)},S,\Delta^r])$. Then for every $\epsilon>0$, we have

$$\frac{1}{k_r}\Big|\Big\{k\leq k_r:\Big\|\frac{u_kN_k(\mathfrak{M}_{(p,w^t,k_f)}\Delta^r x)-l}{\rho},z_1,\cdots,z_{n-1}\Big\|\geq\epsilon\Big\}\Big|$$

$$\geq\frac{1}{k_r}\Big|\Big\{k\in I_r:\Big\|\frac{u_kN_k(\mathfrak{M}_{(p,w^t,k_f)}\Delta^r x)-l}{\rho},z_1,\cdots,z_{n-1}\Big\|\geq\epsilon\Big\}\Big|$$

$$\geq\Big(\frac{\delta}{1+\delta}\Big)\frac{1}{h_r}\Big|\Big\{k\in I_r:\Big\|\frac{u_kN_k(\mathfrak{M}_{(p,w^t,k_f)}\Delta^r x)-l}{\rho},z_1,\cdots,z_{n-1}\Big\|\geq\epsilon\Big\}\Big|.$$

Hence, $[N_k,u,\mathfrak{M}_{(p,w^t,k_f)},S,\Delta^r]\subset[N_k,u,\mathfrak{M}_{(p,w^t,k_f)},S,\Delta^r]_\theta$. □

In the next results we denote the quantity $\frac{u_kN_k(\mathfrak{M}_{(p,w^t,k_f)}\Delta^r x)-l}{\rho}$ by $x_k^{l,\rho}$.

Theorem 3.5. *Let $\theta=(k_r)$ be a lacunary sequence, $\mathcal{M}=(M_k)$ be a Musielak-Orlicz function and $0<h=\inf_k p_k\leq p_k\leq\sup_k p_k=H$. Then $[N_k,\mathcal{M},u,\mathfrak{M}_{(p,w^t,k_f)},\Delta^r,\|\cdot,\cdots,\cdot\|]_\theta\subset[N_k,u,\mathfrak{M}_{(p,w^t,k_f)},S,\Delta^r]_\theta$.*

Proof. Let $x\in[N_k,\mathcal{M},u,\mathfrak{M}_{(p,w^t,k_f)},\Delta^r,\|\cdot,\cdots,\cdot\|]_\theta$. Then there exists a number $\rho>0$ such that $\lim_r\frac{1}{h_r}\sum_{k\in I_r}\Big[M_k\Big(\|x_k^{l,\rho},z_1,\cdots,z_{n-1}\|\Big)\Big]^{p_k}\to0$, as $r\to\infty$. Then given $\epsilon>0$, we have

$$\lim_r\frac{1}{h_r}\sum_{k\in I_r}\Big[M_k\Big(\Big\|x_k^{l,\rho},z_1,\cdots,z_{n-1}\Big\|\Big)\Big]^{p_k}$$

$$\geq\lim_r\frac{1}{h_r}\sum_{\substack{k\in I_r\\ \|x_k^{l,\rho},z_1,\cdots,z_{n-1}\|\geq\epsilon}}\Big[M_k\Big(\Big\|x_k^{l,\rho},z_1,\cdots,z_{n-1}\Big\|\Big)\Big]^{p_k}$$

$$\geq\lim_r\frac{1}{h_r}\sum_{\substack{k\in I_r\\ \|x_k^{l,\rho},z_1,\cdots,z_{n-1}\|\geq\epsilon}}[M_k(\epsilon_1)]^{p_k},\quad\text{where }\ \epsilon/\rho=\epsilon_1$$

$$\geq\lim_r\frac{1}{h_r}\sum_{k\in I_r}\min\Big\{[M_k(\epsilon_1)]^h,[M_k(\epsilon_1)]^H\Big\}$$

$$\geq \lim_r \frac{1}{h_r}\left|\left\{k \in I_r : \left\|x_k^{l,\rho}, z_1, \cdots, z_{n-1}\right\| \geq \epsilon\right\}\right|. \min\left\{[M_k(\epsilon_1)]^h, [M_k(\epsilon_1)]^H\right\}.$$

Hence, $x \in [N_k, u, \mathfrak{M}_{(p,w^t,k_f)}, S, \Delta^r]_\theta$. This completes the proof of the theorem. □

Theorem 3.6. *Let $\theta = (k_r)$ be a lacunary sequence, $\mathcal{M} = (M_k)$ be a Musielak-Orlicz function and $p = (p_k)$ be a bounded sequence, then $[N_k, u, \mathfrak{M}_{(p,w^t,k_f)}, S, \Delta^r]_\theta \subset [N_k, \mathcal{M}, u, \mathfrak{M}_{(p,w^t,k_f)}, \Delta^r, \|\cdot, \cdots, \cdot\|]_\theta$.*

Proof. Let $x \in l_\infty$ and $x_k \to l([N_k, u, \mathfrak{M}_{(p,w^t,k_f)}, S, \Delta^r]_\theta)$. Since $x \in l_\infty$, there is a constant $T > 0$ such that $\|x_k^{l,\rho}, z_1, \cdots, z_{n-1}\| \leq T$ and given $\epsilon > 0$ we have

$$\begin{aligned}
&\lim_r \frac{1}{h_r}\sum_{k\in I_r}\left[M_k\Big(\|x_k^{l,\rho}, z_1, \cdots, z_{n-1}\|\Big)\right]^{p_k} \\
&= \lim_r \frac{1}{h_r}\sum_{\substack{k\in I_r \\ \|x_k^{l,\rho}, z_1,\cdots,z_{n-1}\|\geq\epsilon}}\left[M_k\Big(\|x_k^{l,\rho}, z_1, \cdots, z_{n-1}\|\Big)\right]^{p_k} \\
&\quad + \lim_r \frac{1}{h_r}\sum_{\substack{k\in I_r \\ \|x_k^{l,\rho}, z_1,\cdots,z_{n-1}\|<\epsilon}}\left[M_k\Big(\|x_k^{l,\rho}, z_1, \cdots, z_{n-1}\|\Big)\right]^{p_k} \\
&\leq \lim_r \frac{1}{h_r}\sum_{\substack{k\in I_r \\ \|x_k^{l,\rho}, z_1,\cdots,z_{n-1}\|\geq\epsilon}} \max\left\{\left[M_k\left(\frac{T}{\rho}\right)\right]^h, \left[M_k\left(\frac{T}{\rho}\right)\right]^H\right\} \\
&\quad + \lim_r \frac{1}{h_r}\sum_{\substack{k\in I_r \\ \|x_k^{l,\rho}, z_1,\cdots,z_{n-1}\|<\epsilon}}\left[M_k\left(\frac{\epsilon}{\rho}\right)\right]^{p_k} \\
&\leq \max\left\{[M_k(K)]^h, [M_k(K)]^H\right\}\lim_r \frac{1}{h_r}\left|\left\{k \in I_r : \|x_k^{l,\rho}, z_1, \cdots, z_{n-1}\| \geq \epsilon\right\}\right| \\
&\quad + \max\left\{[M_k(\epsilon_1)]^h, [M_k(\epsilon_1)]^H\right\}, \quad \left(\frac{T}{\rho} = K,\ \frac{\epsilon}{\rho} = \epsilon_1\right).
\end{aligned}$$

Hence, $x \in [N_k, \mathcal{M}, u, \mathfrak{M}_{(p,w^t,k_f)}, \Delta^r, \|\cdot, \cdots, \cdot\|]_\theta$. □

Acknowledgment. The authors extend their appreciation to the Deanship of Scientific Research at King Khalid University for funding this work through research groups program under grant number R.G.P. 1/13/38.

References

[1] M. Başarir, Ş. Konca and E. E. Kara, Some generalized difference statistically convergent sequence spaces in 2-normed space, J. Inequal. Appl., Vol. 2013, Article 177 (2013).

[2] C. Belen and S.A. Mohiuddine, Generalized weighted statistical convergence and application, Appl. Math. Comput., 219 (2013) 9821–9826.

[3] N. L. Braha and M. Et, The sequence space $E_n^q(M,p,s)$ and N_k-lacunary statistical convergence, Banach J. Math. Anal., 7 (2013) 88–96.

[4] R. Çolak, B. C. Tripathy and M. Et, Lacunary strongly summable sequences and q-lacunary almost statistical convergence, Vietnam J. Math., 34 (2006) 129–138.

[5] H. Dutta, On some n-normed linear space valued difference sequences, J. Franklin Inst., 348 (2011) 2876–2883.

[6] M. Et, Generalized Cesàro difference sequence spaces of non-absolute type involving lacunary sequences, Appl. Math. Comput., 219 (2013) 9372–9376.

[7] M. Et and R. Çolak, On generalized difference sequence spaces, Soochow J. Math., 21 (1995) 377-386.

[8] H. Fast, Sur la convergence statistique, Colloq. Math., 2 (1951) 241–244.

[9] A. R. Freedman, J. J. Sember and M. Raphael, Some Cesaro-type summability spaces, Proc. London Math. Soc., 37 (1978) 508–520.

[10] J. A. Fridy, On the statistical convergence, Analysis, 5 (1985) 301–303.

[11] J. A. Fridy and C. Orhan, Lacunary statistical convergence, Pacific J. Math., 160 (1993) 43–51.

[12] S. Gahler, Linear 2-normietre rume, Math. Nachr., 28 (1965) 1–43.

[13] H. Gunawan and M. Mashadi, On n-normed spaces, Int. J. Math. Math. Sci., 27 (2001) 631–639.

[14] H. Kızmaz, On certain sequence spaces, Canad. Math. Bull., 24 (1981) 169-176.

[15] J. Lindenstrauss and L. Tzafriri, On Orlicz sequence spaces, Israel J. Math., 10 (1971) 379-390.

[16] A. Misiak. n-inner product spaces, Math. Nachr., 140 (1989) 299–319.

[17] S. A. Mohiuddine and M. A. Alghamdi, Statistical summability through a lacunary sequence in locally solid Riesz spaces, J. Inequal. Appl., 2012, 2012:225.

[18] S. A. Mohiuddine and B. Hazarika, On strongly almost generalized difference lacunary ideal convergent sequences of fuzzy numbers, J. Comput. Anal Appl., 23(5) (2017) 925–936.

[19] S. A. Mohiuddine, K. Raj, A. Alotaibi, Generalized spaces of double sequences for Orlicz functions and bounded-regular matrices over n-normed spaces, J. Inequal. Appl., Vol. 2014, Article 332 (2014).

[20] M. Mursaleen and S. A. Mohiuddine, On lacunary statistical convergence with respect to the intuitionistic fuzzy normed space, J. Comput. Appl. Math., 233 (2009) 142–149.

[21] M. Mursaleen, Generalized spaces of difference sequences, J. Math. Anal Appl., 203 (1996) 738–745.

[22] J. Musielak, Orlicz spaces and modular spaces, vol. 1034 of Lecture Notes in Mathematics, Springer, Berlin, Germany, 1983.

[23] K. Raj and S. K. Sharma, Some sequence spaces in 2-normed spaces defined by Musielak-Orlicz function, Acta Univ. Sapientiae Math., 3 (2011) 97–109.

[24] T. Šalát, On statictical convergent sequences of real numbers, Math. Slovaca, 30 (1980) 139–150.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 27, NO.5, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC

OSCILLATION ANALYSIS FOR HIGHER ORDER DIFFERENCE EQUATION WITH NON-MONOTONE ARGUMENTS

ÖZKAN ÖCALAN AND UMUT MUTLU ÖZKAN

ABSTRACT. The aim of this paper is to obtain the some new oscillatory conditions for all solutions of higher order difference equation with general argument

$$(\star)\qquad \Delta^m x(n) + p(n)x(\tau(n)) = 0,\ n = 0, 1, \cdots,$$

where $(p(n))$ is a sequence of nonnegative real numbers and $(\tau(n))$ is a sequence of integers and non-monotone.

1. Introduction

Oscillation theory of difference equations has attracted many researchers. In recent years there has been much research activity concerning the oscillation and nonoscillation of solutions of delay difference equations. For these oscillatory and nonoscillatory results, we refer, for instance, $[1-22]$.

Consider the higher order difference equation with general argument

$$\Delta^m x(n) + p(n)x(\tau(n)) = 0,\ n = 0, 1, \cdots, \tag{1.1}$$

where $(p(n))_{n\geq 0}$ is a sequence of nonnegative real numbers and $(\tau(n))_{n\geq 0}$ is a sequence of integers such that

$$\tau(n) \leq n-1 \text{ for all } n \geq 0 \quad \text{and} \quad \lim_{n\to\infty} \tau(n) = \infty. \tag{1.2}$$

Δ denotes the forward difference operator $\Delta x(n) = x(n+1) - x(n)$.

Define

$$r = -\min_{n\geq 0} \tau(n). \quad \text{(Clearly, } k \text{ is a positive integer.)}$$

By a solution of the difference equation (1.1), we mean a sequence of real numbers $(x(n))_{n\geq -r}$ which satisfies (1.1) for all $n \geq 0$.

A solution $(x(n))_{n\geq -r}$ of the difference equation (1.1) is called oscillatory, if the terms $x(n)$ of the sequence are neither eventually positive nor eventually negative. Otherwise, the solution is said to be nonoscillatory.

If $m = 1$, then Eq.(1.1) take the form

$$\Delta x(n) + p(n)x(\tau(n)) = 0,\ n = 0, 1, \cdots. \tag{1.3}$$

In particular, if we take $\tau(n) = n - l$ where $l > 0$, then Eq.(1.3) reduces to

$$\Delta x(n) + p(n)x(n-l) = 0. \tag{1.4}$$

Key words and phrases. Delay difference equation, higher order, oscillation, non-monotone arguments.

Corresponding author : Özkan ÖCALAN; email address: ozkanocalan@akdeniz.edu.tr.

2000 *Math. Subject. Classification: 39A10.*

1

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 27, NO.5, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC

In 1989, Erbe and Zhang [8] proved that each one of the conditions

$$\liminf_{n\to\infty} p(n) > \frac{l^l}{(l+1)^{l+1}} \tag{1.5}$$

and

$$\limsup_{n\to\infty} \sum_{j=n-l}^{n} p(j) > 1 \tag{1.6}$$

is sufficient for all solutions of (1.4) to be oscillatory.

In the same year, 1989, Ladas, Philos and Sficas [11] established that all solutions of (1.4) are oscillatory if

$$\liminf_{n\to\infty} [\frac{1}{l} \sum_{j=n-l}^{n-1} p(j)] > \frac{l^l}{(l+1)^{l+1}}. \tag{1.7}$$

Clearly, condition (1.6) improves to (1.4).

In 1991, Philos [14] extended the oscillation criterion (1.7) to the general case of the Eq.(1.3), by establishing that, if the sequence $(\tau(n))_{n\geq 0}$ is increasing, then the condition

$$\liminf_{n\to\infty} \left[\frac{1}{n-\tau(n)} \sum_{j=\tau(n)}^{n-1} p(j) \right] > \limsup_{n\to\infty} \frac{(n-\tau(n))^{n-\tau(n)}}{(n-\tau(n)+1)^{n-\tau(n)+1}} \tag{1.8}$$

suffices for the oscillation of all solutions of Eq.(1.3).

In 1998, Zhang and Tian [19] obtained that if $(\tau(n))$ is non-decreasing,

$$\lim_{n\to\infty} (n-\tau(n)) = \infty \tag{1.9}$$

and

$$\liminf_{n\to\infty} \sum_{j=\tau(n)}^{n-1} p(j) > \frac{1}{e}, \tag{1.10}$$

then all solutions of Eq.(1.3) are oscillatory.

Later, in 1998, Zhang and Tian [20] obtained that if $(\tau(n))$ is non-decreasing or non-monotone,

$$\limsup_{n\to\infty} p(n) > 0 \tag{1.11}$$

and (1.10) holds, then all solutions of Eq.(1.3) are oscillatory.

In 2008, Chatzarakis, Koplatadze and Stavroulakis [3] proved that if $(\tau(n))$ is non-decreasing or non-monotone, $h(n) = \max_{0\leq s\leq n} \tau(s)$,

$$\limsup_{n\to\infty} \sum_{j=\tau(n)}^{n} p(j) > 1, \tag{1.12}$$

then all solutions of Eq.(1.3) are oscillatory.

In the same year, Chatzarakis, Koplatadze and Stavroulakis [4] proved that if $(\tau(n))$ is non-decreasing or non-monotone, $h(n) = \max_{0\leq s\leq n} \tau(s)$,

$$\limsup_{n\to\infty} \sum_{j=\tau(n)}^{n} p(j) < \infty \tag{1.13}$$

and (1.10) holds, then all solutions of Eq.(1.3) are oscillatory.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 27, NO.5, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC

In 2006, Yan, Meng and Yan [17] obtained that if $(\tau(n))$ is non-decreasing,

$$\liminf_{n\to\infty} \sum_{j=\tau(n)}^{n-1} p(j) > 0 \tag{1.14}$$

and

$$\liminf_{n\to\infty} \sum_{j=\tau(n)}^{n-1} p(j) \left(\frac{j-\tau(j)+1}{j-\tau(j)}\right)^{j-\tau(j)+1} > 1, \tag{1.15}$$

then all solutions of Eq.(1.3) are oscillatory.

Finally, in 2016, Öcalan [16] proved that if $(\tau(n))$ is non-decreasing or non-monotone, $h(n) = \max_{0\le s\le n}\tau(s)$ and (1.15) holds, then all solutions of Eq.(1.3) are oscillatory.

Set

$$k(n) = \left(\frac{n-\tau(n)+1}{n-\tau(n)}\right)^{n-\tau(n)+1}, \quad n \ge 1. \tag{1.16}$$

Clearly

$$e \le k(n) \le 4, \quad n \ge 1. \tag{1.17}$$

Observe that, it is easy to see that

$$\sum_{j=\tau(n)}^{n-1} p(j)k(j) \ge e \sum_{j=\tau(n)}^{n-1} p(j)$$

and therefore condition (1.15) is better than condition (1.10).

In 2006, Zhou [22] studied the following delay difference equation with constant delays

$$\Delta^m x(n) + \sum_{i=1}^{l} p_i(n)x(n-k_i) = 0, \; n = 0, 1, \cdots, \tag{1.18}$$

where $(p_i(n))_{n\ge 0}$ are sequences of nonnegative real numbers and k_i is a positive integer for $i = 1, 2, \cdots, l$. He obtained some new criteria for all solutions of Eq.(1.18) to be oscillatory.

2. Main Results

In this section we investigated the oscillatory behavior of all solutions of Eq.(1.1).

Further, we need the following lemmas proved in [1, 2].

Lemma 2.1. *(Discrete Kneser's Theorem) Let $x(n)$ be defined for $n \ge n_0$, and $x(n) > 0$ with $\Delta^m x(n)$ of constant sing for $n \ge n_0$ and not identically zero. Then, there exists an integer j, $0 \le j \le m$ with $(m+j)$ odd for $\Delta^m x(n) \le 0$ or $(m+j)$ even for $\Delta^m x(n) \ge 0$ and such that*

$$j \le m-1 \textit{ implies } (-1)^{j+i}\Delta^i x(n) > 0, \quad \textit{for all } n \ge n_0, \; j \le i \le m-1,$$

and

$$j \ge 1 \textit{ implies } \Delta^i x(n) > 0, \quad \textit{for all large } n \ge n_0, \; 1 \le i \le j-1.$$

Specially, if $\Delta^m x(n) \le 0$ for $n \ge n_0$, and $(x(n))$ is bounded, then

$$(-1)^{i+1}\Delta^{m-i} x(n) \ge 0, \; \textit{for all } n \ge n_0, \quad 1 \le i \le m-1,$$

and

$$\lim_{n\to\infty} \Delta^i x(n) = 0,\ 1 \le i \le m-1.$$

Lemma 2.2. *Let $x(n)$ be defined for $n \ge n_0$, and $x(n) > 0$ with $\Delta^m x(n) \le 0$ for $n \ge n_0$ and not identically zero. Then, there exists a large $n_1 \ge n_0$ such that*

$$x(n) \ge \frac{(n-n_1)^{m-1}}{(m-1)!}\Delta^{m-1}x(2^{m-j-1}n)\ ;\ \ n \ge n_1,$$

where j is defined in Lemma 2.1. Further, if $x(n)$ is increasing, then

$$x(n) \ge \frac{1}{(m-1)!}\left(\frac{n}{2^{m-1}}\right)^{m-1}\Delta^{m-1}x(n)\ ;\ \ n \ge 2^{m-1}n_1.$$

Set

$$h(n) = \max_{0\le s\le n} \tau(s) \tag{2.1}$$

Clearly, $h(n)$ is nondecreasing, and $\tau(n) \le h(n)$ for all $n \ge 0$. We note that if $\tau(n)$ is nondecreasing, then we have $\tau(n) = h(n)$ for all $n \ge 0$.

Theorem 2.3. *Assume that (1.2) holds. If $(\tau(n))$ is non-decreasing or non-monotone,*

$$\liminf_{n\to\infty} \sum_{j=\tau(n)}^{n-1} p(j)k(j) > (m-1)!, \tag{2.2}$$

where $k(n)$ is defined by (1.16), then every solution of Eq.(1.1) either oscillates or $\lim_{n\to\infty} x(n) = 0$.

Proof. Assume, for the sake of contradiction, that $(x(n))$ is an eventually positive solution of (1.1) and $\lim_{n\to\infty} x(n) > 0$. Then there exists $n_1 \ge n_0$ such that $x(n),\ x(\tau(n)),\ x(h(n)) > 0$, for all $n \ge n_1$. Thus, from Eq.(1.1) we have

$$\Delta^m x(n) = -p(n)x(\tau(n)) \le 0, \quad \text{for all } n \ge n_1. \tag{2.3}$$

By Lemma 2.1, $\Delta^i x(n)$ are eventually of one sign for every $i \in \{1, 2, \ldots, m-1\}$, and $\Delta^{m-1}x(n) > 0$ holds for large n, and there exist two cases to consider: (A) $\Delta x(n) > 0$ and (B) $\Delta x(n) < 0$.

Case A: This says that $(x(n))$ is increasing. By Lemma 2.2, there exists an integer $n_2 \ge n_1$ such that

$$x(n) \ge \frac{1}{(m-1)!}\left(\frac{n}{2^{m-1}}\right)^{m-1}\Delta^{m-1}x(n)\ ,\ \ n \ge n_2$$

and

$$x(\tau(n)) \ge \frac{1}{(m-1)!}\left(\frac{\tau(n)}{2^{m-1}}\right)^{m-1}\Delta^{m-1}x(\tau(n))\ ,\ \ n \ge n_2 \tag{2.4}$$

Letting $y(n) = \Delta^{m-1}x(n)$. So, we have

$$y(n) > 0,\ y(\tau(n)) > 0 \text{ for } n \ge n_2,$$

which implies that

$$\Delta y(n) + p(n)x(\tau(n)) = 0,\ n \ge n_2. \tag{2.5}$$

On the other hand, by (2.4) and since $\lim_{n\to\infty}\tau(n)=\infty$, there exists an integer $n_3 \geq n_2$ such that

$$\begin{aligned} x(\tau(n)) &\geq \frac{1}{(m-1)!}\left(\frac{\tau(n)}{2^{m-1}}\right)^{m-1} y\left(\tau(n)\right) \\ &\geq \frac{1}{(m-1)!} y\left(\tau(n)\right), \quad n \geq n_3. \end{aligned} \tag{2.6}$$

In view of (2.6), Eq.(2.5) gives

$$\Delta y(n) + \frac{1}{(m-1)!} p(n) y\left(\tau(n)\right) \leq 0, \quad n \geq n_3. \tag{2.7}$$

Taking into account that $y(n)$ is nonincreasing and $h(n)$ is nondecreasing, $\tau(n) \leq h(n)$ for all $n \geq 0$, from (2.7) we get

$$\Delta y(n) + \frac{1}{(m-1)!} p(n) y\left(h(n)\right) \leq 0, \quad n \geq n_3. \tag{2.8}$$

It follows that

$$\Delta y(n) + \widetilde{p}(n) y\left(h(n)\right) \leq 0, \quad n \geq n_3, \tag{2.9}$$

where $\widetilde{p}(n) = \frac{p(n)}{(m-1)!}$, which means that inequality (2.9) has an eventually positive solution.

On the other hand, we know from Lemma 2.3 in [16] that

$$\liminf_{n\to\infty} \sum_{j=\tau(n)}^{n-1} p(j)k(j) = \liminf_{n\to\infty} \sum_{j=h(n)}^{n-1} p(j)k(j), \tag{2.10}$$

where $h(n)$ is defined by (2.1).

Therefore, condition (2.2) and (2.10) imply that

$$\liminf_{n\to\infty} \sum_{j=h(n)}^{n-1} \widetilde{p}(j)k(j) = \frac{1}{(m-1)!} \liminf_{n\to\infty} \sum_{j=h(n)}^{n-1} p(j)k(j) > 1 \tag{2.11}$$

Thus, by Theorem 1 in [16], Eq.(2.9) has no eventually positive solution. This is a contradiction.

Case B: Note that, by Lemma 2.1, it is impossible that the case that m is even. In what follows, we only consider the case that m is odd. Case B says that $x(n)$ is decreasing and bounded, and so, $(x(n))$ converges a constant a. By Lemma 2.1, we get

$$(-1)^{i+1}\Delta^{m-i}x(n) > 0, \text{ for all large } n \geq n_1, \quad 1 \leq i \leq m-1, \tag{2.12}$$

and

$$\lim_{n\to\infty} \Delta^{m-1}x(n) = 0. \tag{2.13}$$

By (2.13), there exists an integer $n_4 \geq n_1$ such that

$$0 \leq \Delta^{m-1}x(n) \leq \varepsilon, \text{ for any } \varepsilon > 0, \ n \geq n_4. \tag{2.14}$$

It is obvious that $a > 0$. So, there exists an integer $n_5 \geq n_4$ such that

$$x(n) > \frac{1}{2}a, \ x(\tau(n)) > \frac{1}{2}a, \ n \geq n_5. \tag{2.15}$$

Thus, Eq.(1.1) implies that

$$\Delta^m x(n) + \frac{a}{2}p(n) \leq 0, \ n \geq n_5. \tag{2.16}$$

Summing both sides of (2.16) from n_5 to n, we obtain

$$\Delta^{m-1}x(n+1) - \Delta^{m-1}x(n_5) + \frac{a}{2}\sum_{s=n_5}^{n} p(s) \leq 0, \ n \geq n_5. \tag{2.17}$$

Letting $n \to \infty$, we have

$$\frac{a}{2}\sum_{s=n_5}^{n} p(s) \leq \varepsilon, \ \text{for large } n. \tag{2.18}$$

On the other hand, condition (2.2) says that there exist an integer $n_6 \geq n_5$ such that

$$\sum_{s=\tau(n)}^{n-1} p(s)k(s) > \frac{(m-1)!}{2}, \ n \geq n_6. \tag{2.19}$$

Since $k(n) \leq 4$ for $n \geq 1$, by (2.19) we get

$$\frac{a}{2}\sum_{s=\tau(n)}^{n-1} p(s) \geq \frac{a(m-1)!}{8}, \ \text{for large } n, \tag{2.20}$$

which contradicts (2.18) and (2.20). The proof is completed. □

Theorem 2.4. *Assume that m is even and (1.2) holds. If $(\tau(n))$ is non-decreasing or non-monotone,*

$$\liminf_{n\to\infty} \sum_{j=\tau(n)}^{n-1} \tau^{m-1}(j)p(j)k(j) > 2^{(m-1)^2}(m-1)!, \tag{2.21}$$

where $k(n)$ is defined by (1.16), then every solution of Eq.(1.1) oscillates.

Proof. Assume, for the sake of contradiction, that $(x(n))$ is an eventually positive solution of (1.1). Then there exists $n_1 \geq n_0$ such that $x(n), \ x(\tau(n)), \ x(h(n)) > 0$, for all $n \geq n_1$. According to the proof of Theorem 2.3, there exists a positive integer n_1 such that (2.3) holds. By Lemma 2.1, we have

$$\Delta x(n) > 0$$

which implies $x(n)$ is increasing. In view of proof of Theorem 2.3, we have

$$x(\tau(n)) \geq \frac{1}{(m-1)!}\left(\frac{\tau(n)}{2^{m-1}}\right)^{m-1} y(\tau(n)), \tag{2.22}$$

where $y(n) = \Delta^{m-1}x(n)$. Therefore, from Eq.(2.5) and (2.22), we obtain

$$\Delta y(n) + \frac{1}{(m-1)!}\left(\frac{\tau(n)}{2^{m-1}}\right)^{m-1} p(n)y(\tau(n)) \leq 0, \ n \geq n_2. \tag{2.23}$$

Taking into account that $y(n)$ is nonincreasing and $h(n)$ is nondecreasing, $\tau(n) \leq h(n)$ for all $n \geq 0$, from (2.23) we get,

$$\Delta y(n) + \frac{1}{(m-1)!}\left(\frac{\tau(n)}{2^{m-1}}\right)^{m-1} p(n)y(h(n)) \leq 0, \quad n \geq n_3. \tag{2.24}$$

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 27, NO.5, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC

It follows that

$$\Delta y(n) + \widetilde{p}(n) y\left(h(n)\right) \leq 0, \quad n \geq n_3, \tag{2.25}$$

where $\widetilde{p}(n) = \left(\frac{\tau(n)}{2^{m-1}}\right)^{m-1} \frac{p(n)}{(m-1)!}$, which means that inequality (2.25) has an eventually positive solution.

On the other hand, we know from Lemma 2.3 in [16] that

$$\liminf_{n\to\infty} \sum_{j=\tau(n)}^{n-1} p(j)k(j) = \liminf_{n\to\infty} \sum_{j=h(n)}^{n-1} p(j)k(j), \tag{2.26}$$

where $h(n)$ is defined by (2.1).

Therefore, condition (2.21) and (2.26) imply that

$$\liminf_{n\to\infty} \sum_{j=h(n)}^{n-1} \widetilde{p}(j)k(j) = \frac{1}{(m-1)!}\frac{1}{2^{(m-1)^2}} \liminf_{n\to\infty} \sum_{j=h(n)}^{n-1} \tau^{m-1}(j)p(j)k(j) > 1 \tag{2.27}$$

Thus, by Theorem 1 in [16], Eq.(2.25) has no eventually positive solution. This contradiction completes the proof. □

Now, using (1.16), (1.17), Theorem 2.3 and Theorem 1 in [16], we have the following results immediately.

Corollary 2.5. *Assume that* (1.2) *holds. If* $(\tau(n))$ *is non-decreasing or non-monotone,*

$$\liminf_{n\to\infty} \sum_{j=\tau(n)}^{n-1} p(j) > \frac{1}{e}(m-1)!, \tag{2.28}$$

then every solution of Eq.(1.1) *either oscillates or* $\lim_{n\to\infty} x(n) = 0$.

Corollary 2.6. *Assume that* m *is even and* (1.2) *holds. If* $(\tau(n))$ *is non-decreasing or non-monotone,*

$$\liminf_{n\to\infty} \sum_{j=\tau(n)}^{n-1} \tau^{m-1}(j)p(j) > \frac{2^{(m-1)^2}}{e}(m-1)!, \tag{2.29}$$

then every solution of Eq.(1.1) *oscillates.*

Finally, using the proofs of Theorem 2.3 and Theorem 2.4, and from the Theorem 2.1 in [3], we obtain the following results by removing the proofs.

Theorem 2.7. *Assume that* (1.2) *and* (1.14) *hold. If* $(\tau(n))$ *is non-decreasing or non-monotone,*

$$\limsup_{n\to\infty} \sum_{j=h(n)}^{n} p(j) > (m-1)!, \tag{2.30}$$

where $h(n)$ *is defined by* (2.1), *then every solution of Eq.*(1.1) *either oscillates or* $\lim_{n\to\infty} x(n) = 0$.

Theorem 2.8. *Assume that* m *is even and* (1.2) *holds. If* $(\tau(n))$ *is non-decreasing or non-monotone,*

$$\limsup_{n\to\infty}\sum_{j=h(n)}^{n}\tau^{m-1}(j)p(j) > 2^{(m-1)^2}(m-1)!, \tag{2.31}$$

where $h(n)$ *is defined by* (2.1)*, then every solution of Eq.*(1.1) *oscillates.*

we present an example to show the significance of our new result.

Example 2.1. *Consider the retarded difference equation*

$$\Delta^3 x(n) + \frac{3}{e}x(\tau(n)) = 0, \qquad n \geq 0, \tag{2.32}$$

with

$$\tau(n) = \begin{cases} n-3, & \text{if } n \text{ is even} \\ n-1, & \text{if } n \text{ is odd} \end{cases}.$$

Here, it is clear that (1.2) *is satisfied. By* (2.1)*, we see that*

$$h(n) = \max_{0\leq s\leq n}\tau(s) = \begin{cases} n-2, & \text{if } n \text{ is even} \\ n-1, & \text{if } n \text{ is odd} \end{cases}.$$

Computing, we get

$$\sum_{j=\tau(n)}^{n-1}p(j) = \begin{cases} 6/e, & \text{if } n \text{ is even} \\ 3/e, & \text{if } n \text{ is odd} \end{cases}.$$

Thus

$$\liminf_{n\to\infty}\sum_{j=\tau(n)}^{n-1}p(j) = \frac{3}{e} > \frac{1}{e}(m-1)! = \frac{2}{e},$$

that is, condition (2.28) *of Corollary* 2.5 *is satisfied and therefore every solution of Eq.*(2.32) *either oscillates or* $\lim_{n\to\infty}x(n)=0$.

References

[1] R. P. Agarwal, Difference Equations and Inequalities, Marcel Dekker, New York, 2000.

[2] R. P. Agarwal, S.R. Grace and D. O'Regan, Oscillation Theory for Difference and Functional Differential Equations, Kluwer Academic Publishers, The Netherlands, 2000.

[3] G. E. Chatzarakis, R. Koplatadze, and I. P. Stavroulakis, Oscillation criteria of first order linear difference equations with delay argument, Nonlinear Anal., 68 (2008), 994–1005.

[4] G. E. Chatzarakis, R. Koplatadze, and I. P. Stavroulakis, Optimal oscillation criteria for first order difference equations with delay argument, Pacific J. Math., 235 (2008), 15–33.

[5] G. E. Chatzarakis, Ch. G. Philos, and I. P. Stavroulakis, On the oscillation of the solutions to linear difference equations with variable delay, Electron. J. Differential Equations, No. 50 (2008), 15 pp.

[6] G. E. Chatzarakis, Ch. G. Philos, and I. P. Stavroulakis, An oscillation criterion for linear difference equations with general delay argument, Portugal. Math., (N.S.), 66 (4) (2009), 513–533.

[7] M.-P. Chen and J. S. Yu, Oscillations of delay difference equations with variable coefficients, In Proceedings of the First International Conference on Difference Equations, Gordon and Breach, London 1994, 105–114.

[8] L. H. Erbe and B. G. Zhang, Oscillation of discrete analogues of delay equations, Differential Integral Equations, 2 (1989), 300–309.

[9] I. Györi and G. Ladas, Linearized oscillations for equations with piecewise constant arguments, Differential Integral Equations, 2 (1989), 123–131.

[10] I. Györi and G. Ladas, Oscillation Theory of Delay Differential Equations with Applications, Clarendon Press, Oxford, 1991.

[11] G. Ladas, Ch. G. Philos, and Y. G. Sficas, Sharp conditions for the oscillation of delay difference equations, J. Appl. Math. Simulation, 2 (1989), 101–111.
[12] G. Ladas, Explicit conditions for the oscillation of difference equations, J. Math. Anal. Appl., 153 (1990), 276–287.
[13] G. S. Ladde, V. Lakshmikantham and B. G. Zhang, Oscillation theory of differential equations with deviating arguments, Marcel Dekker, New York, (1987).
[14] Ch. G. Philos, On oscillations of some difference equations, Funkcial. Ekvac., 34 (1991), 157–172.
[15] Ö. Öcalan and S. Ş. Öztürk, An Oscillation Criterion for First Order Difference Equations, Results Math., 68 (2015), no. 1-2, 105–116.
[16] Ö. Öcalan, An improved oscillation criterion for first order difference equations, Bull. Math. Soc. Sci. Math. Roumanie (N.S.), 59(107) (2016), no. 1, 65–73.
[17] W. Yan, Q. Meng and J. Yan, Oscillation criteria for difference equation of variable delays, Dyn. Contin. Discrete Impuls. Syst. Ser. A Math. Anal., 13A (2006), Part 2, suppl., 641–647.
[18] J. S. Yu and Z. C. Wang, Some further results on oscillation of neutral differential equations, Bull. Austral. Math. Soc., 46 (1992) 149–157.
[19] B.G. Zhang and C. J. Tian, Oscillation criteria for difference equations with unbounded delay, Comput. Math. Appl., 35 (4), (1998), 19–26.
[20] B.G. Zhang and C. J. Tian, Nonexistence and existence of positive solutions for difference equations with unbounded delay. Comput. Math. Appl., 36, (1998), 1–8.
[21] B. G. Zhang, X. Z. Yan and X. Y. Liu, Oscillation criteria of certain delay dynamic equations on time scales, J. Difference Equ. Appl., 11(10) (2005), 933–946.
[22] Y. Zhou, Oscillation of higher-order delay difference equations, Adv. Difference Equ., Art. ID 65789 (2006), 7 pp.

Akdeniz University, Faculty of Science, Department of Mathematics, 07058, Antalya, Turkey

E-mail address: ozkanocalan@akdeniz.edu.tr

Afyon Kocatepe University, Faculty of Science and Arts, Department of Mathematics, ANS Campus, 03200, Afyon, Turkey

E-mail address: umut_ozkan@aku.edu.tr

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 27, NO.5, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC

On Orthonormal Wavelet Bases

Richard A. Zalik *

Abstract

Given a multiresolution analysis with one generator in $L^2(\mathbb{R}^d)$, we give a characterization in closed form and in the frequency domain, of all orthonormal wavelets associated to this MRA. Examples are given. This theorem corrects a previous result of the author.

1 Introduction

In what follows $\mathbb{Z}$ will denote the set of integers, and $\mathbb{R}$ the set of real numbers. We will always assume that $\mathbf{A}$ is a dilation matrix preserving the lattice $\mathbb{Z}^d$; that is, $\mathbf{A}\mathbb{Z}^d \subset \mathbb{Z}^d$ and all its eigenvalues have modulus greater than 1; $\mathbf{A}^*$ will denote the transpose of $\mathbf{A}$ and $\mathbf{B} := (A^*)^{-1}$. The underlying space will be $L^2(\mathbb{R}^d)$, where $d \geq 1$ is an integer and $\mathbf{I}$ will stand for the identity matrix. Boldface lowcase letters will denote elements of $\mathbb{R}^d$, which will be represented as column vectors; $\mathbf{x} \cdot \mathbf{y}$ will stand for the standard dot product of the vectors $\mathbf{x}$ and $\mathbf{y}$; $||\mathbf{x}||^2 := \mathbf{x} \cdot \mathbf{x}$.

Let $\mathbf{A} \in \mathbb{R}^{d\times d}$ and $a := |\det \mathbf{A}|$. For every $j \in \mathbb{Z}$ and $\mathbf{k} \in \mathbb{Z}^d$ the dilation operator $D^{\mathbf{A}}$ and the translation operator $T_{\mathbf{k}}$ are defined on $L^2(\mathbb{R}^d)$ by

$$D^{\mathbf{A}} f(\mathbf{t}) := a^{1/2} f(\mathbf{A}\mathbf{t}) \qquad \text{and} \qquad T_{\mathbf{k}} f(\mathbf{t}) := f(\mathbf{t} + \mathbf{k})$$

respectively.

Let $\mathbf{u} = \{u_1, \ldots, u_m\} \subset L^2(\mathbb{R}^d)$; then $T(u_1, \ldots, u_m) = T(\mathbf{u})$, $S(u_1, \ldots, u_m) = S(\mathbf{u})$ and $S(\mathbf{A}; u_1, \ldots, u_m) = S(\mathbf{A}; \mathbf{u})$ are respectively defined by

$$T(\mathbf{u}) := \{T_{\mathbf{k}} u; u \in \mathbf{u}, \mathbf{k} \in \mathbb{Z}^d\}, \qquad S(\mathbf{u}) := \overline{\operatorname{span}}\, T(\mathbf{u}),$$

and

$$S(\mathbf{A}, \mathbf{u}) := \overline{\operatorname{span}}\, \{D^{\mathbf{A}} T_{\mathbf{k}} u; u \in \mathbf{u}, \mathbf{k} \in \mathbb{Z}^d\}.$$

In [5] we formulated a representation theorem for multiresolution analyses having an arbitrary set $u_1, \ldots, u_n$ of scaling functions, i.e., the set of translates of all these functions constitutes an orthonormal basis of V_0. However the proof was based on the implicit (and incorrect) assumption that any such function u_ℓ is contained in $S(\mathbf{A}, u_\ell)$, and it is therefore not valid. The purpose of this paper is to apply the method of proof

*Department of Mathematics and Statistics, Auburn University, AL 36849-5310, zalik@auburn.edu

employed in [5] to prove a representation theorem for MRA's having a single scaling function, and to provide some examples.

A function f will be called $\mathbb{Z}^d$*–periodic* if it is defined on $\mathbb{R}^d$ and $T_{\mathbf{k}}f = f$ for every $\mathbf{k} \in \mathbb{Z}^d$.

The Fourier transform of a function f will be denoted by $\widehat{f}$ or $\mathfrak{F}(f)$. If $f \in L(\mathbb{R}^d)$,

$$\widehat{f}(\mathbf{x}) := \int_{\mathbb{R}^d} e^{-i2\pi\mathbf{x}\cdot\mathbf{t}} f(\mathbf{t})\, d\mathbf{t}.$$

The Fourier transform is extended to $L^2(\mathbb{R}^d)$ in the usual way.

Our starting point and motivation is the following well known characterization in Fourier space of affine MRA orthonormal wavelets in $L^2(\mathbb{R})$ (see e.g. Hernández and Weiss [2], Wojtaszczyk [4]) which, with the definition of Fourier transform we have adopted, may be stated as follows.

Theorem A. *Let φ be a scaling function for a multiresolution analysis M with associated low pass filter p. The following propositions are equivalent:*
(a) ψ is an MRA orthonormal wavelet associated with M.
(b) There is a measurable unimodular $\mathbb{Z}$–periodic function $\mu(x)$ such that

$$\widehat{\psi}(2x) = e^{i2\pi x}\mu(2x)\overline{p(x+1/2)}\widehat{\varphi}(x) \qquad a.e.$$

Recall that a *multiresolution analysis* (MRA) in $L^2(\mathbb{R}^d)$ (generated by $\mathbf{A}$) is a sequence $\{V_j; j \in \mathbb{Z}\}$ of closed linear subspaces of $L^2(\mathbb{R}^d)$ such that:

(i) $V_j \subset V_{j+1}$ for every $j \in \mathbb{Z}$.

(ii) For every $j \in \mathbb{Z}, f(\mathbf{t}) \in V_j$ if and only if $f(\mathbf{At}) \in V_{j+1}$.

(iii) $\bigcup_{j\in\mathbb{Z}} V_j$ is dense in $L^2(\mathbb{R}^d)$.

(iv) $\bigcap_{j\in\mathbb{Z}} V_j = \emptyset$.

(v) There is a function u (called the *scaling function* of the MRA) such that $T(u)$ is an orthonormal basis of V_0.

A finite set of functions $\boldsymbol{\psi} = \{\psi_1, \cdots, \psi_m\} \in L^2(\mathbb{R}^d)$ is called an orthonormal wavelet system if the affine sequence

$$\{D_j^{\mathbf{A}} T_{\mathbf{k}}\psi_\ell; j \in \mathbb{Z}, \mathbf{k} \in \mathbb{Z}^d, \ell = 1, \cdots, m\}$$

is an orthonormal basis of $L^2(\mathbb{R}^d)$.

Let $\boldsymbol{\psi} := \{\psi_1, \cdots, \psi_m\}$ be an orthonormal wavelet system in $L^2(\mathbb{R}^d)$ generated by a matrix $\mathbf{A}$; for $j \in \mathbb{Z}$ we define

$$V_j = \sum_{r<j} S(\mathbf{A}^r; \boldsymbol{\psi}).$$

We say that $\boldsymbol{\psi}$ is *associated* with an MRA, if $M := \{V_j; j \in \mathbb{Z}\}$ is a multiresolution analysis. If this is the case, we also say that $\boldsymbol{\psi}$ is associated with M. Let W_j denote the orthogonal complement of V_j in V_{j+1}. Then it is esily seen that $\boldsymbol{\psi}$ is an orthonormal wavelet associated with M if and only if $T(\boldsymbol{\psi})$ is an orthonormal basis of W_0.

Let $\mathbf{e} := (1, 0, \cdots, 0)^T \in \mathbb{R}^m$ and let $\mathrm{diag}\,\{-\mathrm{e}^{\mathrm{i}\omega}, 1, \cdots, 1\}_{\mathrm{m}}$ denote the $m \times m$ diagonal matrix with $-e^{i\omega}, 1, \cdots, 1$ as its diagonal entries. The following proposition was implicitly established by Jia and Shen in the discussion that follows the proof of [3, Lemma 3.3] (we adopt the convention that $\mathrm{Arg}\ 0 = 0$).

Theorem B. *Let* $\mathbf{b} = (b_1, \cdots, b_m)^T \in \mathbb{C}^m$ *be unimodular,* $\omega := Arg\, b_1$ *and* $\mathbf{q} := \mathbf{b} + e^{i\omega}\mathbf{e}$. *Then the matrix*

$$\mathbf{Q} = (q_{r,k})_{r,k=1}^m := \mathrm{diag}\,\{-\mathrm{e}^{\mathrm{i}\omega}, 1, \cdots, 1\}_{\mathrm{m}} \left[\overline{\mathbf{I} - 2\mathbf{q}\mathbf{q}^*/\mathbf{q}^*\mathbf{q}}\right]$$

is unitary. Moreover

$$q_{r,k} = \begin{cases} b_k & \text{if } r = 1, 1 \le k \le m \\ -\overline{b_r} e^{i\omega} & \text{if } 1 < r \le m, k = 1 \\ \delta_{r,k} - \dfrac{\overline{b_r} b_k}{1 + |b_1|} & \text{if } 1 < r \le m, 1 < k \le m, \end{cases}$$

where $\delta_{r,k}$ *is Krönecker's delta.*

The following proposition is a particular case of [5, Theorem 3].

Lemma C. *Let* $u \in L^2(\mathbb{R}^d)$ and assume that $T(u)$ is an orthonormal sequence. Let $\mathbf{A}$ be a dilation matrix preserving the lattice $\mathbb{Z}^d$, let $\{j_1, \ldots, j_a\}$ be a full collection of representatives of $\mathbb{Z}^d/\mathbf{A}\mathbb{Z}^d$, and let

$$v_k(\mathbf{t}) := a^{1/2} u(\mathbf{A}t + j_k), \quad k = 1, \ldots a. \tag{1}$$

Then $T(v_1, \cdots, v_a)$ is an orthonormal basis of $S(\mathbf{A}; u)$.

Since $\widehat{v}_k(\mathbf{x}) = e^{i2\pi \mathbf{Bx} \cdot j_k} \widehat{u}(\mathbf{Bx})$, a straightforward consequence of Lemma C and [5, Lemma E] is the following

Corollary 1. *Let* $u \in L^2(\mathbb{R}^d)$ *and assume that* $T(u)$ *is an orthonormal sequence. Let* $\mathbf{A}$ *be a dilation matrix preserving the lattice* $\mathbb{Z}^d$, $B := (A^*)^{-1}$, *let* $\{j_1, \ldots, j_a\}$ *be a full collection of representatives of* $\mathbb{Z}^d/\mathbf{A}\mathbb{Z}^d$, *and let* $v_k(\mathbf{t})$ *be defined by* (1). *If* $u \in S(A, u)$, *then there are* $\mathbb{Z}^d$*–periodic functions* $q_k \in L^2(\mathbb{T}^d)$ *such that*

$$\sum_{k=1}^{a} |q_k(\mathbf{x})|^2 = 1 \quad a.e., \tag{2}$$

and

$$\widehat{u}(\mathbf{x}) = \sum_{k=1}^{a} q_k(\mathbf{x}) \widehat{v}_k(\mathbf{x}) = \sum_{k=1}^{a} q_k(\mathbf{x}) e^{i2\pi \mathbf{Bx} \cdot j_k} \widehat{u}(\mathbf{Bx}) = p(B\mathbf{x}) \widehat{u}(B\mathbf{x}), \tag{3}$$

where

$$p(\mathbf{x}) := a^{-1/2} \sum_{k=1}^{a} q_k(A^* \mathbf{x}) e^{i2\pi \mathbf{x}\cdot j_k}.$$

We can now prove

Theorem 1. *Let M be a multiresolution analysis generated by $\mathbf{A}$ with scaling function u, let $v_k(\mathbf{t})$ be defined by (1), $B := (A^*)^{-1}$, and let the functions $q_k(\mathbf{x})$ be $\mathbb{Z}^d$–periodic, in $L^2(\mathbb{T}^d)$, and satisfy (2) and (3). Let*

$$\alpha(\mathbf{x}) := Arg\, q_1(\mathbf{x}), \tag{4}$$

$$w_{r,k}(\mathbf{x}) := \begin{cases} q_k(\mathbf{x}) & \text{if } r = 1, 1 \le k \le a \\ \\ -\overline{q_r(\mathbf{x})} e^{i\alpha(\mathbf{x})} & \text{if } 1 < r \le a, k = 1 \\ \\ \delta_{r,k} - \dfrac{\overline{q_r(\mathbf{x})} q_k(\mathbf{x})}{1 + |q_1(\mathbf{x})|} & \text{if } 1 < r \le a, 1 < k \le a \end{cases} \tag{5}$$

and

$$\widehat{z}_r(\mathbf{x}) := \sum_{k=1}^{a} w_{r,k}(\mathbf{x}) \widehat{v}_k(\mathbf{x}),$$

and let

$$\mathbf{Z}(\mathbf{x}) := (\widehat{z}_2(\mathbf{x}), \ldots, \widehat{z}_a(\mathbf{x}))^T.$$

Then

$$\{\psi_1, \ldots, \psi_{(a-1)}\}$$

is an orthonormal wavelet system associated with M if and only if there exists an $(a-1) \times (a-1)$ unitary matrix function $\mathbf{U}(x)$ such that

$$(\widehat{\psi}_1(\mathbf{x}), \ldots, \widehat{\psi}_{(a-1)}(\mathbf{x}))^T = \mathbf{U}(\mathbf{x}) \mathbf{Z}(\mathbf{x}).$$

Proof. The existence of functions $q_k(\mathbf{x})$ satisfying (2) and (3) is a consequence of Corollary 1. Setting

$$\widehat{\mathbf{v}}(\mathbf{x}) := (\widehat{v}_1(\mathbf{x}), \cdots, \widehat{v}_a(\mathbf{x}))^T$$

and applying Theorem B, we see that

$$(\widehat{z}_1(\mathbf{x}), \cdots \widehat{z}_a(\mathbf{x}))^T = \mathbf{Q}(\mathbf{x}) \widehat{\mathbf{v}}(\mathbf{x}),$$

and that $\mathbf{Q}(\mathbf{x})$ has $(q_1(\mathbf{x}), \cdots, q_a(\mathbf{x}))$ as its first row. Therefore [5, Theorem 8] implies that $\{z_2, \ldots z_a\}$ is an orthonormal wavelet system associated with M, which is equivalent to saying that $S(z_2, \ldots z_a)$ is an orthonormal basis generator of W_0. Applying now [5, Theorem 5], the assertion follows. □

Example 1. Let us verify that Theorem A is a particular case of Theorem 1.

For $d = 1$ and $A = 2$ we have $j_1 = 0$ and $j_2 = 1$, and Corollary 1 implies that

$$p(x) = 2^{-1/2}[q_1(2x) + e^{i2\pi x} q_2(2x)],$$

whence the periodicity of $q_1(x)$ and $q_2(x)$ implies that

$$p(x + 1/2) = 2^{-1/2}[q_1(2x) - e^{i2\pi x} q_2(2x)].$$

On the other hand, since $|q_1(x)|^2+|q_2(x)|^2 = 1$ a.e., (5) implies that $w_{2,1}(x) = -\overline{q_2(x)}e^{i\alpha(x)}$ and

$$w_{2,2}(x) = 1 - \frac{|q_2(x)|^2}{1 + |q_1(x)|} = 1 - \frac{|q_2(x)|^2(1 - |q_1(x)|)}{|q_2(x)|^2} = |q_1(x)|.$$

Since $\mathbf{B} = 1/2$, it follows that $\widehat{v}_1(x) = 2^{-1/2}\widehat{u}(x/2)$ and $\widehat{v}_2(x) = 2^{-1/2}e^{-i\pi x}\widehat{u}(x/2)$, and Theorem 1 implies that

$$\begin{aligned}\widehat{z}_2(x) = 2^{-1/2}[-e^{i\alpha(x)}\overline{q_2(x)} + e^{i\pi x}|q_1(x)|]\widehat{u}(x/2) = \\ 2^{-1/2}e^{i\pi x}e^{i\alpha(x)}[-\overline{q_2(x)}e^{-i\pi x} + e^{-i\alpha(x)}|q_1(x)|]\widehat{u}(x/2) = \\ 2^{-1/2}e^{-i\pi x}e^{i\alpha(x)}[\overline{q_1(x)} - e^{i\pi x}\overline{q_2(x)}]\widehat{u}(x/2),\end{aligned}$$

and therefore

$$\widehat{z}_2(2x) = 2^{-1/2}e^{-i2\pi x}e^{i\alpha(2x)}[\overline{q_1(2x)} - e^{i2\pi x}\overline{q_2(2x)}]\widehat{u}(x) = e^{-i2\pi x}\mu(2x)\overline{p(x + 1/2)}\widehat{u}(x),$$

where $\mu(x) := e^{i\alpha(x)}$ is unimodular and $\mathbb{Z}$-periodic.

Example 2. Let

$$\mathbf{A} := \begin{pmatrix} 0 & 2 \\ -1 & 0 \end{pmatrix}$$

and let $\phi(\mathbf{t})$ be the characteristic function of $[0, 1] \times [0, 1]$. Gröchenig and Madych [1] have shown that ϕ is a scaling function of an MRA generated by the dilation matrix $\mathbf{A}$ and that the function ψ defined by

$$\psi(\mathbf{t}) := \begin{cases} 1 & \text{if } \mathbf{t} \in [0, 1] \times [0, 1/2] \\ -1 & \text{if } \mathbf{t} \in [0, 1] \times [1/2, 1] \\ 0 & \text{otherwise} \end{cases}$$

is a wavelet associated with this MRA. Let us see how this assertion follows from Theorem 1.

Since $\{(0, 0)^T, (1, 0)^T\}$ is a a full collection of representatives of $A/A\mathbb{Z}^2$, from Lemma C we deduce that if $v_1(\mathbf{t}) := 2^{-1/2}\phi(A\mathbf{t})$ and $v_2(\mathbf{t}) := 2^{-1/2}\phi(A\mathbf{t}+(1, 0)^T)$, then $T(v_1, v_2)$ is an orthonormal basis of $S(A, \phi)$, and a straightforward computation shows that

$$\phi(\mathbf{t}) = 2^{-1/2}\left(v_1(\mathbf{t} - (1, 0)^T) + v_2(\mathbf{t} - (1, 1)^T\right),$$

which implies that if $\mathbf{x} = (x_1, x_2)^T$, then

$$\widehat{\phi}(\mathbf{x}) = 2^{-1/2}\left(e^{-i2\pi x_1}\widehat{v}_1(\mathbf{x}) + e^{-i2\pi(x_1+x_2)}\widehat{v}_2(\mathbf{x})\right).$$

Thus $q_1(\mathbf{x}) = 2^{-1/2}e^{-i2\pi x_1}$, $q_2(\mathbf{x}) = 2^{-1/2}e^{-i2\pi(x_1+x_2)}$ and $\alpha(\mathbf{x}) = i2\pi x_1$, and proceeding as in Example 1 we see that

$$w_{2,1}(\mathbf{x}) = -\overline{q_2(x)}e^{i\alpha(x)} = 2^{-1/2}e^{-i2\pi x_2} \quad \text{and} \quad w_{2,2}(\mathbf{x}) = |q_1(\mathbf{x})| = 2^{-1/2}.$$

Thus,

$$\widehat{z_2}(\mathbf{x}) = w_{2,1}(\mathbf{x})\widehat{v}_1(\mathbf{x}) + w_{2,2}(\mathbf{x})\widehat{v}_2(\mathbf{x}) = 2^{-1/2}\left(\widehat{v}_2(\mathbf{x}) - e^{-i2\pi x_2}\widehat{v}_1(\mathbf{x})\right),$$

which by Theorem 1 implies that $\sigma(\mathbf{t})$ is a wavelet associated with A if and only if there is a measurable unimodular $\mathbb{Z}^2$–periodic function $\mu(\mathbf{x})$ such that

$$\widehat{\sigma}(\mathbf{x}) = \mu(\mathbf{x})\widehat{z_2}(\mathbf{x}).$$

In particular, $\widehat{\psi}(\mathbf{x}) = e^{-i2\pi x_1}\widehat{z_2}(\mathbf{x})$.

Example 3. Gröchenig and Madych have also shown in [1] that the characteristic function ϕ of $[0,1]\times[0,1]$ which we considered in the previous example is also a scaling function of an MRA generated by the dilation matrix

$$\mathbf{A} := 2I = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}.$$

Since $a = 4$, from e.g. [5, Theorem H] we know that any orthonormal wavelet associated with this MRA has exactly three generators.. Let us construct an orthonormal wavelet basis using Theorem 1. The vectors $j_1 := (0,0)^T, j_2 := (1,0)^T, j_3 := (0,1)^T$ and $j_4 := (1,1)^T$ are a full collection of representatives of $A/A\mathbb{Z}^2$. Let

$$v_k(\mathbf{t}) := 2\phi(A\mathbf{t} + j_k) = 2\phi(2\mathbf{t} + j_k).$$

Lemma C implies that $T(v_1, v_2, v_3, v_4)$ is an orthonormal basis of $S(A, \phi)$. Moreover, it is easily verified that

$$\phi(\mathbf{t}) = \sum_{k=1}^{4}\phi(2\mathbf{t} - j_k) = (1/2)\sum_{k=1}^{4} v_k(\mathbf{t} - j_k).$$

Since

$$\mathfrak{F}\{v_k(\cdot - j_k)\}(\mathbf{x}) = e^{-i2\pi\mathbf{x}\cdot j_k}\widehat{v_k}(\mathbf{x})$$

we see that

$$\widehat{\phi}(\mathbf{x}) = (1/2)\sum_{k=1}^{4} e^{-i2\pi\mathbf{x}\cdot j_k}\widehat{v}_k(\mathbf{x}),$$

and therefore

$$q_k(\mathbf{x}) = (1/2)e^{-i2\pi\mathbf{x}\cdot j_k}, k = 1, \ldots 4.$$

Since $\alpha(\mathbf{x}) = 0$, (5) implies that

$$w_{r,k}(\mathbf{x}) := \begin{cases} \frac{1}{2}e^{-i2\pi\mathbf{x}\cdot j_k} & \text{if } r = 1, 1 \le k \le 4 \\ -\frac{1}{2}e^{i2\pi\mathbf{x}\cdot j_r} & \text{if } 1 < r \le 4, k = 1 \\ -\frac{1}{6}e^{i2\pi\mathbf{x}\cdot(j_k - j_r)} & \text{if } 1 < r \le 4, 1 < k \le 4, k \ne r. \\ \frac{5}{6} & \text{if } 1 < r \le 4, 1 < k \le 4, k = r. \end{cases}$$

Thus,

$$\widehat{z}_2(\mathbf{x}) = -\frac{1}{2}e^{i2\pi\mathbf{x}\cdot j_2}\widehat{v}_1(\mathbf{x}) + \frac{5}{6}\widehat{v}_2(\mathbf{x}) - \frac{1}{6}e^{i2\pi\mathbf{x}\cdot(j_3-j_2)}\widehat{v}_3(\mathbf{x}) - \frac{1}{6}e^{i2\pi\mathbf{x}\cdot(j_4-j_2)}\widehat{v}_4(\mathbf{x})$$

$$\widehat{z}_3(\mathbf{x}) = -\frac{1}{2}e^{i2\pi\mathbf{x}\cdot j_3}\widehat{v}_1(\mathbf{x}) - \frac{1}{6}e^{i2\pi\mathbf{x}\cdot(j_2-j_3)}\widehat{v}_2(\mathbf{x}) + \frac{5}{6}\widehat{v}_3(\mathbf{x}) - \frac{1}{6}e^{i2\pi\mathbf{x}\cdot(j_4-j_3)}\widehat{v}_4(\mathbf{x})$$

and

$$\widehat{z}_4(\mathbf{x}) = -\frac{1}{2}e^{i2\pi\mathbf{x}\cdot j_4}\widehat{v}_1(\mathbf{x}) - \frac{1}{6}e^{i2\pi\mathbf{x}\cdot(j_2-j_4)}\widehat{v}_2(\mathbf{x}) - \frac{1}{6}e^{i2\pi\mathbf{x}\cdot(j_3-j_4)}\widehat{v}_3(\mathbf{x}) + \frac{5}{6}\widehat{v}_4(\mathbf{x}).$$

i.e.,

$$z_2(t) = -\frac{1}{2}v_1(\mathbf{t} + j_2) + \frac{5}{6}v_2(\mathbf{t}) - \frac{1}{6}v_3(\mathbf{t} + (j_3 - j_2)) - \frac{1}{6}v_4(\mathbf{t} + (j_4 - j_2)),$$

$$z_3(\mathbf{t}) = -\frac{1}{2}v_1(\mathbf{t} + j_3) - \frac{1}{6}v_2(\mathbf{t} + (j_2 - j_3)) + \frac{5}{6}v_3(\mathbf{t}) - \frac{1}{6}v_4(\mathbf{t} + (j_4 - j_3)),$$

and

$$z_4(\mathbf{t}) = -\frac{1}{2}v_1(\mathbf{t} + j_4) - \frac{1}{6}v_2(\mathbf{t} + (j_2 - j_4)) - \frac{1}{6}v_3(\mathbf{t} + (j_3 - j_4)) + \frac{5}{6}v_4(\mathbf{t}).$$

Applying Theorem 1 we conclude that $\{z_2, z_3, z_4\}$ is an orthonormal wavelet system associated with the dilation matrix $\mathbf{A}$, and that $\{\psi_1, \psi_2, \psi_3\}$ is an orthonormal wavelet system associated with $\mathbf{A}$ if and only if there exists a 3×3 unitary matrix function $\mathbf{U}(x)$ such that

$$(\widehat{\psi}_1(\mathbf{x}), \widehat{\psi}_2(\mathbf{x}), \widehat{\psi}_3(\mathbf{x}))^T = \mathbf{U}(\mathbf{x})(\widehat{z}_2(\mathbf{x}), \widehat{z}_3(\mathbf{x}), \widehat{z}_4(\mathbf{x}))^T.$$

References

[1] K. Gröchenig and W. R. Madych, Multiresolution Analysis, Haar Bases, and Self–similar Tilings of R^n, IEEE Trans. Information Theory 38, No.2. March 1992, 556–568.

[2] E. Hernández and G. Weiss, A First Course on Wavelets, CRC Press, Boca Raton, FL, 1996.

[3] R–Q. Jia and Z. Shen, Multiresolution and wavelets, Proc. Edinburgh Mat. Soc. 37 (1994) 271–300.

[4] P. Wojtaszczyk, A Mathematical Introduction to Wavelets, Cambridge University Press, Cambridge, 1997.

[5] R. A. Zalik, Bases of translates and multiresolution analyses, Appl. Comput. Harmon. Anal. 24 (2008) 41–57, Corrigendum, Appl. Comput. Harmon. Anal. 29 (2010), 121.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 27, NO.5, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC

Neutrosophic sets applied to mighty filters in BE-algebras

Jung Mi Ko[1] **and Sun Shin Ahn**[2,*]

[1]*Department of Mathematics, Gangneung-Wonju National University, Gangneung 25457, Korea*
[2]*Department of Mathematics Education, Dongguk University, Seoul 04620, Korea*

Abstract. The notion of a neutrosophic subalgebra of a BE-algebra is introduced and consider characterizations of a neutrosophic subalgebra and a neutrosophic filter. We defined the notion of a neutrosophic mighty filter of a BE-algebra, and investigated some properties of it. We provide conditions for a neutrosophic filter to be a neutrosophic mighty filter.

1. Introduction

In 2007, Kim and Kim [6] introduced the notion of a BE-algebra, and investigated several properties. In [1], Ahn and So introduced the notion of ideals in BE-algebras. They gave several descriptions of ideals in BE-algebras. Y. B. Jun et. al [4] introduced the notions of hesitant fuzzy subalgebras and hesitant fuzzy filters of BE-algebras and investigated their relations and properties. J. S. Han et. al [3] defined the notion of hesitant fuzzy implicative filter of a BE-algebra, and considered some properties of it.

Zadeh [11] introduced the degree of membership/truth (t) in 1965 and defined the fuzzy set. As a generalization of fuzzy sets, Atanassov [2] introduced the degree of nonmembership/falsehood (f) in 1986 and defined the intuitionistic fuzzy set. Smarandache introduced the degree of indeterminacy/neutrality (i) as independent component in 1995 (published in 1998) and defined the neutrosophic set on three components (t, i, f) = (truth, indeterminacy, falsehood). In 2015, neutrosophic set theory is applied to BE-algebra, and the notion of neutrosophic filter is introduced [9]. A new definition of neutrosopic filter is established and some basic properties are presented [12].

In this paper, we introduce the notion of a neutrosophic subalgebra of a BE-algebra and consider characterizations of a neutrosophic subalgebra and a neutrosophic filter. We defined the notion of a neutrosophic mighty filter of a BE-algebra, and investigated some properties of it. We provide conditions for a neutrosophic filter to be a neutrosophic mighty filter.

2. Preliminaries

By a *BE-algebra* ([6]) we mean a system $(X; *, 1)$ of type $(2, 0)$ which the following axioms hold:

(BE1) $(\forall x \in X)\,(x * x = 1)$,

[0] **2010 Mathematics Subject Classification**: 06F35; 03G25; 03B60.
[0] **Keywords**: BE-algebra; ; (mighty) filter; neutrosophic subalgebra; neutrosophic (mighty) filter.
* Correspondence: Tel: +82 2 2260 3410, Fax: +82 2 2266 3409 (S. S. Ahn).
[0]**E-mail**: jmko@gwnu.ac.kr (J. M. Ko); sunshine@dongguk.edu (S. S. Ahn).
[0]This study was supported by Gangneung-Wonju National University.

(BE2) $(\forall x \in X)\,(x * 1 = 1)$,

(BE3) $(\forall x \in X)\,(1 * x = x)$,

(BE4) $(\forall x, y, z \in X)\,(x * (y * z) = y * (x * z)$ (exchange).

We introduce a relation " $\leq$ " on X by $x \leq y$ if and only if $x * y = 1$.

A BE-algebra $(X; *, 1)$ is said to be *transitive* if it satisfies: for any $x, y, z \in X$, $y * z \leq (x * y) * (x * z)$. A BE-algebra $(X; *, 1)$ is said to be *self distributive* if it satisfies: for any $x, y, z \in X$, $x * (y * z) = (x * y) * (x * z)$. Note that every self distributive BE-algebra is transitive, but the converse is not true in general ([6]).

Every self distributive BE-algebra $(X; *, 1)$ satisfies the following properties:

(2.1) $(\forall x, y, z \in X)\,(x \leq y \Rightarrow z * x \leq z * y$ and $y * z \leq x * z)$,

(2.2) $(\forall x, y \in X)\,(x * (x * y) = x * y)$,

(2.3) $(\forall x, y, z \in X)\,(x * y \leq (z * x) * (z * y))$,

Definition 2.1. Let $(X; *, 1)$ be a BE-algebra and let F be a non-empty subset of X. Then F is a *filter* of X ([6]) if

(F1) $1 \in F$;

(F2) $(\forall x, y \in X)(x * y, x \in F \Rightarrow y \in F)$.

F is a *mighty filter* ([8]) of X if it satisfies (F1) and

(F3) $(\forall x, y, z \in X)(z * (y * x), z \in F \Rightarrow ((x * y) * y) * x \in F)$.

Theorem 2.2. ([8]) *A filter F of a BE-algebra X is mighty if and only if*

(2.4) $(\forall x, y \in X)(y * x \in F \Rightarrow ((x * y) * y) * x \in F)$.

Definition 2.3. Let X be a space of points (objects) with generic elements in X denoted by x. A simple valued neutrosophic set A in X is characterized by a truth-membership function $T_A(x)$, an indeterminacy-membership function $I_A(x)$, and a falsity-membership function $F_A(x)$. Then a simple valued neutrosopic set A can be denoted by

$$A := \{\langle x, T_A(x), I_A(x), F_A(x)\rangle | x \in X\},$$

where $T_A(x), I_A(x), F_A(x) \in [0, 1]$ for each point x in X. Therefore the sum of $T_A(x), I_A(x)$, and $F_A(x)$ satisfies the condition $0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3$.

For convenience, "simple valued neutrosophic set" is abbreviated to "neutrosophic set" later.

Definition 2.4. ([10]) A neutrosophic set A is contained in the other neutrosophic B, denoted by $A \subseteq B$, if and only if $T_A(x) \leq T_B(x), I_A(x) \geq I_B(x)$, and $F_A(x) \geq F_B(x)$ for any $x \in X$. Two neutrosophic sets A and B are equal, written as $A = B$, if and only if $A \subseteq B$ and $B \subseteq A$.

Definition 2.5. ([12]) Let A be a neutrosophic set in a BE-algebra X and $\alpha, \beta, \gamma \in [0, 1]$ with $0 \leq \alpha + \beta + \gamma \leq 3$ and an (α, β, γ)-level set of X denoted by $A^{(\alpha,\beta,\gamma)}$ is defined as

$$A^{(\alpha,\beta,\gamma)} = \{x \in X | T_A(x) \geq \alpha, I_A(x) \leq \beta, F_A(x) \leq \gamma\}.$$

3. Neutrosophic subalgebras in BE-algebras

Definition 3.1. A neutrosophic set A in a BE-algebra X is called a *neutrosophic subalgebra* of X if it satisfies:

(NSS) $\min\{T_A(x), T_A(y)\} \leq T_A(x*y), \max\{I_A(x), I_A(y)\} \geq I_A(x*y),$ and $\max\{F_A(x), F_A(y)\} \geq F_A(x*y)$, for any $x, y \in X$.

Example 3.2. Let $X := \{1, a, b, c\}$ be a BE-algebra ([4]) with the following table:

$*$	1	a	b	c
1	1	a	b	c
a	1	1	a	a
b	1	1	1	a
c	1	1	a	1

Define a neutrosophic set A in X as follows:

$$T_A(x) = \begin{cases} 0.83, & \text{if } x \in \{1, a\} \\ 0.13, & \text{otherwise,} \end{cases}$$

$$I_A(x) = \begin{cases} 0.15, & \text{if } x \in \{1, a\} \\ 0.82, & \text{otherwise,} \end{cases}$$

$$F_A(x) = \begin{cases} 0.15, & \text{if } x \in \{1, a\} \\ 0.82, & \text{otherwise.} \end{cases}$$

It is easy to check that A is a neutrosophic subalgebra of X.

Definition 3.3. ([12]) A neutrosophic set A in a BE-algebra X is called a *neutrosophic filter* of X if it satisfies:

(NSF1) $T_A(x) \leq T_A(1), I_A(x) \geq I_A(1),$ and $F_A(x) \geq F_A(1)$, for any $x \in X$;

(NSF2) $\min\{T_A(x), T_A(x*y)\} \leq T_A(y), \max\{I_A(x), I_A(x*y)\} \geq I_A(y),$ and $\max\{F_A(x), F_A(x*y)\} \geq F_A(y)$, for any $x, y \in X$.

Proposition 3.4. *Every neutrosophic filter of a BE-algebra X is a neutrosophic subalgebra of X.*

Proof. Let A be a neutrosophic filter of X. For any $x, y \in X$, we have $\min\{T_A(x), T_A(y)\} \leq \min\{T_A(1), T_A(y)\} = \min\{T_A(y*(x*y)), T_A(y)\} \leq T_A(x*y)$, $\max\{I_A(x), I_A(y)\} \geq \max\{I_A(1), I_A(y)\} = \max\{I_A(y*(x*y)), I_A(y)\} \geq I_A(x*y)$, and $\max\{F_A(x), F_A(y)\} \geq \max\{F_A(1), F_A(y)\} = \max\{F_A(y*(x*y)), F_A(y)\} \geq F_A(x*y)$. Hence A is a neutrosophic subalgebra of X. $\square$

The converse of Proposition 3.4 may not be true in general (see Example 3.5).

Example 3.5. Let $X := \{1, a, b\}$ be a BE-algebra with the following table:

$*$	1	a	b
1	1	a	b
a	1	1	a
b	1	1	1

Define a neutrosophic set A in X as follows: $T_A = \{(1, 0.83), (a, 0.13), (b, 0.16)\}, I_A = \{(1, 0.15), (a, 0.15), (b, 0.82)\}$, and $F_A = \{(1, 0.15), (a, 0.15), (b, 0.82)\}$. It is easy to check that A is a neutrosophic subalgebra of X. But it is not a neutrosophic filter of X, since $\min\{T_A(b * a), T_A(b)\} = \min\{T_A(1), T_A(b)\} = 0.16 \nleq 0.13 = T_A(a)$.

Theorem 3.6. *Let A be a neutrosophic set in a BE-algebra X and let $\alpha, \beta, \gamma \in [0, 1]$ with $0 \le \alpha + \beta + \gamma \le 3$. Then A is a neutrosophic subalgebra of X if and only if all of (α, β, γ)-level set $A^{(\alpha,\beta,\gamma)}$ are subalgebras of X when $A^{(\alpha,\beta,\gamma)} \ne \emptyset$.*

Proof. Assume that A is a neutrosophic subalgebra of X. Let $\alpha, \beta, \gamma \in [0, 1]$ be such that $0 \le \alpha + \beta + \gamma \le 3$ and $A^{(\alpha,\beta,\gamma)} \ne \emptyset$. Let $x, y \in A^{(\alpha,\beta,\gamma)}$. Then $T_A(x) \ge \alpha, T_A(y) \ge \alpha, I_A(x) \le \beta, I_A(y) \le \beta$ and $F_A(x) \le \gamma, F_A(y) \le \gamma$. Using (NSS), we have $\alpha \le \min\{T_A(x), T_A(y)\} \le T_A(x * y)$, $\beta \ge \max\{I_A(x), I_A(y)\} \ge I_A(x * y)$, and $\gamma \ge \max\{F_A(x), F_A(y)\} \ge F_A(x * y)$. Hence $x * y \in A^{(\alpha,\beta,\gamma)}$. Therefore $A^{(\alpha,\beta,\gamma)}$ is a subalgebra of X.

Conversely, all of (α, β, γ)-level set $A^{(\alpha,\beta,\gamma)}$ are subalgebras of X when $A^{(\alpha,\beta,\gamma)} \ne \emptyset$. Assume that there exist $a_t, b_t, a_i, b_i \in X$ and $a_f, b_f \in X$ such that $\min\{T_A(a_t), T_A(b_t)\} > T_A(a_t * b_t), \max\{I_A(a_i), I_A(b_i)\} < I_A(a_i * b_i)$, and $\max\{F_A(a_f), F_A(b_f)\} < F_A(a_f * b_f)$. Then $\min\{T_A(a_t), T_A(b_t)\} \ge t_{\alpha_1} > T_A(a_t * b_t), \max\{I_A(a_i), I_A(b_i)\} \le t_{\alpha_2} < I_A(a_i * b_i)$, and $\max\{F_A(a_f), F_A(b_f)\} \le t_{\alpha_3} < F_A(a_f * b_f)$ for some $t_{\alpha_1} \in (0, 1]$, and $t_{\alpha_2}, t_{\alpha_3} \in [0, 1)$. Hence $a_t, b_t, a_i, b_i, a_f, b_f \in A^{(t_{\alpha_1}, t_{\alpha_2}, t_{\alpha_3})}$, but $a_t * b_t, a_i * b_i, a_f * b_f \notin A^{(t_{\alpha_1}, t_{\alpha_2}, t_{\alpha_3})}$, which is a contradiction. Hence $\min\{T_A(x), T_A(y)\} \le T_A(x * y), \max\{I_A(x), I_A(y)\} \ge I_A(x * y)$, and $\max\{F_A(x), F_A(y)\} \ge F_A(x * y)$ for any $x, y \in X$. Therefore A is a neutrosophic subalgebra of X. □

Since $[0, 1]$ is a completely distributive lattice with respect to the usual ordering, we have the following theorem.

Theorem 3.7. *If $\{A_i | i \in \mathbb{N}\}$ is a family of neutrosopic subalgebras of a BE-algebra X, then $(\{A_i | i \in \mathbb{N}\}, \subseteq)$ forms a complete distributive lattice.*

Proposition 3.8. *If A is a neutrosopic subalgebra of a BE-algebra X, then $T_A(x) \le T_A(1), I_A(x) \ge I_A(1)$, and $F_A(x) \ge F_A(1)$ for all $x \in X$.*

Proof. Straightforward. □

Theorem 3.9. *Let A be a neutrosophic subalgebra of a BE-algebra X. If there exists a sequence $\{a_n\}$ in X such that $\lim_{n\to\infty} T_A(a_n) = 1, \lim_{n\to\infty} I_A(a_n) = 0$, and $\lim_{n\to\infty} F_A(a_n) = 0$, then $T_A(1) = 1, I_A(1) = 0$, and $F_A(1) = 0$.*

Proof. By Proposition 3.8, we have $T_A(x) \le T_A(1), I_A(x) \ge I_A(1)$, and $F_A(x) \ge F_A(1)$ for all $x \in X$. Hence we have $T_A(a_n) \le T_A(1), I_A(a_n) \ge I_A(1)$, and $F_A(a_n) \ge F_A(1)$ for every positive integer n. Therefore $1 = \lim_{n\to\infty} T_A(a_n) \le T_A(1) \le 1, 0 = \lim_{n\to\infty} I_A(a_n) \ge I_A(1) \ge 0$, and $0 = \lim_{n\to\infty} F_A(a_n) \ge F_A(1) \ge 0$. Thus we have $T_A(1) = 1, T_A(1) = 0$, and $F_A(1) = 0$. □

Proposition 3.10. *If every neutrosophic subalgebra A of a BE-algebra X satisfies the condition*

(3.1) $T_A(x * y) \ge T_A(x), I_A(x * y) \le I_A(x), F_A(x * y) \le F_A(x)$, *for any $x, y \in X$,*

then T_A, I_A, and F_A are constant functions.

Proof. It follows from (3.1) that $T_A(x) = T_A(1 * x) \geq T_A(1), I_A(x) = I_A(1 * x) \leq I_A(1)$, and $F_A(x) = F_A(1 * x) \leq F_A(1)$ for any $x \in X$. By Proposition 3.8, we have $T_A(x) = T_A(1), I_A(x) = I_A(1)$, and $F_A(x) = F_A(1)$ for any $x \in X$. Hence T_A, I_A, and F_A are constant functions. □

Proposition 3.11. *Let A be a neutrosophic filter of a BE-algebra X. Then*

(i) $\min\{T_A(x * (y * z)), T_A(y)\} \leq T_A(x * z), \max\{I_A(x * (y * z)), I_A(y)\} \geq I_A(x * z)$, *and* $\max\{F_A(x * (y * z)), F_A(y)\} \geq F_A(x * z)$ *for any* $x, y \in X$.

(ii) $T_A(a) \leq T_A((a * x) * x), I_A(a) \geq I_A((a * x) * x)$, *and* $F_A(a) \geq F_A((a * x) * x)$ *for any* $a, x \in X$.

Proof. (i) Using (BE4) and (NSF2), we have $T_A(x * z) \geq \min\{T_A(y * (x * z)), T_A(y)\} = \min\{T_A(x * (y * z)), T_A(y)\}, I_A(x * z) \leq \max\{I_A(y * (x * z)), I_A(y)\} = \max\{I_A(x * (y * z)), I_A(y)\}$, and $F_A(x * z) \leq \max\{F_A(y * (x * z)), F_A(y)\} = \max\{F_A(x * (y * z)), F_A(y)\}$ for any $x, y \in X$.

(ii) Taking $y := (a * x) * x$ and $x := a$ in (NSF2), we have $T_A((a * x) * x) \geq \min\{T_A(a * ((a * x) * x)), T_A(a)\} = \min\{T_A((a * x) * (a * x)), T_A(a)\} = \min\{T_A(1), T_A(a)\} = T_A(a), I_A((a * x) * x) \leq \max\{I_A(a * ((a * x) * x)), I_A(a)\} = \max\{I_A((a * x) * (a * x)), I_A(a)\} = \max\{I_A(1), I_A(a)\} = I_A(a)$, and $F_A((a * x) * x) \leq \max\{F_A(a * ((a * x) * x)), F_A(a)\} = \max\{F_A((a * x) * (a * x)), F_A(a)\} = \max\{F_A(1), F_A(a)\} = F_A(a)$ for any $a, x \in X$.

□

Theorem 3.12. ([12]) *Let A be a neutrosophic set in a BE-algebra. Then A is a neutrosophic filter of X if and only if it satisfies (NSF1) and*

(3.2) *if* $x \leq y * z$ *for any* $x, y \in X$, *then* $\min\{T_A(x), T_A(y)\} \leq T_A(z), \max\{I_A(x), I_A(y)\} \geq I_A(z)$, *and* $\max\{F_A(x), F_A(y)\} \geq F_A(z)$.

Theorem 3.13. *If every neutrosophic set of a BE-algebra X satisfies (NSF1) and Proposition 3.11(i), then it is a neutrosophic filter of X.*

Proof. Taking $x := 1$ in Proposition 3.11(i) and using (BE3), we get $T_A(z) = T_A(1 * z) \geq \min\{T_A(1 * (y * z)), T_A(y)\} = \min\{T_A(y * z), T_A(y)\}, I_A(z) = I_A(1 * z) \leq \max\{I_A(1 * (y * z)), T_A(y)\} = \max\{I_A(y * z), I_A(y)\}$, and $F_A(z) = F_A(1 * z) \leq \max\{F_A(1 * (y * z)), F_A(y)\} = \max\{F_A(y * z), F_A(y)\}$ for any $y, z \in X$. Hence A is a neutrosophic filter of X. □

Corollary 3.14. *Let A be a neutrosophic set of a BE-algebra X. Then A is a neutrosophic filter of X if and only if it satisfies (NSF1) and Proposition 3.11(i).*

Theorem 3.15. *Let A be a neutrosophic set of a BE-algebra X. Then A is a neutrosophic filter of X if and only if it satisfies the following conditions:*

(i) $T_A(y * x) \geq T_A(x), I_A(y * x) \leq I_A(x)$, *and* $F_A(y * x) \leq F_A(x)$;

(ii) $T_A((a*(b*x))*x) \geq \min\{T_A(a), T_A(b)\}, I_A((a*(b*x))*x) \leq \max\{I_A(a), I_A(b)\}$, *and* $F_A((a*(b*x))*x) \leq \max\{F_A(a), F_A(b)\}$ *for any* $a, b, x \in X$.

Proof. Assume that A is a neutrosophic filter of X. Using (NSF2), we have $T_A(y * x) \geq \min\{T_A(x * (y * x)), T_A(x)\} = \min\{T_A(1), T_A(x)\} = T_A(x), I_A(y * x) \leq \max\{I_A(x * (y * x)), I_A(x)\} = \max\{I_A(1), I_A(x)\} = I_A(x)$, and $F_A(y * x) \leq \max\{F_A(x * (y * x)), F_A(x)\} = \max\{F_A(1), F_A(x)\} = F_A(x)$, for any $x, y \in X$. It follows

from Proposition 3.11 that $T_A((a*(b*x))*x) \geq \min\{T_A((a*(b*x))*(b*x)), T_A(b)\} \geq \min\{T_A(a), T_A(b)\}$, $I_A((a*(b*x))*x) \leq \max\{I_A((a*(b*x))*(b*x)), I_A(b)\} \leq \max\{I_A(a), I_A(b)\}$, and $F_A((a*(b*x))*x) \leq \max\{F_A((a*(b*x))*(b*x)), F_A(b)\} \leq \max\{F_A(a), F_A(b)\}$ for any $x, a, b \in X$.

Conversely, assume that A is a neutrosophic set of X satisfying conditions (i) and (ii). Taking $y := x$ in (i), we have $T_A(1) = T_A(x*x) \geq T_A(x), I_A(1) = I_A(x*x) \leq I_A(x)$ and $F_A(1) = F_A(x*x) \leq F_A(x)$ for any $x \in X$. Using (ii), we get $T_A(y) = T_A(1*y) = T_A(((x*y)*(x*y))*y) \geq \min\{T_A(x*y), T_A(x)\}, I_A(y) = I_A(1*y) = I_A(((x*y)*(x*y))*y) \leq \max\{I_A(x*y), I_A(x)\}, F_A(y) = F_A(1*y) = F_A(((x*y)*(x*y))*y) \leq \max\{F_A(x*y), F_A(x)\}$ for any $x, y \in X$. Hence A is a neutrosophic filter of X. □

4. Neutrosophic mighty filters in BE-algebras

Definition 4.1. A neutrosophic set A in a BE-algebra X is called a *neutrosophic mighty filter* of X if it satisfies (NSF1) and

(NSF3) $\min\{T_A(z*(y*x)), T_A(z)\} \leq T_A(((x*y)*y)*x)), \max\{I_A(z*(y*x)), I_A(z)\} \geq I_A(((x*y)*y)*x)$, and $\max\{F_A(z*(y*x)), F_A(z)\} \geq F_A(((x*y)*y)*x)$ for any $x, y, z \in X$.

Example 4.2. Let $X := \{1, a, b, c, d, 0\}$ be a BE-algebra ([8]) with the following table:

$*$	1	a	b	c	d	0
1	1	a	b	c	d	0
a	1	1	b	c	b	c
b	1	a	1	b	a	d
c	1	a	1	1	a	a
d	1	1	1	b	1	b
0	1	1	1	1	1	1

Define a neutrosophic set A in X as follows:

$$T_A(x) = \begin{cases} 0.83, & \text{if } x \in \{1, b, c\} \\ 0.12, & \text{otherwise,} \end{cases}$$

$$I_A(x) = \begin{cases} 0.14, & \text{if } x \in \{1, b, c\} \\ 0.81, & \text{otherwise,} \end{cases}$$

$$F_A(x) = \begin{cases} 0.14, & \text{if } x \in \{1, b, c\} \\ 0.81, & \text{otherwise.} \end{cases}$$

It is easy to check that A is a neutrosophic mighty filter of X.

Proposition 4.3. *Every neutrosophic mighty filter of a BE-algebra X is a neutrosophic filter of X.*

Proof. Let A be a neutrosophic mighty filter of X. Putting $y := 1$ in (NSF3), we obtain $\min\{T_A(z*(1*x)), T_A(z)\} = \min\{T_A(z*x), T_A(z)\} \leq T_A(((x*1)*1)*x) = T_A(x), \max\{I_A(z*(1*x)), I_A(z)\} = \max\{I_A(z*x), I_A(z)\} \geq I_A(((x*1)*1)*x) = I_A(x)$, and $\max\{F_A(z*(1*x)), F_A(z)\} = \max\{F_A(z*x), F_A(z)\} \geq F_A(((x*1)*1)*x) = F_A(x)$ for any $x, y, z \in X$. Hence A is a neutrosophic filter of X. □

The converse of Proposition 4.3 may be not true in general (see Example 4.4).

Example 4.4. Let $X := \{1, a, b, c, d\}$ be a BE-algebra ([5]) with the following table:

$*$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	b	c	d
b	1	a	1	c	c
c	1	1	b	1	b
d	1	1	1	1	1

Define a neutrosophic set A in X as follows:

$$T_A(x) = \begin{cases} 0.84, & \text{if } x = 1 \\ 0.11, & \text{otherwise,} \end{cases}$$

$$I_A(x) = \begin{cases} 0.13, & \text{if } x = 1 \\ 0.81, & \text{otherwise,} \end{cases}$$

$$F_A(x) = \begin{cases} 0.13, & \text{if } x = 1 \\ 0.81, & \text{otherwise.} \end{cases}$$

Then A is a neutrosophic filter of X, but not a neutrosophic mighty filter of X, since $\min\{T_A(1*(c*a)), T_A(1)\} = T_A(1) = 0.84 \not\leq T_A(((a*c)*c)*a) = T_A(a) = 0.11$.

Theorem 4.5. *Any neutrosophic filter A of a BE-algebra X is mighty if and only if it satisfies the following conditions:*

(4.1) $T_A(y*x) \leq T_A(((x*y)*y)*x), I_A(y*x) \geq I_A(((x*y)*y)*x)$, *and* $F_A(y*x) \geq F_A(((x*y)*y)*x)$ *for any* $x, y \in X$.

Proof. Suppose that a neutrosophic filter A of a BE-algebra X satisfies the condition (4.1). Using (NSF2) and (4.1), we have $\min\{T_A(z*(y*x)), T_A(z)\} \leq T_A(y*x) \leq T_A(((x*y)*y)*x), \max\{I_A(z*(y*x)), I_A(z)\} \geq I_A(y*x) \geq I_A(((x*y)*y)*x)$, and $\max\{F_A(z*(y*x)), F_A(z)\} \geq F_A(y*x) \geq F_A(((x*y)*y)*x)$ for any $x, y \in X$. Hence A is a neutrosophic mighty filter of X.

Conversely, assume that the neutrosophic filter A of X is mighty. Setting $z := 1$ in (NSF3), we have $\min\{T_A(1*(y*x)), T_A(1)\} = T_A(y*x) \leq T_A(((x*y)*y)*x), \max\{I_A(1*(y*x)), I_A(1)\} = I_A(y*x) \geq I_A(((x*y)*y)*x)$, and $\max\{F_A(1*(y*x)), F_A(1)\} = F_A(y*x) \geq F_A(((x*y)*y)*x)$ for any $x, y \in X$. Hence (4.1) holds. □

Proposition 4.6. *Let A be a neutrosophic mighty filter of a BE-algebra X. Denote that $X_T := \{x \in X | T_A(x) = T_A(1)\}, X_I := \{x \in X | I_A(x) = I_A(1)\}$, and $X_F := \{x \in X | F_A(x) = F_A(1)\}$. Then X_T, X_I, and X_F are mighty filters of X.*

Proof. Clearly, $1 \in X_T, X_I, X_F$. Let $z*(y*x), z \in X_T$. Then $T_A(z*(y*x)) = T_A(1), T_A(z) = T_A(1)$. Hence $\min\{T_A(z*(y*x)), T_A(z)\} = T_A(1) \leq T_A(((x*y)*y)*x)$ and so $T_A((x*y)*y)*x) = T_A(1)$. Therefore $((x*y)*y)*x \in X_T$. Thus X_T is a mighty filter of X. Similarly, X_I, X_F are mighty filters of X. □

Theorem 4.7. *Let A, B be neutrosophic filters of a transitive BE-algebra X such that $A \subseteq B$ and $T_A(1) = T_B(1), I_A(1) = I_B(1), F_A(1) = F_B(1)$. If A is mighty, then B is mighty.*

Proof. Let $x, y \in X$. Since A is a neutrosophic mighty filter of a BE-algebra X, by (4.1) and $A\subseteq B$ we have $T_A(1) = T_A(y*((y*x)*x)) \leq T_A(((((y*x)*x)*y)*y)*((y*x)*x)) \leq T_B(((((y*x)*x)*y)*y)*((y*x)*x))$. Since $T_A(1) = T_B(1)$, we get $T_B((y*x)*((((y*x)*x)*y)*y)*x)) = T_B(((((y*x)*x)*y)*y)*((y*x)*x)) = T_B(1)$. It follows from (NSF1) and (NSF2) that

$$\begin{aligned} T_B(y*x) =& \min\{T_B(1), T_B(y*x)\} \\ =& \min\{T_B((y*x)*(((((y*x)*x)*y)*y)*x)), T_B(y*x)\} \\ \leq& T_B(((((y*x)*x)*y)*y)*x). \end{aligned} \tag{4.2}$$

Since X is transitive, we get

$$\begin{aligned} [((((y*x)*x)*y)*y)*x]*&[((x*y)*y)*x] \\ &\geq ((x*y)*y)*((((y*x)*x)*y)*y) \\ &\geq (((y*x)*x)*y)*(x*y) \\ &\geq x*((y*x)*x) \\ &= (y*x)*(x*x) \\ &= (y*x)*1 = 1. \end{aligned}$$

It follows from Theorem 3.12 that $\min\{T_B(((((y*x)*x)*y)*y)*x), T_B(1)\} = T_B(((((y*x)*x)*y)*y)*x) \leq T_B(((x*y)*y)*x)$. Using (4.2), we have $T_B(y*x) \leq T_B(((((y*x)*x)*y)*y)*x) \leq T_B(((x*y)*y)*x)$. Therefore $T_B(y*x) \leq T_B(((x*y)*y)*x)$. Similarly, we have $I_B(y*x) \geq T_B(((x*y)*y)*x)$ and $F_B(y*x) \geq F_B(((x*y)*y)*x)$. By Theorem 4.5, B is a neutrosophic mighty filter of X. □

Theorem 4.8. *Let A be a neutrosophic set in a BE-algebra X and let $\alpha, \beta, \gamma \in [0, 1]$ with $0 \leq \alpha + \beta + \gamma \leq 3$. Then A is a neutrosophic mighty filter of X if and only if all of (α, β, γ)-level set $A^{(\alpha,\beta,\gamma)}$ are mighty filters of X when $A^{(\alpha,\beta,\gamma)} \neq \emptyset$.*

Proof. Assume that A is a neutrosophic mighty filter of X. Let $\alpha, \beta, \gamma \in [0, 1]$ be such that $0 \leq \alpha + \beta + \gamma \leq 3$ and $A^{(\alpha,\beta,\gamma)} \neq \emptyset$. Let $z*(y*x), z \in A^{(\alpha,\beta,\gamma)}$. Then $T_A(z*(y*x)) \geq \alpha, T_A(z) \geq \alpha, I_A(z*(y*x)) \leq \beta, I_A(z) \leq \beta$, and $F_A(z*(y*x)) \leq \gamma, F_A(z) \leq \gamma$. By Definition 4.1, we have $T_A(1) \geq T_A(((x*y)*y)*x) \geq \min\{T_A(z*(y*x)), T_A(z)\} \geq \alpha, I_A(1) \leq I_A(((x*y)*y)*x) \leq \max\{I_A(z*(y*x)), I_A(z)\} \leq \beta$, and $F_A(1) \leq F_A(((x*y)*y)*x) \leq \max\{F_A(z*(y*x)), F_A(z)\} \leq \gamma$. Hence $1, ((x*y)*y)*x \in A^{(\alpha,\beta,\gamma)}$. Therefore $A^{(\alpha,\beta,\gamma)}$ are mighty filters of X.

Conversely, suppose that there exist $a, b, c \in X$ such that $T_A(a) > T_A(1), I_A(b) < I_A(1)$, and $F_A(c) < F_A(1)$. Then there exist $a_t \in (0, 1]$ and $b_t, c_t \in [0, 1)$ such that $T_A(a) \geq a_t > T_A(1), I_A(b) \leq b_t < I_A(1)$ and $F_A(c) \leq c_t < F_A(1)$. Hence $1 \notin A^{(a_t,b_t,c_t)}$, which is a contradiction. Therefore $T_A(x) \leq T_A(1), I_A(x) \geq I_A(1)$ and $F_A(x) \geq F_A(1)$ for all $x \in X$. Assume that there exist $a_t, b_t, c_t, a_i, b_i, c_i \in X$ and $a_f, b_f, c_f \in X$ such that $T_A(((a_t*b_t)*b_t)*a_t) < \min\{T_A(c_t*(b_t*a_t)), T_A(c_t)\}, I_A(((a_i*b_i)*b_i)*a_i) > \max\{I_A(c_i*(b_i*a_i)), I_A(c_i)\}$, and $F_A(((a_f*b_f)*b_f)*a_f) > \max\{F_A(c_f*(b_f*a_f)), F_A(c_f)\}$. Then there exist $s_t \in (0, 1]$ and $s_i, s_f \in [0, 1)$ such that $T_A(((a_t*b_t)*b_t)*a_t) < s_t \leq \min\{T_A(c_t*(b_t*a_t)), T_A(c_t)\}, I_A(((a_i*b_i)*b_i)*a_i) > s_i \geq \max\{I_A(c_i*(b_i*a_i)), I_A(c_i)\}$, and $F_A(((a_f*b_f)*b_f)*a_f) > s_f \geq \max\{F_A(c_f*(b_f*a_f)), F_A(c_f)\}$. Hence $c_t*(b_t*a_t), c_t, c_i*(b_i*a_i), c_i \in A^{(s_t,s_i,s_f)}$ and $c_f*(b_f*a_f), c_f \in A^{(s_t,s_i,s_f)}$ but $((a_t*b_t)*b_t)*a_t, ((a_i*b_i)*b_i)*a_i \notin A^{(s_t,s_i,s_f)}$, and $((a_f*b_f)*b_f)*a_f \notin A^{(s_t,s_i,s_f)}$,

which is a contradiction. Therefore $\min\{T_A(z*(y*x)), T_A(z)\} \leq T_A(((x*y)*y)*x)), \max\{I_A(z*(y*x)), I_A(z)\} \geq I_A(((x*y)*y)*x))$, and $\max\{F_A(z*(y*x)), F_A(z)\} \geq F_A(((x*y)*y)*x))$ for any $x, y, z \in X$. Thus A is a neutrosophic mighty filter of X □

References

[1] S. S. Ahn and K. S. So, *On ideals and upper sets in BE-algerbas,* Sci. Math. Jpn. 68 (2008), 279–285 .

[2] K. Atanassov, *Intuitionistic fuzzy sets,* Fuzzy sets and Systems 20 (1986), 87–96.

[3] J. S. Han and S. S. Ahn, *Hesitant fuzzy implicative filters in BE-algebras*, J. Comput. Anal. Appl. 23 (2017), 530–543.

[4] Y. B. Jun and S. S. Ahn, *On hesitant fuzzy filters in BE-algebras*, J. Comput. Anal. Appl. 22 (2017), 346-358.

[5] Y. B. Jun and S. S. Ahn, *On hesitant fuzzy mighty filters in BE-algebras*, J. Comput. Anal. Appl. 23 (2017), 1112–1119.

[6] H. S. Kim and Y. H. Kim, *On BE-algerbas*, Sci. Math. Jpn. 66 (2007), no. 1, 113–116.

[7] M. Khan, S. Anis, F. Smarandache and Y. B. Jun, *Neutrosophic N-structures and their applications in semigroups*, Ann. Fuzzy Math. Inform., (to appear).

[8] H. R. Lee and S. S. Ahn, Mighty filters in BE-algebras, Honam Mathematical J. 37(2) (2015), 221-233.

[9] A. Rezei, A. B. Saeid, and F. Smarnadache, *Neutrosophic filters in BE-algebras*, Ration Mathematica 29 (2015), 65–79.

[10] F. Smarandache, Neutrosophy, Neutrosophic Probablity, Sets, and Logic, Amer. Res. Press, Rehoboth, USA, 1998.

[11] L. A. Zadeh, *Fuzzy sets*, Information and Control 8 (1965), 338-353.

[12] X. Zhang, P. Yu, F. Smarandache, and C. Park, *Redefined Neutrosophic filters in BE-algebras*, Information, (to submit).

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 27, NO.5, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC

Coupled fixed point of firmly nonexpansive mappings by Mann's iterative processes in Hilbert spaces

TAMER NABIL[a,b1]

[a]King Khalid University, College of Science, Department of Mathematics, 61413, Abha, Saudi Arabia

[b]Suez Canal University, Faculty of Computers and Informatics, Department of Basic Science, Ismailia, Egypt

Abstract

We study the weak convergence of Mann's explicit iteration processes to common coupled fixed point of firmly nonexpansive coupled mappings in Hilbert spaces.Our results extend and generalized the results due to Nabil and Soliman for coupled fixed point approach (T. Nabil and A. H. Soliman, weak convergence theorems of explicit iteration process with errors and applications in optimization, J. Ana. Num. Theor., 5(2017) 81: 89).

Key words and phrases. explicit iteration process, firmly coupled nonexpansive mapping; coupled fixed point; Hilbert space.

AMS Mathematics subject Classification. 47H09,47H10,47H20.

1 Introduction

The study of finding the fixed point of iterative processes has attracted the interest of many researchers due to its applications in physics, optimization, image processing and economics can be recast in terms of a fixed point problem of nonlinear mappings in Hilbert space [[1],[2], [3], [4], [5], [6]]. A lot of this studies consider this mappings as nonexpansive which is defined as: let H be a real Hilbert space and K be a nonempty closed convex subset of H. Then, a mapping R of K into H is called nonexpansive if $\|Rx - Ry\| \leq \|x - y\|$ for all $x, y \in K$. R is called firmly nonexpansive if

$$||Rx - Ry||^2 + ||(Id - R)x - (Id - R)y||^2 \leq ||x - y||^2 \tag{1}$$

[1] $t_3$bdelsadek@yahoo.com

1

for all $x, y \in K$, where $Id : K \to K$ denote the identity operator. We have known that every firmly nonexpansive mapping is a nonexpansive mapping. The finding of common fixed point for iteration process have been investigated since the early 1953 by Mann [7] which consider the following iteration scheme

$$\begin{cases} x_1 \in C \text{ is chosen arbitrarily} \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n) R x_n, \forall\, n \in N \end{cases}$$

where $\{\alpha_n\}$ is a sequence in [0,1]. Several authors studied another types of iteration process such as: Halpern [8] , Bauschke [9] and Xu and Ori [10] . In 2005, Kimura et al. [11], studied the convergence of an iterative scheme to a common fixed point of a finite family of nonexpansive mappings in Banach space.

The problem of finding a common fixed point of families of nonlinear mappings has been investigated by many researchers; see, for instance, ([12]-[17]).

Recently, Chuang and Takahashi [18] defined the new Mann's type iteration process by metric projection from H to K and gave weak convergence theorems for finding a common fixed point of a sequence of firmly nonexpansive mappings in a Hilbert space. More recently, in 2017 Nabil and Soliman [19] studied the weak convergen theorem f a new Mann iterative proesses with errors.

The idea of coupled fixed point was started in 1987 by Guo and Lakshmikantham [20]. Several authors studied the coupled fixed point Theorem See [[21],[22], [23], [24], [25]]

In this work, we prove the weak convergence theorem for finding the coupled fixed points of iteration processes for the families of nonlinear coupled mappings in Hilbert spaces.

2 Firmly nonexpansive coupled mappings

Throughout this paper we denote by N the set of positive integers and strongly (respectively weak) convergence of $\{x_n\}$ to $x \in H$ by $x_n \to x$ (respectively $x_n \rightharpoonup x$). Let H be a Hilbert space . The inner product and the induced norm on H are denoted by $< .,. >$ and $\| \,.\, \|$ respectively. Consider $F(T)$ be the set of fixed points of T (i.e., $F(T) = \{x \in C : Tx = x\}$).

Let $C \neq \emptyset$ be a closed and convex subset of a real Hilbert space H, and consider the coupled mapping

$T : C \times C \to H$. Then $(w_1, w_2) \in C \times C$ is said to be coupled fixed point of T if $T(w_1, w_2) = w_1$ and $T(w_2, w_1) = w_2$, thus we can define the set of all coupled fixed points of T (denoted by $CF(T)$) as : $CF(T) = \{(x, y) \in C \times C : T(x, y) = x, T(y, x) = y\}$.

$T : C \times C \to C$ is said to be nonexpansive coupled mapping (denoted by NCM) if for every (x, y) and $(u, v) \in C \times C$,

$$\|T(x,y) - T(u,v)\| \leq \frac{1}{2}[\|x-u\| + \|y-v\|]$$

;T is said to be firmly nonexpansive coupled mapping (denoted ny $FNCM$) if,

$$\|T(x,y) - T(u,v)\|^2 \leq \frac{1}{2}[\langle x-u, T(x,y) - T(u,v)\rangle + \langle y-v, T(x,y) - T(u,v)\rangle],$$

equivalent;

$$\|T(x,y) - T(u,v)\|^2 \leq \frac{1}{2}\langle x-u+y-v, T(x,y) - T(u,v)\rangle,$$

for all $(x, y), (u, v) \in C \times C$. The following lemma give the relation between NCM and $FNCM$.

Lemma 2.1 Let $C \neq \emptyset$ be subset of real Hilbert space H. If $T : C \times C \to H$ be $FNCM$. Then T is NCM

Proof. Since T is $FNCM$, for all $(x, u), (u, v) \in C \times C$ we get that,

$$\|T(x,y) - T(u,v)\|^2 \leq \frac{1}{2}[\langle x-u, T(x,y) - T(u,v)\rangle + \langle y-v, T(x,y) - T(u,v)\rangle]$$
$$\leq \frac{1}{2}[\|x-u\|\|T(x,y) - T(u,v)\| + \|y-v\|\|T(x,y) - T(u,v)\|].$$

Therefore, we get that;

$$\|T(x,y) - T(u,v)\| \leq \frac{1}{2}[\|x-u\| + \|y-v\|].$$

Thus , T is NCM.

The following example show that the converse of lemma 2.1 is may not be true.

Example 2.1 Let $H = \Re$, and consider $T : \Re \times \Re \to \Re$ such as:for all $(x, y) \in \Re \times \Re$, define $T(x, y) = \frac{1}{2}x$. Let, $(x, y), (u, v) \in \Re \times \Re$, then we have that:

$$\|T(x,y) - T(u,v)\| = \|\frac{1}{2}(x-u)\| \leq \frac{1}{2}[\|x-u\| + \|y-v\|].$$

Thus, T is NCM. However,

$$\langle 1-0-2-0, T(1,-2) - T(0,0)\rangle = \frac{-1}{2} < 2\|T(1,-2) - T(0,0)\|^2.$$

3

Hence, T is not $FNCM$.

Let $C \neq \emptyset$ be closed convex subset of H. Let us recall that : the metric projection of H onto C (denoted by P_C) is defined as the mapping $P_C : H \to C$ such that: for each $x \in H$, there exist a unique $y \in C$ such that: $P_C x = y$ if and only if $\|x - y\| \leq \|x - z\|$ for every $z \in C$. A mapping P_C satisfied some important properties such as:$\|P_C x - P_C y\| \leq \|x - y\|$ for all $x, y \in H$. Also $\|P_C x - P_C y\|^2 \leq \frac{1}{2}\langle x - y, P_C x - P_C y\rangle$, for all $x, y \in H$. The following lemma give one of useful properties of metric projection mapping.

Lemma 2.2 [18]. Let $C \neq \phi$ be closed and convex subset of a Hilbert space H, and let P_C be the metric projection from H onto C. Then $\langle x - P_C x, P_C x - y\rangle \geq 0, \forall x \in H, y \in C$.

Let C be a nonempty, closed and convex subset of a Hilbert space H. Let $\{T_n : C \times C \to H\}$ be a $FNCM$. Then we say that $\{T_n\}$ satisfies then resolvent coupled property (denoted by RCP) if there exist a NCM, $T : C \times C \to H$ and two natural numbers n_0 and k such that: $||x - T(x, y)|| \leq k||x - T_n(x, y)||$ and $||y - T(y, x)|| \leq k||y - T_n(y, x)||$ for all $x, y \in C$ and $n \in N$ with $n \geq n_0$ and $CF(T) = \cap_{n=1}^{\infty} CF(T_n)$. The next example give sequence of mapping which satisfy $FRCP$.

Example 2.2. Let $H = \Re$ and $C = [0, 2.]$ Define $T_1 : C \times C \to \Re$ and $T_2 : C \times C \to \Re$ such as:

$$T_1(x,y) = \begin{cases} 0 & \text{if } x \in [0, \frac{3}{2}], y \in [0, 2], \\ 0 & \text{if } x \in [0, 2], y \in [0, \frac{3}{2}], \\ \frac{1}{2}(x+y) - \frac{3}{2} & \text{if } x \in (\frac{3}{2}, 2], y \in (\frac{3}{2}, 2], \end{cases}$$

and

$$T_2(x,y) = \begin{cases} 0 & \text{if } x \in [0, 1], y \in [0, 2], \\ 0 & \text{if } x \in [0, 2], y \in [0, 1], \\ \frac{1}{2}(x+y) - 1 & \text{if } x \in (1, 2], y \in (1, 2], \end{cases}$$

let $T_{2n-1}(x, y) = T_1(x, y)$ and $T_{2n}(x, y) = T_2(x, y)$ for all $n \in N$. Therefore,it is clear that: $CF(T_1) = CF(T_2) = \{(0, 0)\}$. Now , If $(x, y), (u, v) \in [\frac{3}{2}, 2] \times [\frac{3}{2}, 2]$, we get that,

4

$$\|T_1(x,y)-T_1(u,v)\|^2 = \|\frac{1}{2}(x+y)-\frac{1}{2}(u-v)\|^2$$
$$= \frac{1}{4}\|x-u+y-v\|^2 = \frac{1}{4}\langle x-u+y-v, x-u+y-v\rangle$$
$$= \frac{1}{2}\langle x-u+y-v, \frac{1}{2}(x-u)+\frac{1}{2}(y-v)\rangle$$
$$= \frac{1}{2}\langle x-u+y-v, T_1(x,y)-T_1(u,v)\rangle. \tag{2}$$

In the other hand, If $(x,y),(u,v)$ in other region , we get that the same result. Also, if $(x,y),(u,v)\in[1,2]\times[1,2]$, we have that:

$$\|T_2(x,y)-T_2(u,v)\|^2 = \|\frac{1}{2}(x+y)-\frac{1}{2}(u+v)\|^2$$
$$= \frac{1}{4}\|(x-u)+(y-v)\|^2 = \frac{1}{2}\langle x-u+y-v, \frac{1}{2}(x-u)+\frac{1}{2}(y-v)\rangle$$
$$= \frac{1}{2}\langle x-u+y-v, \frac{1}{2}(x+y-1)-\frac{1}{2}(u+v-1)\rangle.$$
$$= \frac{1}{2}\langle x-u+y-v, T_2(x,y)-T_2(u,v)\rangle. \tag{3}$$

By the same method, we can prove that: if $(x,y),(u,v)$ in other regions of the mapping T_2 we get the same above results. Thus, T_1 and T_2 are $FNCM$. Let $T(x,y)=T_1(x,y)$. Thus, T is NCM , $CFT=\{(0,0)\}$ and:

$$\|x-T(x,y)\|\le 2\|x-T_n(x,y)\|$$

Also, we get that:

$$\|y-T(y,x)\|\le 2\|y-T_n(y,x)\|$$

Then $\{T_n\}$ satisfies a RCP.

Lemma 2.3. Let C be a nonempty, closed and convex subset of a real Hilbert space H and let $T: C\times C\to C$ be even mapping in the second variable (i.e. $T(x,-y)=T(x,y)$, for all $(x,y)\in C\times C$) and $FNCM$ with $CF(T)\neq\emptyset$. Then $\langle x-T(x,y), T(x,y)-w_1\rangle\ge 0$ and $\langle y-T(y,x), T(y,x)-w_2\rangle\ge 0$ for all $(x,y\in C\times C$ and $(w_1,w_2)\in CF(T))$.

Proof. Since $(w_1,w_2)\in CF(T)$,we get that: for all $(x,y)\in C\times C$,

$$\|T(x,y)-T(w_1,w_2)\|^2 = \|T(x,y)-w_1\|^2 \le \frac{1}{2}\langle x-w_1+y-w_2, T(x,y)-w_1\rangle.$$

5

Therefore,

$$\|T(x,-y)-T(w_1,-w_2)\|^2 = \|T(x,y)-w_1\|^2 \leq \frac{1}{2}\langle x-w_1-y+w_2, T(x,y)-w_1\rangle.$$

Thus, we have that:

$$\begin{aligned}
&\langle x-T(x,y), T(x,y)-w_1\rangle = \langle x-w_1+y-w_2-T(x,y)+w_1-y+w_2, T(x,y)-w_1\rangle \\
&= \langle x-w_1+y-w_2, T(x,y)-w_1\rangle + \langle -T(x,y)+w_1-y+w_2, T(x,y)-w_1\rangle \\
&\geq 2\|T(x,y)-w_1\|^2 + \langle -T(x,y)+w_1-y+w_2, T(x,y)-w_1\rangle \\
&= \langle 2T(x,y)-2w_1, T(x,y)-w_1\rangle + \langle -T(x,y)+w_1-y+w_2, T(x,y)-w_1\rangle \\
&= \langle T(x,y)+w_1-y+w_2, T(x,y)-w_1\rangle \\
&= \langle T(x,y)-x, T(x,y)-w_1\rangle + \langle x-w_1-y+w_2, T(x,y)-w_1\rangle \qquad (4)
\end{aligned}$$

Hence , we get that:

$$2\langle x-T(x,y), T(x,y)-w_1\rangle \geq \langle x-w_1+y-w_2, T(x,y)-w_1\rangle \geq 2\|T(x,y)-w_1\|^2 \geq 0. \qquad (5)$$

Similarly, we can prove that: $\langle y-T(y,x), T(y,x)-w_2\rangle \geq 0$.

Definition 2.1 [26]. A space X is said to satisfy Opial's condition if for each sequence $\{x_n\}$ in X which $x_n \rightharpoonup x$, we have $\forall\ y \in X, y \neq x$ the following:

(i) $\liminf\limits_{n\to\infty} ||x_n - x|| < \liminf\limits_{n\to\infty} ||x_n - y||$,

(ii) $\limsup\limits_{n\to\infty} ||x_n - x|| < \limsup\limits_{n\to\infty} ||x_n - y||$.

We recall that: every Hilbert space has Opial's property [26] .

3 Main results

In this section, we prove the main weak convergence theorems for families of $FNCM$ in Hilbert spaces. To prove it, we use the following Lemma.

Lemma 3.1.([27]) Let T be a closed convex subset of a real Hilbert space H. Let T be a nonexpansive non self-mapping of K into H such that $F(T) \neq \emptyset$. Then $F(T) = F(P_K T)$.

Now, we prove the main theorem in this paper.

Theorem 3.1. Let $C \neq \phi$ be closed and convex subset of a Hilbert space H. Consider $\{T_n\} : C \times C \to H$ be a sequence of $FNCM$ and be even mappings in the second variable (i.e. $T_n(x, -y) = T_n(x, y)$, for all $(x, y) \in C \times C$) with $S := \bigcap_{n=1}^{\infty} CF(T_n) \neq \phi$. Let $\{\alpha_n\}$ be a sequence of real numbers in (0,2). Let $\{(x_n, y_n)\}$ be a sequence in $C \times C$ defined by:

$$\begin{cases} (x_1, y_1) \in C \text{ is chosen arbitrarily} \\ x_{n+1} := P_c((1-\alpha_n)x_n + \alpha_n T_n(x_n, y_n)), \forall\, n \in N, \\ y_{n+1} := P_c((1-\alpha_n)y_n + \alpha_n T_n(y_n, x_n)), \forall\, n \in N. \end{cases}$$

If $\{T_n\}$ satisfies RCP and $\liminf_{n\to\infty} \alpha_n(2-\alpha_n) > 0$, then $(x_n, y_n) \rightharpoonup (\overline{x}, \overline{y})$ where $(\overline{x}, \overline{y}) \in \bigcap_{n=1}^{\infty} CF(T_n)$.

Proof. Let $(w_1.w_2) \in S$, now we will prove that : $P_C(w_1) = w_1$, and $P_C(w_2) = w_2$. Consider the mapping : $f_{w_2} : C \to H$ such that: $f_{w_2} = T_1(x, w_2)$. Thus, we have that: $f_{w_2}(w_1) = T_1(w_1, w_2) = w_1$. Then, w_1 is fixed point of f_{w_2}. Therefore, let $x, y \in C$. Then, we get that:

$$\begin{aligned} &\|f_{w_2}(x) - f_{w_2}(y)\|^2 = \|T_1(x, w_2) - T_2(y, w_2)\|^2 \\ &= \frac{1}{2}\langle x - y, f_{w_2}(x) - f_{w_2}(y)\rangle \\ &\leq \frac{1}{2}\|x - y\|\|f_{w_2}(x) - f_{w_2}(y)\| \end{aligned} \tag{6}$$

Hence, we have that:

$$\|f_{w_2}(x) - f_{w_2}(y)\| \leq \frac{1}{2}\|x - y\| \leq \|x - y\|.$$

Then, f_{w_2} is nonexpansive mapping . By applying lemma 3.1 , we get that: $P_C(w_1) = w_1$. By the same method, let $f_{w_1} : C \to H$, which defined as: $f_{w_1}(x) = T_1(x, w_1)$. It is clear that: $f_{w_1}(w_2) = T_1(w_2, w_1) = w_2$. Thus, w_2 is fixed point of the mapping f_{w_1} and therefore f_{w_1} is nonexpansive. Then , we get that: $P_C(w_2) = w_2$.

Also, by applying lemma 2.3, we get that: $\langle x_n - T_n(x_n, y_n), T_n(x_n, y_n) - w_1\rangle \geq 0$ and

$\langle y_n - T_n(y_n,x_n), T_n(y_n,x_n) - w_2\rangle \geq 0$, for all $n \in N$. Then, we get that:

$$\|x_{n+1} - w_1\|^2 = \|P_C((1-\alpha_n)x_n + \alpha_n T_n(x_n,y_n)) - P_C(w_1)\|^2 \leq \|(1-\alpha_n)x_n + \alpha_n T_n(x_n,y_n) - w_1\|^2$$

$$= \|(x_n - w_1) + \alpha_n(T_n(x_n,y_n) - x_n)\|^2$$

$$= \langle (x_n - w_1) + \alpha_n(T_n(x_n,y_n) - x_n), (x_n - w_1) + \alpha_n(T_n(x_n,y_n) - x_n)\rangle$$

$$= \langle x_n - w_1, x_n - w_1\rangle + 2\alpha_n\langle x_n - w_1, T_n(x_n,y_n) - x_n\rangle + \alpha_n^2\langle T_n(x_n,y_n) - x_n, T_n(x_n,y_n) - x_n\rangle$$

$$= \|x_n - w_1\|^2 + \alpha_n^2\|x_n - T_n(x_n,y_n)\|^2 - 2\alpha_n\langle x_n - w_1, T_n(x_n,y_n) - x_n\rangle$$

$$= \|x_n - w_1\|^2 + \alpha_n^2\|x_n - T_n(x_n,y_n)\|^2 + 2\alpha_n\langle x_n - T_n(x_n,y_n), T_n(x_n,y_n) - x_n\rangle$$

$$-2\alpha_n\langle T_n(x_n,y_n) - w_1, T_n(x_n,y_n) - x_n\rangle$$

$$\leq \|x_n - w_1\|^2 - \alpha_n(2-\alpha_n)\|x_n - T_n(x_n,y_n)\|^2$$

for all $n \in N$. By doing the same steps , we get also:

$$\|y_{n+1} - w_2\|^2 \leq \|y_n - w_2\|^2 - \alpha_n(2-\alpha_n)\|y_n - T_n(y_n,x_n)\|^2;$$

for all $n \in N$. Then we have that, $\{x_n\}$ and $\{y_n\}$ are bounded sequence in C, , therefore, $\lim_{n\to\infty}\|x_n - w_1\|$ exist and $\lim_{n\to\infty}\|y_n - w_2\|$ exist. Therefore, we get that:

$$\lim_{n\to\infty} \alpha_n(2-\alpha_n)\|x_n - T_n(x_n,y_n)\| = 0.$$

Also, we have that:

$$\lim_{n\to\infty} \alpha_n(2-\alpha_n)\|y_n - T_n(y_n,x_n)\| = 0,$$

and since $\lim_{n\to\infty}\alpha_n(2-\alpha_n) > 0$, then, we have that:

$$\lim_{n\to\infty}\|x_n - T_n(x_n,y_n)\| = 0, \lim_{n\to\infty}\|y_n - T_n(y_n,x_n)\| = 0.$$

Since $\{T_n\}$ satisfies the RCP , then there exist NCM $T : C \times C \to C$ and $n_0, k \in N$ such that:

$$\|x - T(x,y)\| \leq k\|x - T_n(x,y)\|, \|y - T(y,x)\| \leq k\|y - T_n(y,x)\|,$$

therefore , we get that:

$$\|x_n - T(x_n,y_n)\| \leq k\|x_n - T_n(x_n,y_n)\|, \|y_n - T(y_n,x_n)\| \leq k\|y_n - T_n(y_n,x_n)\|,$$

8

for every $n \geq 0$. Then we have that :

$$\lim_{n\to\infty} \|x_n - T(x_n, y_n)\| = 0, \ \lim_{n\to\infty} \|y_n - T(y_n, x_n)\| = 0.$$

Since $\{x_n\}$ and $\{y_n\}$ are bounded, then there exist subsequences $\{x_{n_{k_1}}\}$ of $\{x_n\}$, $\{y_{n_{k_2}}\}$ of $\{y_n\}$ and $(u_1, u_2) \in C \times C$ such that : $x_{n_{k_1}} \rightharpoonup u_1$ and $y_{n_{k_2}} \rightharpoonup u_2$. By applying one of the useful property of the Hilbert space , we get:

$$\|T(u_1, u_2) - x_n\|^2 = \|T(u_1, u_2) - u_1\|^2 + 2\langle T(u_1, u_2) - u_1, u_1 - x_n\rangle + \|u_1 - x_n\|^2$$

Since, $\{x_n\}$ convergent weakly to u_1, then we obtain that:

$$\lim_{n\to\infty} \langle T(u_1, u_2) - u_1, u_1 - x_n\rangle = 0.$$

Hence, we find that:

$$\lim_{n\to\infty} \sup \|T(u_1, u_2) - x_n\|^2 = \|T(u_1, u_2) - u_1\|^2 + \lim_{n\to\infty} \sup \|u_1 - x_n\|^2.$$

Also , using the condition of coupled firmly non-expansive , we get that:

$$\lim_{n\to\infty} \|T(u_1, u_2) - x_n\| \leq \lim_{n\to\infty} \|T(u_1, u_2) - T(x_n, y_n)\| + \lim_{n\to\infty} \|x_n - T(x_n, y_n)\| \leq \lim_{n\to\infty} \| u_1 - x_n \| .$$

Thus, we have that:

$$\| T(u_1, u_2) - u_1 \|^2 + \lim_{n\to\infty} \sup \|u_1 - x_n\|^2 \leq \lim_{n\to\infty} sup \| u_1 - x_n \|^2 .$$

Then, we have that:

$$\| T(u_1, u_2) - u_1 \|^2 = 0.$$

Therefore, we get that: $T(u_1, u_2) = u_1$ and similarly we can prove that: $T(u_2, u_1) = u_2$. Thus , it clear that : $(u_1, u_2) \in S$. Now we prove that $\{(x_n, y_n)\} \rightharpoonup (\overline{x}, \overline{y}) \in S$. Let, $\{x_{n_l}\}$ and $\{x_{n_m}\}$ be subsequences of $\{x_n\}$ which converge weakly to $u, v \in C$ respectively. If $u \neq v$, from the the Opial property,

$$\begin{aligned} &\lim_{l\to\infty} \|x_{n_l} - u\| < \lim_{l\to\infty} \|x_{n_l} - v\| = \lim_{m\to\infty} \|x_{n_m} - v\| \\ &< \lim_{m\to\infty} \|x_{n_m} - u\| = \lim_{l\to\infty} \|x_{n_l} - u\||. \end{aligned} \tag{7}$$

9

This is contradiction. Therefore, $x_n \rightharpoonup \overline{x}$. By the same method, we can prove $y_n \rightharpoonup \overline{y}$. Thus $(x_n, y_n) \rightharpoonup (\overline{x}, \overline{y}) \in S$.

Corollary 3.1. Let $C \neq \phi$ be closed and convex subset of a Hilbert space H. Consider $T : C \times C \to H$ be $FNCM$ and even mapping in the second variable (i.e. $T(x, -y) = T(x, y)$, for all $(x, y) \in C \times C$) with $CF(T) \neq \phi$. Let $\{\alpha_n\}$ be a sequence of real numbers in (0,2). Let $\{(x_n, y_n)\}$ be a sequence in $C \times C$ defined by:

$$\begin{cases} (x_1, y_1) \in C \text{ is chosen arbitrarily} \\ x_{n+1} := P_c((1-\alpha_n)x_n + \alpha_n T(x_n, y_n)), \forall\, n \in N, \\ y_{n+1} := P_c((1-\alpha_n)y_n + \alpha_n T(y_n, x_n)), \forall\, n \in N. \end{cases}$$

If $\liminf_{n\to\infty} \alpha_n(2-\alpha_n) > 0$, then $(x_n, y_n) \rightharpoonup (\overline{x}, \overline{y})$ where $(\overline{x}, \overline{y}) \in CF(T)$.

Lemma 3.2. Let $C \neq \phi$ be closed and convex subset of a Hilbert space H. Consider $\{T_n\} : C \times C \to H$ be a sequence of $FNCM$ and be even mappings in the second variable (i.e. $T_n(x, -y) = T_n(x, y)$, for all $(x, y) \in C \times C$). Suppose that : $\sum_{n=1}^{n=\infty} \sup\{\|T_{n+1}(x, y) - T_n(x, y)\| < \infty : (x, y) \in C \times C\}$. Then $\{T_n(x, y)\}$ converges strongly to some point of $C \times C$. In the other hand, if $T : C \times C \to C$ defined by: $T(x, y) = \lim_{n\to\infty} T_n(x, y)$, for all $(x, y) \in C \times C$. Then $\lim_{n\to\infty} \sup\{\|T(x, y) - T_n(x, y)\| : (x, y) \in C \times C\} = 0$.

Proof. First, we will prove that $\{T_n(x, y)\}$ is Cauchy sequence for all $(x, y) \in C \times C$. Let $i, j \in N$ and $i > j$. we get that:

$$\begin{aligned} &\|T_i(x, y) - T_j(x, y)\| \leq \| \sup\{\|T_i(x, y) - T_j(x, y)\| : (x, y) \in C \times C\} \\ &\leq \sup\{\|T_i(x, y) - T_{i-1}(x, y)\| : (x, y) \in C \times C\} + \sup\{\|T_{i-1}(x, y) - T_j(x, y)\| : (x, y) \in C \times C\} \leq .... \\ &\leq \sum_{i=1}^{\infty} \sup\{\|T_{n+i}(x, y) - T_n(x, y)\| : (x, y) \in C \times C\} \end{aligned}$$

Let $i \to \infty$, Then we get that $\{T_n(x, y)\}$ is a Cauchy sequence. Thus $\{T_n(x, y)\}$ converges strongly to some point of $C \times C$. Also, we have :

$$\|T(x, y) - T_j(x, y)\| \leq \sum_{i=1}^{\infty} \sup\{\|T_{n+i}(x, y) - T_n(x, y)\| : (x, y) \in C \times C\}$$

Thus, we have that: $\lim_{j\to\infty} \sup\{\|T(x, y) - T_j(x, y)\| : (x, y) \in C \times C\} = 0$.

Theorem 3.2. Let $C \neq \phi$ be closed and convex subset of a Hilbert space H. Consider $\{T_n\} : C \times C \to H$ be a sequence of $FNCM$ and be even mappings in the second variable (i.e. $T_n(x, -y) = T_n(x, y)$, for all

$(x, y) \in C \times C)$ with $S := \bigcap_{n=1}^{\infty} CF(T_n) \neq \phi$. Let $\{\alpha_n\}$ be a sequence of real numbers in (0,2). Let $\{(x_n, y_n)\}$ be a sequence in $C \times C$ defined by:

$$\begin{cases} (x_1, y_1) \in C \text{ is chosen arbitrarily} \\ x_{n+1} := P_c((1-\alpha_n)x_n + \alpha_n T_n(x_n, y_n)), \forall\, n \in N, \\ y_{n+1} := P_c((1-\alpha_n)y_n + \alpha_n T_n(y_n, x_n)), \forall\, n \in N. \end{cases}$$

If $\{T_n\}$ satisfies the property: $\sum_{n-1}^{n=\infty} \sup\{\|T_{n+1}(x, y) - T_n(x, y)\| < \infty : (x, y) \in C \times C\}$ and $\liminf_{n\to\infty} \alpha_n(2 - \alpha_n) > 0$, then $(x_n, y_n) \rightharpoonup (\overline{x}, \overline{y})$ where $(\overline{x}, \overline{y}) \in \bigcap_{n=1}^{\infty} CF(T_n)$.

Proof. First, we will apply lemma 3.2. Define $T : C \times C \to H$ by $T(x, y) = \lim_{n\to\infty} T_n(x, y)$

$$\begin{aligned} &\|T(x, y) - T(u, v)\| = \| \lim_{n\to\infty} T_n(x, y) - \lim_{n\to\infty} T_n(u, v)\| \\ &= \lim_{n\to\infty} \|T_n(x, y) - T_n(u, v)\| \leq \lim_{n\to\infty} \frac{1}{2}(\|x - u\| + \|y - v\|). \end{aligned} \tag{8}$$

For all $(x, y), (u, v) \in C \times C$. Hence T is a NCM . Therefore, we get that:

$$\lim_{n\to\infty} \sup\{\|T(x, y) - T_n(x, y)\| : (x, y) \in B\} = 0, \tag{9}$$

for each bounded subset B of $C \times C$. Then by doing the same steps as in Theorem 3.1, we get that

$$\|x_n - w_1\|^2 \leq \|x_n - w_1\|^2 - \alpha_n(2 - \alpha_n)\|x_n - T_n(x_n, y_n)\|^2. \tag{10}$$

therefore,

$$\|y_n - w_2\|^2 \leq \|y_n - w_2\|^2 - \alpha_n(2 - \alpha_n)\|y_n - T_n(y_n, x_n)\|^2$$

Thus , we have that:

$$\lim_{n\to\infty} \|T(x_n, y_n) - T_n(x_n, y_n)\| = 0. \tag{11}$$

Then, we get the following:

$$\|x_n - T(x_n, y_n)\| \leq \|x_n - T_n(x_n, y_n)\| + \|T(x_n, y_n) - T_n(x_n, y_n)\|.$$

therefore, we get that:

$$\lim_{n\to\infty} \|x_n - T(x_n, y_n)\| = 0. \tag{12}$$

11

By doing the same step we can prove that:

$$\lim_{n\to\infty} \|y_n - T(y_n, x_n)\| = 0.$$

Agian , by doing the same steps as the proof of Theorem 3.1, we get the proof of Theorem 3.2.

Let C be a nonempty closed convex subset of a Hilbert space H and let $\{T_n\}$ and Γ be two families of NCM mappings of $C \times C$ into C and even in the second variable, such that: $\emptyset \neq CF(\Gamma) = \bigcap_{n=1}^{\infty} CF(T_n)$, where $CF(T_n)$ is the set of all coupled fixed points of $\{T_n\}$ and $CF(\Gamma)$ is the set of all common coupled fixed points of Γ. We gave the following Condition.

Condition 3.1. For each bounded sequence $\{(x_n, y_n)\}$ of $C \times C$, if we have that: $\lim_{n\to\infty} \|x_n - T_n(x_n, y_n)\| = 0$ and $\lim_{n\to\infty} \|y_n - T_n(y_n, x_n)\| = 0$, then $\lim_{n\to\infty} \|x_n - T(x_n, y_n)\| = 0$ and $\lim_{n\to\infty} \|y_n - T(y_n, x_n)\| = 0$ for all $T \in \Gamma$.

Theorem 3.3. Let H be a Hilbert space, C be a nonempty, closed and convex subset of H. Consider $\{T_n\}$: $C \times C \to C$ be a sequence of $FNCM$ mappings. Let Γ be a family of NCM of $C \times C$ into C , which satisfies $\emptyset \neq CF(\Gamma) \subseteq \bigcap_{n=1}^{\infty} CF(T_n)$ and condition (3.1). Let $\{\alpha_n\}$ be a sequence of real numbers in (0,2), and $\{(x_n, y_n)\}$ be a sequence in $C \times C$ defined by:

$$\begin{cases} (x_1, y_1) \in C \text{ is chosen arbitrarily} \\ x_{n+1} := P_c((1-\alpha_n)x_n + \alpha_n T_n(x_n, y_n)), \forall\, n \in N, \\ y_{n+1} := P_c((1-\alpha_n)y_n + \alpha_n T_n(y_n, x_n)), \forall\, n \in N. \end{cases}$$

If $\liminf_{n\to\infty} \alpha_n(2-\alpha_n) > 0$, then $(x_n, y_n) \rightharpoonup (\overline{x}, \overline{y})$ where $(\overline{x}, \overline{y}) \in \bigcap_{n=1}^{\infty} CF(T_n)$.

Proof. By doing the same steps as in the proof of Theorem 3.1, we get $\{(x_n, y_n)\}$ is bounded and

$$\lim_{n\to\infty} \|x_n - T_n(x_n, y_n)\| = 0,$$

also,

$$\lim_{n\to\infty} \|y_n - T_n(y_n, x_n)\| = 0.$$

By condition (3.1),

$$\lim_{n\to\infty} \|x_n - T(x_n, y_n)\| = 0, \lim_{n\to\infty} \|y_n - T(y_n, x_n)\| = 0,$$

12

for all $T \in \Gamma$. Since $\{(x_n, y_n)\}$ is bounded , there exist a subsequence $\{(x_{n_k}, y_{n_k})\}$ of $\{(x_n, y_n)\}$ and $(u_1, u_2) \in C \times C$ such that: $x_{n_k} \to u_1$ and $y_{n_k} \to u_2$. By lemma 2.6, we have that $(u_1, u_2) \in CF(T)$ for all $T \in \Gamma$. Thus we have that: $(u_1, u_2) \in CF(\Gamma) \subseteq \bigcap_{n=1}^{\infty} CF(T_n)$. Then the same steps as in the proof of Theorem 3.1 lead to $(x_n, y_n) \rightharpoonup (\overline{x}, \overline{y})$, where $(\overline{x}, \overline{y}) \in \bigcap_{n=1}^{\infty} CF(T_n)$.

Acknowledgments

The author would like to express their gratitude to King Khalid University, Saudi Arabia, for providing administrative and technical support.

References

[1] P. Combettes, Strong convergence of block-iterative outer approximation methods of convex optimization, SIAM Journal On Control and Optimization, 38 (2000) 538 -565.

[2] O.Chadli, Q. Ansari and S. Al-Homidan, Existence of solution for nonlinear implicit differential equationsL An equilibrium problems approach, Numerical Functional Analysis and OPtimization, 37 (2016) 1385 -1419.

[3] L. Arias, P. Combettes, J. Pesquet and N. Pustelink, Proximal algorithms for multicomponent image recovery problem, Journal of Mathematical Imaging and Vision, 41 (2011) 3 -22.

[4] O. Chadi, I. Konnov and J. Yao, Descent methods for equilbrium problems in a Banach space, Comput. Math. Appl. , 41 (1999) 435 -453.

[5] S. Atsushiba and W. Takahashi, Strong convergence theorems for a finite family of nonexpansive mappings and applications, Indian J. Math., 38 (2000) 538 -565.

[6] C. Roland and R. Varadhan, New iterative schemes for nonlinear fixed point problems with applications to problems with bifurcations and incomplete-data problems, Applied Numerical Mathematics, 55 (2005) 215 -226.

[7] W. R. Mann, Mean value methods in iteration, Proc. Amer. Math. Soc. 4 (1953) 504 -510.

[8] B. Halpern, Fixed points of nonexpanding maps, Bull. Amer. Math. Soc. 73 (1967) 957-961.

[9] H. H. Bauschke, The approximation of compositions of nonexpansive mappings in Hilbert space, J. Math. Anal. Appl. 202 (1996) 250- 159.

[10] H. K. Xu and R. G. Ori, An implicit iteration process for nonexpansive mappings, Numer. Funct. Anal. And Optimiz. 22 (2001) 767- 773.

[11] Y. Kimura, W. Takahashiand M. Toyoda, Convergence to common fixed points of a finite family of nonexpansive mappings, Arch. Math. 84 (2005) 350-363.

[12] W. M. Kozlowski, Fixed point iteration processes for asymptotic point-wise nonexpansive mappingsin Banach spaces, J. Math. Anal. Appl. 377 (2011) 43-52.

[13] J. Balooee, Weak and strong convergence theorems of modified Lshikawa iteration for infinitely countable family of pointwise asymptotically nonexpansivemappings in Hilbert spaces, Arab journal of Mathematical Sciences, 17 (2011) 153-169.

[14] H. Manaka and W. Takahashi, Weak convergence theorems for maximal monotone operators with nonspreading mappings in a Hilbert spaces, CUBO A Mathemtical Joutnal, 13 (2011) 11-24.

[15] S. Suantai, W. Cholamjiak and P. Cholamjiak, An implicit iteration process for solving a fixed point problem of a finite family of multi-valued mappings in Banach spaces, Applied Mathematics Letters, 25 (2012) 1656-1660.

[16] Y. Shehu, Convergence theorems for maximal montone operators and fixed point problems in Banach spaces, applied Mathematics and Computation, 239 (2014) 285-298 .

[17] L. C. Ceng, C. T. Pang and C. F. Wen, Implicit and explicit iterative methods for mixed equilibria with constraints of system generalized equilbtia and Hierarchical fixed point problem, Journal of Inequalities and Applications(2015) 2015: 280.

[18] C. S. Chuang and W. Takahashi, Weak convergence Theorems for families of nonlinear mappings with generalized parameters, Numerical Functional Analysis and Optimization, 36(2015) 41: 54.

[19] T. Nabil and A. H. Soliman, weak convergence theorems of explicit iteration process with errors and applications in optimization, J. Ana. Num. Theor., 5(2017) 81: 89.

[20] D. Guo and V. Lakshmikantham, Coupled fixed points of nonlinear operators with applications, Non-linear Anal., 11(1987) 623: 632.

[21] T. Bhaskar and V. Lakshmikantham, Fixed point theorems in partially ordered metric spaces and applications, Non-linear Anal., 65(2006) 1379: 1393.

[22] V. Lakshmikantham and L. Ciric, Coupled fixed point theorems for nonlinear contractions in partially ordered metric spaces, Non-linear Anal., 70(2009) 4341: 4341.

[23] N. Luong and N. Thuan , Coupled fixed point theorems for mixed monotone mappings and an application to integral equations, Computers and Mathematics with Applications, 62(2011) 4238: 4248.

[24] W. Sintunavarat and P. Kumam , Coupled fixed point results for nonlinear integral equations , Journal of Egyptian Mathematical Society, 21(2013) 266: 272.

[25] A. H. Soliman , Coupled fixed poit theorem for nonexpansive one parameter semigroup , J. Adv. Math. Stud., 7(2014) 28: 37 .

[26] Z. Opial, Weak convergence of the sequence of successive approximations for nonexpansive mappings, Bull. Amer. Soc., 73(1967) , 591- 597.

[27] S. Matsushita and D. Kuroiwa, Approximation of fixed points of nonexpansive nonself mappings, Sci. Math. Jpn. 57(2003) 171-176.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 27, NO.5, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC

Dynamics of the zeros of analytic continued the second kind q-Euler polynomial

Cheon Seoung Ryoo

Department of Mathematics, Hannam University, Daejeon 306-791, Korea

Abstract : In this paper we study that the second kind q-Euler numbers $E_{n,q}$ and q-Euler Euler polynomials $E_{n,q}(x)$ are analytic continued to $E_q(s)$ and $E_q(s,w)$. We investigate the new concept of dynamics of the zeros of analytic continued polynomials. Finally, we observe an interesting phenomenon of 'scattering' of the zeros of $E_q(s,w)$.

Key words : Second kind Euler polynomial, Euler Zeta function, Analytic Continuation, complex zeros, dynamics.

2000 Mathematics Subject Classification : 11B68, 11S40, 11S80.

1. Introduction

Several mathematicians have studied the Bernoulli numbers and polynomials, Euler numbers and polynomials, q-Bernoulli numbers and polynomials, q-Euler numbers and polynomials, the second kind Euler numbers and polynomials(see [1-11]). These numbers and polynomials posses many interesting properties and arising in many areas of mathematics and physics. Throughout this paper, we always make use of the following notations:$\mathbb{N} = \{1, 2, 3, \cdots\}$ denotes the set of natural numbers, $\mathbb{N}_0 = \{0, 1, 2, 3, \cdots\}$ denotes the set of nonnegative integers, $\mathbb{Z}$ denotes the set of integers, $\mathbb{R}$ denotes the set of real numbers, $\mathbb{C}$ denotes the set of complex numbers. We introduced the second kind q-Euler numbers $E_{n,q}$ and polynomials $E_{n,q}(x)$ and investigate their properties(see [6]). Let q be a complex number with $|q| < 1$. We define the second kind q-Euler numbers $E_{n,q}$ and polynomials $E_{n,q}(x)$ as follows:

$$F_q(t) = \frac{2e^t}{qe^{2t}+1} = \sum_{n=0}^{\infty} E_{n,q}\frac{t^n}{n!}, \tag{1}$$

$$F_q(x,t) = \left(\frac{2e^t}{qe^{2t}+1}\right)e^{xt} = \sum_{n=0}^{\infty} E_{n,q}(x)\frac{t^n}{n!}. \tag{2}$$

By the above definition (2) and Cauchy product, we have

$$\begin{aligned}\sum_{l=0}^{\infty} E_{l,q}(x)\frac{t^l}{l!} &= \left(\frac{2e^t}{e^{2t}+1}\right)e^{xt} = \sum_{n=0}^{\infty} E_{n,q}\frac{t^n}{n!}\sum_{m=0}^{\infty} x^m\frac{t^m}{m!}\\ &= \sum_{l=0}^{\infty}\left(\sum_{n=0}^{l} E_{n,q}\frac{t^n}{n!}x^{l-n}\frac{t^{l-n}}{(l-n)!}\right) = \sum_{l=0}^{\infty}\left(\sum_{n=0}^{l}\binom{l}{n}E_{n,q}x^{l-n}\right)\frac{t^l}{l!}.\end{aligned}$$

By using comparing coefficients $\dfrac{t^l}{l!}$, we have the following theorem.

Theorem 1. For $n \in \mathbb{N}_0$, one has

$$E_{n,q}(x) = \sum_{k=0}^{n}\binom{n}{k}E_{k,q}x^{n-k}.$$

By Theorem 1 and some calculations, we have

$$\begin{aligned}\int_a^b E_{n,q}(x)dx &= \sum_{l=0}^{n}\binom{n}{l}E_{l,q}\int_a^b x^{n-l}dx = \sum_{l=0}^{n}\binom{n}{l}E_{l,q}\left.\frac{x^{n-l+1}}{n-l+1}\right|_a^b\\ &= \frac{1}{n+1}\sum_{l=0}^{n+1}\binom{n+1}{l}E_{l,q}\left. x^{n-l+1}\right|_a^b.\end{aligned}$$

By Theorem 1, we get

$$\int_a^b E_{n,q}(x)dx = \frac{E_{n+1,q}(b)-E_{n+1,q}(a)}{n+1}. \tag{3}$$

Since $E_{n,q}(0)=E_{n,q}$, by (3), we have the following theorem.

Theorem 2. For $n\in\mathbb{N}$, one has

$$E_{n,q}(x) = E_{n,q} + n\int_0^x E_{n-1,q}(t)dt.$$

By using computer, the second kind q-Euler polynomials $E_{n,q}(x)$ can be determined explicitly. A few of them are

$$\begin{aligned}
E_{0,q}(x) &= \frac{2}{1+q},\\
E_{1,q}(x) &= \frac{2}{(1+q)^2} - \frac{2q}{(1+q)^2} + \frac{2x}{(1+q)},\\
E_{2,q}(x) &= \frac{4}{(1+q)^3} - \frac{8q}{(1+q)^3} + \frac{4q^2}{(1+q)^3} - \frac{2}{(1+q)^2} - \frac{2q}{(1+q)^2} + \frac{4x}{(1+q)^2} - \frac{4qx}{(1+q)^2} + \frac{2x^2}{(1+q)}.
\end{aligned}$$

2. Analytic Continuation of the second kind q-Euler numbers and the q-Euler Zeta function

By using the second kind q-Euler numbers and polynomials, the second kind q-Euler zeta function and Hurwitz q-Euler zeta functions are defined. From (1), we note that

$$\left.\frac{d^k}{dt^k}F_q(t)\right|_{t=0} = 2\sum_{n=0}^{\infty}(-1)^n q^n(2n+1)^k = E_{k,q}, (k\in\mathbb{N}).$$

By using the above equation, we are now ready to define the second kind q-Euler zeta functions.

Definition 3. For $s\in\mathbb{C}$ with $\mathrm{Re}(s)>0$, define the second kind q-Euler zeta function by

$$\zeta_E(s) = 2\sum_{n=0}^{\infty}\frac{(-1)^n q^n}{(2n+1)^s}.$$

Notice that the Euler zeta function can be analytically continued to the whole complex plane, and these q-zeta function have the values of the q-Euler numbers at negative integers. That is, the second kind q-Euler numbers are related to the second kind q-Euler zeta function as

$$\zeta_{E,q}(-k) = E_{k,q}.$$

By using (2), we note that

$$\frac{d^k}{dt^k}F_q(x,t)\bigg|_{t=0} = 2\sum_{n=0}^{\infty}(-1)^n q^n (2n+x+1)^k, (k \in \mathbb{N}), \tag{4}$$

and

$$\left(\frac{d}{dt}\right)^k \left(\sum_{n=0}^{\infty} E_{n,q}(x)\frac{t^n}{n!}\right)\bigg|_{t=0} = E_{k,q}(x), \text{ for } k \in \mathbb{N}. \tag{5}$$

By (4) and (5), we are now ready to define the Hurwitz q-Euler zeta functions.

Definition 4. We define the Hurwitz q-zeta function $\zeta_{E,q}(s,x)$ for $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 0$ by

$$\zeta_{E,q}(s,x) = 2\sum_{n=0}^{\infty}\frac{(-1)^n q^n}{(2n+x+1)^s}.$$

Note that $\zeta_{E,q}(s,x)$ is a meromorphic function on $\mathbb{C}$. Relation between $\zeta_{E,q}(s,x)$ and $E_{k,q}(x)$ is given by the following theorem.

Theorem 5. For $k \in \mathbb{N}$, we have

$$\zeta_{E,q}(-k,x) = E_{k,q}(x). \tag{6}$$

We now consider the function $E_q(s)$ as the analytic continuation of the second kind q-Euler numbers. From the above analytic continuation of the second kind q-Euler numbers, we consider

$$\begin{aligned} &E_{n,q} \mapsto E_q(s), \\ &\zeta_{E,q}(-n) = E_{n,q} \mapsto \zeta_{E,q}(-s) = E_q(s). \end{aligned} \tag{7}$$

All the second kind q-Euler number $E_{n,q}$ agree with $E_q(n)$, the analytic continuation of the second

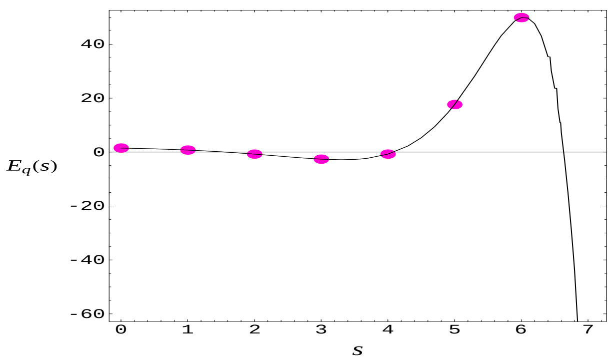


Figure 1: The curve $E_q(s)$ runs through the points of all $E_{n,q}$

kind q-Euler numbers evaluated at n(see Figure 1). Consider

$$E_{n,q} = E_q(n) \text{ for } n \geq 0 \tag{8}$$

In Figure 1, we choose $q = 1/3$. In fact, we can express $E'_q(s)$ in terms of $\zeta'_{E,q}(s)$, the derivative of $\zeta_{E,q}(s)$. Consider

$$\begin{aligned} &E_q(s) = \zeta_{E,q}(-s), \\ &E'_q(s) = -\zeta'_{E,q}(-s) \\ &E'_q(2n+1) = -\zeta'_{E,q}(-2n-1) \text{ for, } n \in \mathbb{N}_0. \end{aligned} \tag{9}$$

From the relation (9), we can define the other analytic continued half of the second kind q-Euler numbers

$$\begin{aligned} &E_q(s) = \zeta_{E,q}(-s), \quad E_q(-s) = \zeta_{E,q}(s) \\ &\Rightarrow E_q(-n) = \zeta_{E,q}(n), n \in \mathbb{N}. \end{aligned} \tag{10}$$

By (10), we have

$$\lim_{n\to\infty} E_q(-n) = \zeta_{E,q}(n) = 2.$$

The curve $E_q(s)$ runs through the points $E_{-n,q} = E_q(-n)$ and grows ~ 2 asymptotically as $-n \to \infty$(see Figure 2).

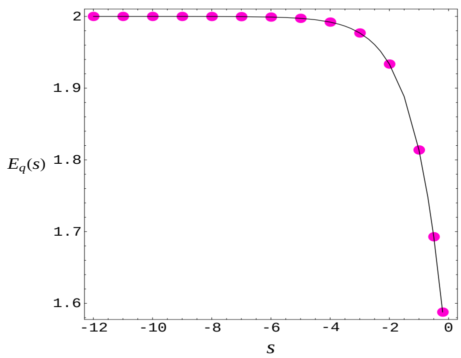


Figure 2: The curve $E_q(s)$ runs through the points $E_{-n,q}$ for $q = \frac{1}{3}$

3. Dynamics of the zeros of analytic continued polynomials

Our main purpose in this section is to investigate the new concept of dynamics of the zeros of analytic continued polynomials. Let $\Gamma(s)$ be the gamma function. The analytic continuation can be then obtained as

$$\begin{aligned} &n \mapsto s \in \mathbb{R}, x \mapsto w \in \mathbb{C}, \\ &E_{k,q} \mapsto E_q(k+s-[s]) = \zeta_{E,q}(-(k+(s-[s]))), \\ &\binom{n}{k} \mapsto \frac{\Gamma(1+s)}{\Gamma(1+k+(s-[s]))\Gamma(1+[s]-k)} \\ &\Rightarrow E_{n,q}(w) \mapsto E_q(s,w) = \sum_{k=-1}^{[s]} \frac{\Gamma(1+s)E_q(k+s-[s])w^{[s]-k}}{\Gamma(1+k+(s-[s]))\Gamma(1+[s]-k)} \\ &= \sum_{k=0}^{[s]+1} \frac{\Gamma(1+s)E_q((k-1)+s-[s])w^{[s]+1-k}}{\Gamma(k+(s-[s]))\Gamma(2+[s]-k)}, \end{aligned} \tag{11}$$

where $[s]$ gives the integer part of s, and so $s-[s]$ gives the fractional part. By (11), we obtain analytic continuation of the second kind q-Euler polynomials for $q=1/3$. Consider

$$\begin{aligned}
&E_{0,q}(w)\approx 1.5,\\
&E_q(1,w)\approx 0.75+1.5w,\\
&E_q(2,w)\approx -0.75+1.5w+1.5w^2,\\
&E_q(2.2,w)\approx -1.14137+1.13863w+1.84171w^2+0.15595w^3,\\
&E_q(2.4,w)\approx -1.54674+0.60395w+2.13491w^2+0.38568w^3,\\
&E_q(2.6,w)\approx -1.94844-0.12719w+2.33741w^2+0.69096w^3,\\
&E_q(2.8,w)\approx -2.32024-1.07449w+2.39690w^2+1.06697w^3,\\
&E_q(3,w)\approx -2.625-2.25w+2.25w^2+1.5w^3.
\end{aligned}\tag{12}$$

By using (12), we plot the deformation of the curve $E_q(2,w)$ into the curve of $E_q(3,w)$ via the real analytic continuation $E_q(s,w), 2\le s\le 3, w\in\mathbb{R}$(see Figure 3). In [6], we observe that $E_q(n,w), w\in$

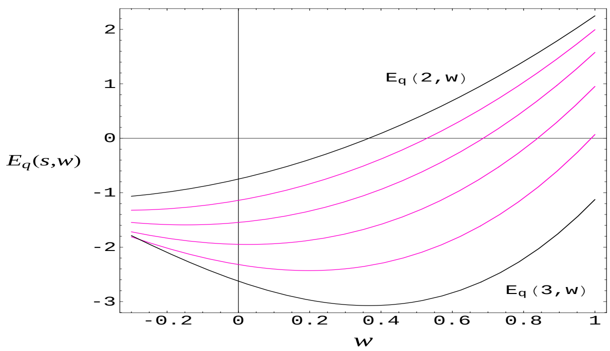


Figure 3: The curve of $E_q(s,w), 2\le s\le 3, -0.3\le w\le 1$

$\mathbb{C}$, has $Im(w)=0$ reflection symmetry analytic complex functions(see Figure 4). The zeros of $E_q(n,w)$ will also inherit these symmetries.

$$\text{If } E_q(n,w_0)=0, \text{ then } E_q(n,w_0^*)=0,$$

where $*$ denotes complex conjugation.

For $n\in\mathbb{N}_0$, it is easy to deduce that the second kind q-Euler polynomials $E_{n,q}(x)$ satisfy

$$\sum_{n=0}^{\infty}E_{n,q^{-1}}(-x)\frac{(-t)^n}{n!}=\frac{2e^{-t}}{q^{-1}e^{-2t}+1}e^{(-x)(-t)}=\frac{2qe^t}{e^{2t}+1}e^{xt}=q\sum_{n=0}^{\infty}E_{n,q}(x)\frac{t^n}{n!}.$$

By using comparing coefficients $\frac{t^n}{n!}$ in the above equation, we have the following theorem.

Theorem 6 (Theorem of complement). For any positive integer n, we have

$$E_{n,q}(x)=(-1)^nq^{-1}E_{n,q^{-1}}(-x).\tag{13}$$

The question is as follows: what happens with the reflexive symmetry (13), when one considers the second kind q-Euler polynomials? Prove that $E_q(n,w), w\in\mathbb{C}$, has not $Re(w)=0$ reflection

symmetry analytic complex functions(see Figure 4). Next, we investigate the beautiful zeros of the $E_q(s,w)$ by using a computer. We plot the zeros of $E_q(s,w)$ for $s = 9, 9.3, 9.7, 10, q = 1/3$, and $w \in \mathbb{C}$(Figure 4). In Figure 4(top-left), we choose $s = 9$. In Figure 4(top-right), we choose $s = 9.3$.

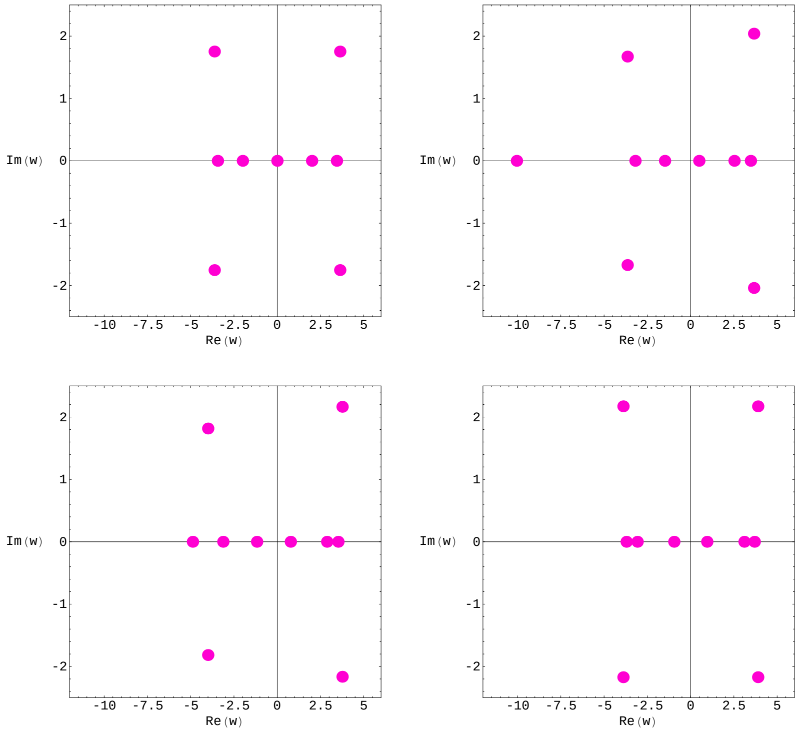


Figure 4: Zeros of $E_q(s,w)$ for $s = 9, 9.3, 9.7, 10$

In Figure 4(bottom-left), we choose $s = 9.7$. In Figure 4(bottom-right), we choose $s = 10$.

Stacks of zeros of $E_q(s,w)$ for $s = n + 1/3, 1 \leq n \leq 50$, forming a 3D structure are presented(Figure 5).

In Figure 5(top-right), we draw y and z axes but no x axis in three dimensions. In Figure 5(bottom-left), we draw x and y axes but no z axis in three dimensions. In Figure 5(bottom-right), we draw x and z axes but no y axis in three dimensions. However, we observe that $E_q(n,w), w \in \mathbb{C}$, has $Im(w) = 0$ reflection symmetry analytic complex functions(see Figure 4 and Figure 5).

Our numerical results for approximate solutions of real zeros of $E_q(s,w), q = 1/3$, are displayed. We observe a remarkably regular structure of the complex roots of the second kind q-Euler polynomials. We hope to verify a remarkably regular structure of the complex roots of the second kind q-Euler polynomials(Table 1).

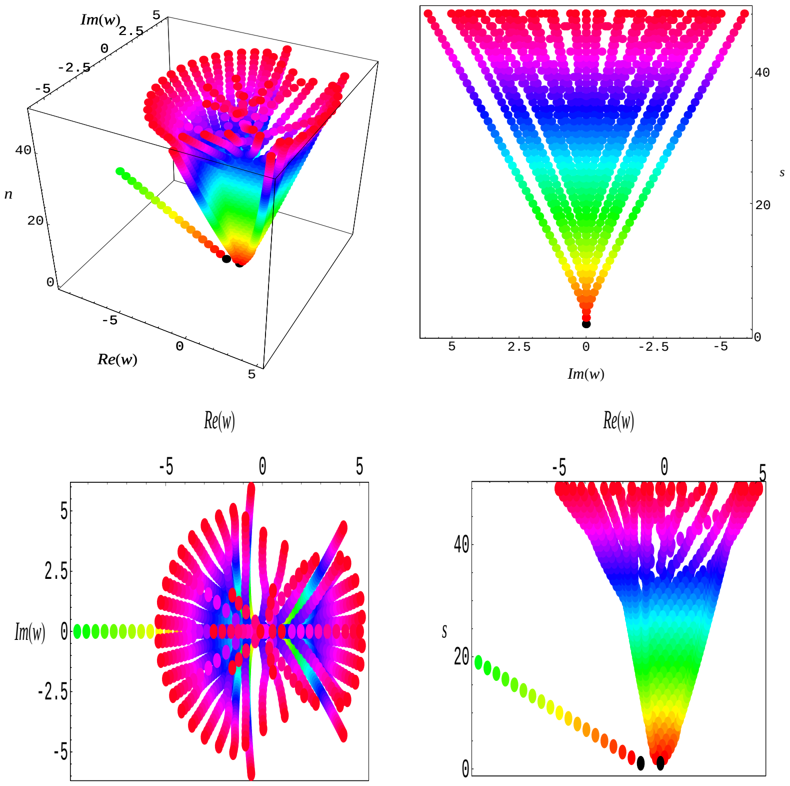


Figure 5: Stacks of zeros of $E_q(s,w)$ for $1 \leq n \leq 50$

Table 1. Numbers of real and complex zeros of $E_q(s,w)$

s	real zeros	complex zeros
1.5	2	0
2.5	3	0
3.5	4	0
4.5	3	2
5.5	4	2
6.5	5	2
7.5	6	2
8.5	3	6
9	3	6
9.3	4	6
9.5	4	6
9.8	4	6
10	4	6

Next, we calculated an approximate solution satisfying $E_q(s,w), q = 1/3, w \in \mathbb{R}$. The results

are given in Table 2.

Table 2. Approximate solutions of $E_q(s,w)=0, w\in\mathbb{R}$

s	w
6	$-2.25291,\quad -0.499167,\quad 1.50121,\quad 2.89899$
6.5	$-8.19021,\quad -1.97235,\quad -0.106447,\quad 1.90361,\quad 3.03711$
7	$-2.65744,\quad -1.71446,\quad 0.286584,\quad 2.31062,\quad 3.2536$
7.5	$-9.25827,\quad -2.51685,\quad -1.32105,\quad 0.679634,\quad 2.83991,\quad 3.19538$
8	$-0.927418,\quad 1.07258$
8.5	$-10.3265,\quad -0.534533,\quad 1.46541$
9	$-2.1399,\quad -0.141641,\quad 1.85831$
9.2	$-35.7141,\quad -1.98173,\quad 0.0155236,\quad 2.01523$
9.5	$-11.3949,\quad -1.74785,\quad 0.251276,\quad 2.2499$
9.7	$-6.68645,\quad -1.59132,\quad 0.408446,\quad 2.40587$
10	$-3.09896,\quad -1.3558,\quad 0.644202,\quad 2.64146$

In Figure 6, we plot the real zeros of the the second kind q-Euler polynomials $E_q(s,w)$ for $s=n+\dfrac{1}{3}, 1\le n\le 30, q=1/3$, and $w\in\mathbb{C}$ (Figure 7). In Figure 6(right), we choose $E_q(s,w)$ for

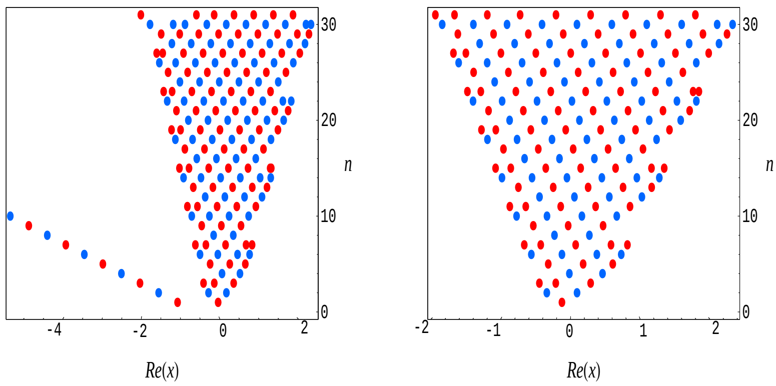


Figure 6: Real zeros of $E_q(s,w)$

$s=n+\dfrac{1}{3}, 1\le n\le 30$. In Figure 6(left), we choose $E_q(n,w)$ for $1\le n\le 30$.

The second kind q-Euler polynomials $E_{n,q}(w)$ is a polynomials of degree n. Thus, $E_{n,q}(w)$ has n zeros and $E_{n+1,q}(w)$ has $n+1$ zeros. When discrete n is analytic continued to continuous parameter s, it naturally leads to the question: How does $E_q(s,w)$, the analytic continuation of $E_{n,q}(w)$, pick up an additional zero as s increases continuously by one? This introduces the exciting concept of the dynamics of the zeros of analytic continued polynomials-the idea of looking at how the zeros

move about in the w complex plane as we vary the parameter s. To have a physical picture of the motion of the zeros in the complex w plane, imagine that each time, as s increases gradually and continuously by one, an additional real zero flies in from positive infinity along the real positive axis, gradually slowing down as if " it is flying through a viscous medium ". More studies and results in this subject we may see references [5], [6], [7], [10].

Acknowledgement: This work was supported by the National Research Foundation of Korea(NRF) grant funded by the Korea government(MEST) (No. 2017R1A2B4006092).

REFERENCES

1. R. P. Agarwal, J. Y. Kang, C. S. Ryoo, Some properties of (p, q)-tangent polynomials, J. Computational Analysis and Applications, **24** (2018), 1439-1454.

2. I. N. Cangul, H. Ozden, and Y. Simsek, A new approach to q-Genocchi numbers and their interpolation functions, Nonlinear Analysis, **71** (2009), 793-799.

3. M. S. Kim, S. Hu, On p-adic Hurwitz-type Euler Zeta functions, J. Number Theory (132) (2012), 2977-3015.

4. T. Kim, C.S. Ryoo, L.C. Jang, S.H. Rim, Exploring the q-Riemann Zeta function and q-Bernoulli polynomials, Discrete Dynamics in Nature and Society, **2005**, 171-181.

5. C. S. Ryoo, A numerical computation of the roots of q-Euler polynomials, J. Comput. Anal. Appl., **12** (2010), 148-156.

6. C. S. Ryoo, A numerical computation of the structure of the roots of the second kind q-Euler polynomials, J. Comput. Anal. Appl., **14** (2012), 321-327.

7. C. S. Ryoo, Analytic Continuation of Euler Polynomials and the Euler Zeta Function Discrete Dynamics in Nature and Society, Volume **2014** (2014), Article ID 568129, 6 pages.

8. C. S. Ryoo, On the (p, q)-analogue of Euler zeta function, J. Appl. Math. & Informatics, **35** (2017), 303-311.

9. C. S. Ryoo, On degenerate q-tangent polynomials of higher order, J. Appl. Math. & Informatics **35** (2017), 113-120.

10. C. S. Ryoo, R .P. Agarwal, Some identities involving q-poly-tangent numbers and polynomials and distribution of their zeros, Advances in Difference Equations **2017:213** (2017), 1-14.

11. Y. Simesk, V. Kurt and D. Kim, New approach to the complete sum of products of the twisted (h, q)-Bernoulli numbers and polynomials, J. Nonlinear Math. Phys., **14** (2007), 44-56.

Remarks on the blow-up for damped Klein-Gordon equations with a gradient nonlinearity *

Hongwei Zhang, Jian Dang, Qingying Hu
(Department of Mathematics, Henan University of Technology, Zhengzhou 450001, China)

Abstract We consider initial boundary value problem for a class of damped Klein-Gordon type wave equations with a gradient nonlinearity and derive sufficient conditions for finite time blow-up of its solutions. To prove blow-up of the solution, we use eigenfunction method combining with a modification of Glassey's inequality. This extend the early results.

Keywords Klein-Gordon equations; blow-up; initial-boundary value problem; gradient nonlinearity

AMS Classification (2010): 35L20,35B44.

1 Introduction

The aim of this paper is to give some sufficient conditions for blow-up of solutions to the following damped Klein-Gordon type wave equations with a gradient nonlinearity

$$u_{tt} - \Delta u + cu_t = f(u, \nabla u),\ in\ \Omega \times (0, T), \tag{1.1}$$

$$u(x, t) = 0,\ x \in \partial\Omega, t \in (0, T), \tag{1.2}$$

$$u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x),\ x \in \Omega; \tag{1.3}$$

where Ω is a bounded domain in R^n with sufficiently smooth boundary $\partial\Omega$, $f(u, \nabla u) = a|u|^{p-1}u+ b|\nabla u|^q$,$p, q > 1, a, b \in R, ab < 0$, and $c > 0$.

Nonlinear wave equations of the form (1.1) arise in differential geometry, controllability theory of partial differential equations, and in various areas of physics(see [1] and its references). The derivative Klein-Gordon type wave problem (1.1)-(1.3) can be viewed as a simplification of the Boussinesq equation [2, 3, 4, 1] with higher order spatial derivative terms appearing neither in the linear part nor in the nonlinearity. It belongs to the family of nonlinear wave equations of the form $u_{tt}+Au = \rho(u)\nabla u+g(u)$. This family of wave equations have as an important subclass the Yang-Mills-type equations with $\rho(u) = u$ and $g(u) = u^3$. Yang-Mills-type wave equations have the same scaling as the cubic nonlinear wave equation, but are more difficult technically because of the derivative term $u\nabla u$. Other important examples of the type equations include the Maxwell-Klein-Gordon and Yang-Mills-Higgs equations in the Lorenz gauge at least, as well as the simplified model equations of these (see [1]). If $b = 0$ and $a \neq 0$, then equation (1.1) is the standard Klein-Gordon wave problem. The standard Klein-Gordon wave problem in the critical exponent has been studied by many authors. In this case, the blowup behavior of solutions is by

*Corresponding author:Zhang H.W., Email: whz661@163.com

now fairly well understood, and various sufficient conditions for blowup have been provided and qualitative properties have also been investigated (see for example [5, 6, 7, 8, 9, 10, 11, 12, 13, 14], to cite just a few). However, very little is known in literature concerning the asymptotic dynamics exhibited by the derivative Klein-Gordon type wave equations of the form (1.1) and in space dimensions greater than one or two (see [1]). Recently, D'Abbicco [15] proved global existence of small data solutions to the following Cauchy problem for the doubly dissipative wave equation with power nonlinearity $|\nabla u|^p$:

$$u_{tt} - \Delta u + u_t - \Delta u_t = |\nabla u|^p,$$

for $p > 1 + \frac{1}{n+1}$, in any space dimension $n \geq 1$, and he also derive optimal energy estimates and $L^1 - L^1$ estimates for the solution to the semilinear problems. Willie[1] studied a nonlinear wave problem of the form

$$u_{tt} - \Delta u + du_t = -\rho|\nabla u|^2 + \gamma|u|^{p-1}u, \rho \geq 0, \gamma > 0,$$

its linear problem well-posedness, behaviour of the spectrum of the wave differential operator in varied damping and diffusion constants, as well as the asymptotic dynamics defined by the derivative Klein-Gordon type wave problem.

We mention also some related mathematical work involving the derivative nonlinearity term in the literature. Ebihara [16, 17, 18] established global existence of classical solutions and asymptotic behavior of solutions of the following nonlinear wave equation

$$u_{tt} - \Delta u = f(u, u_t, \nabla u), \tag{1.4}$$

where $f(u, u_t, \nabla u) = -u^p - |\nabla u|^{2r} - u_t^q$ (or $f(u, u_t, \nabla u) = -u^p|\nabla u|^q u_t^r$, here $p, q, r > 0$). When $f(u, u_t, \nabla u) = -a(x)\beta(u_t, \nabla u)$ in (1.4), where $\beta(\lambda_1, \lambda_2, ..., \lambda_n)\lambda_1 \geq 0$, Slemrod [19], Vancostenoble [20] and Haraux [21] proved the weak asymptotic stabilization of solutions. Quite recently, Nakao [22, 23, 24, 25, 26] considered the nonlinear wave equations of the form

$$u_{tt} - \Delta u + \rho(x, u_t) = f(u, u_t, \nabla u), \tag{1.5}$$

and he proved the global existence and decay of solutions.

On the other hand, relatively little is known on the blowup for nonlinearities with a dependence on spatial derivatives of u. As far as we know, the previous studies of blow-up of solutions of (1.1) were performed in [27, 28, 29, 30]. In [27], Sideris gives blow-up of small data solutions in finite time for the Cauchy problem in three dimensions when the nonlinear gradient term $a|u|^{p-1}u + b|\nabla u|^q$ in (1.1) is replaced with term $f(u, u_t, \nabla u) = a^2|\nabla u|^2 + b^2|\Delta u|^2$. To our knowledge, this is the first blow-up result for nonlinear wave equation when the nonlinear perturbation term depends on the derivatives of u. Then the result was extended by Schaeffer [28] and Rammaha [29, 30]. However, very little is known in the literature concerning the blow-up of solutions for initial boundary problem of equation (1.1) and such a method in [27, 28, 29, 30] cannot applied this case. Levine [7] has pointed that the eigenfunction method can easily be modified to include nonlinear terms of the form $f(u, \nabla u)$ provided that for all $s \in R^1, p \in R^n, f(s, p) \geq G(s)$, where $G(s)$ is a convex function and the function $G(s)$ satisfy the

following conditions:(1)$G(s)-(\lambda+1)s$ is nonnegative and nondecreasing on (s_0,∞) for some $s_0>0$; (2)$\int_0^s G(\rho)d\rho-\frac{1}{2}s^2$ is nondecreasing on (s_0,∞);(3)$[\int_0^s G(\rho)d\rho-\frac{1}{2}s^2]^{\frac{-1}{2}}$ is integrable at $+\infty$ for s. However, when $f(u,\nabla u)=a|u|^{p-1}u+b|\nabla u|^q, a,b\in R$, we can't find any function $G(s)$ such that $f(s,p)\geq G(s)$.

Motivated by the eigenfunction method in [5, 7], the main purpose of this paper is to give sufficient conditions for finite time blow-up of solutions for the initial boundary value problem of equation (1.1) under certain conditions. We will generalize Glassey's inequality (Lemma 1.1 in [5],and see also [7]), and get sufficient conditions for blow-up of solutions to problem (1.1)-(1.3) for various $a,b\in R$ and $ab<0$ by eigenfunction method. In this sense, we extend the result [5, 7]. This method applies also to the case of the equation (1.1) with Neumann boundary condition and it remains valid for more general equation

$$u_{tt}-\Delta u+cu_t=|u|^{p-1}u+f(u,|\nabla u|), \tag{1.6}$$

where f is locally Lipschitz continuous and satisfies certain growth condition (see remark 2.4).

2 Main results

Throughout this paper we assume all function spaces are considered over real field and their notations and definitions are same as those [31]. By the usual Galerkin method and similar to the proof in [16], we can obtain regular solution in the local sense. Now we extend Lemma 1.1 in [5](see also [7]) to the following lemma, which play an essential role in this paper.

Lemma 1 Let $\phi(t)\in C^2$ satisfy

$$\phi_{tt}+k_1\phi_t\geq h(\phi), \quad t\geq 0 \tag{2.1}$$

with $\phi(0)=\alpha>0, \phi_t(0)=\beta>0$, where $k_1>0$. Suppose that $h(s)\geq 0$ for all $s\geq\alpha$. If $\delta_0=k_1\int_\alpha^{+\infty}[\beta^2+2\int_\alpha^s h(\rho)d\rho]^{-\frac{1}{2}}ds<1$, then $\phi_t(t)>0$ where $\phi_t(t)$ exists and $\lim_{t\to T^-}\phi(t)=+\infty$ where $T\leq T^*=-\frac{1}{k_1}ln(1-\delta_0)$.

Proof Because $\phi(0)=\alpha>0$ and $\phi_t(0)=\beta>0$ then there exist an interval $[0,T_0)$ such that $\phi_t(t)>0$ and $\phi(t)>\alpha$ for $t\in[0,T_0)$. If it is false, let

$$t_1=\inf\{t:\phi(t)=\alpha\}, t_2=\inf\{t:\phi_t(t)=0\}.$$

If $t_2<t_1$, taking into account the condition (2.1) and the fact that $h(s)\geq 0$ for all $s\geq\alpha$, we have

$$\frac{d}{dt}(e^{k_1t}\phi_t)=e^{k_1t}(\phi_{tt}+k_1\phi_t)\geq e^{k_1t}h(\phi)>0.$$

Thus $\phi_t(t_2)>e^{-k_1t_2}\phi_t(0)>0$, which contradicts $\phi_t(t_2)=0$, and so we have $t_2\geq t_1$. Furthermore, we have $\phi_t(t)>0$ for $t\in[0,t_1)$. In this case, we get that $\phi(t_1)=\phi(0)+\int_0^{t_1}\phi_t(s)ds>\phi(0)=\alpha>0$, this is a contradiction of the fact $\phi(t_1)=\alpha$. Thus, there exist an interval $[0,T_0)$ such that $\phi_t(t)>0$ and $\phi(t)>\alpha$ for $t\in[0,T_0)$.

A multiplication of (2.1) by $2e^{2k_1t}\phi_t(t)$ gives

$$2e^{2k_1t}\phi_t\phi_{tt}+2k_1e^{2k_1t}(\phi_t)^2\geq 2e^{2k_1t}h(\phi)\phi_t,$$

that is,

$$\frac{d}{dt}[e^{2k_1t}(\phi_t)^2] \geq 2e^{2k_1t}h(\phi)\phi_t \geq 2h(\phi)\phi_t = 2\frac{d}{dt}\int_\alpha^\phi h(s)ds. \quad (2.2)$$

Integrating (2.2) from 0 to t yields

$$e^{2k_1t}(\phi_t)^2 - (\phi_t(0))^2 \geq 2\int_\alpha^\phi h(s)ds,$$

since $\phi_t > 0$, hence

$$\phi_t \geq e^{-k_1t}(\beta^2 + 2\int_\alpha^\phi h(s)ds)^{-\frac{1}{2}}. \quad (2.3)$$

We may separate variables and integrate over $(0,t)$ to obtain

$$1 - e^{-k_1t} \leq k_1\int_\alpha^{+\infty}(\beta^2 + 2\int_\alpha^y h(s)ds)^{-\frac{1}{2}}dy = \delta_0.$$

Therefore we get the result.

We consider the following spectral problem

$$\Delta w + \lambda w = 0 \ in \ \Omega, \quad (2.4)$$

$$w = 0, \ on \ \partial\Omega. \quad (2.5)$$

It is well known that problem (2.4)-(2.5) has the smallest eigenvalue $\lambda_1 > 0$ and the corresponding normalized eigenfunction $w_1 > 0$ in Ω, $\int_\Omega w_1(x)dx = 1$. Then we denote

$$k_0 = (\int_\Omega \frac{|\nabla w_1|^{\frac{q}{q-1}}}{w_1^{1/(q-1)}}dx)^{\frac{q-1}{q}}. \quad (2.6)$$

Theorem 2 Suppose $q > 1, a = 0$ and $b > 0$. Let $u(x,t)$ be a regular solution of problem (1.1)-(1.3). Suppose that the following conditions are satisfied:

$$\int_\Omega u_0(x)w_1(x)dx = \alpha, \int_\Omega u_1(x)\psi_1(x)dx = \beta,$$

where $\alpha > \frac{k_0^{q/(q-1)}}{\lambda_1} > 0, \beta > 0$, and that $(\frac{\lambda_1}{k_0})^q s^q - \lambda_1 s$ is a nongeative, nondecreasing function for $s \geq \alpha$. If $\delta_1 = c\int_\alpha^{+\infty}[\beta^2 + 2\int_\alpha^s[(\frac{\lambda_1}{k_0})^q\rho^q - \lambda_1\rho]d\rho]^{-\frac{1}{2}}ds < 1$, then the solution of problem (1.1)-(1.3) blows up in a finite time.

Proof Let

$$U(t) = \int_\Omega u(x,t)w_1(x)dx.$$

Then $U(0) = \alpha > 0, U_t(0) = \beta > 0$ and as it follows from (1.1)-(1.3), $U(t)$ satisfies

$$U_{tt} + cU_t + \lambda_1 U = \int_\Omega |\nabla u|^q w_1 dx. \quad (2.7)$$

By (2.4) and Holder inequality, we get

$$\lambda_1 U \leq |\int_\Omega u(x,t)\lambda_1 w_1(x)dx| = |\int_\Omega u(x,t)\Delta w_1(x)dx|$$
$$= |\int_\Omega \nabla u\nabla w_1 dx| \leq \int_\Omega |\nabla u||\nabla w_1|dx = \int_\Omega(|\nabla u|w_1^{1/q})\frac{|\nabla w_1|}{w_1^{1/q}}dx$$
$$\leq (\int_\Omega \frac{|\nabla w_1|^{\frac{q}{q-1}}}{w_1^{1/(q-1)}}dx)^{\frac{q-1}{q}}(\int_\Omega |\nabla u|^q w_1 dx)^{\frac{1}{q}} = k_0(\int_\Omega |\nabla u|^q w_1 dx)^{\frac{1}{q}},$$

4

that is to say

$$\int_\Omega |\nabla u|^q w_1 dx \geq (\frac{\lambda_1}{k_0})^q U^q. \tag{2.8}$$

Therefore, from (2.7) and inequality(2.2), we obtain the ordinary differential inequality

$$U_{tt} + cU_t \geq (\frac{\lambda_1}{k_0})^q U^q - \lambda_1 U, \tag{2.9}$$

with $U(0) = \alpha > 0, U_t(0) = \beta > 0$. Denote $h(s) = (\frac{\lambda_1}{k_0})^q s^q - \lambda_1 s$, since $h(s) > 0$ for $s \geq \alpha$, it follows from Lemma 6 that $\lim\limits_{t\to T_0^-} U(t) = \infty$, for some $T_0 \leq T^* = -\frac{1}{c}ln(1-\delta_1)$. Furthermore, since $U(t) > 0$, we have $U(t) = |U(t)| \leq sup_\Omega|u(x,t)| \int_\Omega w_1 dx \leq sup_\Omega|u(x,t)|$, and we get $\lim\limits_{t\to T_0^-} ||u||_p^p = \infty, \forall 1 \leq p \leq \infty$, for some $T_0 \leq T^* = -\frac{1}{c}ln(1-\delta_1)$, which proves the theorem.

Theorem 3 Suppose $q \geq 2, 0 < p < 2, a < 0$ and $b > 0$. Let $u(x,t)$ be a regular solution of problem (1.1)-(1.3). Suppose that the following conditions are satisfied:

$$\int_\Omega u_0(x) w_1(x) dx = \alpha_0, \int_\Omega u_1(x)\psi_1(x)dx = \beta_0,$$

where $\beta_0 > 0$ and α_0 is the positive root of the equation $b(\frac{\lambda_1}{k_0})^q s^q - |a|s^p - \lambda_1 s = 0$. If $\delta_2 = c\int_\alpha^{+\infty}[\beta^2 + 2\int_\alpha^s[(\frac{\lambda_1}{k_0})^q\rho^q - |a|\rho^p - \lambda_1\rho]d\rho]^{-\frac{1}{2}}ds < 1$, then the solution of problem (1.1)-(1.3) blows up in a finite time.

Proof Let

$$U(t) = \int_\Omega u(x,t)w_1(x)dx.$$

Then $U(0) = \alpha_0 > 0, U_t(0) = \beta_0 > 0$ and as it follows from (1.1)-(1.3), $U(t)$ satisfies

$$U_{tt} + cU_t + \lambda_1 U = a\int_\Omega |u|^p w_1 dx + b\int_\Omega |\nabla u|^q w_1 dx. \tag{2.10}$$

Then (2.8) and the inequality $\int_\Omega |u|^p w_1 dx \geq U^p$ yield the ordinary differential inequality

$$U_{tt} + cU_t \geq b(\frac{\lambda_1}{k_0})^q U^q - |a|U^p - \lambda_1 U = h_2(U), \tag{2.11}$$

with $U(0) = \alpha_0 > 0, U_t(0) = \beta_0 > 0$. Since $h_2(s) > 0$ for $s \geq \alpha_0$, then the rest of the proof is similar to the proof of Theorem 2 and the proof is complete.

Theorem 4 Suppose $p \geq 2, 0 < q < 2, b < 0$ and $a > 0$. Let $u(x,t)$ be a regular solution of problem (1.1)-(1.3). Suppose that the following conditions are satisfied:

$$\int_\Omega u_0(x)w_1(x)dx = \alpha_1, \int_\Omega u_1(x)\psi_1(x)dx = \beta_1,$$

where $\beta_1 > 0$ and α_1 is the positive root of the equation $as^p - |b|(\frac{\lambda_1}{k_0})^q s^q - \lambda_1 s = 0$. If $\delta_3 = c\int_\alpha^{+\infty}[\beta^2 + 2\int_\alpha^s[a\rho^p - |b|(\frac{\lambda_1}{k_0})^q\rho^q - \lambda_1\rho]d\rho]^{-\frac{1}{2}}ds < 1$, then the solution of problem (1.1)-(1.3) blows up in a finite time.

Proof Similar to the proof Theorem 3, $U(t)$ satisfies

$$U_{tt} + cU_t + \lambda_1 U = a \int_\Omega |u|^p w_1 dx + b \int_\Omega |\nabla u|^q w_1 dx, \tag{2.12}$$

with $U(0) = \alpha_1 > 0, U_t(0) = \beta_1 > 0$, and then we have

$$U_{tt} + cU_t \geq aU^p - |b|(\frac{\lambda_1}{k_0})^q U^q - \lambda_1 U = h_3(U), \tag{2.13}$$

with $U(0) = \alpha_1 > 0, U_t(0) = \beta_1 > 0$. Since $h_3(s) > 0$ for $s \geq \alpha_0$, then the rest of the proof is similar to the proof of Theorem 3 and the proof is complete.

Remark 1 By Theorem 2-Theorem 4, we can also prove that the blowup result holds under the similar initial conditions for the case $a > 0, p > 2, p > q$ or $b > 0, q > 2, q > p$.

Remark 2 The same results hold if the boundary condition is of the form $a\frac{\partial u}{\partial n} + bu = 0$.

Remark 3 The results remain true when $\triangle u$ is replaced by p-Laplace operator $div(|\nabla u|^p \nabla u)$.

Remark 4 The method remains valid for more general equation (1.6), where f is locally Lipschitz continuous and satisfies the growth condition $f(u, |\nabla u|) \leq C(1 + |u|^k + |\nabla u|^q)$.

ACKNOWLEDGEMENTS This work is supported by the National Natural Science Foundation of China (No.11601122).

References

[1] R. Willie. Spectral analysis, an integral mild solution formula and asymptotic dynamics of the derivative Klein-Gordon type wave equation. Journal of Abstract Differential Equations and Applications, 2011, 2(1):54-83.

[2] D. Henry. Geometric theory of semilinear parabolic problems. Lecture notes in mathematics 840. Springer Verlag. New York 1981.

[3] G. Sell, and Y. You. Dynamics of evolutionary equations. Applied Mathematical Sciences 143. Springer-Verlag 2002.

[4] Y. You. Global dynamics of 2D Boussinesq equations. Nonlinear Analysis. T.M.A., 1997, 30: 4643-4654.

[5] R. T. Glassey. Blow-up theorems for nonlinear wave equations. Mathematische Zeitschrift, 1973, 132(3): 183-203.

[6] H.A.Levine. Instability and nonexistence of global solutions to nonlinear wave equations of the form $Pu_{tt} = Au + F(u)$. Transactions of the American Mathematical Society, 1974, 192: 1-21.

[7] H.A.Levine. Nonexistence of global solutions to some properly and improperly posed problem of mathematical physics: The method of unbounded fourier coefficients. Mathematische Annalen, 1975, 214(3): 205-220.

[8] L.E.Payne, D.Sattinger. Saddle points and instability of nonlinear hyperbolic equations. Israel Journal of Mathematics, 1975, 22(3-4): 273-303.

[9] V.K.Kalantarov, O.A.Ladyzhenskaya. The occurrence of collapse for quasilinear equations of parabolic and hyperbolic types. Journal of Soviet Mathematics, 1978, 10(1): 77-102.

[10] V. Georgiev, G.Todorova. Existence of a solution of the wave equation with nonlinear damping and source terms. Journal of Differential Equations, 1994, 109: 295-308.

[11] S. A. Messaoudi. Blow-up in a nonlinear damped wave equation. Mathematische Nachrichten, 2001, 231(1): 105-111.

[12] V.Barbu, I.Lasiecka, M.A.Rammaha. On nonlinear wave equations with degenerate damping and source terms. Transactions of the American Mathematical Society, 2005, 357(7): 2571-2611.

[13] E.Vitillaro. Global nonexistence theorems for a class of evolution equationwith dissipation. Archive for Rational Mechanics and Analysis, 1999, 149(2): 155-182.

[14] M.O.Korpusov, A.A.Panin. Blow-up of solutions of an abstract Cauchy problem for a formally hyperbolic equation with double non-linearity. Izvestiya: Mathematics, 2014, 78(5): 937-985.

[15] Marcello D'Abbicco. $L^1 - L^1$ estimates for a doubly dissipative semilinear wave equation. Nonlinear Differential Equations and Applications, 2017, 24(1):article5,1-23.

[16] Y. Ebihara. On the local classical solution of the mixed problem for nonlinear wave equation. Mathematical reports of College of General Education, Kyushu University, 1974, 9(2): 43-65.

[17] Y. Ebihara. On the equations with variable coefficients $u_{tt} - u_{xx} = F(x, t, u, u_x, u_t)$. Funkcialaj Ekvacioj, 1977, 20(1): 77-95.

[18] Y. Ebihara. Nonlinear wave equations with variable coefficients. Funkcialaj Ekvacioj, 1979, 22(2): 143-159.

[19] M. Slemrod. Weak asymptotic decay via a "relaxed invariance principle" for a wave equation with nonlinear, nonmonotone damping. Proceedings of the Royal Society of Edinburgh Sect. A, 1989, 113(1): 87-97.

[20] J. Vancostenoble. Weak asymptotic decay for a wave equation with gradient dependent damping. Asymptotic Analysis,2001, 26: 1-20.

[21] A. Haraux. Remarks on weak stabilization of semilinear wave equations. ESAIM: Control, Optimisation and Calculus of Variations, 2001,6: 553-560.

[22] M. Nakao. Existence of global decaying solution to the Cauchy problem for a nonlinear dissipative wave equations of Klein-Gordon type with a derivative nonlinearity. Funkcialaj Ekvacioj, 2012, 55: 457-477.

[23] M. Nakao. Global existence and decay for nonlinear dissipative wave equations with a derivative nonlinearity. Nonlinear Analysis, 2012, 75: 2236-2248.

[24] M.Nakao. Existence of global decaying solutions to the exterior problem for the Klein-Gordon equation with a nonlinear localized dissipation and a derivative nonlinearity. Journal of Differential Equations, 2013, 255: 3940-3970.

[25] M. Nakao. Global solutions to the initial-boundary value problem for the quasilinear viscoelastic equation with a derivative nonlinearity. Opuscula Mathatics, 2014, 34(3): 569-590.

[26] M. Nakao. Existence of global decaying solutions to the initial boundary value problem for the quasilinear wave equation of p-Laplacian type with Kelvin-Voigt dissipation and a derivative nonlinearity. Kyushu Journal of Mathematics, 2016,70: 63-82.

[27] T. Sideris. Global behavior of solutions to nonloinear wave equations in three dimensions. Communications in Partial Differential Equations, 1983, 8(12):1291-1323.

[28] J. Schaeffer. Finite-time blow-up for $u_{tt} - \Delta u = H(u_r, u_t)$ in two space dimensions. Communications in Partial Differential Equations, 1986, 11(5):513-543.

[29] M.A. Rammaha. Finite-time blow-up for nonloinear wave equations in high dimensions. Communications in Partial Differential Equations, 1987, 12(6):677-700.

[30] M.A. Rammaha. On the blowing up solutions to nonlinear wave equations in two space dimensions. Journal fur Die Reine Und Angewandte Mathematik, 1988, 391: 55-64.

[31] J.L.Lions. Quelques methodes de resolution des problemes aur limites non lineaires. Dunod Gauthier-villars, Paris,1969.

The γ-fuzzy topological semigroups and γ-fuzzy topological ideals

Cheng-Fu Yang

(School of Mathematics and Statistics of Hexi University, Zhangye Gansu,734000, P. R. China)

Abstract: Based on the concepts of semigroup and Chang's fuzzy topological space, this paper gives the defines of the γ-fuzzy topological semigroups, γ-fuzzy topological left ideals (γ-fuzzy topological right ideals, γ-fuzzy topological intrinsic ideals and γ-fuzzy topological double ideals) and discusses the fuzzy continuous homomorphic image and the fuzzy continuous homomorphic inverse image of them.

Keywords: Fuzzy topological space; γ-fuzzy topological semigroup; γ-fuzzy topological ideal; F-continuous; homomorphic image and homomorphic inverse image

1.Introduction

Since Zadeh [15] introduced fuzzy sets and fuzzy set operations in 1965. The concept of fuzzy sets has been widely used in various fields. For example, in 1968, Chang [2] applied the fuzzy set to topological space to give fuzzy topological space. After that, Pu and Liu [9,10] introduced neighborhood structure of a fuzzy point, moore-smith convergence and product and quotient spaces in fuzzy topological space. Afterwards Rosenfeld [12] formulated the elements of the theory of fuzzy groups and Foster [4] introduced the fuzzy topological groups. In 2011, Tanay et al. [13] gave the notion of fuzzy soft topological spaces and studied neighborhood and interior of a fuzzy soft set and then used these to characterize fuzzy soft open sets. Then Nazmul and Samanta [8] introduced the fuzzy soft topological groups. Subsequently, Coker [3] used the notion of intuitionistic fuzzy sets gave by Atanassov in [1] to introduce the notion of intuitionistic fuzzy topological spaces and obtained several preservation properties and some characterizations concerning fuzzy compactness and fuzzy connectedness. After that, Kul [6] introduced the intuitionistic fuzzy topological groups.

Recently, Rajesh gave the notion of γ-fuzzy topological group in [11] and discussed the connection between fuzzy topological group and γ-fuzzy topological group. Based on this idea, in this paper, we give the concepts of the γ-fuzzy topological semigroups, γ-fuzzy topological left ideals (right ideals, intrinsic ideals and double ideals) and then discuss the homomorphic image and inverse image of them.

2.Preliminary

Definition 2.1.[15] A fuzzy set $\tilde{A}$ in X is a set of ordered pairs:

$$\tilde{A} = \{(x, \tilde{A}(x)) : x \in X\}$$

Where $\tilde{A}(x): X \to I = [0,1]$ is a mapping and $\tilde{A}(x)$ states the grade of belongness of x in $\tilde{A}$. The family of all fuzzy sets in *X* is denoted by I^X.

Particularly, the fuzzy set

$$x_\lambda(y) = \begin{cases} \lambda, if\ x = y \\ 0, otherwise \end{cases}, \forall y \in X$$

is called a fuzzy point in *X*, denoted by x_λ.

Deflnition 2.2.[15] Let $\tilde{A}$, $\tilde{B}$ be two fuzzy sets of I^X

1) $\tilde{A}$ is contained in $\tilde{B}$ if and only if $\tilde{A}(x) \le \tilde{B}(x)$ for every $x \in X$.

2)The union of $\tilde{A}$ and $\tilde{B}$ is a fuzzy set $\tilde{C}$, denoted by $\tilde{A} \bigcup \tilde{B} = \tilde{C}$, whose membership function $\tilde{C}(x) = \tilde{A}(x) \vee \tilde{B}(x)$ for every $x \in X$.

3)The intersection of $\tilde{A}$ and $\tilde{B}$ is a fuzzy set $\tilde{C}$, denoted by $\tilde{A} \bigcap \tilde{B} = \tilde{C}$, whose membership function $\tilde{C}(x) = \tilde{A}(x) \wedge \tilde{B}(x)$ for every $x \in X$.

4)The complement of $\tilde{A}$ is a fuzzy set, denoted by $\tilde{A}^c$, whose membership function $\tilde{A}^c(x) = 1 - \tilde{A}(x)$ for every $x \in X$.

Definition 2.3.[2] Let *X*, *Y* be two nonempty sets, *f* a function from *X* to *Y* and $\tilde{B}$ a fuzzy set in *Y* with membership function $\tilde{B}(y)$. Then the inverse of $\tilde{B}$, written as $f^{-1}(\tilde{B})$, is a fuzzy set in *X* whose membership function is defined by $f^{-1}(\tilde{B})(x) = \tilde{B}(f(x))$ for all *x* in *X*.

Conversely, let $\tilde{A}$ be a fuzzy set in *X* with membership function $\tilde{A}$(x). The image of $\tilde{A}$, written as f ($\tilde{A}$), is a fuzzy set in *Y* whose membership function is given by

$$f(\tilde{A})(y) = \begin{cases} \vee_{x \in f^{-1}(y)} \tilde{A}(x), if\ f^{-1}(y) \ne \phi \\ 0, \qquad otherwise, \end{cases} \quad \text{for } \forall y \in Y$$

where $f^{-1}(y) = \{x \mid f(x) = y\}$.

Proposition 2.4.[2] Let *f* be a function from *X* to *Y*. Then:

(1) $f^{-1}[\tilde{B}^c] = [f^{-1}\tilde{B}]^c$ for any fuzzy set $\tilde{B}$ in *Y*.

(2) $f[A^c] \supset [f(A)]^c$ for any fuzzy set $\tilde{A}$ in *X*.

(3) $\tilde{B}_1 \subset \tilde{B}_2 \Rightarrow f^{-1}[\tilde{B}_1] \subset f^{-1}[\tilde{B}_2]$, where $\tilde{B}_1$, $\tilde{B}_2$ are fuzzy sets in *Y*.

(4) $\tilde{A}_1 \subset \tilde{A}_2 \Rightarrow f[\tilde{A}_1] \subset f[\tilde{A}_2]$, where $\tilde{A}_1$, $\tilde{A}_2$ are fuzzy sets in *X*.

(5) $\tilde{B} \supset f[f^{-1}[\tilde{B}]]$ for any fuzzy set $\tilde{B}$ in Y.

(6) $\tilde{A} \subset f[f^{-1}[\tilde{A}]]$ for any fuzzy set $\tilde{A}$ in X.

Proposition 2.5. Let f be a function from X to Y. Then:

(1) $f(\tilde{A} \cap \tilde{B}) \subseteq f(\tilde{A}) \cap f(\tilde{B})$ and $f(\tilde{A} \cup \tilde{B}) = f(\tilde{A}) \cup f(\tilde{B})$ for any $\tilde{A}, \tilde{B} \in I^X$.

(2) $f^{-1}(\tilde{A} \cap \tilde{B}) = f^{-1}(\tilde{A}) \cap f^{-1}(\tilde{B})$ and $f^{-1}(\tilde{A} \cup \tilde{B}) = f^{-1}(\tilde{A}) \cup f^{-1}(\tilde{B})$ for any $\tilde{A}, \tilde{B} \in I^X$.

Proof. This proposition can be directly verified by the Definition 2.3.

Next example shows that $f(\tilde{A} \cap \tilde{B}) \supseteq f(\tilde{A}) \cap f(\tilde{B})$ for any $\tilde{A}, \tilde{B} \in I^X$ has not hold.

Example. Let $X = \{a,b,c,d\}$, $Y = \{x,y\}$, $\tilde{A} = \frac{0.3}{a} + \frac{0.2}{b} + \frac{0.6}{c} + \frac{0.1}{d}$, $\tilde{B} = \frac{0.5}{a} + \frac{0.8}{b} + \frac{0.1}{c} + \frac{0.3}{d}$. Define $f: X \to Y$ as $f(a) = f(b) = f(c) = x, f(d) = y$. Then $\tilde{A} \cap \tilde{B} = \frac{0.3}{a} + \frac{0.2}{b} + \frac{0.1}{c} + \frac{0.1}{d}$, $f(\tilde{A} \cap \tilde{B}) = \frac{0.3}{x} + \frac{0.1}{y}$, $f(\tilde{A}) = \frac{0.6}{x} + \frac{0.1}{y}$,

$f(\tilde{B}) = \frac{0.8}{x} + \frac{0.3}{y}$, $f(\tilde{A}) \cap f(\tilde{B}) = \frac{0.6}{x} + \frac{0.1}{y}$. Thus $f(\tilde{A} \cap \tilde{B}) \supseteq f(\tilde{A}) \cap f(\tilde{B})$ for any $\tilde{A}, \tilde{B} \in I^X$ has not hold.

3.Fuzzy topological space

Definition 3.1.[2] A fuzzy topology is a family τ of fuzzy sets in X which satisfies the following conditions:

(1) $\tilde{0}, \tilde{1} \in \tau$;

(2)If $\tilde{A}, \tilde{B} \in \tau$,then $\tilde{A} \cap \tilde{B} \in \tau$;

(3) If $\tilde{A}_i \in \tau, \forall i \in \Gamma$,then $\bigcup_{i \in \Gamma} \tilde{A}_i \in \tau$;

τ is called a fuzzy topology for X, and the pair (X, τ) is called a fuzzy topological space, or fts for short. Every member of τ is called a τ-open fuzzy set. A fuzzy set is τ-closed if and only if its complement is τ-open. In the sequel, when no confusion is likely to arise, we shall call a τ-open (τ-closed) fuzzy set simply an open (closed) set.

Proposition 3.2. Let X be a nonempty set. If τ and J are two fuzzy topologicals for X, then $\tau \cap J$ is a fuzzy topology for X, where $\tau \cap J = \{A \cap B \mid A \in \tau, B \in J\}$.

Proof. Straightforward.

Proposition 3.3. Let *X*, *Y* be two nonempty sets, *f* be an one-to-one mapping from *X* to *Y*. If τ is a fuzzy topological for *X*, then $f(\tau)$ is a fuzzy topology for *Y*, where $f(\tau)=\{f(A)\mid A\in\tau\}$.

Proof. (1) Obviously, $f(\tilde{0})=\tilde{0}$, $f(\tilde{1})=\tilde{1}$;

(2) If $\tilde{A},\tilde{B}\in f(\tau)$, then there exist $\tilde{A}_1,\tilde{B}_1\in\tau$ such that $\tilde{A}=f(\tilde{A}_1)$ and $\tilde{B}=f(\tilde{B}_1)$ respectively. $\forall y\in Y$,

$$(\tilde{A}\cap\tilde{B})(y)=(f(\tilde{A}_1)\cap f(\tilde{B}_1))(y)=f(\tilde{A}_1)(y)\wedge f(\tilde{B}_1)(y)=(\vee_{x\in f^{-1}(y)}\tilde{A}_1(x))\wedge(\vee_{x\in f^{-1}(y)}\tilde{B}_1(x))$$

$$=\vee_{x\in f^{-1}(y)}(\tilde{A}_1(x)\wedge\tilde{B}_1(x))=\vee_{x\in f^{-1}(y)}(\tilde{A}_1\cap\tilde{B}_1)(x)=f(\tilde{A}_1\cap\tilde{B}_1)(y).$$

This means $\tilde{A}\cap\tilde{B}=f(\tilde{A}_1\cap\tilde{B}_1)$. Since $\tilde{A}_1\cap\tilde{B}_1\in\tau$, thus $f(\tilde{A}_1\cap\tilde{B}_1)\in f(\tau)$

(3) If $\tilde{A}_i\in f(\tau),\forall i\in\Gamma$, then for any $i\in\Gamma$ there exists a $\tilde{A}_1'\in\tau$ such that $\tilde{A}_i=f(\tilde{A}_i')$. And then

$$(\bigcup_{i\in\Gamma}\tilde{A}_i)(y)=\vee_{i\in\Gamma}(\tilde{A}_i(y))=\vee_{i\in\Gamma}(f(\tilde{A}_i')(y))=\vee_{i\in\Gamma}(\vee_{x\in f^{-1}(y)}\tilde{A}_1'(x))$$

$$=\ \vee_{x\in f^{-1}(y)}(\vee_{i\in\Gamma}\tilde{A}_1'(x))=\vee_{x\in f^{-1}(y)}((\bigcup_{i\in\Gamma}\tilde{A}_1')(x))=f(\bigcup_{i\in\Gamma}\tilde{A}')(y).$$

This means $\bigcup_{i\in\Gamma}\tilde{A}_i=f(\bigcup_{i\in\Gamma}\tilde{A}')$. Since $\bigcup_{i\in\Gamma}\tilde{A}_i\in\tau$, thus $f(\bigcup_{i\in\Gamma}\tilde{A}')\in f(\tau)$. This completes the proof.

Proposition 3.4. Let *X*, *Y* be two nonempty sets and f a mapping from *X* to *Y*. If τ is a fuzzy topological for *Y*, then $f^{-1}(\tau)$ is a fuzzy topology for *X*, where $f^{-1}(\tau)=\{f^{-1}(\tilde{A})\mid\tilde{A}\in\tau\}$.

Proof. (1) Obviously, $f^{-1}(\tilde{0})=\tilde{0}$, $f^{-1}(\tilde{1})=\tilde{1}$;

(2)If $\tilde{A},\tilde{B}\in f^{-1}(\tau)$,then there exist $\tilde{A}_1,\tilde{B}_1\in\tau$, such that $\tilde{A}=f^{-1}(\tilde{A}_1)$ and $\tilde{B}=f^{-1}(\tilde{B}_1)$ respectively. $\forall x\in X$,

$$(\tilde{A}\cap\tilde{B})(x)=(f^{-1}(\tilde{A}_1)\cap f^{-1}(\tilde{B}_1))(x)=f^{-1}(\tilde{A}_1)(x)\wedge f^{-1}(\tilde{B}_1)(x)=\tilde{A}_1(f(x))\wedge\tilde{B}_1(f(x))$$

$$=(\tilde{A}_1\cap\tilde{B}_1)(f(x))=f^{-1}(\tilde{A}_1\cap\tilde{B}_1)(x).$$

This means $\tilde{A}\cap\tilde{B}=f^{-1}(\tilde{A}_1\cap\tilde{B}_1)$. Since $\tilde{A}_1\cap\tilde{B}_1\in\tau$, thus $\tilde{A}\cap\tilde{B}=f^{-1}(\tilde{A}_1\cap\tilde{B}_1)\in f^{-1}(\tau)$.

(3) If $\tilde{A}_i\in f^{-1}(\tau),\forall i\in\Gamma$, then for any $i\in\Gamma$ there exists a $\tilde{A}_1'\in\tau$ such that $\tilde{A}_i=f^{-1}(\tilde{A}_i')$. And then

$$(\bigcup_{i\in\Gamma}\tilde{A}_i)(x)=\vee_{i\in\Gamma}(\tilde{A}_i(x))=\vee_{i\in\Gamma}(f^{-1}(\tilde{A}_i')(x))=\vee_{i\in\Gamma}(\tilde{A}_1'(f(x)))$$

$$=\vee_{i\in\Gamma}(\tilde{A}_1'(f(x)))=(\bigcup_{i\in\Gamma}\tilde{A}_1')(f(x))=f^{-1}(\bigcup_{i\in\Gamma}\tilde{A}_1')(x).$$

This means $\bigcup_{i\in\Gamma}\tilde{A}_i=f^{-1}(\bigcup_{i\in\Gamma}\tilde{A}_1')$. Since $\bigcup_{i\in\Gamma}\tilde{A}_1'\in\tau$, thus $\bigcup_{i\in\Gamma}\tilde{A}_i=f^{-1}(\bigcup_{i\in\Gamma}\tilde{A}_1')\in f^{-1}(\tau)$.

This complete the proof.

Definition 3.5.[2] A function *f* from a fts (X,τ) to a fts (Y,U) is *F*-continuous iff the inverse of each open set in *Y* is open set in *X*.

Definition 3.6.[9] Let $\tilde{A}$ be a fuzzy set in (X, τ) and the union of all the open sets contained in $\tilde{A}$ is called the interior of $\tilde{A}$, denoted by $\tilde{A}^O$. Evidently $\tilde{A}^O$ is the largest open set contained in $\tilde{A}$ and $\tilde{A}^{OO}$ = $\tilde{A}^O$.

Proposition 3.7.[2] Let $\tilde{A}$ be a fuzzy set in a fts (X, τ). Then $\tilde{A}$ is open iff $\tilde{A}$ = $\tilde{A}^O$.

Definition 3.8.[9] The intersection of all the closed sets containing $\tilde{A}$ is called the closure of $\tilde{A}$, denoted by $\bar{\tilde{A}}$. Obviously $\bar{\tilde{A}}$ is the smallest closed set containing $\tilde{A}$ and $\bar{\bar{\tilde{A}}}$ = $\bar{\tilde{A}}$.

By the definitions of the interior and closure, obviously $\tilde{A}^O$ ⊂ $\tilde{A}$ ⊂ $\bar{\tilde{A}}$.

Proposition 3.9.[9] Let $\tilde{A}$ be a fuzzy set in a fts (X, τ). Then $\tilde{A}$ is closed iff $\tilde{A}$ = $\bar{\tilde{A}}$.

Proposition 3.10. Let $\tilde{A}$ be a fuzzy set in a fts (X, τ).

(1) If $\tilde{A}$ ⊂ $\tilde{B}$, then $\tilde{A}^O$ ⊂ $\tilde{B}^O$.

(2) If $\tilde{A}$ ⊂ $\tilde{B}$, then $\bar{\tilde{A}}$ ⊂ $\bar{\tilde{B}}$.

Proof. According to the definition can be directly proved.

Proposition 3.11.[10] Let f : (X, τ)→ (Y, U) be a function, then the ollowing are equivalent:

(1) *f* is *F*-continuous.

(2) For every closed set $\tilde{A}$ in Y, $f^{-1}(\tilde{A})$ is closed set in X.

(3) For any fuzzy set $\tilde{A}$ in X, $f(\overline{\tilde{A}}) \subset \overline{f(\tilde{A})}$.

(4) For any fuzzy set $\tilde{B}$ in Y, $\overline{f^{-1}(\tilde{B})} \subset f^{-1}(\overline{\tilde{B}})$.

Proposition 3.12. Let f : $(X, \tau) \to (Y, U)$ be a function; then the following are equivalent:

(1) f is F-continuous.

(2) For any fuzzy set $\tilde{B}$ in Y, $f^{-1}(\tilde{B}^O) \subset (f^{-1}(\tilde{B}))^o$.

Proof. (1)$\Rightarrow$ (2). For any fuzzy set $\tilde{B}$ in Y, by the definition of the interior and f is F-continuous, this means $f^{-1}(\tilde{B}^O)$ is an open set in X. On the other hand, since $\tilde{B}^O \subset \tilde{B}$, by (3) of Proposition 2.4, $f^{-1}(\tilde{B}^O) \subset f^{-1}(\tilde{B})$. Considering $(f^{-1}(\tilde{B}))^o$ is the union of all the open sets contained in $f^{-1}(\tilde{B})$, thus $f^{-1}(\tilde{B}^O) \subset (f^{-1}(\tilde{B}))^o$.

(2) $\Rightarrow$ (1). Let $\tilde{B}$ be any open fuzzy set in Y, then $\tilde{B}^O = \tilde{B}$. By condition, $f^{-1}(\tilde{B}) = f^{-1}(\tilde{B}^O) \subset (f^{-1}(\tilde{B}))^o$. On the other hand, since $f^{-1}(\tilde{B}) \supset (f^{-1}(\tilde{B}))^O$, thus $f^{-1}(\tilde{B}) = (f^{-1}(\tilde{B}))^o$. This means $f^{-1}(\tilde{B})$ is an open set in X, thus f is F-continuous.

Proposition 3.13. Let (X, τ) be a fts and f : $X\to Y$ be an one-to-one F-continuous mapping, then the following are hold:

(1) For any fuzzy set $\tilde{A}$ in X, $f(\tilde{A}^O) = (f(\tilde{A}))^o$.

(2) For any fuzzy set $\tilde{A}$ in X, $f((\overline{\tilde{A}})^O) \subset (\overline{f(\tilde{A})})^o$

Proof. According to the previous conclusion, $(Y, f(\tau))$ is a fts.

(1) For any fuzzy set $\tilde{A}$ in X, since $\tilde{A}^O = \bigcup\left\{\tilde{B} \mid \tilde{B} \subset \tilde{A}, \tilde{B} \in \tau\right\}$, by (1) of Proposition 2.5, $f(\tilde{A}^O) = \bigcup\left\{f(\tilde{B}) \mid f(\tilde{B}) \subset f(\tilde{A}), f(\tilde{B}) \in f(\tau)\right\}$ .On the other hand, by the definition of the interior, $(f(\tilde{A}))^O = \bigcup\left\{f(\tilde{B}) \mid f(\tilde{B}) \subset f(\tilde{A}), f(\tilde{B}) \in f(\tau)\right\}$. Thus $f(\tilde{A}^O) = (f(\tilde{A}))^o$.

(2) For any fuzzy set $\tilde{A}$ in X, since $\overline{\tilde{A}}^O = \bigcup\left\{\tilde{B} \mid \tilde{B} \subset \overline{\tilde{A}}, \tilde{B} \in \tau\right\}$, by (1) of Proposition 2.5, thus $f((\overline{\tilde{A}})^O) = \bigcup\left\{f(\tilde{B}) \mid f(\tilde{B}) \subset f(\overline{\tilde{A}}), f(\tilde{B}) \in f(\tau)\right\}$.Considering f is a F-continuous, thus for any $f(\tilde{B}) \subset f(\overline{\tilde{A}})$ and $f(\tilde{B}) \in f(\tau)$, by (3) of

Proposition 3.11, $f(\tilde{B}) \subset f(\overline{\tilde{A}}) \subset \overline{f(\tilde{A})}$ holds. By the definition of the interior of $(\overline{f(\tilde{A})})^o$ and (1) of Proposition 3.10, and then $f(\tilde{B}) = (f(\tilde{B}))^O \subset (\overline{f(\tilde{A})})^o$, thus $f((\overline{\tilde{A}})^O) \subset (\overline{f(\tilde{A})})^o$.

Definition 3.14.[5, 11] A fuzzy set $\tilde{A}$ in fts (X, τ) is said to be fuzzy γ-open if $\tilde{A} \subset (\overline{\tilde{A}})^o \cup \overline{(\tilde{A})^O}$. The complement of a fuzzy γ-open set is called a fuzzy γ-closed set. The family of all fuzzy γ-open sets of X is denoted by *γO(X)*.

Proposition 3.15. Let (X, τ) be a fts and $f: X \to Y$ be an one-to-one F-continuous mapping. If $\tilde{A}$ is a γ-open set in X, then $f(\tilde{A})$ is a γ-open set in Y.

Proof. Since $\tilde{A}$ is a γ-open set in X, then $\tilde{A} \subset (\overline{\tilde{A}})^o \cup \overline{(\tilde{A})^O}$. And then $f(\tilde{A}) \subset f((\overline{\tilde{A}})^o \cup \overline{(\tilde{A})^O}) = f((\overline{\tilde{A}})^o) \cup f(\overline{(\tilde{A})^O})$. By Proposition 3.11 and Proposition 3.13, $f(\overline{(\tilde{A})^O}) \subset \overline{f(\tilde{A}^O)} = \overline{(f(\tilde{A}))^O}$ and $f((\overline{\tilde{A}})^o) \subset (\overline{f(\tilde{A})})^O$. This means $f(\tilde{A}) \subset f((\overline{\tilde{A}})^o \cup \overline{(\tilde{A})^O}) = f((\overline{\tilde{A}})^o) \cup f(\overline{(\tilde{A})^O}) \subset (\overline{f(\tilde{A})})^o \cup \overline{(f(\tilde{A}))^o}$. By the Definition of the fuzzy γ-open set, $f(\tilde{A})$ is a γ-open set in Y.

Proposition 3.16. Let (Y, J) be a fts and $f^{-1}: Y \to X$ be an one-to-one F-continuous. If $\tilde{B}$ is a γ-open set in Y, then $f^{-1}(\tilde{B})$ is a γ-open set in X.

Proof. Let f^{-1} as f in proposition 3.15, the proof is similar to proposition 3.15.

Definition 3.17. A fuzzy set $\tilde{A}$ in a fts (X, τ) is called a γ-neighborhood of fuzzy point x_λ, if there exists a γ-open set $\tilde{B} \in \tau$ such that $x_\lambda \in \tilde{B} \subset \tilde{A}$. The family consisting of all γ-neighborhoods of x_λ is called the system of γ-neighborhoods of x_λ.

Definition 3.18. A fuzzy point x_λ is said to be quasi-coincident with a fuzzy set $\tilde{A}$, denoted by $x_\lambda \, q \, \tilde{A}$, if $\lambda + \tilde{A}(x) > 1$. A fuzzy set $\tilde{A}$ is said to be a Q_γ-neighborhood x_λ of if there exists a γ-open set $\tilde{B} \in \tau$ such that $x_\lambda \, q \, \tilde{B} \subset \tilde{A}$. The family consisting of all the Q_γ-neighborhoods of x_λ is called the system of Q_γ-neighborhoods of x_λ.

Proposition 3.19. Let (X , τ) be a fts and f an one-to-one F-continuous mapping from X to Y. If U is a Q_γ-neighborhood of a in X, then $f(U)$ is a Q_γ-neighborhood

of $f(a_\lambda)=[f(a)]_\lambda$ in fts $(Y, f(\tau))$.

Proof. In order to avoid confusion, here record $y=f(a)$. Without losing generality, let $U\in\gamma O(X)$, Since

$$f(U)(f(a))+\lambda=f(U)(y)+\lambda=\underset{x\in f^{-1}(y)}{\vee}U(x)+\lambda>U(a)>1,$$

this means $[f(a)_\lambda qf(U)]$ .Considering $f(U)\in\gamma O(Y)$, thus $f(U)$ is a Q_γ-neighborhood of $[f(a)]_\lambda$.

Proposition 3.20. Let (Y, J) be a fts and f^{-1} an one-to-one *F*-continuous mapping from Y to X. If V is a Q_γ-neighborhood of $(f(a))_\lambda$ in Y, then $f^{-1}(V)$ is a Q_γ-neighborhood of a_λ in fts $(X, f^{-1}(J))$.

Proof. Without losing generality, let $V\in\gamma O(Y)$, Since

$$f^{-1}(V)(a)+\lambda=V(f(a))+\lambda>1,$$

this means $a_\lambda q f^{-1}(V)$. Considering $f^{-1}(V)\in\gamma O(X)$, thus $f^{-1}(V)$ is a Q_γ-neighborhood of a_λ.

4 γ-Fuzzy topological semigroup

Definition 4.1.[7] Let X be a semigroup and $\tilde{A}$, $\tilde{B}$ two fuzzy sets in X. $\tilde{A}\tilde{B}$ is defined as a fuzzy set in X, which membership function is as follows:

$$\tilde{A}\tilde{B}(x)=\vee_{x_1x_2=x}(\tilde{A}(x_1)\wedge\tilde{B}(x_2))\quad for\quad x\in X.$$

Proposition 4.2. Let X, Y be two semigroups and f an epimorphism from X to Y. If $\tilde{A}$, $\tilde{B}$ are any two the fuzzy sets in X, then $f(\tilde{A}\tilde{B})=f(\tilde{A})f(\tilde{B})$.

Proof. For any $y\in Y$, since

$$f(\tilde{A}\tilde{B})(y)=\underset{x\in f^{-1}(y)}{\vee}(\tilde{A}\tilde{B})_{(x)}=\underset{x\in f^{-1}(y)}{\vee}(\underset{x_1x_2=x}{\vee}(\tilde{A}(x_1)\wedge\tilde{B}(x_2)))=\underset{x_1x_2=x}{\vee}(\underset{x\in f^{-1}(y)}{\vee}(\tilde{A}(x_1)\wedge\tilde{B}(x_2)))$$

$$=\underset{x_1x_2=x}{\vee}(\underset{x_1x_2\in f^{-1}(y)}{\vee}(\tilde{A}(x_1)\wedge\tilde{B}(x_2)))=\underset{f(x_1x_2)=f(x)=y}{\vee}(\underset{x_1x_2\in f^{-1}(y)}{\vee}(\tilde{A}(x_1)\wedge\tilde{B}(x_2)))$$

$$=\underset{f(x_1)f(x_2)=f(x)=y}{\vee}(\underset{x_1x_2\in f^{-1}(y)}{\vee}(\tilde{A}(x_1)\wedge\tilde{B}(x_2)))=\underset{y_1y_2=y}{\vee}(\underset{x_1x_2\in f^{-1}(y)}{\vee}(\tilde{A}(x_1)\wedge\tilde{B}(x_2)))$$

$$=\underset{y_1y_2=y}{\vee}(\underset{x_1x_2\in f^{-1}(y_1y_2)}{\vee}(\tilde{A}(x_1)\wedge\tilde{B}(x_2)))=\underset{y_1y_2=y}{\vee}(\underset{x_1x_2\in f^{-1}(y_1)f^{-1}(y_2)}{\vee}(\tilde{A}(x_1)\wedge\tilde{B}(x_2)))$$

$$=\underset{y_1y_2=y}{\vee}((\underset{x_1x_2\in f^{-1}(y_1)f^{-1}(y_2)}{\vee}(\tilde{A}(x_1))\wedge(\underset{x_1x_2\in f^{-1}(y_1)f^{-1}(y_2)}{\vee}\tilde{B}(x_2)))$$

$$=\underset{y_1y_2=y}{\vee}((\underset{x_1\in f^{-1}(y_1)}{\vee}(\tilde{A}(x_1))\wedge(\underset{x_2\in f^{-1}(y_2)}{\vee}\tilde{B}(x_2)))$$

$$= \vee_{y_1 y_2 = y} (f(\tilde{A})(y_1) \wedge f(\tilde{B})(y_2)) = f(\tilde{A}\tilde{B})(y),$$

thus $f(\tilde{A}\tilde{B}) = f(\tilde{A}) f(\tilde{B})$.

Proposition 4.3. Let *X*, *Y* be two semigroups and *f* a monomorphism from *X* to *Y*. If $\tilde{C}, \tilde{D}$ are any two the fuzzy sets in *Y*, then $f^{-1}(\tilde{C}\tilde{D}) = f^{-1}(\tilde{C}) f^{-1}(\tilde{D})$.

Proof. For any $x \in X$, since

$$f^{-1}(\tilde{C}\tilde{D})(x) = \tilde{C}\tilde{D}(f(x)) = \vee_{y_1 y_2 = f(x)} (\tilde{C}(y_1) \wedge \tilde{D}(y_2)) = \vee_{f^{-1}(y_1) f^{-1}(y_2) = x} (\tilde{C}(y_1) \wedge \tilde{D}(y_2))$$

$$= \vee_{x_1 x_2 = x} (\tilde{C}(f(x_1)) \wedge \tilde{D}(f(x_2))) = \vee_{x_1 x_2 = x} (f^{-1}(\tilde{C}(x_1)) \wedge f^{-1}(\tilde{D}(x_2))) = (f^{-1}(\tilde{C}) f^{-1}(\tilde{D}))(x),$$

thus $f^{-1}(\tilde{C}\tilde{D}) = f^{-1}(\tilde{C}) f^{-1}(\tilde{D})$.

Definition 4.4. Let *X* be a semigroup and (X, τ) a fts. Then (X, τ) is called a γ-fuzzy topological semigroup, or γ-ftsg for short, if for all $a, b \in X$ and any Q_γ-neighborhood W of fuzzy point $(ab)_\lambda$ there exist Q_γ-neighborhoods U of a_λ and V of b_λ such that $UV \subset W$.

Proposition 4.5. Let *X, Y* be two semigroups and (X, τ) a γ-ftsg. If *f* is an one-to-one *F*-continuous homomorphic mapping from *X* to *Y*, then $(Y, f(\tau))$ is a γ-ftsg.

Proof. By Proposition 3.3, $(Y, f(\tau))$ is a fts. For any Q_γ-neighborhood *W* of fuzzy point $(ab)_\lambda$ in *Y*, according to Proposition 3.20, $f^{-1}(W)$ is a Q_γ-neighborhood of fuzzy point $f^{-1}((ab)_\lambda)$ in *X*. Since (X, τ) is a γ-ftsg, thus there exist Q_γ-neighborhoods *U* of $f^{-1}(a_\lambda)$ and *V* of $f^{-1}(b_\lambda)$ such that $UV \subset f^{-1}(W)$, and then $f(UV) \subset W$. By proposition 3.19, $f(U)$ and $f(V)$ is the Q_γ-neighborhoods of a_λ and b_λ respectively. Again by Proposition 4.2, $f(U)f(V) = f(UV) \subset W$. Thus $(Y, f(\tau))$ is a γ-ftsg.

Proposition 4.6. Let *X, Y* be two semigroups and (*Y, J*) a γ-ftsg. If f^{-1} is an

one-to-one *F*-continuous homomorphic mapping from *X* to *Y,* then $(X, f^{-1}(J))$ is a γ-ftsg.

Proof. By Proposition 3.4, $f^{-1}(J)$ is a fts. For any Q$_\gamma$-neighborhood *W* of fuzzy point $(ab)_\lambda$ in *X*, according to Proposition 3.19, $f(W)$ is a Q$_\gamma$-neighborhood *W* of fuzzy point $f((ab)_\lambda)$ in *Y*. Since (*Y, J*) is a γ-ftsg, thus there exist Q$_\gamma$-neighborhoods *U* of $f(a_\lambda)$ and *V* of $f(b_\lambda)$ such that $UV \subset f(W)$, and then $f^{-1}(UV) \subset W$. By proposition 3.20, $f^{-1}(U)$ and $f^{-1}(V)$ is the Q$_\gamma$-neighborhoods of a and b respectively.Again by Proposition 4.3, $f^{-1}(U)f^{-1}(V) = f^{-1}(UV) \subset W$. Thus $(X, f^{-1}(J))$ is a γ-ftsg.

5.γ-Fuzzy topological ideal

Definition 5.1. Let *X* be a semigroup and (X,τ) a fts. Then (X,τ) is called a -fuzzy topological left ideal (right ideal), if for all $a,b \in X$ and any Q$_\gamma$-neighborhood W of fuzzy point $(ab)_\lambda$ there exists Q$_\gamma$-neighborhood *U* of b_λ (Q$_\gamma$-neighborhood V of a_λ) such that $U \subset W$ $(V \subset W)$.

Definition 5.2. Let *X* be a semigroup and (X,τ) a fts. Then (X,τ) is called a γ-fuzzy topological intrinsic ideal (double ideal), if for all $a,b,c \in X$ and any Q$_\gamma$-neighborhood *W* of fuzzy point $(abc)_\lambda$ there exists Q$_\gamma$-neighborhood *U* of b_λ (Q$_\gamma$-neighborhood *U* of a_λ and Q -neighborhood V of c_λ) such that $U \subset W$ (such that $UV \subset W$).

Proposition 5.3. Let *X, Y* be two semigroups and (X,τ) a -fuzzy topological left ieal (right ieal, intrinsic ideal, double ideal). If f is an one-to-one *F*-continuous homomorphic mapping from *X* to *Y*, then $(Y, f(\tau))$ is a γ-fuzzy topological left ieal (right ieal, intrinsic ideal, double ideal).

Proof. Similar to the proof of Proposition 4.5.

Proposition 5.4. Let *X, Y* be two semigroups and (*Y, J*) a γ-fuzzy topological left ieal (right ieal, intrinsic ideal, double ideal). If f^{-1} is an one-to-one *F*-continuous homomorphic mapping from *X* to *Y*, then $(X, f^{-1}(J))$ is a γ-fuzzy topological left ieal (right ieal, intrinsic ideal, double ideal).

Proof. Similar to the proof of Proposition 4.6.

References

[1]K. T. Atanassov, Intuitionistic fuzzy sets, Fuzzy Sets Syst. 20(1) (1986) 87-96.

[2]C. L. Chang, Fuzzy Topological Spaces, J. Math. Anal. Appl 24(1968)182-190.

[3]D. Coker, An introduction to intuitionistic fuzzy topological space, Fuzzy sets and systems, 1997, 81-89.

[4]D. H. Foster, fuzzy topological groups, J. Math. Anal. Appl. 67(1979)549-564.

[5]I. M. Hanafy, Fuzzy -open sets and fuzzy -continuity, J. Fuzzy Math., 7(2)(1999)419-430.

[6]Kul Hur, Young Bae Jun, Jang Hyun Ryou, Intuitionistic fuzzy topological groups. Honam Math. J. 26(2004)163-192.

[7]J. L. Ma, C. H. Yu, Fuzzy topological groups. Fuzzy Sets and Systems 12(1984)289-299.

[8]S. Nazmul, S. K. Samanta, Fuzzy Soft Topological Groups, Fuzzy Inf. Eng. 6(2014)71-92.

[9]Pu Pao-ming and Liu Ying-ming, Fuzy topology I, J. Math. Anal. Appl. 76(1980)571-599.

[10]Pu Pao ming and Liu Ying ming, Fuzy topology II, J. Math. Anal. Appl. 77(1980)20-37.

[11]N. Rajesh, A New Type of Fuzzy Topological Groups, The journal of fuzzy mathematics, 21(1)(2013)99-103.

[12]A. Rosenfeld, fuzzy groups, J. Math. Anal. Appl. 35(1971)512-517.

[13]B. Tanay and M. Burc Kandemir, Topological structure of fuzzy soft sets, Comp. Math. Appl. 61(2011)2952-2957.

[14]C. H. Yu, On fuzzy topological groups, Fuzzy Sets and Systems 23(1987)281-287.

[15]L. A. Zadeh, Fuzzy Sets, Inform. Control 8 (1965) 338-353.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 27, NO.5, 2019, 849-863

The Behavior and Closed Form of the Solutions of Some Difference Equations

E. M. Elsayed[1,2], and Hanan S. Gafel[1]

[1]Mathematics Department, Faculty of Science, King Abdulaziz University,
P. O. Box 80203, Jeddah 21589, Saudi Arabia.

[2]Department of Mathematics, Faculty of Science, Mansoura University, Mansoura 35516, Egypt.

E-mail: emelsayed@yahoo.com, h-s-g2006@hotmail.com.

ABSTRACT:

In this paper, we will investigate the local stability of the equilibrium points, global attractor, boundedness and the form of the solutions for the following equations

$$x_{n+1} = \frac{Ax_n x_{n-3}}{Bx_n + Cx_{n-2}} \text{ and } x_{n+1} = \frac{Ax_n x_{n-3}}{Bx_n - Cx_{n-2}},$$

where the coefficients A, B and C are real positive numbers, and the initial conditions x_{-3}, x_{-2}, x_{-1}, x_0 are arbitrary non zero real numbers.

Keywords: difference equations, global attractor, local stability, equilibrium point, boundedness.

Mathematics Subject Classification: 39A10.

1. Introduction

The study of difference equations has been growing continuously for the last decade. This is largely due to the fact that difference equations manifest themselves as mathematical models describing real life situations in probability theory ,quelling theory, statistical problems, stochastic time series. combinatorial analysis number theory, geometry, electrical network, quanta in radiation, genetics in biology ,economics, psychology. sociology, etc. In fact, now it occupies a central position in applicable analysis and will no doubt continue to play an important role in mathematics as a whole see [1]-[20]. The purpose of this article is to investigate the global attractivity of the equilibrium point, and the asymptotic behavior of the solutions of the following difference equations

$$x_{n+1} = \frac{Ax_n x_{n-3}}{Bx_n + Cx_{n-2}}. \tag{1}$$

$$x_{n+1} = \frac{Ax_n x_{n-3}}{Bx_n - Cx_{n-2}}. \tag{2}$$

Where the initial conditions x_{-3}, x_{-2}, x_{-1}, x_0 are arbitrary positive real numbers, and A, B, C are positive constants. We obtain the form of the solution of some special cases of equation (1) and (2) and some numerical simulation to the equation are given to illustrate our results.

Lemma 1.1. Let I be some interval of real numbers and let $f: I^{k+1} \to I$, be a continuously differentiable function. Then for every initial conditions $x_{-k}, \dots, x_{-1}, x_0 \in I, k \in \{1, 2, 3, \dots\}$, the difference equation

$$x_{n+1} = f(x_n, x_{n-1}, \dots, x_{n-k}), \quad n = 0, 1, 2, \dots, \tag{3}$$

has a unique solution $\{x_n\}_{n=-k}^{\infty}$.

Difinition 1.1. A point $\bar{x} \in I$ is called an **equilibrium point** of equation (3) if $\bar{x} = f(\bar{x}, \bar{x}, \dots, \bar{x})$. That is, $x_n = \bar{x}$ for $n \geq 0$, is a solution of equation (3), or equivalently, $\bar{x}$ is a fixed point of f.

Difinition 1.2. The equilibrium point $\bar{x}$ of equation (3) is called **locally stable** if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for all $x_{-k}, x_{-k+1}, \dots, x_{-1}, x_0 \in I$ with $|x_{-k} - \bar{x}| + |x_{-k+1} - \bar{x}| + \dots + |x_0 - \bar{x}| < \delta$, we have $|x_n - \bar{x}| < \varepsilon, \forall n \geq -k$.

Difinition 1.3. The equilibrium point $\bar{x}$ of equation (3) is called **locally asymptotically stable** if it is locally stable and if there exists $\gamma > 0$ such that for all $x_{-k}, x_{-k+1}, \dots, x_{-1}, x_0 \in I$ with

$$|x_{-k} - \bar{x}| + |x_{-k+1} - \bar{x}| + \dots + |x_0 - \bar{x}| < \gamma,$$

we have $\lim_{n\to\infty} x_n = \bar{x}$.

Difinition 1.4. The equilibrium point $\bar{x}$ of equation (3) is called **global attractor** if for all x_{-k}, $x_{-k+1}, \dots, x_{-1}, x_0 \in I$, we have $\lim_{n\to\infty} x_n = \bar{x}$.

Difinition 1.5. The equilibrium point $\bar{x}$ of equation (3) is called **global asymptotically stable** if it is locally stable and a global attractor of equation (3).

The equilibrium point $\bar{x}$ of equation (3) is **unstable** if it is not locally stable.

The linearized equation of equation (3) about the equilibrium point $\bar{x}$ is the linear difference equation

$$y_{n+1} = \sum_{i=0}^{k} \frac{\partial f(\bar{x}, \bar{x}, \dots, \bar{x})}{\partial x_{n-i}} y_{n-i} \tag{4}$$

Now suppose that the characteristic equation associated with equation (4) is

$$p(\lambda) = p_0 \lambda^k + p_1 \lambda^{k-1} + \dots + p_{k-1}\lambda + p_k = 0,$$

where $p_i = \frac{\partial f(\bar{x}, \bar{x}, \dots, \bar{x})}{\partial x_{n-i}}$.

Theorem A. [32] Suppose that $p_i \in R$, $i = 1,2,3, \dots k$ and $\mathrm{k} \in \{0,1,2, \cdots\}$. Then $\sum_{i=1}^{k} |p_i| < 1$, is a sufficient condition for the **asymptotic stability** of the difference equation

$$x_{n+k} + p_1 x_{n+k-1} + \cdots + p_k x_n = 0, \qquad n = 0,1,2, \dots \ .$$

Theorem B. [33] Let $[a, b]$ be an interval of real numbers and assume that $g: [a, b]^3 \to [a, b]$ is a continuous function satisfying the following properties:

(a) $g(x, y, z)$ is non-decreasing in x and z in $[a, b]$ for each $y \in [a, b]$, and is non-increasing in $y \in [a, b]$ for each *x* and *z* in $[a, b]$;

(b) If $(m, M) \in [a, b] \times [a, b]$ is a solution of the system

$$M = g(M, m, M) \quad and \quad m = g(m, M, m).$$

Then $m = M$. Then equation $x_{n+1} = g(x_n, x_{n-1}, x_{n-2})$, has a unique equilibrium $\bar{x} \in [a, b]$ and every solution of this equation converges to $\bar{x}$.

Theorem C. [33] Let $[a, b]$ be an interval of real numbers and assume that $g: [a, b]^3 \to [a, b]$ is a continuous function satisfying the following properties:

(a) $g(x, y, z)$ is non-decreasing in y and z in $[a, b]$ for each $x \in [a, b]$, and is non-increasing in $x \in [a, b]$ for each *y* and *z* in $[a, b]$;

(b) If $(m, M) \in [a, b] \times [a, b]$ is a solution of the system

$$M = g(m, M, M) \quad and \quad m = g(M, m, m).$$

Then $m = M$. Then equation $x_{n+1} = g(x_n, x_{n-1}, x_{n-2})$, has a unique equilibrium $\bar{x} \in [a, b]$ and every solution of this equation converges to $\bar{x}$.

Theorem D. [33] Let $[a, b]$ be an interval of real numbers and assume that $g: [a, b]^3 \to [a, b]$ is a continuous function satisfying the following properties:

(a) $g(x, y, z)$ is non-decreasing in x in $[a, b]$ for each $y \ and \ z \in [a, b]$, and is non-increasing in x and $z \in [a, b]$ for each *y* in $[a, b]$;

(b) If $(m, M) \in [a, b] \times [a, b]$ is a solution of the system

$$M = g(m, M, m) \quad and \quad m = g(M, m, M).$$

Then $m = M$. Then equation $x_{n+1} = g(x_n, x_{n-1}, x_{n-2})$, has a unique equilibrium $\bar{x} \in [a, b]$ and every solution of this equation converges to $\bar{x}$.

Many researchers have investigated the behavior of the solution of difference equation for example: Cinar [5-7] has got the solutions of the following difference

$$x_{n+1} = \frac{x_{n-1}}{1 + a x_n x_{n-1}}, \quad x_{n+1} = \frac{x_{n-1}}{-1 + a x_n x_{n-1}}, \qquad x_{n+1} = \frac{x_{n-1}}{1 + b x_n x_{n-1}}.$$

Elabbasy et al. [10] studied the behavior of the difference equation

$$x_{n+1} = a x_n - \frac{b x_n}{c x_n - d x_{n-1}}.$$

El-Metwally et al. [12] investigated the asymptotic behavior of the population model

$$x_{n+1} = \alpha + \beta x_{n-1} e^{-x_n}.$$

Karatas et al. [28] got the form of the solution of the difference equation

$$x_{n+1} = \frac{x_{n-5}}{1 + x_{n-2} x_{n-5}}.$$

Zayed and El-Moneam [45] deal with the behavior of the following rational recursive sequence

$$\mathrm{x}_{n+1} = \frac{\alpha + \beta \mathrm{x}_n + \gamma \mathrm{x}_{n-1}}{\mathrm{A} + \mathrm{B}\mathrm{x}_n + \mathrm{C}\mathrm{x}_{n-1}}.$$

Wang and Zhang [38] considered the sufficient and necessary condition for the existence and uniqueness of the initial value problem of difference equations of higher order. In addition, they investigated the local stability, asymptotic behavior, periodicity and oscillation of solutions for the same equation. See also [21]-[47].

2. The Behavior of Equation (1)

This section will examine the behavior of solutions of equation (1). The constants A, B and C within the equation are real positive numbers.

2.1. Local Stability of Equation (1)

In this subsection, we explore the local stability character of the solution of equation (1).

Equation (1) make sure has a unique equilibrium point is set as follows:

$$\bar{\mathrm{x}} = \frac{\mathrm{A}\overline{\mathrm{x}\mathrm{x}}}{\mathrm{B}\bar{\mathrm{x}} + \mathrm{C}\bar{\mathrm{x}}} \quad \Rightarrow \quad \bar{\mathrm{x}} = 0.$$

Then the unique equilibrium point is $\bar{x} = 0$ if $B + C \neq A$.

Theorem 2.1. Assume that $A(B + 3C) < (B + C)^2$, then the equilibrium point of equation (1) is locally asymptotically stable.

Proof: Let $f: (0, \infty)^3 \to (0, \infty)$ be a function define by

$$f(u, v, w) = \frac{Auw}{Bu + Cv}. \tag{5}$$

Thus, it follows that

$$\frac{\partial f}{\partial u} = \frac{ACvw}{(Bu + Cv)^2}, \quad \frac{\partial f}{\partial v} = \frac{-ACuw}{(Bu + Cv)^2}, \qquad \frac{\partial f}{\partial w} = \frac{Au}{Bu + Cv}.$$

As it can be seen

$$\frac{\partial f}{\partial u}\Big|_{\bar{x}=0} = \frac{AC}{(B + C)^2}, \qquad \frac{\partial f}{\partial v}\Big|_{\bar{x}=0} = \frac{-AC}{(B + C)^2}, \quad \frac{\partial f}{\partial w}\Big|_{\bar{x}=0} = \frac{A}{B + C}.$$

Then the linearized equation associated with equation (1) about $\bar{x} = 0$ is

$$y_{n+1} - \frac{AC}{(B + C)^2} y_n + \frac{AC}{(B + C)^2} y_{n-2} - \frac{A}{B + C} y_{n-3} = 0,$$

and it associated characteristic equation is

$$\lambda^4 - \frac{AC}{(B + C)^2}\lambda^3 + \frac{AC}{(B + C)^2}\lambda - \frac{A}{B + C} = 0.$$

It follows by theorem A that equation (1) is asymptotically stable if

$$\left| \frac{AC}{(B + C)^2} \right| + \left| \frac{AC}{(B + C)^2} \right| + \left| \frac{A}{B + C} \right| < 1,$$

thus,

$$A(B + 3C) < (B + C)^2.$$

Therefore, the proof is complete.

2.2. Global Attractivity of the Equilibrium Point of Equation (1)

The global attractivity character of solutions of equation (1) will be investigated in this section.

Theorem 2.2. The equilibrium point of equation (1) is global attractor if $B \neq A$.

Proof. Let p, q are real numbers and suppose that $f: [p, q]^3 \to [p, q]$ be a function define by equation $f(u, v, w) = \frac{Auw}{Bu+Cv}$, then we can easily see that the function increasing in u, w and decreasing in v. Assume that (m, M) is a solution of the system

$$M = f(M, m, M) \quad \text{and } m = f(m, M, m).$$

Then from equation (1), we see that

$$M = \frac{AM^2}{BM + Cm}, \qquad m = \frac{Am^2}{Bm + CM}$$

$$\Rightarrow BM^2 + CMm = AM^2, \quad Bm^2 + CMm = Am^2.$$

Formerly

$$(B - A)(M - m)(M + m) = 0.$$

Then $M = m$ if $B \neq A$. Therefore, it can be concluded from Theorem B that $\bar{x}$ is a global attractor.

2.3. Boundedness of the Solutions of Equation (1)

The boundedness of the solutions of Equation (1) will be discussed in this section.

Theorem 2.3. Every solution of equation (1) is bounded if $\frac{A}{B} < 1$.

Proof. Let $\{x_n\}_{n=-3}^{\infty}$ be a solution of equation (1). It follows from equation (1) that

$$x_{n+1} = \frac{Ax_n x_{n-3}}{Bx_n + Cx_{n-2}} \leq \frac{Ax_n x_{n-3}}{Bx_n} = \frac{A}{B} x_{n-3}, \qquad \text{for all } n \geq 1.$$

By using a comparison, we can write the right hand side as follows $y_{n+1} = \frac{A}{B} y_{n-3}$.

So $y_n = \left(\frac{A}{B}\right)^n K$, K is constant, and this equation is locally asymptotically stable if $\frac{A}{B} < 1$, and converges to the equilibrium point $\bar{y} = 0$. Thus the solution of equation (1) is bounded.

2.4. Special Case of Equation (1)

In this subsection, the solution of the fourth order difference equation will be presented here

$$x_{n+1} = \frac{x_n x_{n-3}}{x_n + x_{n-2}}. \tag{6}$$

Such that the initial conditions $x_{-3}, x_{-2}, x_{-1}, x_0$ are arbitrary non zero real numbers.

Theorem 2.4. The solution of equation (6) is given by the following formulas for $n = 0,1,2, \ldots$

$$x_{10n-3} = \frac{(abc)^n d^{n+1}}{(a+c)^n (b+d)^n (ab+ad+bc)^n}, \qquad x_{10n-2} = \frac{(abd)^n c^{n+1}}{(a+c)^n (b+d)^n (ab+ad+bc)^n},$$

$$x_{10n-1} = \frac{(acd)^n b^{n+1}}{(a+c)^n (b+d)^n (ab+ad+bc)^n}, \qquad x_{10n} = \frac{(bcd)^n a^{n+1}}{(a+c)^n (b+d)^n (ab+ad+bc)^n},$$

$$x_{10n+1} = \frac{(ad)^{n+1} (bc)^n}{(a+c)^{n+1} (b+d)^n (ab+ad+bc)^n}, \qquad x_{10n+2} = \frac{(acd)^{n+1} b^n}{(a+c)^n (b+d)^n (ab+ad+bc)^{n+1}},$$

$$x_{10n+3} = \frac{(cbd)^{n+1} a^n}{(a+c)^{n+1} (b+d)^{n+1} (ab+ad+bc)^n}, x_{10n+4} = \frac{(bca)^{n+1} d^n}{(a+c)^n (b+d)^n (ab+ad+bc)^{n+1}},$$

$$x_{10n+5} = \frac{(abd)^{n+1} c^n}{(a+c)^{n+1} (b+d)^{n+1} (ab+ad+bc)^n}, x_{10n+6} = \frac{(cd)^{n+1} b^n a^{n+2}}{(a+c)^{n+1} (b+d)^n (ab+ad+bc)^{n+1}},$$

where $x_{-3} = d, x_{-2} = c, x_{-1} = b, x_0 = a$.

Proof. By using mathematical induction, we can prove as follow. For $n = 0$ the result holds. Assume that the result holds for $n - 1$, as follows

$$x_{10n-7} = \frac{(cbd)^n a^{n-1}}{(a+c)^n (b+d)^n (ab+ad+bc)^{n-1}}, x_{10n-6} = \frac{(bca)^n d^{n-1}}{(a+c)^{n-1} (b+d)^{n-1} (ab+ad+bc)^n}$$

$$x_{10n-5} = \frac{(abd)^n c^{n-1}}{(a+c)^n (b+d)^n (ab+ad+bc)^{n-1}}, x_{10n-4} = \frac{(cd)^n b^{n-1} a^{n+1}}{(a+c)^n (b+d)^{n-1} (ab+ad+bc)^n}.$$

We see from equation (6) that

$$x_{10n-3} = \frac{x_{10n-4} x_{10n-7}}{x_{10n-4} + x_{10n-6}}$$

$$= \frac{(cd)^n b^{n-1} a (acbd)^n a^{n-1}}{(a+c)^{2n} (b+d)^{2n-1} (ab+ad+bc)^{2n-1}} \div \left[\frac{(acbd)^n b^{-1} a (a+c)^{-1} + (bcad)^n d^{-1}}{(a+c)^{n-1} (b+d)^{n-1} (ab+ad+bc)^n}\right]$$

$$= \frac{(cd)^n b^{n-1} a^n \div [b^{-1} d^{-1} (a+c)^{-1} (da + b(a+c)]}{(a+c)^{n+1} (b+d)^n (ab+ad+bc)^{n-1}} = \frac{(acb)^n d^{n+1}}{(a+c)^n (b+d)^n (ab+ad+bc)^n}.$$

$$x_{10n-2} = \frac{x_{10n-3} x_{10n-6}}{x_{10n-3} + x_{10n-5}}$$

$$= \frac{(acbd)^{2n}}{(a+c)^{2n-1} (b+d)^{2n-1} (ab+ad+bc)^{2n}} \div \left[\frac{(acbd)^n d (ab+ad+bc)^{-1} + (abcd)^n c^{-1}}{(a+c)^n (b+d)^n (ab+ad+bc)^{n-1}}\right]$$

$$= \frac{(acbd)^n \div (ab+ad+bc)^{-1} c^{-1} [dc + (ab+ad+bc)]}{(a+c)^{n-1} (b+d)^{n-1} (ab+ad+bc)^{n+1}} = \frac{(abd)^n c^{n+1}}{(a+c)^n (b+d)^n (ab+ad+bc)^n}.$$

Also, the other relations can be proved similarly. The proof is completed.

2.5. Numerical Examples

In this subsection, numerical examples which represent different types of solutions to equation (1). Are considered to confirm the results.

Example 5.1. We assume the initial condition as follows: $x_{-3} = 14,\ x_{-2} = 2,\ x_{-1} = 7,\ x_0 = 5$ and the constants $A = 2,\ B = 4,\ C = 1$. See Fig. 1.

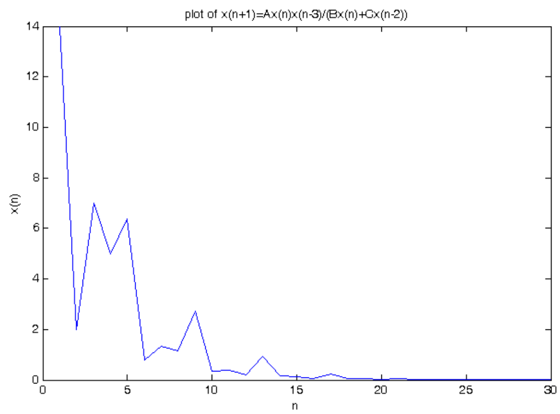


Figure 1.

Example 5.2. See Fig. 2 since we put $x_{-3} = 4,\ x_{-2} = 2,\ x_{-1} = 7,\ x_0 = 5$ and the constants $A = 12,\ B = 4,\ C = 3$.

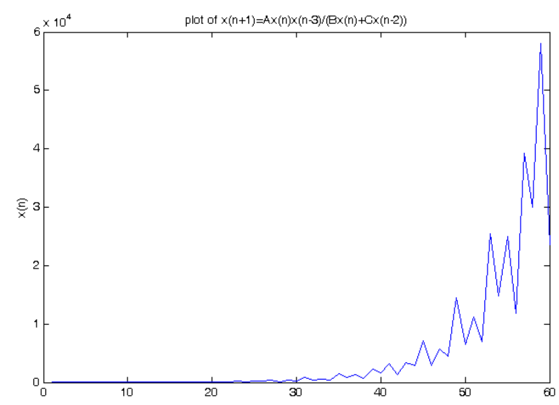


Figure 2.

3. The Behavior of Equation (2)

This section will examine the behavior of solutions of equation (2). The constants A, B and C within the equation are real positive numbers.

3.1. Local Stability of Equation (2)

In this subsection, we explore the local stability character of the solution of equation (2). Equation (2) make sure a unique equilibrium point is set as follows:

$$\bar{x} = \frac{A\overline{xx}}{B\bar{x} - C\bar{x}} \quad \Rightarrow \quad \bar{x} = 0.$$

Then the unique equilibrium point is $\bar{x} = 0$ if $A + C \neq B$.

Theorem 3.1. Assume that $A(B - 3C) < (B - C)^2$, then the equilibrium point of equation (2) is locally asymptotically stable.

Proof: Let $f: (0, \infty)^3 \to (0, \infty)$ be a function define by

$$f(u, v, w) = \frac{Auw}{Bu - Cv}. \tag{7}$$

Thus, it follows that

$$\frac{\partial f}{\partial u} = \frac{-ACvw}{(Bu - Cv)^2}, \qquad \frac{\partial f}{\partial v} = \frac{ACuw}{(Bu - Cv)^2}, \qquad \frac{\partial f}{\partial w} = \frac{Au}{Bu - Cv}.$$

As it can be seen

$$\frac{\partial f}{\partial u}\Big|_{\bar{x}=0} = \frac{-AC}{(B - C)^2}, \qquad \frac{\partial f}{\partial v}\Big|_{\bar{x}=0} = \frac{AC}{(B - C)^2}, \qquad \frac{\partial f}{\partial w}\Big|_{\bar{x}=0} = \frac{A}{B - C}.$$

Then the linearized equation associated with equation (2) about $\bar{x} = 0$ is

$$y_{n+1} + \frac{AC}{(B - C)^2} y_n - \frac{AC}{(B - C)^2} y_{n-2} - \frac{A}{B - C} y_{n-3} = 0,$$

and it associated characteristic equation is

$$\lambda^4 + \frac{AC}{(B - C)^2}\lambda^3 - \frac{AC}{(B - C)^2}\lambda - \frac{A}{B - C} = 0.$$

It follows by theorem A that equation (2) is asymptotically stable if

$$\left|\frac{AC}{(B-C)^2}\right| + \left|\frac{AC}{(B-C)^2}\right| + \left|\frac{A}{B-C}\right| < 1,$$

or

$$A(B-3C) < (B-C)^2.$$

Therefore, the proof is complete.

3.2. Global Attractivity of the Equilibrium Point of Equation (2)

The global attractivity character of solutions of equation (2) will be investigated in this section.

Theorem 3.2. The equilibrium point of equation (2) is global attractor if $C \neq A$.

Proof. Let p, q are real numbers and suppose that $f: [p,q]^3 \to [p,q]$ be a function define by $f(u,v,w) = \frac{Auw}{Bu-Cv}$. Then we can easily see that the function $f(u,v,w)$ decreasing in u and increasing in v. So we have two cases we prove case (1) and case (2) is similar and so will be omitted.

Case (1):- If $Bu - Cv > 0$, then we can easily see that the function $f(u,v,w)$ increasing in w. Assume that (m, M) is a solution of the system

$$M = f(m, M, M) \quad \text{and } m = f(M, m, m).$$

Then from equation (2), we see that

$$M = \frac{AMm}{Bm - CM}, \qquad m = \frac{AMm}{BM - Cm}$$

$$\Rightarrow BMm - CM^2 = AMm, \quad BMm - Cm^2 = AMm.$$

Formerly $C(M-m)(M+m) = 0$. Then $M = m$. Therefore, it can be concluded from Theorem C that $\bar{x}$ is a global attractor.

3.3. Special Case of Equation (2)

In this section, we study the following special cases of equation (2) where the constants A, B and C are integers numbers. The solution of the fourth order difference equation will be presented here

$$x_{n+1} = \frac{x_n x_{n-3}}{x_n - x_{n-2}}. \tag{8}$$

Such that the initial conditions $x_{-3}, x_{-2}, x_{-1}, x_0$ arearbitrary nonzero realnumbers.

Theorem 3.3. The solution of equation (8) is given by the following formulas for $n = 0,1,2,...$

$$x_{10n-3} = \frac{(abc)^n d^{n+1}}{(a-c)^n (b-d)^n (bc - ab + ad)^n}, \qquad x_{10n-2} = \frac{(abd)^n c^{n+1}}{(a-c)^n (b-d)^n (bc - ab + ad)^n},$$

$$x_{10n-1} = \frac{(acd)^n b^{n+1}}{(a-c)^n (b-d)^n (bc-ab+ad)^n}, \quad x_{10n} = \frac{(bcd)^n a^{n+1}}{(a-c)^n (b-d)^n (bc-ab+ad)^n},$$

$$x_{10n+1} = \frac{(ad)^{n+1} (bc)^n}{(a-c)^{n+1} (b-d)^n (bc-ab+ad)^n}, \quad x_{10n+2} = \frac{(acd)^{n+1} b^n}{(a-c)^n (b-d)^n (bc-ab+ad)^{n+1}},$$

$$x_{10n+3} = \frac{(cbd)^{n+1} a^n}{(a-c)^{n+1} (b-d)^{n+1} (bc-ab+ad)^n}, x_{10n+4} = \frac{(bca)^{n+1} d^n}{(a-c)^n (b-d)^n (bc-ab+ad)^{n+1}},$$

$$x_{10n+5} = \frac{(abd)^{n+1} c^n}{(a-c)^{n+1} (b-d)^{n+1} (bc-ab+ad)^n}, x_{10n+6} = \frac{(cd)^{n+1} b^n a^{n+2}}{(a-c)^{n+1} (b-d)^n (bc-ab+ad)^{n+1}}.$$

Proof. As the proof of Theorem 2.4 and will be omitted.

3.4. Numerical Examples

Example 3.4. We suppose that initial condition are taken as follows: $x_{-3} = 14,\ x_{-2} = 32,\ x_{-1} = -7,\ x_0 = 5$ and the constants $A = 12,\ B = 3,\ C = 8$. See Fig. 3.

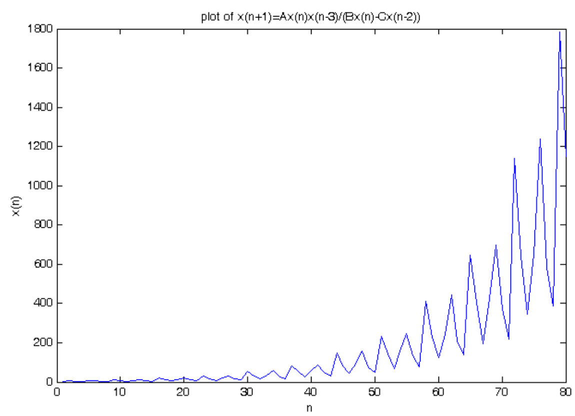


Figure 3.

Example 3.5. The Figure 4 shows the behavior of the solutions of equation (2) when $x_{-3} = 1.55,\ x_{-2} = 2.20,\ x_{-1} = 5.45,\ x_0 = 7$ and the constants $A = 4,\ B = 2,\ C = 3$. See Fig. 4.

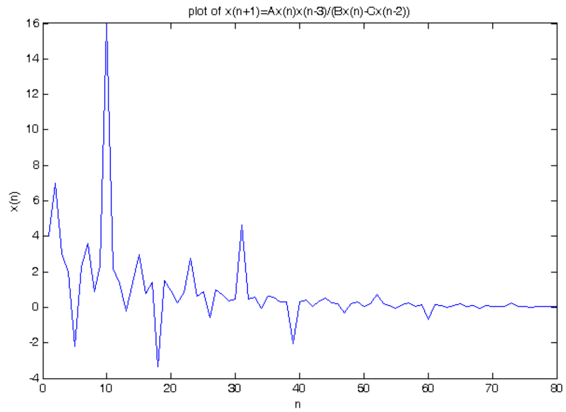


Figure 4.

References

[1] R. Abo-Zeid, On the oscillation of a third order rational difference equation, J. Egyptian Math. Soc., 23 (2015), 62–66.

[2] R. Abo-Zeid, Global behavior of a fourth-order difference equation with quadratic term, Boletn de la Sociedad Matemtica Mexicana, (2017), 1-8, doi.org/10.1007/s40590-017-0180-8.

[3] M. Aloqeili, Dynamics of a rational difference equation, Appl. Math. Comp., 176 (2), (2006), 768-774.

[4] A. Bilgin and M. R. S. Kulenovic, Global Attractivity for Nonautonomous Difference Equation via Linearization, Journal of Computational Analysis and Applications, 23 (7) (2017), 1311-1322.

[5] C. Cinar ,On the positive solutions of the difference equation $x_{n+1} = \frac{x_{n-1}}{1+ax_nx_{n-1}}$, Appl. Math. Comp., 158 (3) (2004), 809-812 .

[6] C. Cinar ,On the positive solutions of the difference equation $x_{n+1} = \frac{x_{n-1}}{-1+ax_nx_{n-1}}$, Appl. Math. comp., 158(3) (2004), 79W797 .

[7] C. Cinar ,On the positive solutions of the difference equation $x_{n+1} = \frac{x_{n-1}}{1+bx_nx_{n-1}}$, Appl. Math. Comp., 156 (2004), 587-590 .

[8] Q. Din, Asymptotic behavior of an anti-competitive system of second-order difference equations, Journal of the Egyptian Mathematical Society, 24 (1) (2016), 37-43.

[9] Q. Din, Global stability and Neimark-Sacker bifurcation of a host-parasitoid model, International Journal of Systems Science, 48 (6) (2017), 1194-1202.

[10] E. M. Elabbasy, H. El-Metwally and E. M. Elsayed, On the difference equation $x_{n+1} = ax_n - \frac{bx_n}{cx_n - dx_{n-1}}$, Adv. Differ. Equ., Volume 2006,(2006), Article ID 82579, 1-10.

[11] E. M. Elabbasy, H. El-Metwally and E. M. Elsayed, Qualitative behavior of higher order difference equation, Soochow Journal of Mathematics, 33 (4) (2007). 861-873.

[12] H. El-Metwally. E. A. Grove, G. Ladas, R. Levins, and M. Radin, On the difference equation $x_{n+1} = \alpha + \beta x_{n-1}e^{-x_n}$, Nonlinear Analysis : Theory, Methods & Applications. 17 (7) (2003), 4623—1634.

[13] H. El-Metwally, I. Yalcinkaya, and C. Cinar, Global Stability of an Economic Model, Utilitas Mathematica, 95 (2014), 235-244.

[14] E. M. Elsayed, On the solution of some difference equations, European J. Pure Appl. Math., 4 (3) (2011), 287-303 .

[15] E. M. Elsayed, Solution and Attractivity for a Rational Recursive Sequence, Discrete Dynamics in Nature and Society, Volume 2011, Article ID 982309, 17 pages.

[16] E. M. Elsayed, Solutions of Rational Difference System of Order Two, Mathematical and Computer Modelling, 55 (2012), 378--384.

[17] E. M. Elsayed, On the solutions and periodic nature of some systems of difference equations, International Journal of Biomathematics, 7 (6) (2014), 1450067, (26 pages).

[18] E. M. Elsayed, On a max type recursive sequence of order three, Miskolc Mathematical Notes, 17 (2) (2016), 837--859.

[19] E. M. Elsayed, Dynamics and behavior of a higher order rational difference equation, Journal of Nonlinear Sciences and Applications (JNSA), 9 (4) (2016), 1463-1474.

[20] E. M. Elsayed and Abdul Khaliq, Global attractivity and periodicity behavior of a recursive sequence, J. Comp. Anal. Appl, 22 (2017), 369-379.

[21] E. M. Elsayed and A. M. Ahmed, Dynamics of a three-dimensional systems of rational difference equations, Mathematical Methods in The Applied Sciences, 39 (5) (2016), 1026--1038.

[22] E. M. Elsayed and Asma Alghamdi, Dynamics and Global Stability of Higher Order Nonlinear Difference Equation, Journal of Computational Analysis and Applications, 21 (3) (2016), 493-503.

[23] E. M. Elsayed and T. F. Ibrahim, Periodicity and solutions for some systems of nonlinear rational difference equations, Hacettepe Journal of Mathematics and Statistics, 44 (6) (2015), 1361--1390.

[24] E. M. Elsayed and T. F. Ibrahim, Solutions and Periodicity of a Rational Recursive Sequences of Order Five, Bulletin of the Malaysian Mathematical Sciences Society, 38 (1) (2015), 95-112.

[25] S. Ergin and R. Karata, On the Solutions of the Recursive Sequence $x_{n+1} = ax_{n-k} /a- x_{n-i}$, Thai Journal of Mathematics, 14 (2) (2016), 391–397.

[26] Y. Halim, global character of systems of rational difference equations, Electronic Journal of Mathematical Analysis and Applications, 3 (1) (2015), 204-214.

[27] G. Jawahar and P. Gopinath, Some Properties of Oscillation of Second Order Neutral Delay Difference Equations, International Journal of Dynamics of Fluids, 13 (1) (2017), 47-51.

[28] R. Karatas, C. Ciliar and D. Simsek, On positive solutions of the difference equation $x_{n+1} = \frac{x_{n-5}}{1+x_{n-2}x_{n-5}}$, Int. J. Contemp. Math. Sci., 1 (10) (2006), 495-500.

[29] A. Khaliq, and E. M. Elsayed, The Dynamics and Solution of some Difference Equations, Journal of Nonlinear Sciences and Applications, 9 (3) (2016), 1052-1063.

[30] A. Q. Khan, and M. N. Qureshi, Qualitative behavior of two systems of higher□order difference equations, Mathematical Methods in the Applied Sciences, 39 (11) (2016), 3058-3074.

[31] T. Khyat, E. Pilav and M. R. S. Kulenovic, The Invariant Curve Caused by Neimark–Sacker Bifurcation of a Perturbed Beverton–Holt Difference Equation, International Journal of Difference Equations, 12 (2) (2017), 267–280.

[32] V. L. Kocic and G. Ladas, Global Behavior of Nonlinear Difference Equations of Higher Order with Applications, Kluwer Academic Publishers, Dordrecht, 1993.

[33] M. R. S. Kulenovic and G. Ladas, Dynamics of Second Order Rational Difference Equations with Open Problems and Conjectures, Chapman & Hall / CRC Press, 2001.

[34] L. Li and P. Zhang, Unbounded solutions of an iterative-difference equation, Acta Univ. Sapientiae, Mathematica, 9 (1) (2017), 224–234.

[35] O. Ocalan, Global dynamics of a non-autonomous rational difference equation, Journal of applied mathematics & informatics, 32 (5-6) (2014), 843-848.

[36] U. Saeed, M. Ozair, T. Hussain and Q. Din, Fractional-Order Vector-Host Disease Model, Dynamics of Continuous, Discrete and Impulsive Systems Series B: Applications & Algorithms, 24 (2017), 97-111.

[37] N. Touafek and E. M. Elsayed, On a second order rational systems of difference equation, Hokkaido Mathematical Journal, 44 (1) (2015), 29--45.

[38] Q. Wang and Q. Zhang, Dynamics of A Higher-Order Rational Difference Equation, Journal of Applied Analysis and Computation, 7 (2) (2017), 770–787.

[39] I. Yalçınkaya, Global asymptotic stability in a rational equation, Selçuk Journal of Applied Mathematics, Summer-Autumn, 6 (2) (2005), 59-68.

[40] I. Yalçınkaya, On the difference equation $x_{n+1} = \alpha + \frac{x_{n-m}}{x_n^k}$, Discrete Dynamics in Nature and Society, Vol. 2008, Article ID 805460, 8 pages, doi: 10.1155/2008/ 805460.

[41] I. Yalçınkaya, C. Cinar and M. Atalay, On the solutions of systems of difference equations, Adv. Differ. Equ., Vol. 2008, Article ID 143943, 9 pages.

[42] I. Yalcinkaya, A. E. Hamza, and C. Cinar, Global Behavior of a Recursive Sequence, Selçuk J. Appl. Math., 14 (1) (2013), 3-10.

[43] Y. Yazlik, E. M. Elsayed, N. Taskara, On the Behaviour of the Solutions of Difference Equation

Systems, Journal of Computational Analysis and Applications, 16 (5) (2014), 932-941.

[44] E. M. E. Zayed, On the dynamics of the nonlinear rational difference equation, Dynamics of Continuous, Discrete and Impulsive Systems Series A: Mathematical Analysis 22 (2015), 61-71.

[45] E. M. E. Zayed and M. A. El-Moneam, On the rational recursive sequence $x_{n+1} = \frac{\alpha+\beta x_n+\gamma x_{n-1}}{A+Bx_n+Cx_{n-1}}$, Communications on Applied Nonlinear Analysis, 12 (4) (2005), 15-28.

[46] Q. Zhang, W. Zhang, J. Liu and Y. Shao, On a Fuzzy Logistic Difference Equation, WSEAS Transactions on Mathematics, 13 (2014), 282-290.

[47] Q. Zhang, and W. Zhang, On the System of Nonlinear Rational Difference Equations, International Journal of Mathematical, Computational, Physical, Electrical and Computer Engineering, 8 (4) (2014), 692-695.

Convexity and Monotonicity of Certain Maps Involving Hadamard Products and Bochner Integrals for Continuous Fields of Operators

Pattrawut Chansangiam*

Department of Mathematics, Faculty of Science,
King Mongkut's Institute of Technology Ladkrabang,
Bangkok 10520, Thailand.

Abstract

We investigate the convexity and the monotonicity of certain maps involving Hadamard products and Bochner integrals for continuous fields of Hilbert space operators. Their special cases and consequences are then discussed. In particular, we obtain certain arithmetic mean-harmonic mean, Jensen, and Fiedler type inequalities.

Keywords: Hadamard product, tensor product, continuous field of operators, Bochner integral
Mathematics Subject Classifications 2010: 26D15, 474A63, 46G10, 47A80.

1 Introduction

Throughout, let $\mathfrak{B}(\mathbb{H})$ be the algebra of bounded linear operators on a complex separable Hilbert space $\mathbb{H}$. The positive cone $\mathfrak{B}(\mathbb{H})^+$ of $\mathfrak{B}(\mathbb{H})$ consists of all positive operators on $\mathbb{H}$. The identity operator is denoted by I, where the underlying space should be clear from contexts. The spectrum of $A \in \mathfrak{B}(\mathbb{H})$ is written as $\mathrm{sp}(A)$. For self-adjoint operators A and B, the situation $A \geqslant B$ means that $A - B \in \mathfrak{B}(\mathbb{H})^+$. If A is an invertible positive operator, we write $A > 0$. The operator norm of $A \in \mathfrak{B}(\mathbb{H})$ is denoted by $\|A\|$. The notation $\|\cdot\|_{\infty,X}$ is used for the supremum norm on the set X.

The *Hadamard product* of A and B in $\mathfrak{B}(\mathbb{H})$ is defined to be the bounded linear operator $A \circ B$ satisfying

$$\langle (A \circ B)e, e\rangle = \langle Ae, e\rangle\langle Be, e\rangle \quad \text{for all } e \in \mathcal{E}. \tag{1.1}$$

*Corresponding author. Email: pattrawut.ch@kmitl.ac.th

Here, $\mathcal{E}$ is a fixed countable orthonormal basis for $\mathbb{H}$. This definition is independent on a choice of the orthonormal basis. In [7], it was shown that there is a positive linear map Φ taking the tensor product $A \otimes B$ to the Hadamard product $A \circ B$ for any $A, B \in \mathfrak{B}(\mathbb{H})$. Indeed, the map Φ is given by $\Phi(X) = Z^* X Z$ where $Z : \mathbb{H} \to \mathbb{H} \otimes \mathbb{H}$ is the isometry defined on the basis $\mathcal{E}$ by

$$Ze \;=\; e \otimes e \quad \text{for all } e \in \mathcal{E}. \tag{1.2}$$

From the condition (1.1), the Hadamard product is commutative, bilinear, and positivity preserving. When $\mathbb{H}$ is the finite-dimensional space $\mathbb{C}^n$, the Hadamard product for square complex matrices is just a principal submatrix of their Kronecker product, and it can be computed easily as the entrywise product.

In the literature, there are many results concerning Hadamard products for matrices/operators, see e.g. [5, 8, 10]. A well known result is Fiedler's inequality:

Theorem 1.1 ([6]). *For any positive definite matrix A, we have*

$$A \circ A^{-1} \;\geqslant\; I.$$

Theorem 1.1 can be extended in the following way:

Theorem 1.2 ([11]). *For each $i = 1, 2, \dots, n$, let A_i be a positive definite matrix and X_i a positive semidefinite matrix of the same size. Then the map*

$$\alpha \mapsto \sum_{i=1}^{n} X_i^{1/2} A_i^{\alpha} X_i^{1/2} \circ \sum_{i=1}^{n} X_i^{1/2} A_i^{-\alpha} X_i^{1/2}$$

is increasing on $[0, \infty)$.

In this paper, we shall investigate the convexity and the monotonicity of an integral map

$$\alpha \mapsto \int_\Omega X_t^* A_t^{\alpha} X_t \, d\mu(t) \circ \int_\Omega X_t^* A_t^{-\alpha} X_t \, d\mu(t) \tag{1.3}$$

where α is a real constant. Here, $(A_t)_{t \in \Omega}$ and $(X_t)_{t \in \Omega}$ are two operator-valued maps parametrized by a locally compact Hausdorff space Ω. Some interesting special cases of this map are discussed. Moreover, we obtain certain arithmetic mean - harmonic mean (AM-HM), Jensen, and Fiedler type inequalities as consequences. When we set Ω to be a finite space endowed with the counting measure, our results are reduced to the corresponding discrete inequalities. In particular, these include Theorems 1.1 and 1.2.

The paper is structured as follows. In Section 2, we set up basic notations, and discuss Bochner integrability of continuous field of operators on a locally compact Hausdorff space. The main part of the paper, Section 3, discusses convexity and monotonicity of the map (1.3) and its interesting special cases. As consequences, we obtain certain AM-GM, Jensen, and Fielder type inequalities in the last section.

2

2 Continuous field of operators on a locally compact Hausdorff space

In this section, we provide fundamental background on continuous fields of operators and their integrability. See, e.g., [1, 3, 12] for more information.

Let Ω be a locally compact Hausdorff space endowed with a Radon measure μ. A family $(A_t)_{t\in\Omega}$ of operators in $\mathfrak{B}(\mathbb{H})$ is said to be a *continuous field of operators* if the parametrization $t \mapsto A_t$ is norm-continuous on Ω. If, in addition, the norm function $t \mapsto \|A_t\|$ is Lebesgue integrable on Ω, then we can form the Bochner integral $\int_\Omega A_t \, d\mu(t)$ which is the unique operator in $\mathfrak{B}(\mathbb{H})$ such that

$$\phi\left(\int_\Omega A_t \, d\mu(t)\right) = \int_\Omega \phi(A_t)\, d\mu(t)$$

for every bounded linear functional ϕ on $\mathfrak{B}(\mathbb{H})$ (see e.g. [14, pp. 75-78]).

In what follows, suppose further that the measure μ on Ω is finite. Next, we shall prove the Bochner integrability of an operator-valued map involving a continuous field of operators (Proposition 2.3). To do this we need some auxiliary results about functional calculus and vector-valued integration.

Lemma 2.1. *Let Δ be a nonempty compact subset of $\mathbb{C}$ and $f : \Delta \to \mathbb{C}$ a continuous function. Let $\mathcal{A}$ be the subset of $\mathfrak{B}(\mathbb{H})$ consisting of all normal operators whose spectra are contained in Δ. Then the map $\Psi : \mathcal{A} \to \mathfrak{B}(\mathbb{H})$, $A \mapsto f(A)$ is continuous. Here, $f(A)$ is the continuous functional calculus of f on the spectrum of A.*

Proof. See [4, Lemma 2.1]. □

Lemma 2.2. *Let $(\mathbb{X}, \|\cdot\|_{\mathbb{X}})$ be a Banach space, and let (Γ, ν) be a finite measure space. Suppose that $f : \Gamma \to \mathbb{X}$ is a measurable function. Then f is Bochner integrable if and only if its norm function $\|f\|$ is Lebesgue integrable, i.e.,*

$$\int_\Gamma \|f\| \, d\nu < \infty.$$

Here, $\|f\|$ is defined by $\|f\|(x) = \|f(x)\|_{\mathbb{X}}$ for any $x \in \mathbb{X}$.

Proof. See e.g. [1, Theorem 11.44]. □

Now we are in a position to prove the Bochner integrability of a map related to the map (1.3).

Proposition 2.3. *Let $(A_t)_{t\in\Omega}$ be a continuous field of normal operators in $\mathfrak{B}(\mathbb{H})$ such that $\operatorname{sp}(A_t) \subseteq [m, M]$ for all $t \in \Omega$. Let $(X_t)_{t\in\Omega}$ be a bounded continuous field of operators in $\mathfrak{B}(\mathbb{H})$. For any continuous function $f : [m, M] \to \mathbb{C}$, we can form the Bochner integral*

$$\int_\Omega X_t^* f(A_t) X_t \, d\mu(t).$$

In addition, if $f([m, M]) \subseteq [0, \infty)$, then this operator is positive.

Proof. Let $K > 0$ be such that $\|X_t\| \leqslant K$ for all $t \in \Omega$. By Lemma 2.2, it suffices to prove the Lebesgue integrability of the norm function $t \mapsto \|X_t^* f(A_t) X_t\|$. We shall show that the map $t \mapsto X_t^* f(A_t) X_t$ is continuous and bounded. Since $t \mapsto A_t$ is continuous, the map $t \mapsto f(A_t)$ is continuous on Ω by Lemma 2.1, and hence so is the map $t \mapsto X_t^* f(A_t) X_t$. For boundedness, we have that for each $t \in \Omega$,

$$\begin{aligned} \|X_t^* f(A_t) X_t\| &\leqslant \|X_t^*\| \cdot \|f(A_t)\| \cdot \|X_t\| \\ &\leqslant \|X_t\|^2 \cdot \|f\|_{\infty,[m,M]} \\ &\leqslant K^2 \|f\|_{\infty,[m,M]}. \end{aligned}$$

Now, suppose that f is positive on $[m, M]$. Then $f(A_t)$ is a positive operator for all $t \in \Omega$. Hence the resulting integral is positive since the integrand is positive. □

Remark 2.4. For convenience to all results in this paper, we may assume that Ω is a compact Hausdorff space. In this case, any Radon measure on Ω is always finite. It follows that every continuous field of operators on Ω is automatically bounded, and hence is Bochner integrable.

Lemma 2.5. *Let $\mathbb{X}$ and $\mathbb{Y}$ be Banach spaces and let (Γ, ν) be a measure space. Suppose that a function $f : \Gamma \to \mathbb{X}$ is Bochner integrable. If $T : \mathbb{X} \to \mathbb{Y}$ be a bounded linear operator, then the composition $T \circ f$ is Bochner integrable and*

$$\int_\Gamma (T \circ f)\, d\nu \;=\; T\left(\int_\Gamma f\, d\nu\right).$$

Proof. See e.g. [1, Lemma 11.45]. □

The next property will be used to prove the main result of the paper.

Proposition 2.6. *Let $(A_t)_{t\in\Omega}$ be a bounded continuous field of operators in $\mathfrak{B}(\mathbb{H})$. For any $X \in \mathfrak{B}(\mathbb{H})$, we have*

$$\int_\Omega A_t\, d\mu(t) \circ X \;=\; \int_\Omega (A_t \circ X)\, d\mu(t). \tag{2.1}$$

Proof. By Lemma 2.2, the map $t \mapsto A_t$ is Bochner integrable on Ω since it is continuous and bounded. Note that the map $T \mapsto T \circ X$ is a bounded linear operator from $\mathfrak{B}(\mathbb{H})$ to itself. It follows from Lemma 2.5 that the map $t \mapsto A_t \circ X$ is Bochner integrable on Ω and the property (2.1) holds. □

3 Convexity and Monotonicity of certain maps for Hadamard products of operators

In this section, we consider convexity and monotonicity of the map

$$\alpha \mapsto \int_\Omega X_t^* A_t^\alpha X_t\, d\mu(t) \circ \int_\Omega X_t^* A_t^{-\alpha} X_t\, d\mu(t)$$

where α is a real constant. We start with an auxiliary result.

4

Lemma 3.1. *For each $A > 0$, the map $\alpha \mapsto A^\alpha + A^{-\alpha}$ is convex on $\mathbb{R}$, increasing on $[0, \infty)$, decreasing on $(-\infty, 0]$ and attaining its minimum at $\alpha = 0$.*

Proof. The convexity of the map $F(\alpha) = A^\alpha + A^{-\alpha}$ means that for each $\alpha, \beta \in \mathbb{R}$ and $\omega \in (0, 1)$, we have $F((1-\omega)\alpha+\omega\beta) \leqslant (1-\omega)F(\alpha)+\omega F(\beta)$ or equivalently,

$$A^{(1-\omega)\alpha+\omega\beta} + A^{-((1-\omega)\alpha+\omega\beta)} \leqslant (1-\omega)(A^\alpha + A^{-\alpha}) + \omega(A^\beta + A^{-\beta}). \qquad (3.1)$$

Indeed, for each fixed $x > 0$, consider the function $f(\alpha) = x^\alpha + x^{-\alpha}$ in a real variable α. Then

$$f''(\alpha) = (\ln x)^2 (x^\alpha + x^{-\alpha}) > 0, \quad \alpha \in \mathbb{R}.$$

It follows that f is convex on $\mathbb{R}$, i.e., for each $\alpha, \beta \in \mathbb{R}$ and $\omega \in (0, 1)$ we have

$$x^{(1-\omega)\alpha+\omega\beta} + x^{-((1-\omega)\alpha+\omega\beta)} \leqslant (1-\omega)(x^\alpha + x^{-\alpha}) + \omega(x^\beta + x^{-\beta}). \qquad (3.2)$$

Applying the functional calculus on the spectrum of A yields the desired inequality (3.1). Note also that $f'(\alpha) = \alpha(x^{\alpha-1} - x^{-\alpha-1})$ for each $\alpha \in \mathbb{R}$. Hence, f is increasing on $[0, \infty)$, decreasing on $(-\infty, 0]$ and attaining its minimum at $\alpha = 0$. Similarly, applying the functional calculus yields the desired results. $\square$

A proof of a part of this fact in matrix context was given in [13], using diagonalization.

Theorem 3.2. *Let $(A_t)_{t \in \Omega}$ be a continuous field of positive operators in $\mathfrak{B}(\mathbb{H})$ such that $\operatorname{sp}(A_t) \subseteq [m, M] \subseteq (0, \infty)$ for all $t \in \Omega$. Let $(X_t)_{t \in \Omega}$ be a bounded continuous field of operators in $\mathfrak{B}(\mathbb{H})$. Then the map*

$$\alpha \mapsto \int_\Omega X_t^* A_t^\alpha X_t \, d\mu(t) \circ \int_\Omega X_t^* A_t^{-\alpha} X_t \, d\mu(t) \qquad (3.3)$$

is convex on $\mathbb{R}$, increasing on $[0, \infty)$, decreasing on $(-\infty, 0]$ and attaining its minimum at $\alpha = 0$.

Proof. Denote this map by F. Proposition 2.3 asserts the Bochner integrability of the map $t \mapsto X_t^* A_t^\alpha X_t$ for each $\alpha \in \mathbb{R}$. For each $\alpha \in \mathbb{R}$, we have by Proposition 2.6 and Fubini's theorem for Bochner integrals [2] that

$$\begin{aligned} F(\alpha) &= \int_\Omega \left(X_t^* A_t^\alpha X_t \circ \int_\Omega X_s^* A_s^{-\alpha} X_s \, d\mu(s) \right) d\mu(t) \\ &= \iint_{\Omega^2} (X_t^* A_t^\alpha X_t \circ X_s^* A_s^{-\alpha} X_s) \, d\mu(s) \, d\mu(t) \qquad (3.4) \\ &= \frac{1}{2} \iint_{\Omega^2} (X_t^* A_t^\alpha X_t \circ X_s^* A_s^{-\alpha} X_s) + (X_t^* A_t^{-\alpha} X_t \circ X_s^* A_s^\alpha X_s) \, d\mu(s) \, d\mu(t). \end{aligned}$$

5

Then, appealing the isometry Z defined by (1.2), we have

$$\begin{aligned} F(\alpha) &= \frac{1}{2}\iint_{\Omega^2} Z^*\left[(X_t^* A_t^\alpha X_t \otimes X_s^* A_s^{-\alpha} X_s) + (X_t^* A_t^{-\alpha} X_t \otimes X_s^* A_s^\alpha X_s)\right] Z \\ &\quad d\mu(s)\, d\mu(t) \\ &= \frac{1}{2}\iint_{\Omega^2} Z^*(X_t \otimes X_s)^* \left[(A_t \otimes A_s^{-1})^\alpha + (A_t \otimes A_s^{-1})^{-\alpha}\right](X_t \otimes X_s) Z \\ &\quad d\mu(s)\, d\mu(t). \end{aligned} \tag{3.5}$$

Now, for each $\alpha, \beta \in \mathbb{R}$ and $\omega \in (0,1)$, we have from Lemma 3.1 and (3.5) that

$$\begin{aligned} &F((1-\omega)\alpha + \omega\beta) \\ &\leqslant \frac{1}{2}\iint_{\Omega^2} Z^*(X_t \otimes X_s)^*\big[(1-\omega)\{(A_t \otimes A_s^{-1})^\alpha + (A_t \otimes A_s^{-1})^{-\alpha}\} \\ &\qquad + \omega\{(A_t \otimes A_s^{-1})^\beta + (A_t \otimes A_s^{-1})^{-\beta}\}\big](X_t \otimes X_s) Z d\mu(s)\, d\mu(t) \\ &= \frac{1}{2}\iint_{\Omega^2} Z^*\big[(1-\omega)(X_t^* A_t^\alpha X_t \otimes X_s^* A_s^{-\alpha} X_s + X_t^* A_t^{-\alpha} X_t \otimes X_s^* A_s^\alpha X_s) \\ &\qquad + \omega(X_t^* A_t^\beta X_t \otimes X_s^* A_s^{-\beta} X_s + X_t^* A_t^{-\beta} X_t \otimes X_s^* A_s^\beta X_s)\big] Z d\mu(s)\, d\mu(t) \\ &= \frac{1-\omega}{2}\iint_{\Omega^2} (X_t^* A_t^\alpha X_t \circ X_s^* A_s^{-\alpha} X_s + X_t^* A_t^{-\alpha} X_t \circ X_s^* A_s^\alpha X_s) d\mu(s)\, d\mu(t) \\ &\qquad + \frac{\omega}{2}\iint_{\Omega^2} (X_t^* A_t^\beta X_t \circ X_s^* A_s^{-\beta} X_s + X_t^* A_t^{-\beta} X_t \circ X_s^* A_s^\beta X_s) d\mu(s)\, d\mu(t) \\ &= (1-\omega)F(\alpha) + \omega F(\beta). \end{aligned}$$

Therefore, F is convex. In the rest, it suffices to show that F is increasing on $[0,\infty)$ since the Hadamard product is commutative. It follows from (3.5) and Lemma 3.1 that for $0 \leqslant \alpha \leqslant \beta$,

$$\begin{aligned} F(\alpha) &\leqslant \frac{1}{2}\iint_{\Omega^2} Z^*(X_t \otimes X_s)^* \left[(A_t \otimes A_s^{-1})^\beta + (A_t \otimes A_s^{-1})^{-\beta}\right](X_t \otimes X_s) Z\, d\mu(s)\, d\mu(t) \\ &= \frac{1}{2}\iint_{\Omega^2} Z^*\left[(X_t^* A_t^\beta X_t \otimes X_s^* A_s^{-\beta} X_s) + (X_t^* A_t^{-\beta} X_t \otimes X_s^* A_s^\beta X_s)\right] Z\, d\mu(s)\, d\mu(t) \\ &= \iint_{\Omega^2} (X_t^* A_t^\beta X_t \circ X_s^* A_s^{-\beta} X_s)\, d\mu(s)\, d\mu(t). \end{aligned}$$

From (3.4), the right-hand side is equal to $F(\beta)$. Thus, F is increasing on $[0,\infty)$. ☐

In the rest of section, we discuss certain special cases of Theorem 3.2.

Corollary 3.3. *Let $(A_t)_{t\in\Omega}$ and $(B_t)_{t\in\Omega}$ be two bounded continuous field of positive operators in $\mathfrak{B}(\mathbb{H})$ such that $\mathrm{sp}(A_t) \subseteq [m,M] \subseteq (0,\infty)$ and $A_t B_t = B_t A_t$ for all $t \in \Omega$. Then the map*

$$\alpha \mapsto \int_\Omega A_t^\alpha B_t\, d\mu(t) \circ \int_\Omega A_t^{-\alpha} B_t\, d\mu(t) \tag{3.6}$$

is convex on $\mathbb{R}$*, increasing on* $[0,\infty)$*, decreasing on* $(-\infty,0]$ *and attaining its minimum at* $\alpha=0$*.*

Proof. Set $X_t = B_t^{1/2}$ for each $t \in \Omega$. Then $(X_t)_{t\in\Omega}$ is a continuous field by Lemma 2.1. The family $(X_t)_{t\in\Omega}$ is bounded due to the boundedness of $(B_t)_{t\in\Omega}$. The result now follows from Theorem 3.2. □

An interesting special case of Corollary 3.3 is when $B_t = f(A_t)$ where f is a complex-valued continuous function on $[m,M]$. In this case, the field $(B_t)_{t\in\Omega}$ is bounded since

$$\|f(A_t)\| \leqslant \|f\|_{\infty,[m,M]}$$

for all $t \in \Omega$. Hence we obtain monotonicity information of the map

$$\alpha \mapsto \int_\Omega A_t^\alpha f(A_t)\,d\mu(t) \circ \int_\Omega A_t^{-\alpha} f(A_t)\,d\mu(t).$$

In particular, when $f(x) = x^\lambda$, we get the following result.

Corollary 3.4. *For any* $\lambda \in \mathbb{R}$*, the map*

$$\alpha \mapsto \int_\Omega A_t^{\lambda+\alpha}\,d\mu(t) \circ \int_\Omega A_t^{\lambda-\alpha}\,d\mu(t)$$

is convex on $\mathbb{R}$*, increasing on* $[0,\infty)$*, decreasing on* $(-\infty,0]$ *and attaining its minimum at* $\alpha=0$*.*

The next result is also a special case of Theorem 3.2 in which the weights are scalars.

Corollary 3.5. *Let* $(A_t)_{t\in\Omega}$ *be a continuous field of positive operators in* $\mathfrak{B}(\mathbb{H})$ *such that* $\mathrm{sp}(A_t) \subseteq [m,M] \subseteq (0,\infty)$ *for all* $t\in\Omega$*. For any bounded continuous function* $w : \Omega \to [0,\infty)$*, the map*

$$\alpha \mapsto \int_\Omega w(t) A_t^\alpha\,d\mu(t) \circ \int_\Omega w(t) A_t^{-\alpha}\,d\mu(t) \tag{3.7}$$

is convex on $\mathbb{R}$*, increasing on* $[0,\infty)$*, decreasing on* $(-\infty,0]$ *and attaining its minimum at* $\alpha=0$*.*

Proof. Set $X_t = \sqrt{w(t)}I$ for all $t \in \Omega$ in Theorem 3.2. We see that $(X_t)_{t\in\Omega}$ is a bounded continuous field of operators. □

Corollary 3.6. *Let* $f : \Omega \to \mathbb{C}$ *and* $g : \Omega \to [0,\infty)$ *be bounded continuous functions such that* $\mathrm{Range}(f) \subseteq [m,M] \subseteq (0,\infty)$*. Then the map*

$$\alpha \mapsto \int_\Omega g f^\alpha\,d\mu \int_\Omega g f^{-\alpha}\,d\mu$$

is convex on $\mathbb{R}$*, increasing on* $[0,\infty)$*, decreasing on* $(-\infty,0]$ *and attaining its minimum at* $\alpha=0$*.*

7

Proof. Set $\mathbb{H} = \mathbb{C}$ in Corollary 3.3. ☐

A discrete version of Theorem 3.2 is obtained in the next corollary, which is an operator extension of Theorem 1.2.

Corollary 3.7. *For each $i = 1, 2, \ldots, n$, let $A_i, X_i \in \mathfrak{B}(\mathbb{H})$ be such that A_i is positive and invertible. Then the map*

$$\alpha \mapsto \sum_{i=1}^{n} X_i^* A_i^{\alpha} X_i \circ \sum_{i=1}^{n} X_i^* A_i^{-\alpha} X_i$$

is convex on $\mathbb{R}$, increasing on $[0, \infty)$, decreasing on $(-\infty, 0]$ and attaining its minimum at $\alpha = 0$.

Proof. In Theorem 3.2, set Ω to be the finite space $\{1, 2, \ldots, n\}$ equipped with the counting measure. ☐

4 AM-GM, Jensen, and Fiedler type inequalities

From the main result (Theorem 3.2), we get three interesting inequalities. The first consequence is an integral version of the weighted arithmetic-harmonic mean inequality for bounded continuous function defined on a locally compact Hausdorff space:

Corollary 4.1. *Let f be a bounded continuous function defined on Ω such that* Range$(f) \subseteq [m, M] \subseteq (0, \infty)$. *Let $w : \Omega \to [0, \infty)$ be a weight function, i.e., $\int_{\Omega} w \, d\mu = 1$. We obtain the following bound for the weight integral of f:*

$$\|wf\|_1 \geqslant \frac{1}{\|w/f\|_1}. \tag{4.1}$$

Here, $\|\cdot\|_1$ denotes the L^1-norm on Ω.

Proof. Setting $\mathbb{H} = \mathbb{C}$ in Corollary 3.5 yields that the function

$$\alpha \mapsto \int_{\Omega} wf^{\alpha} \, d\mu \int_{\Omega} \frac{w}{f^{\alpha}} \, d\mu$$

is increasing on $[0, \infty)$. In particular, this implies that

$$\int_{\Omega} wf \, d\mu \int_{\Omega} \frac{w}{f} \, d\mu \geqslant \left(\int_{\Omega} w \, d\mu \right)^2 = 1.$$

Now, since $\int \frac{w}{f} \, d\mu \geqslant \int wM^{-1} \, d\mu = M^{-1} > 0$, the inequality (4.1) follows. ☐

The second consequence is a Jensen type inequality for a continuous field of strictly positive operators.

8

Corollary 4.2. *Let* $(A_t)_{t\in\Omega}$ *be a continuous field of positive operators in* $\mathfrak{B}(\mathbb{H})$ *such that* $\operatorname{sp}(A_t) \subseteq [m, M] \subseteq (0,\infty)$ *for all* $t \in \Omega$. *Suppose that* $\mu(\Omega) = 1$. *Then*

$$\int_\Omega A_t^2 \, d\mu(t) \circ I \;\geqslant\; \int_\Omega A_t \, d\mu(t) \circ \int_\Omega A_t \, d\mu(t).$$

Proof. Form Corollary 3.4, the map

$$G(\alpha) \;\equiv\; \int_\Omega {A_t}^{1+\alpha} \, d\mu(t) \circ \int_\Omega {A_t}^{1-\alpha} \, d\mu(t)$$

is increasing on $[0,\infty)$. In particular, we have $G(1) \geqslant G(0)$, which is the desired inequality. □

Corollary 4.2 may be regarded as a Jensen type inequality for continuous field of operators (cf. [9]). Indeed, the case $\mathbb{H} = \mathbb{C}$ of this corollary can be described as follows. Suppose that (Ω, μ) is a probability space. For any continuous function $f : \Omega \to (0,\infty)$, we have

$$\int_\Omega f^2 \, d\mu \;\geqslant\; \left(\int_\Omega f \, d\mu\right)^2,$$

which is Jensen's inequality for the convex function $\phi(x) = x^2$.

The final result is an operator extension of Fiedler's inequality (Theorem 1.1).

Corollary 4.3. *For each invertible positive operator* A*, we have*

$$A \circ A^{-1} \;\geqslant\; I. \tag{4.2}$$

Proof. The case $n = 1$ in Corollary 3.7 says that the map $\alpha \mapsto A^\alpha \circ A^{-\alpha}$ has a minimum at $\alpha = 0$. It follows that $A^\alpha \circ A^{-\alpha} \geqslant I$ for any $\alpha \in \mathbb{R}$. Replacing A with $A^{1/\alpha}$ yields the inequality (4.2). □

Acknowledgements

This research is supported by King Mongkut's Institute of Technology Ladkrabang Research Fund.

References

[1] C. D. Aliprantis, K. C. Border, *Infinite Dimensional Analysis*, Springer-Verlag, New York, 2006.

[2] W. M. Bogdanowicz, Fubini's theorem for generalized Lebesgue-Bochner-Stieltjes integral. *Proc. Jpn. Acad.*, 42, 979-983 (1966).

[3] P. Chansangiam, Certain integral inequalities involving tensor products, positive linear maps, and operator means. *J. Inequal. Appl.*, Article 121, (2016), DOI 10.1186/ s13660-016-1063-7.

[4] P. Chansangiam, Kantorovich type integral inequalities for tensor products of continuous fields of positive operators. *J. Comput. Anal. Appl.*, 25, 1385-1397 (2018).

[5] R. Drnovšek, Inequalities on the spectral radius and the operator norm of Hadamard products of positive operators on sequence spaces. *Banach J. Math. Anal.*, 10(4), 800-814 (2016).

[6] M. Fiedler, Uber eine ungleichung fur positiv definite matrizen. *Math. Nachr.*, 23, 197-199 (1961)

[7] J. I. Fujii, The Marcus-Khan theorem for Hilbert space operators. *Mathematica Japonica*, 41, 531-535 (1995).

[8] J. I. Fujii, M. Nakamura, Y. Seo, Ando's theorem for Hadamard products and operator means. *Sci. Math. Jpn.*, e-2006, 603-608 (2006).

[9] F. Hansen, J. Pečarić, I. Perić, Jensen's operator inequality and its converses. *Math. Scand.*, 100, 61-73 (2007).

[10] K. Kitamura, R. Nakamoto, Schwarz inequalities on Hadamard products. *Sci. Math. Jpn.*, 1(2), 243-246 (1998).

[11] J. S. Matharu, J. S. Aujla, Hadamard product versions of the Chebyshev and Kantorovich inequalities, *J. Inequal. Pure Appl. Math.*, 10, Article 51 (2009).

[12] M. S. Moslehian, Chebyshev type inequalities for Hilbert space operators. *J. Math. Anal. Appl.*, 420(1), 737-749 (2014).

[13] M. S. Moslehian, J. S. Matharu, J. S. Aujla, Non-commutative Callebaut inequality. *Linear Algebra Appl.*, 436(9), 3347-3353 (2012).

[14] G. K. Pedersen, *Analysis Now*, Springer-Verlag, New York, 1989.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 27, NO.5, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC

Fibonacci periodicity and Fibonacci frequency

Hee Sik Kim[1], J. Neggers[2] and Keum Sook So[3,*]

[1]**Department of Mathematics, Research Institute for Natural Sciences, Hanyang University, Seoul, 04763, Korea**

[2]**Department of Mathematics, University of Alabama, Tuscaloosa, AL 35487-0350, U. S. A**

[3,*]**Department of Statistics and Financial Informatics, Hallym University, Chuncheon, 24252, Korea**

Abstract. In this paper we introduce the notion of Fibonacci periodicity modulo n, denoting this period by the function $\widehat{F}(n)$. We note that $\widehat{F}(n)$ is an integral multiple of a fundamental frequency $\widehat{f}(n)$, where the ratio $\widehat{F}(n)/\widehat{f}(n)$ is a power of 2 for a collection of observed values of n. It is demonstrated that if a, b are natural numbers with $gcd(a, b) = 1$, then $\widehat{F}(n) = lcm\{\widehat{F}(a), \widehat{F}(b)\}$ and thus that $\widehat{F}$ is a non-trivial example of a function which we refer to as radical. From observations it also seems clear that $\frac{\widehat{F}(p^{s+1})}{\widehat{F}(p^s)} = p$ for primes p.

1. Introduction and Preliminaries

Fibonacci-numbers have been studied in many different forms for centuries and the literature on the subject is consequently incredibly vast. One of the amazing qualities of these numbers is the variety of mathematical models where they play some sort of role and where their properties are of importance in elucidating the ability of the model under discussion to explain whatever implications are inherent in it. The fact that the ratio of successive Fibonacci numbers approaches the Golden ratio (section) rather quickly as they go to infinity probably has a good deal to do with the observation made in the previous sentence. Surveys and connections of the type just mentioned are provided in [1] and [2] for a very minimal set of examples of such texts, while in [3] an application (observation) concerns itself with a theory of a particular class of means which has apparently not been studied in the fashion done there by two of the authors the present paper. Surprisingly novel perspectives are still available.

Kim and Neggers [6] showed that there is a mapping $D : M \to DM$ on means such that if M is a Fibonacci mean so is DM, that if M is the harmonic mean, then DM is the arithmetic mean, and if M is a Fibonacci mean, then $\lim_{n\to\infty} D^n M$ is the golden section mean.

In [5] Han et al. discussed Fibonacci functions on the real numbers $\mathbf{R}$, i.e., functions $f : \mathbf{R} \to \mathbf{R}$ such that for all $x \in \mathbf{R}$, $f(x+2) = f(x+1)+f(x)$, and developed the notion of Fibonacci functions using the concept of f-even and f-odd functions. Moreover, they showed that if f is a Fibonacci function then $\lim_{x\to\infty} \frac{f(x+1)}{f(x)} = \frac{1+\sqrt{5}}{2}$. The present authors [8] discussed Fibonacci functions using the (ultimately) periodicity and we also discuss the exponential Fibonacci functions. Especially, given a non-negative real-valued function, we obtain several exponential Fibonacci functions.

The present authors [9] introduced the notions of Fibonacci (co-)derivative of real-valued functions, and found general solutions of the equations $\triangle(f(x)) = g(x)$ and $(\triangle + I)(f(x)) = g(x)$. Moreover, they [10] defined and studied a function $F : [0, \infty) \to \mathbf{R}$ and extensions $F : \mathbf{R} \to \mathbf{C}$, $\widetilde{F} : \mathbf{C} \to \mathbf{C}$ which are continuous and such that if $n \in \mathbf{Z}$, the set of all integers, then $F(n) = F_n$, the n^{th} Fibonacci number based on $F_0 = F_1 = 1$. If x is not an integer and $x < 0$, then $F(x)$ may be a complex number, e.g., $F(-1.5) = \frac{1}{2} + i$. If $z = a + bi$,

[0*] Correspondence: Tel.: +82 33 248 2011, Fax: +82 33 256 2011 (K. S. So).

then $\widetilde{F}(z) = F(a) + iF(b-1)$ defines complex Fibonacci numbers. In connection with this function (and in general) they defined a Fibonacci derivative of $f : \mathbf{R} \to \mathbf{R}$ as $(\triangle f)(x) = f(x+2) - f(x+1) - f(x)$ so that if $(\triangle f)(x) \equiv 0$ for all $x \in \mathbf{R}$, then f is a (real) Fibonacci function. A complex Fibonacci derivative $\widetilde{\triangle}$ is given as $\widetilde{\triangle} f(a+bi) = \triangle f(a) + i \triangle f(b-1)$ and its properties are discussed in same detail.

The notion of Fibonacci means was introduced by

$$M(x,y) = \frac{a(x+y) + 2bxy}{2a + b(x+y)}$$

where $M(x,x) = \dfrac{2ax + 2bx^2}{2a + 2bx} = x$ provided $2a + 2bx \neq 0$ ([6]).

Particular cases are $a > 0, b = 0, M(x,y) = \frac{x+y}{2}$, the average (arithmetic mean), $a = 0, b > 0, M(x,y) = \frac{2xy}{x+y}$, the harmonic mean, and if $q = \frac{1+\sqrt{5}}{2}$, $M_q(x,y) = \frac{q(x+y)+2xy}{2q+(x+y)}$, the golden section mean. Hence both the harmonic mean, the arithmetic mean and golden section mean are special cases of the Fibonacci mean.

The golden section mean $M_q(x,y)$ is defined by $M_q(x,y) = \frac{q(x+y)+2xy}{2q+(x+y)}$ where $q = \frac{1+\sqrt{5}}{2}$, and we define $M_{q^*}(x,y)$ by $\frac{q^*(x+y)+2xy}{2q^*+(x+y)}$ where $q^* = \frac{1-\sqrt{5}}{2}$, which is called a *conjugate golden section mean.*

It was shown that: if $M(x,y) = \frac{a(x+y)+2bxy}{2a+b(x+y)}$ is a Fibonacci mean and if $M(x,y) = DM(x,y)$, then either $M(x,y) = M_q(x,y)$ or $M(x,y) = M_{q^*}(x,y)$.

2. Fibonacci frequency

Given a positive integer $n \geq 2$, let $\widehat{F}(n) = m$ provided $F_k \equiv F_{k+m} \pmod{n}$ for all positive integers k, where m is the smallest positive integer with this property and F_k is the k^{th} Fibonacci number relative to arbitrary inputs $F_1 = a, F_2 = b$, non-negative integers.

For example, for $n = 2$ we have with inputs $a = 1, b = 1$:

$$1,\ 1,\ 0,\ 1, 1, 0, \cdots$$

whence $\widehat{F}(2) \geq 3$. Also, $a, b, a+b, a, b, a+b, \cdots$ shows that $\widehat{F}(2) \leq 3$. Thus, combining these observations we establish that:

Proposition 2.1. $\widehat{F}(2) = 3$.

For $n = 3$, $a = 1, b = 1$ yields a lower bound computation is:

$$1, 1, 2, 0, 2, 2, 1, 0, 1, 1, 2, 0, 2, 2, 1, 0, 1, \cdots$$

and $\widehat{F}(3) \geq 8$. Also, $a, b, a+b, a+2b, 2a, 2b, 2a+2b, 2a+b, a, b, \cdots$ and $\widehat{F}(3) \leq 8$. Hence, it follows that:

Proposition 2.2. $\widehat{F}(3) = 8$.

When we consider the method of proof of the above two propositions, we note that the pattern $0, 1, 1, 2, \cdots$ corresponds to a pattern $\alpha a + \beta b, a, b, a+b, \cdots$ in the second sequence where $\alpha + \beta \equiv 0 \pmod{n}$, and that if $\cdots, s, \cdots$ is any term in the $a = 1, b = 1$ sequence, then $\cdots, s, \cdots$ corresponds to $\cdots, \lambda a + \mu b, \cdots$ where $\lambda + \mu \equiv s \pmod{n}$. Hence if $s = 1$, then we have $\lambda \equiv 1 \pmod{n}, \mu \equiv 0 \pmod{n}$ or $\lambda \equiv 0 \pmod{n}, \mu \equiv 1 \pmod{n}$, i.e., in the first case we find the input a, whereas in the second case we find the input b. Notice that $\alpha a + \beta b, a, b, a+b, \cdots$ with $\alpha + \beta \equiv 0 \pmod{n}$ means $(\alpha+1)a + \beta b \equiv b \pmod{n}$, $\beta \equiv -\alpha \pmod{n}$, $(\alpha+1)(a-b) \equiv 0$

(mod n), and $a - b$ arbitrary means $\alpha + 1 \equiv 0 \pmod{n}$, i.e., $\alpha \equiv n - 1 \pmod{n}$, $\beta \equiv 1 \pmod{n}$. Hence, the sequence looks like $(n-1)a + b, a, b, \cdots$. If we continue the construction by including one more term, then $\lambda a + \mu b, (n-1)a + b, a, b, \cdots$ yields $(\lambda + (n-1))a + (\mu + 1)b \equiv a \pmod{n}$ and $(\lambda + n - 2)a + (\mu + 1)b \equiv 0 \pmod{n}$. Hence $a = 0$ yields $\mu \equiv n - 1 \pmod{n}$ and $\lambda + n - 2 \equiv \lambda - 2 \equiv 0 \pmod{n}$, i.e., $\lambda \equiv 2 \pmod{n}$, i.e., the sequence is $\cdots, 2a + (n-1)b, (n-1)a + b, a, b, \cdots$. Letting $a = b = 1$, the corresponding pattern is $\cdots, 2 + (n-1) \equiv 1, (n-1) + 1 \equiv 0, 1, 1, \cdots$. Thus, if this occurs at the m^{th} location, then $\widehat{F}(n) \geq m$ from $a = 1, b = 1$ and $\widehat{F}(n) \leq m$ from a, b unspecified, whence $\widehat{F}(n) = m$. We thus obtain:

Theorem 2.3. *To determine $\widehat{F}(n)$ it suffices to take $a = 1, b = 1$, and construct the Fibonacci sequence modulo n until the pattern $\cdots, 1, 0$ is obtained. If the sequence has m terms, then $\widehat{F}(n) = m$.*

Suppose for example that we wish to determine $\widehat{F}(4)$. Using Theorem 2.3 we let $a = 1, b = 1$, whence the sequence is

$$1, 1, 2, 3, 1, 0,$$

and $\widehat{F}(4) = 6$. Note that $\widehat{F}(4)/\widehat{F}(2) = 6/3 = 2$.

As another example consider the computation of $\widehat{F}(9)$. Again, using Theorem 3.3 and $a = 1, b = 1$, we obtain the following sequence:

$$1, 1, 2, 3, 5, 8, 4, 3, 7, 1, 8, 0, 8, 8, 7, 6, 4, 1, 5, 6, 2, 8, 1, 0$$

and $\widehat{F}(9) = 24$. Note that $\widehat{F}(9)/\widehat{F}(3) = 24/8 = 3$.

In this case we also note that the first 0 shows up after 12 steps. Accordingly we take $\widehat{f}(9) = 12$ and $\widehat{F}(9) = \widehat{f}(9)\widehat{m}(9), \widehat{F}(9) = 2$. We consider $\widehat{f}(9)$ to be the *fundamental frequency* and $\widehat{m}(9)$ to be the *multiplicity*, with $\widehat{F}(9)$ the *Fibonacci frequency* of the integer 9 (≥ 2).

Fibonacci numbers have been studied in great detail over many years and the literature on the subject is quite substantial with entire books on the subject dedicated to their study and the study of these numbers also meriting chapters in books on number theory ([1, 2]). Recently, two of the authors of this paper were able to make a different but not entirely surprising connection between Fibonacci numbers and the Golden Section than the usual one ([3]).

If $a = b = 1$, then it is well-known that $F_m \mid F_n$ if and only if $m \mid n$ for example. Another known result is the following: For any prime p, there are infinitely many Fibonacci numbers which are divisible by p and these are equally spaced in the Fibonacci sequence. The case $\widehat{f}(3) = 4$ is an instance of this observation. Our point of view allows us to consider a more general situation and obtain some results on relationships connecting various values of $\widehat{F}(n)$ and to make some conjectures on these relationships which appear to be interesting.

3. Radical functions

In the number-theoretical setting, a function f on the natural numbers is multiplicative if $\gcd(a, b) = 1$ implies $f(ab) = f(a)f(b)$. Certainly any function for which $f(xy) = f(x)f(y)$ is multiplicative. The Euler-phi-function is multiplicative in the number-theoretical sense without being multiplicative in the strict sense.

Given a natural number $m = p_1^{r_1} \cdots p_n^{r_n}$, where the p_i's are distinct primes in the factorization of m, we let $rad(m) = p_1 \cdots p_n$, according to conventional ring-theoretical practice. Thus, for natural numbers m_1, m_2, we find that $rad(m_1 m_2) = lcm\{rad(m_1), rad(m_2)\}$. This function is an example of functions on the natural numbers satisfying the following "multiplicative" condition: a function f on the natural numbers is a *radical function* if $\gcd(a, b) = 1$ implies $f(ab) = lcm\{f(a), f(b)\}$.

When we check the table in the previous section, we observe that in the available examples it is true that $\gcd(a, b) = 1$ implies $\widehat{F}(ab) = lcm\{\widehat{F}(a), \widehat{F}(b)\}$. For example, $\widehat{F}(4) = 6, \widehat{F}(5) = 20, \widehat{F}(20) = 60, \gcd(4, 5) = 1$ and $\widehat{F}(20) = lcm\{6, 20\} \neq 120$, i.e., $\widehat{F}$ is not a multiplicative function in the number-theoretical sense. Thus, it is our goal in this section to prove that $\widehat{F}$ is a radical function in the sense described above.

Lemma 3.1. *If $d|n$, then $\widehat{F}(d) \leq \widehat{F}(n)$.*

Proof. Since $d|n$, there exists an integer q such that $n = dq$. If we let $\widehat{F}(n) = m$, then $F_k \equiv F_{k+m} \pmod{n}$ for any integer k, so that $F_{k+m} - F_k = nu = dqu$ for some $u \in Z$, i.e., $d|F_{k+m} - F_k$. This means that $\widehat{F}(d) \leq m = \widehat{F}(n)$. □

Lemma 3.2. *If $d|n$, then $\widehat{F}(d)|\widehat{F}(n)$.*

Proof. Using Division Algorithm, we have $\widehat{F}(n) = \widehat{F}(d)q + r$ for some $q, r \in Z$, where $0 \leq r < \widehat{F}(d)$. Let $\widehat{F}(n) := m$ and $\widehat{F}(d) := t$. Then $m = qt + r$ and hence $F_k \equiv F_{k+m} \pmod{n}$, so that $n|F_{k+m} - F_k$. Since $d|n$, we have $d|\, F_{k+m} - F_k$. We claim that $F_{k+r} \equiv F_{k+qt+r} \pmod{d}$. Since $\widehat{F}(d) := t$,

$$F_k \equiv F_{k+t} \pmod{d} \tag{1}$$

for any natural number $k \in N$. If we take $k := k + t$ in (1), then $F_{k+t} \equiv F_{(k+t)+t} \equiv F_{k+2t} \pmod{d}$. Similarly, we obtain

$$F_k \equiv F_{k+(q-1)t} \pmod{d} \tag{2}$$

for any natural number $k \in N$ and natural number $q > 1$. If we replace k by $k+r$ in (2), then $F_{k+r} \equiv F_{k+r+(q-1)t} \equiv F_{k+r+qt} \pmod{d}$. Hence $F_k \equiv F_{k+m} \equiv F_{k+qt+r} \equiv F_{k+r} \pmod{d}$. Since $\widehat{F}(d) = t$ is the smallest positive integer with this property and $0 \leq t < t$, we have $r = 0$, i.e., $m = qt$, proving the assertion. □

Theorem 3.3. *If $n = ab$, where a, b are natural numbers with $gcd(a, b) = 1$, then $\widehat{F}(n) = lcm\{\widehat{F}(a), \widehat{F}(b)\}$.*

Proof. Suppose that $n = ab$, where a, b are natural numbers with $gcd(a, b) = 1$. Then $\widehat{F}(a)|\widehat{F}(n)$, $\widehat{F}(b)|\widehat{F}(n)$ by Lemma 4. Hence $lcm\{\widehat{F}(a), \widehat{F}(b)\} \leq \widehat{F}(n)$. If we let $m := lcm\{\widehat{F}(a), \widehat{F}(b)\}$, then there exists natural numbers r, s such that $m = r\widehat{F}(a) = s\widehat{F}(b)$ where $gcd(r, s) = 1$. Let $\alpha := \frac{m}{r}, \beta := \frac{m}{s}$. Then $F_k \equiv F_{k+\alpha} \pmod{a}$, $F_k \equiv F_{k+\beta} \pmod{b}$ for any positive integer k. Hence $F_k \equiv F_{k+r\alpha} \equiv F_{k+r\frac{m}{r}} \equiv F_{k+m} \pmod{a}$. Similarly, $F_k \equiv F_{k+s\beta} \equiv F_{k+s\frac{m}{s}} \equiv F_{k+m} \pmod{b}$ for any positive integer k. This means that $a|F_k - F_{k+m}, b|F_k - F_{k+m}$. Since $gcd(a, b) = 1$, it follows that $F_k \equiv F_{k+m} \pmod{ab}$, i.e., $F_{k+m} \equiv F_k \pmod{n}$, so that $\widehat{F}(n) \leq m$ by the minimality property of $\widehat{F}$, proving the theorem. □

Corollary 3.4. *Let a, b, c are natural numbers which are relatively prime. Then $\widehat{F}(abc) = lcm\{\widehat{F}(a), \widehat{F}(b), \widehat{F}(c)\}$.*

Corollary 3.5. *Let a, b are natural numbers which are relatively prime. Then*

$$gcd\{\widehat{F}(a), \widehat{F}(b)\} = \frac{\widehat{F}(a)\widehat{F}(b)}{\widehat{F}(ab)}$$

Example 3.6. $\widehat{F}(1147) = \widehat{F}(1517) = 760$ using the table above along with Theorem 3.3. Indeed, $1147 = 31 \cdot 37$ and $1517 = 41 \cdot 37$, $\widehat{F}(31) = \widehat{F}(41) = 40$, $\widehat{F}(37) = 76$ and $lcm\{40, 76\} = 760$. It is of course true that F_{760} is not a small integer.

4. Powers of primes

From the table given above, it is not immediately clear that there is any pattern to the values of $\widehat{F}(p)$, where p is a prime. However, in all cases we have seen, the following properties holds:

Conjecture 4.1. For any prime p,

$$\frac{\widehat{F}(p^{s+1})}{\widehat{F}(p^s)} = p$$

Thus, for example $\widehat{F}(27)/\widehat{F}(9) = 72/24 = 3$ and $\widehat{F}(25)/\widehat{F}(5) = 5$. Accepting Conjecture 4.1 as true, we note that if $\eta(n) = \widehat{F}(n)/n$, then $\eta(p^{s+1}) = \widehat{F}(p^{s+1})/p^{s+1} = p\widehat{F}(p^s)/p^{s+1} = \widehat{F}(p^s)/p^s = \eta(p^s) = \cdots = \eta(p)$. For example, $\eta(27) = \eta(3) = \widehat{F}(3)/3 = 8/3 = 72/27$. If $n = p^{r+1}q^{s+1}$, then

$$\begin{aligned}
\widehat{F}(n) &= lcm\{\widehat{F}(p^{r+1}, \widehat{F}(q^{s+1}\} \\
&= lcm\{p^r\widehat{F}(p), q^s\widehat{F}(q)\} \\
&= \frac{p^r q^s \widehat{F}(p)\widehat{F}(q)}{\gcd\{p^r\widehat{F}(p), q^s\widehat{F}(q)\}} \\
&= \frac{n\widehat{F}(p)\widehat{F}(q)}{\gcd\{\widehat{F}(p^{r+1}), \widehat{F}(q^{s+1})\}pq}
\end{aligned}$$

and thus

$$\eta(n) = \frac{\widehat{F}(p)\widehat{F}(q)}{pq\gcd\{\widehat{F}(p^{r+1}), \widehat{F}(q^{s+1})\}}$$

Now, $pq = rad(n)$. Continuing in the same fashion, if $m = p_1^{r_1} \cdots p_n^{r_n}$, then we find that

$$\eta(m) = \frac{\widehat{F}(p_1) \cdots \widehat{F}(p_n)}{rad(m)\gcd\{\widehat{F}(p_1^{r_1}), \cdots, \widehat{F}(p_n^{r_n})\}}$$

Global properties of the function $\widehat{F}(n)$ may then be gathered in the function:

$$Z_\eta(s) = \sum_{n=2}^{\infty} \frac{\widehat{F}(n)}{n^s},$$

so that $Z_\eta(1) = \sum_{n=2}^{\infty} \frac{\widehat{F}(n)}{n} = \sum_{n=2}^{\infty} \eta(n)$, which does not appear to converge for $s = 1$, but may well converge for complex variables with $Re(s)$ sufficiently large. Other generating functions may also be constructed such as $\sum_{n=2}^{\infty} \widehat{F}(n)z^n$, $\sum_{n=2}^{\infty} \eta(n)z^n$, $\sum_{n=2}^{\infty} \widehat{F}(n)\frac{z^n}{n!}$, etc..

Given $\widehat{F}(n)$ for $n \geq 2$, define $\widehat{F}(1) = 1$ and for positive integers a, b with $\gcd(a, b) = 1$, let $\widehat{F}(\frac{a}{b})$ satisfying the following equation:

$$\widehat{F}(\frac{a}{b}) = \frac{lcm\{\widehat{F}(a), \widehat{F}(b)\}}{lcm\{\widehat{F}(b), \widehat{F}(b^2)\}}$$

Thus, if $b = 1$, then $\widehat{F}(\frac{a}{1}) = \widehat{F}(a)/1 = \widehat{F}(a)$. Also, $\widehat{F}(\frac{1}{b}) = \widehat{F}(b)/lcm\{\widehat{F}(b), \widehat{F}(b^2)\}$. Thus, if $b = p$ is a prime and if Conjecture 4.1 is accepted, then $\widehat{F}(\frac{1}{p}) = \widehat{F}(p)/p\widehat{F}(p) = 1/p$. The meaning or interpretation of the values of $\widehat{F}$ on fractions is not quite clear. It does however demonstrate that the function $\widehat{F}$ defined on integers $n \geq 2$ has extensions to the positive rationals, the one described here being one of them. Since $mn = (-m)(-n)$, it makes sense to define $\widehat{F}(q) = \widehat{F}(-q)$ for rationals $q > 0$. Also, since we expect $\widehat{F}(\frac{a}{b})$ to be "near zero" if $\frac{a}{b}$ is "near zero", $\widehat{F}(0) = 0$ appears to be a sensible decision also.

For irrational values α, the definition of $\widehat{F}(\alpha)$ could be as follows: if we define $S(n, \alpha) := \sup\{F(q) \,|\, q \in Q \cap [\alpha - \frac{1}{n}, \alpha + \frac{1}{n}]\}$, then $0 \leq S(n+1, \alpha) \leq S(n, \alpha)$ and hence $\lim_{n\to\infty} S(n, \alpha) = \inf_{n\in\omega} S(n, \alpha)$. Since $\cap_{n\in\omega} S(n, \alpha) = \{\alpha\}$, it follows that this permits us to define $\widehat{F}(\alpha)$ for α an irrational number. If $\widehat{F}(\alpha) = \infty$, then $S(n, \alpha) = \infty$ for all integers n. Thus, if this is the case, there is a sequence of rational numbers $\{q_i\}_{i=1}^{\infty}$ such that $\lim_{i\to\infty} q_i = \alpha$ and at the same time $\lim_{i\to\infty} \widehat{F}(q_i) = \alpha$ and at the same time $\lim_{i\to\infty} \widehat{F}(q_i) = \infty$. We conjecture the following:

Conjecture 4.2. *There is no sequence* $\{q_i\}_{i=1}^{\infty}$ *of rational numbers such that* $\lim_{i\to\infty} q_i = \alpha$ *and* $\lim_{i\to\infty} \widehat{F}(q_i) = \infty$.

Given that the conjecture holds, $\widehat{F}(\alpha)$ is defined for irrational values of α as well, i.e., the domain of $\widehat{F}$ is the real numbers.

5. Comments

In this paper we have considered several aspects of the sequence of Fibonacci numbers with inputs a, b arbitrary related to the periodicity of such a sequence modulo n. Because of the plenitude of relations known to exist among various Fibonacci numbers it was not surprising that patterns would be observed. We were pleased to discover that there were numerous relationships to be found, even if not all of them are explainable. The most mysterious values are those for $\widehat{F}(p)$ where p is an arbitrary prime. Thus, $\widehat{F}(29) = 14, \widehat{F}(31) = 40$, which insists on announcing that from the "Fibonacci point of view" there is a "big difference' between these two primes in the twin-prime couple. Also, given an integer n, then fact that $\widehat{F}(n^2)/\widehat{F}(n) \neq n$, suffices to identify it as a composite number without knowing anything about any factorization of n. Thus, e.g., $\widehat{F}(36)/\widehat{F}(6) = 24/24 = 1$. Since Fibonacci numbers grow rather quickly, this observation may prove useful in the exercise of primality testing. Also, if $\widehat{F}(n^2)/\widehat{F}(n) = n$, then, although this does not guarantee (maybe) that n is a prime, it seems that it ought to greatly improve the probability that it is.

6. Appendix

n	$\widehat{F}(n)$	$\widehat{f}(n)$	n	$\widehat{F}(n)$	$\widehat{f}(n)$	n	$\widehat{F}(n)$	$\widehat{f}(n)$	n	$\widehat{F}(n)$	$\widehat{f}(n)$
2	3	3	3	8	4	4	6	6	5	20	5
6	24	12	7	16	8	8	12	6	9	24	12
10	60	15	11	10	10	12	24	12	13	28	7
14	48	24	15	40	20	16	24	12	17	36	9
18	24	12	19	18	18	20	60	30	21	16	8
22	30	30	23	48	24	24	24	12	25	100	25
26	84	21	27	72	36	28	48	24	29	14	14
30	120	60	31	30	30	32	48	24	33	40	20
34	36	9	35	80	40	36	24	12	37	76	19
38	18	18	39	56	28	40	60	30	41	40	20
42	48	24	43	88	44	44	30	30	45	120	60
46	48	24	47	32	16	48	24	12	49	112	56
50	300	75	51	72	36	52	84	42	53	108	27
54	72	36	55	20	10	56	48	24	57	72	36
58	42	42	59	58	58	60	120	60	61	60	15
62	30	30	63	48	24	64	96	48	65	140	35
66	120	60	67	136	68	68	36	18	69	48	24
70	240	120	71	70	70	72	24	12	73	148	37
74	228	57	75	200	100	76	18	18	77	80	40
78	168	84	79	78	78	80	120	60	81	216	108
82	120	60	83	168	84	84	48	24	85	180	45
86	264	132	87	56	28	88	60	30	89	44	11
90	120	60	91	112	56	92	48	24	93	120	60
94	96	48	95	180	90	96	48	24	97	196	49
98	336	168	99	120	60	100	300	150			

References

[1] K. Atanassove et. al, New Visual Perspectives on Fibonacci numbers, World Sci. Pub. Co., New Jersey, 2002.

[2] R. A. Dunlap, The Golden Ratio and Fibonacci Numbers, World Scientific, New Jersey, 1997.

[3] J. S. Han, H. S. Kim, J. Neggers, *The Fibonacci norm of a positive integer n- observations and conjectures* -, Int. J. Number Th. **6** (2010), 371-385.

[4] J. S. Han, H. S. Kim and J. Neggers, *Fibonacci sequences in groupoids*, Advances in Difference Equations 2012 **2012**:19 (doi:10.1186/1687-1847-2012-19).

[5] J. S. Han, H. S. Kim and J. Neggers, *On Fibonacci functions with Fibonacci numbers*, Advances in Difference Equations 2012 **2012**:126 (doi:10.1186/1687-1847-2012-126).

[6] H. S. Kim and J. Neggers, On Fibonacci Means and Golden Section Mean, Computers and Mathematics with Applications **56** (2008), 228-232.

[7] H. S. Kim, J. Neggers and K. S. So, *Generalized Fibonacci sequences in groupoids*, Advances in Difference Equations 2013 **2013**:26 (doi:10.1186/1687-1847-2013-26).

[8] H. S. Kim, J. Neggers and K. S. So, *On Fibonacci functions with periodicity*, Advances in Difference Equations 2014 **2014**:293. (doi:10.1186/1687-1847-2014-293).

[9] H. S. Kim, J. Neggers and K. S. So, *On continuous Fibonacci functions*, J. Comput. & Appl. Math. **24** (2018), 1482-1490.

[10] H. S. Kim, J. Neggers and K. S. So, *On Fibonacci derivative equations*, J. Comput. & Appl. Math. **24** (2018), 628-635.

The weighted moving averages for a series of fuzzy numbers based on non-additive measures with $\sigma - \lambda$ rules†

Zeng-Tai Gong[a,*], Wen-Jing Lei[a,b]
[a] College of Mathematics and Statistics, Northwest Normal University, Lanzhou, 730070, China
[b] School of Economics and Management, Tongji University, Shanghai, 200092, China

Abstract Non-additive measure theory is an important mathematical tool to deal with inter-dependent or interactive information. The concept of fuzzy number provides an effective means of describing vague and uncertain system. The aim of this study is to integrate moving average with non-additive measures with $\sigma - \lambda$ rules under fuzzy environment. That is, the moving average for a series of fuzzy numbers based on non-additive measures with $\sigma - \lambda$ rules is proposed. Further, its specific calculation is invested and some properties are discussed. In particular, triangular fuzzy numbers about this method are also discussed. Finally, an example is given to illustrate our results.
Keywords: Fuzzy number; Fuzzy measure; Moving average.

1. Introduction

Non-additive measure theory, as an extension of classical measure theory for the study of inter-dependent or interactive information, was proposed by Sugeon [18] by replacing additivity with monotonicity. Many studies have focused on theoretical aspects and applications of non-additive measures. Asahina [1] studied implication relationship among six continuity conditions and two null-additivity conditions with respect to non-additive measures. Li [8] discussed four versions of Egoroff's theorem in non-additive measure theory by using special condition. In particular, the Choquet integral with respect to non-additive measures has lbeen successfully applied in decision-making [23, 19], information fusion [6], economic theory [17] and so on.

Considering the inherent uncertain and imprecise of information in practical life, another key mathematical structure is introduced to model uncertain and incomplete systems, which is called fuzzy number, proposed by Zadeh [25], on the basis of fuzzy sets [24]. Fuzzy number has been investigated intensively by researches from various aspects. Gong [5] generalized convexity from vector-valued maps to n-dimensional fuzzy number-valued functions. Saeidifar [16] introduced the concepts of nearest weighted interval and point approximations of a fuzzy number. And Wang [22] applied triangular fuzzy number to study management knowledge performance evaluation.

Moving average is that, given a series of numbers and fixed subset size, the first element of the moving average is obtained by taking the average of the initial fixed subset of the number series [2]. The moving average has been widely applied in time series analysis [20], cloud computing [14] signal processing and financial mathematics, etc. However, when we use moving average to make forecasting, it is not reasonable to assume that all data are real data before we elicit them from practical data, fuzzy data may exit, such as in financial and sociological application. So we need to take the vagueness of the universe of importance. Furthermore, there is interaction among data in real application. The aim of this paper is to propose the moving average for a series of fuzzy numbers based on non-additive measures with $\sigma - \lambda$ rules. In particular, triangular fuzzy numbers about this method are also discussed.

The structure of this paper is as follows. In Section 2, we review some basic cponcepts and properties about non-additive measure with $\sigma - \lambda$ rules and fuzzy numbers. And the definition of conduct between a non-negative matrix and fuzzy number vector is given to make our analysis possible. In Section 3, we propose the moving average for a series of fuzzy numbers based on non-additive measures with $\sigma - \lambda$ rules.

†Supported by National Natural Science Fund of China (61763044, 11461062).
*Corresponding author. Tel.: +8613993196400.
Email addresses: zt-gong@163.com(Zeng-Tai Gong), leiwenjingbz@163.com(Wen-Jing Lei).

In Section 4, the calculation of the weighted moving averages for fuzzy-number based on a non-additive measure with $\sigma-\lambda$ rules is invested and some properties are discussed. The paper ends with conclusion in In Section 5.

2. Preliminaries

Throughout this study, R^m denotes the m-dimension real Euclidean space and $R^+=(0,\infty)$.
Definition 2.1 [18, 10, 3]. Let X denote a nonempty set and $\mathscr{A}$ a $\sigma-$ algebra on the X. A set function μ is referred to as a regular fuzzy measure if

(1) $\mu(\emptyset)=0$;

(2) $\mu(X)=1$;

(3) for every A and $B\in\mathscr{A}$ such that $A\subseteq B$, $\mu(A)\leq\mu(B)$.

Definition 2.2 [18, 10, 3]. g_λ is called a fuzzy measure based on $\sigma-\lambda$ rules if it satisfies

$$g_\lambda\left(\bigcup_{i=1}^{\infty}A_i\right)=\begin{cases}\dfrac{1}{\lambda}\left\{\prod\limits_{i=1}^{\infty}[1+\lambda g_\lambda(A_i)]-1\right\}, & \lambda\neq 0,\\ \sum\limits_{i=1}^{\infty}g_\lambda(A_i), & \lambda=0,\end{cases}$$

where $\lambda\in(-\frac{1}{\sup\mu},\infty)\bigcup\{0\}$, $\{A_i\}\subset\mathscr{A}$, $A_i\cap A_j=\emptyset$ for all $i,j=1,2,\cdots$ and $i\neq j$.

Particularly, if $\lambda=0$, then g_λ is a classic probability measure.

A regular fuzzy measure μ is called Sugeno measure based on $\sigma-\lambda$ rules if μ satisfies $\sigma-\lambda$ rules, briefly denoted as g_λ. The fuzzy measure denoted in this paper is Sugeno measure.
Remark 2.1. In the Definition, if $n=2$, then

$$\mu\left(A\cup B\right)=\begin{cases}\mu(A)+\mu(B)+\lambda\mu(A)\mu(B), & \lambda\neq 0,\\ \mu(A)+\mu(B), & \lambda=0.\end{cases}$$

Remark 2.2. If X be a finite set, for any subset A of X, then

$$g_\lambda\left(A\right)=\begin{cases}\dfrac{1}{\lambda}\left\{\prod\limits_{x\in A}[1+\lambda g_\lambda(\{x\})]-1\right\}, & \lambda\neq 0,\\ \sum\limits_{i=1}^{\infty}g_\lambda(\{x\}), & \lambda=0.\end{cases}$$

Remark 2.3 [3]. If X be a finite set, then the parameter λ of a regular Sugeno measure based on $\sigma-\lambda$ rules is determined by the equation

$$\prod_{i=1}^{n}(1+\lambda g_{\lambda i})=1+\lambda.$$

Let $g_\lambda(\{x_i\})=g_i, i=1,2,...,m$, then $g_\lambda(A_i)$ is obtained from the following recurrence relation

$$g_\lambda(A_m)=g_\lambda(\{x_m\})=g_m,\quad g_\lambda(A_i)=g_\lambda(A_{i+1})+\lambda g_i g_\lambda(A_{i+1}), 1\leq i<m.$$

Let $\tilde{A}(x)\in\tilde{E}, r\in(0,1]$ and $[\tilde{A}]^r=\{x\in R: u_{\tilde{A}}(X)\geq r\}$. If $\tilde{A}$ satisfies

(1) $\tilde{A}$ is a normal fuzzy set, i.e., an $x_0\in R$ exists such that $u_{\tilde{A}}(x_0)=1$;

(2) $\tilde{A}$ is a convex fuzzy set, i.e., $u_{\tilde{A}}(\lambda x+(1-\lambda)y)\geq\min\{u_{\tilde{A}}(x),u_{\tilde{A}}(y)\}$ for any $x,y\in R$, and $\lambda\in(0,1]$;

(3) $\tilde{A}$ is a upper semicontinuous fuzzy set;

(4) $[A]^0=\overline{X\in R: u_{\tilde{A}}(x)>0}=\overline{\bigcup_{r\in(0,1]}[A]^r}$ is compact, where $\bar{A}$ denotes the closure of A.

Then, $\tilde{A}$ is called a fuzzy number. We use $\tilde{E}$ to denote the fuzzy number space [9].

It is clear that each $x \in R$ can be consider as a fuzzy number $\tilde{A}$ defined by

$$u_{\tilde{A}}(x) = \begin{cases} 1, & x = A, \\ 0, & otherwise. \end{cases}$$

Given any two fuzzy numbers $\tilde{A}_1, \tilde{A}_2, k, k_1 k_2 \geq 0$, the operational rules are as follows:
(1) $k(\tilde{A}_1 + \tilde{A}_2) = k\tilde{A}_1 + k\tilde{A}_2$,
(2) $k_1(k_2\tilde{A}_1) = (k_1k_2)\tilde{A}_1$,
(3) $(k_1 + k_2)\tilde{A}_1) = k_1\tilde{A}_1 + k_2\tilde{A}_1$.

Lemma 2.1 [11, 12, 9]. For a fuzzy set $\tilde{A}$, it satisfy the following equation

$$\tilde{A} = \bigcup_{r \in [0,1]} (r^* \bigcap [\tilde{A}]^r),$$

where r^* denotes the fuzzy set whose membership function is constant function r.

Let $\tilde{A}, \tilde{B} \in \tilde{E}, k \in \mathbb{R}$, the addition and scalar conduct are defined by

$$[\tilde{A} + \tilde{B}]^r = [\tilde{A}]^r + [\tilde{B}]^r, \qquad [k\tilde{A}]^r = k[\tilde{A}]^r,$$

respectively, where $[\tilde{A}]^r = \{x : u_{\tilde{A}}(x) \geqslant r\} = [A^-(r), A^+(r)]$, for any $r \in (0, 1]$.

Lemma 2.2 [11, 12, 9]. If $\tilde{A} \in \tilde{E}$, then
(1) $[\tilde{A}]^r$ is a nonempty bounded closed interval for any $r \in (0, 1]$;
(2) $[\tilde{A}]^{r_1} \supset [\tilde{A}]^{r_2}$ where $0 \leqslant r_1 \leqslant r_2 \leqslant 1$;
(3) if $r_n > 0$ and $\{r_n\}$ converging increasingly to $r \in (0, 1]$, then

$$\bigcap_{n=1}^{\infty} [\tilde{A}]^{r_n} = [\tilde{A}]^r.$$

Conversely, if for any $r \in [0, 1]$, there exists $B_r \subset \mathbb{R}$ satisfying $(1) - (3)$, then there exists a unique $\tilde{A} \in \tilde{E}$ such that $[\tilde{A}]^r = A^r, r \in (0, 1]$, and $[\tilde{A}]^0 = \overline{\bigcup_{r \in (0,1]} [\tilde{A}]^r} \subset B_0$.

Definition 2.3 [21]. A triangle fuzzy number $\tilde{A}$ is a fuzzy number with piecewise linear membership function $\tilde{A}$ defined by

$$u_{\tilde{A}}(x) = \begin{cases} \dfrac{x - a_l}{a_m - a_l}, & a_l \leq x \leq a_m, \\ \dfrac{a_n - x}{a_n - a_m}, & a_m < x \leq a_n, \\ 0, & otherwise, \end{cases}$$

which can be indicated as a triplet (a_l, a_m, a_n).

Given any two triangle fuzzy numbers $\tilde{x}_i = (x_i - \delta_{i,1}, x_i, x_i + \delta_{i,1}), \tilde{x}_j = (x_j - \delta_{j,1}, x_j, x_j + \delta_{j,1}))$, and $k \geq 0$, the operational rules are as follows:
(1) $\tilde{x}_i + \tilde{x}_j = (x_i - \delta_{i,1} + x_j - \delta_{j,1}, x_i + x_j, x_i + \delta_{i,1} + x_j + \delta_{j,1})$,
(2) $k \cdot \tilde{x}_i = (kx_i - k\delta_{i,1}, kx_i, kx_i + k\delta_{i,2})$.

Definition 2.4. Given a nonnegative matrix $P = [p_{ij}]$ and a fuzzy-number vector $\tilde{X}$, if $P \in R_+^{m \times m}$ and $\tilde{X} = [\tilde{x}_1, \tilde{x}_2, \cdots, \tilde{x}_m]^\intercal \in \tilde{E}^m$(The $^\intercal$ denotes the conjugate transpose of a vector or a matrix.), then the product of P and X is defined as follows:

$$P\tilde{X}_{n-1} = \begin{bmatrix} \sum\limits_{j=1}^{m} p_{ij}\tilde{x}_j \\ \vdots \\ \sum\limits_{j=1}^{m} p_{mj}\tilde{x}_j \end{bmatrix}.$$

3. The weighted moving averages for fuzzy-number based on a non-additive measure with $\sigma-\lambda$ rules

Definition 3.1. Let $(\tilde{x}_1,\tilde{x}_2,\cdots,\tilde{x}_m)\in\tilde{E}^m$, $(t_1,t_2,\cdots,t_m)\in R^m$, and g_λ be fuzzy measure satisfying $\delta-\lambda$ rules. Denote $A_i=\{t_i,t_{i+1},\cdots,t_m\}$, $i=1,2,...,m$, $A_{m+1}=\emptyset$. Then the weighted moving averages for fuzzy-number based on a non-additive measure with $\sigma-\lambda$ rules is defined as follows:

$$\tilde{x}_n=(g_\lambda(A_1)-g_\lambda(A_2))\tilde{x}_{n-m}+(g_\lambda(A_2)-g_\lambda(A_3))\tilde{x}_{n-m+1}+\cdots+(g_\lambda(A_m)-g_\lambda(A_{m+1}))\tilde{x}_{n-1},$$

where $n>m$.

When we use moving average to make forecasting, it is not reasonable to assume that all data are real data before we elicit them from practical data, fuzzy data may exit, such as in financial and sociological application. So we need to take the vagueness of the universe of importance. Furthermore, there is interaction among data in real application.

Remark 3.1. If $\lambda=0$, and $\tilde{x_i}$ is a special fuzzy number, namely, real number, $i=1,2,\cdots$, the weighted moving averages for a series of fuzzy numbers based on non-additive measures with $\sigma-\lambda$ rules degenerates to the classic weighted moving average in Ref. [2].

Theorem 3.1. Let $(\tilde{x}_1,\tilde{x}_2,\cdots,\tilde{x}_m)\in\tilde{E}^m$, $(t_1,t_2,\cdots,t_m)\in R^m$, and g_λ be fuzzy measure satisfying $\delta-\lambda$ rules. Let $A_i=\{t_i,t_{i+1},\cdots,t_m\}$, $i=1,2,...,m$, $A_{m+1}=\emptyset$, $\tilde{X}_n=[\tilde{x}_n,\tilde{x}_{n+1},\cdots,\tilde{x}_{n+m-1}]^\mathsf{T}$, then

$$\tilde{X}_n=\boldsymbol{P}\tilde{X}_{n-1}=\boldsymbol{P}^2\tilde{X}_{n-2}=\cdots=\boldsymbol{P}^{n-1}\tilde{X}_1,$$

$n=1,2,3,...$, where

$$\boldsymbol{P}=\begin{bmatrix} 0 & 1 & 0 & \cdots & 0\\ 0 & 0 & 1 & \cdots & 0\\ \vdots & \cdots & \cdots & \cdots & \vdots\\ 0 & 0 & 0 & \cdots & 1\\ g_\lambda(A_1)-g_\lambda(A_2) & g_\lambda(A_2)-g_\lambda(A_3) & g_\lambda(A_3)-g_\lambda(A_4) & \cdots & g_\lambda(A_m)-g_\lambda(A_{m+1}) \end{bmatrix}.$$

Proof. Based on Definition 3.1 and the operational rules of fuzzy numbers, we have

$$\boldsymbol{P}\tilde{X}_{n-1}=\begin{bmatrix} 0 & 1 & \cdots & 0\\ 0 & 0 & \cdots & 0\\ \vdots & \cdots & \cdots & \vdots\\ 0 & 0 & \cdots & 1\\ g_\lambda(A_1)-g_\lambda(A_2) & g_\lambda(A_2)-g_\lambda(A_3) & \cdots & g_\lambda(A_m)-g_\lambda(A_{m+1}) \end{bmatrix}\begin{bmatrix}\tilde{x}_{n-1}\\ \tilde{x}_n\\ \vdots\\ \tilde{x}_{n+m-3}\\ \tilde{x}_{n+m-2}\end{bmatrix}=$$

$[\tilde{x}_n,\tilde{x}_{n+1},\cdots,\tilde{x}_{n+m-3},(g_\lambda(A_1)-g_\lambda(A_2))\tilde{x}_{n-1}+(g_\lambda(A_2)-g_\lambda(A_3))\tilde{x}_n+\cdots+(g_\lambda(A_m)-g_\lambda(A_{m+1}))\tilde{x}_{n+m-2}]^\mathsf{T}$.

And we know that

$$(g_\lambda(A_1)-g_\lambda(A_2))\tilde{x}_{n-1}+(g_\lambda(A_2)-g_\lambda(A_3))\tilde{x}_n+\cdots+(g_\lambda(A_m)-g_\lambda(A_{m+1}))\tilde{x}_{n+m-2}=\tilde{x}_{n+m-1},$$

This follows that

$$\boldsymbol{P}\tilde{X}_{n-1}=\tilde{X}_n.$$

The proof is complete. □

Theorem 3.1. Let $(\tilde{x}_1,\tilde{x}_2,\cdots,\tilde{x}_m)\in\tilde{E}^m$, $(t_1,t_2,\cdots,t_m)\in R^m$, and g_λ be fuzzy measure satisfying $\delta-\lambda$ rules. Let $A_i=\{t_i,t_{i+1},\cdots,t_m\}$, $i=1,2,...,m$, $A_{m+1}=\emptyset$, $\tilde{X}_n=[\tilde{x}_n,\tilde{x}_{n+1},\cdots,\tilde{x}_{n+m-1}]^\mathsf{T}$, and $\tilde{X}_n^-(r)$ and $\tilde{X}_n^+(r)$ as follows

$$\tilde{X}_n^-(r)=[\tilde{x}_n^-(r),\tilde{x}_{n+1}^-(r),\cdots,\tilde{x}_{n+m-1}^-(r)]^\mathsf{T},$$

$$\tilde{X}_n^+(r)=[\tilde{x}_n^+(r),\tilde{x}_{n+1}^+(r),\cdots,\tilde{x}_{n+m-1}^+(r)]^\mathsf{T},$$

where $[\tilde{x_i}]_r=[x_i^-(r),x_i^+(r)]$. Then

$$\tilde{X}_n^-(r)=\boldsymbol{P}\tilde{X}_{n-1}^-(r)=\boldsymbol{P}^2\tilde{X}_{n-2}^-(r)=\cdots=\boldsymbol{P}^{n-1}\tilde{X}_1^-(r),$$

$$\tilde{X}_n^+(r) = \boldsymbol{P}\tilde{X}_{n-1}^+(r) = \boldsymbol{P}^2\tilde{X}_{n-2}^+(r) = \cdots = \boldsymbol{P}^{n-1}\tilde{X}_1^+(r),$$

where $n = 1, 2, 3, ...$, and $\boldsymbol{P}$ is the same matrix in Theorem 3.1.

Proof. Based on Theorem 3.1, we have

$$\boldsymbol{P}\tilde{X}_{n-1}^-(r) = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \cdots & \cdots & \vdots \\ 0 & 0 & \cdots & 1 \\ g_\lambda(A_1) - g_\lambda(A_2) & g_\lambda(A_2) - g_\lambda(A_3) & \cdots & g_\lambda(A_m) - g_\lambda(A_{m+1}) \end{bmatrix} \begin{bmatrix} \tilde{x}_{n-1}^-(r) \\ \tilde{x}_n^-(r) \\ \vdots \\ \tilde{x}_{n+m-3}^-(r) \\ \tilde{x}_{n+m-2}^-(r) \end{bmatrix} =$$

$$[\tilde{x}_n^-(r), \cdots, \tilde{x}_{n+m-3}^-(r), (g_\lambda(A_1) - g_\lambda(A_2))\tilde{x}_{n-1}^-(r) + \cdots + (g_\lambda(A_m) - g_\lambda(A_{m+1}))\tilde{x}_{n+m-2}^-(r)]^\intercal.$$

By Definition 3.1, we get

$$(g_\lambda(A_1) - g_\lambda(A_2))\tilde{x}_{n-1} + (g_\lambda(A_2) - g_\lambda(A_3))\tilde{x}_n^-(r) + \cdots + (g_\lambda(A_m) - g_\lambda(A_{m+1}))\tilde{x}_{n+m-2}^-(r) = \tilde{x}_{n+m-1}^-(r).$$

This follows that

$$\boldsymbol{P}\tilde{X}_{n-1}^-(r) = \tilde{X}_n^-(r).$$

Similarly, we can prove that

$$\boldsymbol{P}\tilde{X}_{n-1}^+(r) = \tilde{X}_n^+(r).$$

The proof is complete. □

Theorem 3.2. Let $(\tilde{x}_1, \tilde{x}_2, \cdots, \tilde{x}_m) \in \tilde{E}^m$, $(t_1, t_2, \cdots, t_m) \in R^m$, and g_λ be fuzzy measure satisfying $\delta - \lambda$ rules. Let $A_i = \{t_i, t_{i+1}, \cdots, t_m\}$, $i = 1, 2, ..., m$, $A_{m+1} = \emptyset$, $\tilde{X}_n = [\tilde{x}_n, \tilde{x}_{n+1}, \cdots, \tilde{x}_{n+m-1}]^\intercal$,

If $\tilde{x}_i$ is a triangle fuzzy number, and $\tilde{x}_i = (x_i - \delta_{i,1}, x_i, x_i + \delta_{i,2})$, $i = 1, 2, \cdots$, then

$$\begin{aligned}\tilde{X}_n^-(r) &= [\tilde{x}_n^-(r), \tilde{x}_{n+1}^-(r), \cdots, \tilde{x}_{n+m-1}^-(r)]^\intercal \\ &= [\delta_{n,1}r + x_n - \delta_{n,1}, \delta_{n+1,1}r + x_n - \delta_{n+1,1}, \cdots, \delta_{n+m-1,,1}r + x_{n+m-1} - \delta_{n+m-1,1}]^\intercal,\end{aligned}$$

$$\begin{aligned}\tilde{X}_n^+(r) &= [\tilde{x}_n^+(r), \tilde{x}_{n+1}^+(r), \cdots, \tilde{x}_{n+m-1}^+(r)]^\intercal \\ &= [-\delta_{n,2}r + x_n + \delta_{n,2}, -\delta_{n+1,2}r + x_{n+1} + \delta_{n+1,2}, \cdots, -\delta_{n+m-1,2}r + x_{n+m-1} + \delta_{n+m-1,2}]^\intercal.\end{aligned}$$

Proof. Based on the operational rules we have

$$\begin{aligned}\tilde{X}_n^-(r) &= [\tilde{x}_n^-(r), \tilde{x}_{n+1}^-(r), \cdots, \tilde{x}_{n+m-1}^-(r)]^\intercal \\ &= [\delta_{n,1}r + x_n - \delta_{n,1}, \delta_{n,1}r + x_n - \delta_{n,1}, \cdots, \delta_{n+m-1,1}r + x_{n+m-1} - \delta_{n+m-1,1}]^\intercal, \\ \tilde{X}_n^+(r) &= [\tilde{x}_n^+(r), \tilde{x}_{n+1}^+(r), \cdots, \tilde{x}_{n+m-1}^+(r)]^\intercal \\ &= [-\delta_{n,2}r + x_n + \delta_{n,2}, -\delta_{n,2}r + x_n + \delta_{n,2}, \cdots, -\delta_{n+m-1,2}r + x_{n+m-1} + \delta_{n+m-1,2}]^\intercal.\end{aligned}$$

The proof is complete. □

4. The calculation of the weighted moving averages for fuzzy-number based on a non-additive measure with $\sigma - \lambda$ rules

Lemma 4.1 [15]. Let$(d_1, d_2, \cdots, d_m) \in R_+^m$ and set

$$q(x) = x^m - d_1 x^{m-1} - \cdots - d_m.$$

Suppose that $\gcd\{k \in \{1, 2, \cdots, m\} : d_k > 0\} = 1$, where the greatest common division of a set S is denoted by $\gcd(S)$. Then q has a unique positive root$r$. Moreover, the algebraic multiplicity of r is 1, coinciding with the geometric multiplicity of r, and the modulus of every other root of q is strictly less than r.

Lemma 4.2 [13]. Let$B \in C^{m\times m}$, where C denotes plural numbers. Then the following holds

(1) $\{B^n\}$ converges to nonzero matrix if and only if 1 is a eigenvalue of B, whose algebraic multiplicities and geometric multiplicities coincide, and every other eigenvalues of B has modulus strictly less than 1;

(2) If $\rho(B) = \max\limits_{\lambda\in\sigma(B)} |\lambda| = 1$ is a eigenvalue of B whose algebraic multiplicity and geometric multiplicity of 1 coincide, equal to 1, with right-hand and left-hand eigenvalue x and $y^{\intercal}$ respectively, then

$$\lim_{n\to\infty} B^n = \frac{xy^{\intercal}}{y^{\intercal}x},$$

where $\sigma(B)$ is the set of eigenvalues of B.

Theorem 4.1. Let $(\tilde{x}_1, \tilde{x}_2, \cdots, \tilde{x}_m) \in \tilde{E}^m$, $(t_1, t_2, \cdots, t_m) \in R^m$, and g_λ be fuzzy measure satisfying $\delta - \lambda$ rules. Let $A_i = \{t_i, t_{i+1}, \cdots, t_m\}$, $i = 1, 2, ..., m$, $A_{m+1} = \emptyset$, $\tilde{X}_n = [\tilde{x}_n, \tilde{x}_{n+1}, \cdots, \tilde{x}_{n+m-1}]^{\intercal}$, For the matrix **P** satisfying the following recurrence relation in Theorem 3.1

$$\tilde{X}_n = \boldsymbol{P}\tilde{X}_{n-1} = \boldsymbol{P}^2\tilde{X}_{n-2} = \cdots = \boldsymbol{P}^{n-1}\tilde{X}_1,$$

if gcd $\{i \in \{1, 2, \cdots, m\} | g_\lambda(A_i) - g_\lambda(A_{i+1}) > 0\} = 1$, then $\lim\limits_{n\to\infty} \boldsymbol{P}^n$ exists, and

$$\lim_{n\to\infty} \boldsymbol{P}^n = \frac{ea^{\intercal}}{a^{\intercal}e} = eb^{\intercal},$$

where $e = \sum\limits_{i=1}^{m} e_k = [1, 1, \cdots, 1]^{\intercal} \in R^m$, e_k is the ith standard unit column vector,

$a = [a_1, a_2, \cdots, a_m]^{\intercal}$, $b = [b_1, b_2, \cdots, b_m]^{\intercal}$, $a_k = \sum\limits_{i=1}^{k}(g_\lambda(A_i) - g_\lambda(A_{i+1}))$,

$b_k = \frac{a^{\intercal}e_k}{a^{\intercal}e} = \frac{a_k}{\sum\limits_{i=1}^{m} a_i} = \frac{g_\lambda(A_1)-g_\lambda(A_{k+1})}{mg_\lambda(A_1)-\sum\limits_{i=2}^{m} g_\lambda(A_i)}, k = 1, 2, 3, ..., m.$

Proof. For matrix $\boldsymbol{P}$, its characteristic polynomial is $p(t) = \det(t\boldsymbol{Id} - \boldsymbol{P})$, where $\boldsymbol{Id}$ is the unit matrix of order m . It is easy to obtain

$$p(t) = t^m - (g_\lambda(A_m) - g_\lambda(A_{m+1}))t^{m-1} - \cdots - (g_\lambda(A_2) - g_\lambda(A_3))t - (g_\lambda(A_1) - g_\lambda(A_2)).$$

Since $\sum\limits_{i=1}^{m}(g_\lambda(A_i) - g_\lambda(A_{i+1})) = g_\lambda(A_1) = 1$, $t = 1$ is a positive root of $p(t)$. Note that

$$\gcd\{k \in \{1, 2, \cdots, m\} : g_\lambda(A_i) - g_\lambda(A_{i+1}) > 0\} = 1.$$

According to Lemma 4.1, we can obtain $t = 1$ is the unique root of $p(t)$, whose algebraic multiplicity and geometric multiplicity of 1 are both equal to 1, and the modulus of every other root of q is strictly less than r.

Let x be the right-hand eigenvector of matrix $\boldsymbol{P}$ with respect to eigenvalue 1, then $\boldsymbol{P}x = x$. By using the elementary line transformation and the first elementary column transformation to matrix P, we can obtain

$$\begin{aligned} &\boldsymbol{Id} - \boldsymbol{P} \\ = &\begin{bmatrix} 1 & -1 & 0 & \cdots & 0 \\ 0 & 1 & -1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & -1 \\ g_\lambda(A_2) - g_\lambda(A_1) & g_\lambda(A_3) - g_\lambda(A_2) & g_\lambda(A_4) - g_\lambda(A_3) & \cdots & 1 - (g_\lambda(A_{m+1}) - g_\lambda(A_m)) \end{bmatrix} \\ &\to \cdots \to \begin{bmatrix} 1 & 0 & 0 & \cdots & -1 \\ 0 & 1 & 0 & \cdots & -1 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & -1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}. \end{aligned}$$

Hence, a basic systeme of solutions for homogeneous linear equation set $(\boldsymbol{Id}-\boldsymbol{P})x=[0,0,\cdots,0]^{\intercal}$ is determined. It follows that the right-hand eigenvalue of $\boldsymbol{P}$ with respect to 1 is $x=[1,1,\cdots,1]^{\intercal}=e$.

Let $y^{\intercal}$ be the left-hand eigenvector of matrix $\boldsymbol{P}$ with respect to eigenvalue 1, then $y^{\intercal}\boldsymbol{P}=y^{\intercal}$. By using the elementary line transformation and the first elementary column transformation to matrix $\boldsymbol{Id}-\boldsymbol{P}^{\intercal}$, we can obtain

$$\boldsymbol{Id}-\boldsymbol{P}^{\intercal}=\begin{bmatrix} 1 & 0 & 0 & \cdots & g_\lambda(A_2)-g_\lambda(A_1) \\ 0 & 1 & 0 & \cdots & g_\lambda(A_3)-g_\lambda(A_2) \\ \vdots & \cdots & \cdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & g_\lambda(A_m)-g_\lambda(A_{m-1}) \\ 0 & 0 & 0 & \cdots & 1-(g_\lambda(A_m)-g_\lambda(A_{m+1})) \end{bmatrix}$$

$$\to\cdots\to\begin{bmatrix} 1 & 0 & 0 & \cdots & g_\lambda(A_2)-g_\lambda(A_1) \\ 0 & 1 & 0 & \cdots & g_\lambda(A_3)-g_\lambda(A_1) \\ \vdots & \cdots & \cdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & g_\lambda(A_m)-g_\lambda(A_1) \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

Thus, a basic system of solutions for homogeneous linear equation set $(\boldsymbol{Id}-\boldsymbol{P}^{\intercal})y=[0,0,\cdots,0]$ is determined as follows:

$$[g_\lambda(A_1)-g_\lambda(A_2),\ g_\lambda(A_1)-g_\lambda(A_3),\ \cdots,\ g_\lambda(A_1)-g_\lambda(A_m)]^{\intercal},$$

It follows that the left-hand eigenvalue of $\boldsymbol{P}$ with respect to 1 is $a^{\intercal}=[a_1,a_2,\cdots,a_m], a_k=\sum\limits_{i=1}^{k}(g_\lambda(A_i)-g_\lambda(A_{i+1})),\ k=1,2,3,...,m$. According to Lemma 4.2(1), we know that $\{\boldsymbol{P}^n\}$ converges to a nonzero matrix. Combing Lemma 4.2(2), we can get

$$\lim_{n\to\infty}\boldsymbol{P}^n=\frac{ea^{\intercal}}{a^{\intercal}e}=eb^{\intercal}.$$

The proof is complete. $\square$

Theorem 4.2. Let $(\tilde{x}_1,\tilde{x}_2,\cdots,\tilde{x}_m)\in\tilde{E}^m$, $(t_1,t_2,\cdots,t_m)\in R^m$, and g_λ be fuzzy measure satisfying $\delta-\lambda$ rules. Let $A_i=\{t_i,t_{i+1},\cdots,t_m\}$, $i=1,2,...,m$, $A_{m+1}=\emptyset$, $\tilde{X}_n=[\tilde{x}_n,\tilde{x}_{n+1},\cdots,\tilde{x}_{n+m-1}]^{\intercal}$. For the matrix P satisfying the recurrence relation in Theorem 3.1

$$\tilde{X}_n=\boldsymbol{P}\tilde{X}_{n-1}=\boldsymbol{P}^2\tilde{X}_{n-2}=\cdots=\boldsymbol{P}^{n-1}\tilde{X}_1,$$

if gcd $\{i\in\{1,2,\cdots,m\}:g_\lambda(A_i)-g_\lambda(A_{i+1})>0\}=1$, then $\lim\limits_{n\to\infty}\tilde{x}_n$ exists, and

$$\lim_{n\to\infty}\tilde{x}_n=\sum_{i=1}^{m}b_i\tilde{x}_i,$$

where $e=\sum\limits_{i=1}^{m}e_k=[1,1,\cdots,1]^{\intercal}\in\mathbb{R}^{m\times1}$, e_k is the ith standard unit column vector,

$a=[a_1,a_2,\cdots,a_m]^{\intercal}$, $b=[b_1,b_2,\cdots,b_m]^{\intercal}$, $a_k=\sum\limits_{i=1}^{k}(g_\lambda(A_i)-g_\lambda(A_{i+1}))$,

$b_k=\frac{a^{\intercal}e_k}{a^{\intercal}e}=\frac{a_k}{\sum\limits_{i=1}^{m}a_i}=\frac{g_\lambda(A_1)-g_\lambda(A_{k+1})}{mg_\lambda(A_1)-\sum\limits_{i=2}^{m}g_\lambda(A_i)}, k=1,2,3,...,m.$

Proof. Since

$$\tilde{X}_n=\boldsymbol{P}\tilde{X}_{n-1}=\boldsymbol{P}^2\tilde{X}_{n-2}=\cdots=\boldsymbol{P}^{n-1}\tilde{X}_1,$$

we have

$$\lim_{n\to\infty}[\tilde{x}_n,\tilde{x}_{n+1},\cdots,\tilde{x}_{n+m-1}]^{\intercal}=\lim_{n\to\infty}\boldsymbol{P}^{n-1}[\tilde{x}_n,\tilde{x}_{n+1},\cdots,\tilde{x}_{n+m-1}]^{\intercal}.$$

then, by Theorem 4.2, we can get

$$\lim_{n\to\infty}[\tilde{x}_n,\tilde{x}_{n+1},\cdots,\tilde{x}_{n+m-1}]^{\intercal}=er^{\intercal}[\tilde{x}_n,\tilde{x}_{n+1},\cdots,\tilde{x}_{n+m-1}]^{\intercal},$$

i.e. $\lim_{n\to\infty} \tilde{x}_n$ is determined by the operation of the first row of $\lim_{n\to\infty} \boldsymbol{P}^{n-1}$ and $\tilde{X}_1 = [\tilde{x}_1, \tilde{x}_2, \cdots, \tilde{x}_m]^\intercal$. It follows that

$$\lim_{n\to\infty} \tilde{x}_n = \frac{a^\intercal x}{a^\intercal e} = \frac{\sum\limits_{i=1}^{m} a_i \tilde{x}_i}{\sum\limits_{i=1}^{m} a_i} = \sum_{i=1}^{m} b_i \tilde{x}_i.$$

The proof is complete. $\square$

In moving weighted average, the weight of the information contained in the data is not the same, and is independent of each other, so to identify the data of each phase is not reasonable. And introducing the non-additive measure into the moving weighted average is of practical significance.

Example 4.1. Given a closing stock prices system over 5 days. The closing prices of each day is denoted as $\tilde{x}_i$, $(\tilde{x}_1, \tilde{x}_2, \cdots, \tilde{x}_5) \in \tilde{E}^5$, and every $\tilde{x}_i$ is a triangle fuzzy number, $\tilde{x}_i = (x_i - \delta_{i,1}, x_i, x_i + \delta_{i,2})$, $i = 1, 2, \cdots, 5$. Suppose $(t_1, t_2, \cdots, t_5) \in R^5$, $A_i = \{t_i, t_{i+1}, \cdots, t_5\}$, $i = 1, 2, ..., 5$, $A_6 = \emptyset$. The value and the weight of each $\tilde{x}_i$ is shown in Table 1, $i = 1, 2, \cdots, 5$, then we can get the closing stock price over 10 days and some relevant results.

Day	Closing stock price	g_λ
1	(19,20,21)	0.1
2	(21,22,23)	0.2
3	(23,24,25)	0.3
4	(24,25,26)	0.15
5	(22,23,24)	0.175

Table 1: The closing stock prices over 5 days.

According to Remark 2.3 again, we know that $\prod\limits_{i=1}^{5}(1 + \lambda g_{\lambda i}) = 1 + \lambda$, hence we can gain $\lambda = 0.218$.

Then, by Remark 2.3, we have

$g_\lambda(A_1) = 1, \quad g_\lambda(A_2) = \frac{1}{\lambda}\left\{\prod\limits_{i=2}^{5}[1 + \lambda g_\lambda(\{x\})] - 1\right\} = 0.88,$

$g_\lambda(A_3) = \frac{1}{\lambda}\left\{\prod\limits_{i=3}^{5}[1 + \lambda g_\lambda(\{x\})] - 1\right\} = 0.65$, $g_\lambda(A_4) = \frac{1}{\lambda}\left\{\prod\limits_{i=4}^{5}[1 + \lambda g_\lambda(\{x\})] - 1\right\} = 0.33,$

$g_\lambda(A_5) = g_\lambda(\{x_5\}) = 0.175, \quad g_\lambda(A_6) = 0.$

By Definition 3.1, we have

$$\begin{aligned}\tilde{x}_6 = (&\sum_{i=1}^{5}((x_i - \delta_{i,1})(g_\lambda(A_i) - g_\lambda(A_{i+1})),\\ &\sum_{i=1}^{5} x_i(g_\lambda(A_i) - g_\lambda(A_{i+1})), \sum_{i=1}^{5}(x_i + \delta_{i,2})(g_\lambda(A_i) - g_\lambda(A_{i+1}))),\\ =\ &(22.04, 23.04, 24.04).\end{aligned}$$

Similarly, we can also calculate $\tilde{x}_n, n = 7, 8, 9, 10$, with respect to fuzzy measure g_λ on A, shown in Table 2. And by Theorem 4.1 and Theorem 4.3, we have

$$\begin{aligned}\lim_{n\to\infty} \boldsymbol{P}^n &= \frac{ea^\intercal}{a^\intercal e}\\ &= \frac{1}{0.12 + 0.35 + 0.67 + 0.825 + 1}\begin{bmatrix} 0.12 & 0.35 & 0.67 & 0.825 & 1\\ 0.12 & 0.35 & 0.67 & 0.825 & 1\\ 0.12 & 0.35 & 0.67 & 0.825 & 1\\ 0.12 & 0.35 & 0.67 & 0.825 & 1\\ 0.12 & 0.35 & 0.67 & 0.825 & 1 \end{bmatrix}\end{aligned}$$

Day	Closing stock price	g_λ
1	(19,20,21)	0.1
2	(21,22,23)	0.2
3	(23,24,25)	0.3
4	(24,25,26)	0.15
5	(22,23,24)	0.175
6	(22.04,23.04,24.04)	
7	(22.76,23.76,24.76)	
8	(22.72,23,72,24.72)	
9	(22.5,23.5,24.5)	
10	(22.45,23.45,24.45)	

Table 2: The closing stock prices over 10 days.

$$= \begin{bmatrix} 0.04 & 0.11 & 0.23 & 0.28 & 0.34 \\ 0.04 & 0.11 & 0.23 & 0.28 & 0.34 \\ 0.04 & 0.11 & 0.23 & 0.28 & 0.34 \\ 0.04 & 0.11 & 0.23 & 0.28 & 0.34 \\ 0.04 & 0.11 & 0.23 & 0.28 & 0.34 \end{bmatrix},$$

$$\lim_{n\to\infty} \tilde{x}_n = \frac{a^\intercal \tilde{X}_1}{a^\intercal e} = \frac{1}{0.12+0.35+0.67+0.825+1}(66.84, 69.865, 72.77) = (22.54, 23.56, 24.54),$$

where $e = \sum\limits_{i=1}^{5} e_k = [1, 1, \cdots, 1]^\intercal \in R^{5\times 1}$, e_k is the ith standard unit column vector,

$a_1 = g_\lambda(A_1) - g_\lambda(A_2) = 0.12$, $a_2 = \sum\limits_{i=1}^{2}(g_\lambda(A_i) - g_\lambda(A_{i+1})) = 0.35$, $a_3 = \sum\limits_{i=1}^{3}(g_\lambda(A_i) - g_\lambda(A_{i+1})) = 0.67$, $a_4 = \sum\limits_{i=1}^{4}(g_\lambda(A_i) - g_\lambda(A_{i+1})) = 0.825$, $a_5 = \sum\limits_{i=1}^{5}(g_\lambda(A_i) - g_\lambda(A_{i+1})) = 0.1$.

Here when n is infinite, the forecasting value of x_n will become a stable value (22.54,23.56,24.54) by the weighted moving averages for a series of fuzzy numbers based on non-additive measures with $\sigma - \lambda$ rules.

5. Conclusion

In this paper, the moving average for a series of fuzzy numbers was proposed by means of non-additive measures with $\sigma - \lambda$ rules and fuzzy number. Meanwhile, the special case, i,e. the moving average for a series of triangular fuzzy numbers based on non-additive measures with $\sigma-\lambda$ were also discussed. Further, the calculation of the weighted moving averages for fuzzy-number based on a non-additive measure with $\sigma - \lambda$ rules was invested and some properties were discussed. Finally, an example was given to illustrate the practical importance of the main results.

References

[1] S. Asahina, K. Uchino, T. Murofushi, Relationship among continuity conditions and null-additivity conditions in non-additive measure theory. 157(5) (2004) 691-698.

[2] H.H. Bauschke, J. Sarada, X. Wang, On moving averages, Journal of Convex Analysis 21 (2014) 219-235.

[3] L. Chen, Z.T. Gong, Genetic algorithm optimization for determing fuzzy measures from fuzzy data, Journal of Applied Mathematics 2013(3) (2013) 1-11.

[4] G. Choquet. Theory of Capacities[J]. Annual instuitute Fourier, 1954, 5: 131-295.

[5] Z.T. Gong, S.X. Hai, Convexity of n-dimensional fuzzy number-valued functions and its applications, Fuzzy Sets and Systems 295 (2016) 2016 19-36.

[6] Z.T. Gong, L. Chen, G. Duan, Choquet Integral of Fuzzy-Number-Valued Functions: The Differentiability of the Primitive with respect to Fuzzy Measures and Choquet Integral Equations, Abstract and Applied Analysis 2014(3) (2014) 1-11.

[7] M. Grabisch, New algorithm for identifying fuzzy measures and its application to pattern recognition, In Proceedings of the IEEE International Conference on Fuzzy Systems (IFES ’ 95) Yokohama, Japan (1995) 145-150.

[8] J. Li, M. Yasuda, On Egoroff's theorems on finite monotone non-additive measure spaceFuzzy Sets and Systems 153(1) (2005) 71-78 .

[9] M. Ma, On embedding problems of fuzzy number spaces: part 5, Fuzzy Sets Syst 55 (1993) 313 – 318.

[10] T. Murofushi, M. Sugeno, An interpretation of fuzzy measures and the Choquet integral as an integral with respect to a fuzzy measure, Fuzzy Sets and Systems 29 (1989) 201-227.

[11] C.V. Negoita, D.A. Ralescu, Application of Fuzzy Sets to Systems Analysis Wiley, New York, 1975.

[12] O. Kaleva, Fuzzy differential equations, Fuzzy Sets Syst 24 (1987) 301 – 317.

[13] C.D. Meyer, Matrix Analysis and Applied linear Algebra, SIAM, Philadelphia, 2000.

[14] P. V, C. Nelson Kennedy Babu, Moving average fuzzy resource scheduling for virtualized cloud data services, Computer Standards and Interfaces 50 (2017) 251 – 257.

[15] A.M. Ostrowski, Solution of Equation and Systems of Equations, Academic Press, New York and London 1966.

[16] A. Saeidifar, E. Pasha, The possibilistic moments of fuzzy numbers and their applications, Journal of Computational and Applied Mathematics 223(2) (2009) 1028-1042.

[17] D. Schmeidler, Subjective probability and expected utility without additivity, Econometrica 57(3) (1989) 571 – 587.

[18] M. Sugeno, Theory of fuzy integral and its application, Doctorial Dissertation, Tokyo Institute of Technology, 1974.

[19] C.Q. Tan, X.H. Chen. Intuitionistic fuzzy Choquet integral operator for multi-criteria decision making, Expert Systems with Applications 37 (2010) 149-157.

[20] C.P. Tsokos, K-th moving, weighted and exponential moving average for time series forecasting models, European Journal of Pure and Applied Mathematics 3 (2010) 406-416.

[21] P.Z. Wang, Fuzzy Set Theory and Application, Shanghai Science and Technology Press, Shanghai, 1983(in Chinese).

[22] Y.L. Wang, J.G. Zheng, Knowledge management performance evaluation based on triangular fuzzy number, Procedia Engineering 7 (2010) 38-45.

[23] J.Q. Wang, Overview on fuzzy multi-criteria decision-making approach, Control and Decision 23 (2008) 601-606.

[24] L.A. Zadeh, Fuzzy Sets, Information and Control 8(3) (1965) 338-353.

[25] L.A. Zadeh, The concept of a linguistic variable and its application approx- imate reasoning, Information Sciences 8(3) (1975) 199-249.

A Periodic Observer Based Stabilization Synthesis Approach for LDP Systems based on iteration *

Lingling Lv †, Wei He ‡, Zhe Zhang §, Lei Zhang ¶, Xianxing Liu ‖

Abstract

The stabilization problem of state observer based for linear discrete-time periodic (LDP) system and its robust consideration are discussed in this paper. It is proved that the periodic controller and the full-dimensional periodic state observer can be designed separately. Based on the well-known CG-algorithm for matrix equation Ax = b as well as applying the lifting technique and algebraic operations, an iterative algorithm for both periodic observer gains and periodic state feedback gains can be generated simultaneously. By optimizing the free parameter matrix in the proposed algorithm, a robust stabilization algorithm based on periodic observer for LDP systems is presented. One numerical example is worked out to illustrate the effect of the proposed approaches.

Keywords: Linear discrete-time periodic (LDP) systems; periodic state observers; stabilization; iterative method.

1 Introduction

The controller design requires us to master the state characteristics of the system. However, it is impractical to direct measure all state variables precisely in practical applications. So it requires us to make reliable estimates of the states that cannot be measured directly. The state observer is also called state reconstruction. The basic design idea is to design a state equivalent to the original system and use the designed state equivalent to the original state (see [1]-[2] and references therein). Especially, full-dimensional state observer in the construction idea is based on the original observed coefficient matrix in accordance with the same structure to establish a copy system. The difference between the observed system y and the copy system output $\widehat{y}$ is taken as a fixed variable and fed back to the input of the integrator group in the copy system to form a closed-loop system (see [3]-[5] and references therein). The design of observer has always been a research hot topic in control theory and control engineering, one can see [6, 7, 8] and references therein for instance.

Because of its extensive applications in cyclostationary process, multirate digital control, economics and management, biology, etc., and advantages of improving control performance by using periodic controllers, linear discrete periodic systems have been paid renewed attentions in the control theory community(see [9]-[11] and the references therein). The stabilization problem of dynamic systems has a fundamental importance in engineering, and hence it is among the most studied problems in modern control theory. Particularly, the

*This work is supported by the Programs of National Natural Science Foundation of China (Nos. U1604148, 11501200, 61402149), Innovative Talents of Higher Learning Institutions of Henan (No. 17HASTIT023), China Postdoctoral Science Foundation (No. 2016M592285).

†1. College of Environment and Planning, Henan University, Kaifeng, 475004, P. R. China. 2. Institute of Electric power, North China University of Water Resources and Electric Power, Zhengzhou 450011, P. R. China. Email: lingling_lv@163.com (Lingling Lv).

‡Institute of electric power, North China University of Water Resources and Electric Power, Zhengzhou 450011, P. R. China. Email: 13298369256@163.com (Wei He).

§Institute of electric power, North China University of Water Resources and Electric Power, Zhengzhou 450011, P. R. China. Email: zhe_Zhang5218@163.com (Zhe Zhang)

¶Institute of Data and Knowledge Engineering, School of Computer and Information Engineering, Henan University, Kaifeng 475004, P.R. China. Email: zhanglei@henu.edu.cn (Lei Zhang).

‖Computer and Information Engineering College, Henan University, Kaifeng 475004, P. R. China. Email: Liuxianxing@henu.edu.cn (Xianxing Liu). Corresponding author.

stabilization of periodic motions of dynamic systems has drawn much attention over the past years (see [12]-[16] and references therein). In [14], LMI based conditions for stabilization via static periodic state feedback as well as via static periodic output feedback are presented, and the problem of quadratic stabilization in the presence of either norm-bounded or polytopic parameter uncertainty is also treated. The output stabilization problem for discrete-time linear periodic systems is solved in [15], where both the state-feedback control law and the state-predictor are based on a suitable time-invariant state-sampled reformulation associated with a periodic system. In addition, utilizing parametric poles assignment algorithm and robust performance index, an algorithm of robust stabilization based on periodic observers is proposed in [16].

In this paper, the problem of stabilization of discrete-time periodic systems based on state observer is transformed into the solution of the corresponding matrix equations, and a neat iterative algorithm is given based on the well-known conjugate gradient algorithm. Initially, we consider the stabilization problem for linear discrete-lime periodic systems without disturbances and give the expected algorithm. On this basis, in case that uncertain disturbances exist in the system parameters, a robust control algorithm for purpose of stabilization is also derived.

Notation 1 *The superscripts "*T*" and "* -1*" stand for matrix transposition and matrix inverse, respectively;* $\mathbb{R}^n$ *denotes the n-dimensional Euclidean space;* $\overline{i,j}$ *represents the integer set* $\{i, i+1, \ldots, j-1, j\}$*,* $\mathrm{tr}(A)$ *means the trace of matrix A. Norm* $\|A\|$ *is a Frobenius norm of matrix A.* $\Lambda(A)$ *means the eigenvalue set of matrix A and* Ψ_A *denotes the monodromy matrix* $A_{T-1}A_{T-2}\cdots A_0$ *with period T.*

2 Preliminaries

Consider the completely observable and completely reachable LDP systems with the following state space representation

$$\begin{cases} x_{t+1} = A_t x_t + B_t u_t \\ y_t = C_t x_t \end{cases} \tag{1}$$

where $t \in \mathbb{Z}$, the set of integers, $x_t \in \mathbb{R}^n$, $u_t \in \mathbb{R}^r$ and $y_t \in \mathbb{R}^m$ are respectively the state vector, the input vector and the output vector, A_t, B_t, C_t are matrices of compatible dimensions satisfying

$$A_{t+T} = A_t,\ B_{t+T} = B_t,\ C_{t+T} = C_t.$$

In case that the state of system (1) can be measured, by periodic feedback control law

$$u_t = -K_t x_t + v(t), \quad K_{t+T} = K_t, \quad K_t \in \mathbb{R}^{r\times n} \tag{2}$$

where v_t is the reference input, we can obtain the following combined system with period T

$$\begin{cases} x_{t+1} = (A_t - B_t K_t) x_t + B_t v_t \\ y_t = C_t x_t \end{cases} \tag{3}$$

When there exists some restrictions in practice, the state of system (1) can not be gotten by hardware, but the input u_t and the output y_t can be measured. In this case, we need build another periodic system which can give an asymptotic estimation of system states. The system with the following form can be adopted:

$$\hat{x}_{t+1} = A_t \hat{x}_t + B_t u_t - L_t (C_t \hat{x} - y_t) \tag{4}$$

where $\hat{x} \in \mathbb{R}^n$ and $L(t) \in \mathbb{R}^{n\times m}$, $t \in \mathbb{Z}$ are real matrices of period T. Obviously, equation 4 has the following equivalent presentation:

$$\hat{x}_{t+1} = (A_t - L_t C_t)\hat{x}_t + B_t u_t + L_t y_t \tag{5}$$

Integrating (4) and (3) gives the following augmented system:

$$\begin{cases} \begin{bmatrix} x_{t+1} \\ \hat{x}_{t+1} \end{bmatrix} = \begin{bmatrix} A_t & B_t K_t \\ L_t C_t & \widetilde{A}_t - B_t K_t \end{bmatrix} \begin{bmatrix} x_t \\ \hat{x}_t \end{bmatrix} + \begin{bmatrix} B_t \\ B_t \end{bmatrix} v_t \\ y_t = \begin{bmatrix} C_t & 0 \end{bmatrix} \begin{bmatrix} x_t \\ \hat{x}_t \end{bmatrix} \end{cases} \tag{6}$$

where $\widetilde{A}_t = A_t - L_t C_t$.

Then the problem of stabilization based on periodic observer for LDP system (1) can be represented as

Problem 1 *Given a completely reachable and completely observable LDP system (1), find periodic matrix $K(t) \in \mathbb{R}^{r\times n}, t \in \overline{0, T-1}$ and $L(t) \in \mathbb{R}^{n\times m}, t \in \overline{0, T-1}$, such that the augmented system (6) is asymptotically stable.*

When the system is disturbed by external environment, the closed loop system matrix will deviate from the nominal matrix $\widetilde{A}_t$, which can be generally expressed as

$$A_t - B_tK_t \mapsto A_t + \Delta_{a,t} - (B_t + \Delta_{b,t}) K_t,\ t \in \overline{0, T-1},$$

$$A_t + L_tC_t \mapsto A_t + \Delta_{a,t} + L_t (C_t + \Delta_{c,t}),\ t \in \overline{0, T-1},$$

in which $\Delta_{a,t} \in \mathbb{R}^{n\times n}$, $\Delta_{b,t} \in \mathbb{R}^{n\times r}$, $\Delta_{c,t} \in \mathbb{R}^{m\times n}, t \in \overline{0, T-1}$ are random small perturbations. Thus, the problem of robust observer design for linear discrete-time periodic system (1) can be portrayed as

Problem 2 *Consider the completely observable and completely reachable linear discrete-time periodic system (1), seek the periodic matrix $K(t) \in \mathbb{R}^{r\times n}, t \in \overline{0, T-1}$ and $L_t \in \mathbb{R}^{n\times m}, t \in \overline{0, T-1}$, such that the following conditions are met:*

1. *The augmented system (6) is asymptotically stable;*
2. *Eigenvalues of the augmented system (6) are as insensitive as possible to small perturbations on systems matrices.*

3 Main result

The first thing to consider is the existence condition for a periodic state observer and a periodic state feedback controller. To do this, we would like to give the following theorem firstly.

Theorem 1 *For a given completely observable and completely reachable LDP system (1), the transfer function of the closed-loop system (6) is equal to the transfer function of the closed-loop system (3).*

Proof. It is easy to calculate that the transfer function of the closed-loop system (3) is:

$$G(s) = C_t(sI - A_t - B_tK_t)^{-1}B_t \tag{7}$$

Let

$$P_t = \begin{bmatrix} I & 0 \\ -I & I \end{bmatrix}.$$

It is easily computed that

$$P_t^{-1} = \begin{bmatrix} I & 0 \\ I & I \end{bmatrix}.$$

Noticing the coefficient matrices of system (6), we can obtain that

$$P_t \begin{bmatrix} A_t & B_tK_t \\ -L_tC_t & \widetilde{A}_t + B_tK_t \end{bmatrix} P_t^{-1} = \begin{bmatrix} A_t + B_tK_t & B_tK_t \\ 0 & \widetilde{A}_t \end{bmatrix},$$

$$P \begin{bmatrix} B_t \\ B_t \end{bmatrix} = \begin{bmatrix} B_t \\ 0 \end{bmatrix},$$

$$\begin{bmatrix} C_t & 0 \end{bmatrix} P_t^{-1} = \begin{bmatrix} C_t & 0 \end{bmatrix}.$$

Obviously, system (6) is algebra equivalent to the following system:

$$\left(\begin{bmatrix} A_t + B_tK_t & B_tK_t \\ 0 & \widetilde{A}_t \end{bmatrix}, \begin{bmatrix} B_t \\ 0 \end{bmatrix}, \begin{bmatrix} C_t & 0 \end{bmatrix} \right) \tag{8}$$

Since the systems which are algebra equivalent to each other have the same transfer function, we only need to prove that the transfer function of system (8) is as shown in (7). By noticing

$$\begin{bmatrix} sI - A_t - B_tK_t & B_tM_t \\ 0 & sI - \widetilde{A}_t \end{bmatrix}^{-1} = \begin{bmatrix} (sI - A_t - B_tK_t)^{-1} & * \\ 0 & (sI - \widetilde{A}_t)^{-1} \end{bmatrix} \tag{9}$$

the transfer function corresponding to (8) can be calculated as

$$\begin{aligned} \bar{G}(s) &= \begin{bmatrix} C_t & 0 \end{bmatrix} \begin{bmatrix} sI - A_t - B_tK_t & B_tK_t \\ 0 & SI - \widetilde{A}_t \end{bmatrix}^{-1} \begin{bmatrix} B_t \\ 0 \end{bmatrix} \\ &= C_t(sI - A_t - B_tK_t)^{-1}B_t \end{aligned}$$

which is exactly equal to the transfer function of system (6). Thus the proof is accomplished. ■

According to theorem 1, the introduction of periodic state observer has no influence on the desired poles of the closed-loop systems via periodic state feedback. Similarly, the introduction of periodic state feedback has no influence on the designed poles of observer. In this point, the LDP systems keep pace with the linear time invariant systems. Therefore, for the stabilization problem of LDP systems based on periodic observer, the periodic state feedback controller and periodic observer can be designed separately. In the following, poles assignment techniques are adopt to realize the desired purpose.

Let Γ_1 and Γ_2 be the predetermined set of poles of the close-loop system (3) and (5) respectively, which are both symmetric with respect to the real axis. Let $\bar{F}_j^{\mathrm{K}}, \bar{F}_j^{\mathrm{L}} \in \mathbb{R}^{n\times n}$ be the T-periodic matrix satisfying $\Lambda(\Psi_{\bar{F}^{\mathrm{K}}}) = \Gamma_1$ and $\Lambda(\Psi_{\bar{F}^{\mathrm{L}}}) = \Gamma_2$, respectively. Clearly, to make system (3) and (5) possess the pole set Γ_1 and Γ_2 if and only if there exists a T-periodic invertible matrix X_j and Y_j such that

$$X_{j+1}^{-1}(A_j - B_jK_j)X_j = -F_j^{\mathrm{K}}. \tag{10}$$

and

$$Y_{j+1}^{-1}(A_j^{\mathrm{T}} - C_j^{\mathrm{T}}L_j^{\mathrm{T}})Y_j = -F_j^{\mathrm{L}}. \tag{11}$$

where $F_j^{\mathrm{K}} = -\bar{F}_j^{\mathrm{K}}, F_j^{\mathrm{L}} = -\bar{F}_j^{\mathrm{L}}, j \in \overline{0, T-1}$. Obviously, equations (10) and (11) can be rewritten as the following periodic Sylvester matrices:

$$A_jX_j - B_jK_jX_j = -X_{j+1}F_j^{\mathrm{K}}, \tag{12}$$

and

$$A_j^{\mathrm{T}}Y_j - C_j^{\mathrm{T}}L_j^{\mathrm{T}}Y_j = -Y_{j+1}F_j^{\mathrm{L}}, \tag{13}$$

Next, an iterative algorithm of stabilization problem based on periodic observer via periodic state feedback is presented firstly, and its correctness will be strictly verified in the subsequence.

Algorithm 1 *(Periodic CG-based Algorithm of problem 1)*

1. *Let* $F_j^{\mathrm{K}} \in \mathbb{R}^{n\times n}, F_j^{\mathrm{L}} \in \mathbb{R}^{n\times n}, j \in \overline{0, T-1}$ *be a real periodic matrix, which satisfies* $\Lambda(\Psi_{F_j^{\mathrm{K}}}) = \Gamma_1$ *and* $\Lambda(\Psi_{F_j^{\mathrm{K}}}) \bigcap \Lambda(\Psi_{A_j}) = 0$*;* $\Lambda(\Psi_{F_j^{\mathrm{L}}}) = \Gamma_2$ *and* $\Lambda(\Psi_{F_j^{\mathrm{L}}}) \bigcap \Lambda(\Psi_{A_j^{\mathrm{T}}}) = 0$*. Further, let* $G_j = K_jX_j \in \mathbb{R}^{r\times n}, D_j = L_j^{\mathrm{T}}Y_j \in \mathbb{R}^{m\times n}$ *are real parametric matrix such that periodic matrix pair* (F_j^{K}, G_j) *and* (F_j^{L}, D_j) *is completely observable.*

2. *Set tolerance* ε*; Choose arbitrary initial periodic matrix* $X_j(0) \in \mathbb{R}^{n\times n}, Y_j(0) \in \mathbb{R}^{n\times n}$*,* $j \in \overline{0, T-1}$*; Calculated as follows:*

$$\begin{aligned} Q_j(0) &= B_jG_j - A_jX_j(0) - X_{j+1}(0)F_j^{\mathrm{K}}, \\ W_j(0) &= C_j^{\mathrm{T}}D_j - A_j^{\mathrm{T}}Y_j(0) - Y_{j+1}(0)F_j^{\mathrm{L}}; \\ R_j(0) &= A_j^{\mathrm{T}}Q_j(0) + Q_{j-1}(0)(F_{j-1}^{\mathrm{K}})^{\mathrm{T}}; \\ N_j(0) &= A_jW_j(0) + W_{j-1}(0)(F_{j-1}^{\mathrm{L}})^{\mathrm{T}}; \\ P_j(0) &= -R_j(0); \\ H_j(0) &= -N_j(0); \\ t &:= 0. \end{aligned}$$

4

3. *If* $\sum_{j=0}^{T-1} \|R_j(t)\| \leq \varepsilon$ *and* $\sum_{i=0}^{T-1} \|N_j(t)\| \leq \varepsilon$*, stop; else, go to next step.*

4. *While* $\sum_{j=0}^{T-1} \|R_j(t)\| \geq \varepsilon$ *and* $\sum_{i=0}^{T-1} \|N_j(t)\| \geq \varepsilon$*, calculate*

$$\begin{aligned}
\alpha_j(t) &= \frac{\sum_{j=0}^{T-1} \operatorname{tr}\left[P_j^{\mathrm{T}}(t)R_j(t)\right]}{\sum_{j=0}^{T-1} \|A_jP_j(t)+P_{j+1}(t)B_j\|^2};\\
\beta_j(t) &= \frac{\sum_{j=0}^{T-1} \operatorname{tr}\left[H_j^{\mathrm{T}}(t)N_j(t)\right]}{\sum_{j=0}^{T-1} \left\|A_j^T H_j(t)+H_{j+1}(t)C_j^{\mathrm{T}}\right\|^2};\\
X_j(t+1) &= X_j(t)+\alpha_j(t)P_j(t);\\
Y_j(t+1) &= Y_j(t)+\beta_j(t)H_j(t);\\
Q_j(t+1) &= B_jG_j - A_jX_j(t+1) - X_{j+1}(t+1)F_j^{\mathrm{K}};\\
W_j(t+1) &= C_j^{\mathrm{T}}D_j - A_j^TY_j(t+1) - Y_{j+1}(t+1)F_j^{\mathrm{L}};\\
R_j(t+1) &= A_j^{\mathrm{T}}Q_j(t+1)+Q_{j-1}(t+1)(F_{j-1}^{\mathrm{K}})^{\mathrm{T}},\\
N_j(t+1) &= A_jW_j(t+1)+W_{j-1}(t+1)(F_j^{\mathrm{L}})^{\mathrm{T}};\\
P_j(t+1) &= -R_j(t+1)+\frac{\sum_{j=0}^{T-1}\|R_j(t+1)\|^2}{\sum_{j=0}^{T-1}\|R_j(t)\|^2}P_j(t);\\
H_j(t+1) &= -N_j(t+1)+\frac{\sum_{j=0}^{T-1}\|N_j(t+1)\|^2}{\sum_{j=0}^{T-1}\|N_j(t)\|^2}H_j(t);\\
t &= t+1;
\end{aligned}$$

5. *Let* $X_j = X_j(t)$*,*$Y_j = Y_j(t)$*. The real periodic matrix* K_j *and* L_j *can be obtained as*

$$\begin{aligned}
K_j &= G_jX_j^{-1}, j\in\overline{0,T-1},\\
L_j &= (D_jY_j^{-1})^{\mathrm{T}}, j\in\overline{0,T-1}.
\end{aligned}$$

Remark 1 *The main part of the algorithm does not contain nested loops, so the computational complexity of the algorithm is* $O(n)$*.*

Next, the convergence and correctness of the algorithm are proved.

Lemma 1 *For sequences* $\{R_j(k)\}$*,* $\{P_j\}(k)$*,*$\{N_j(k)\}$*,* $\{H_j(k)\}$*,* $j\in\overline{0,T-1}$*, the following relations hold for* $k\geq 0$*:*

$$\sum_{j=0}^{T-1}\operatorname{tr}\left[R_j^{\mathrm{T}}(k+1)P_j(k)\right]=0,\quad \sum_{j=0}^{T-1}\operatorname{tr}\left[N_j^{\mathrm{T}}(k+1)H_j(k)\right]=0, \tag{14}$$

$$\sum_{j=0}^{T-1}\operatorname{tr}\left[R_j^{\mathrm{T}}(k)P_j(k)\right]+\sum_{j=0}^{T-1}\|R_j(k)\|^2=0,\quad \sum_{j=0}^{T-1}\operatorname{tr}\left[N_j^{\mathrm{T}}(k)H_j(k)\right]+\sum_{j=0}^{T-1}\|N_j(k)\|^2=0 \tag{15}$$

$$\sum_{k>0}\frac{\left(\sum_{j=0}^{T-1}\|R_j(k)\|^2\right)^2}{\sum_{j=0}^{T-1}\|P_j(k)\|^2}<\infty,\quad \sum_{k>0}\frac{\left(\sum_{j=0}^{T-1}\|N_j(k)\|^2\right)^2}{\sum_{j=0}^{T-1}\|H_j(k)\|^2}<\infty \tag{16}$$

Proof. By the expression of $R_j(k+1)$ in Algorithm 1, the following deduction is established.

$$\begin{aligned}
R_j(k+1) &= A_j^{\mathrm{T}} Q_j(k+1) + Q_{j-1}(k+1)(F_{j-1}^{\mathrm{K}})^{\mathrm{T}} \\
&= A_j^{\mathrm{T}} \left(C_j G_j - A_j X_j(k+1) - X_{j+1}(k+1) F_j^{\mathrm{K}}\right) \\
&\quad + \left(C_{j-1} G_{j-1} - A_{j-1} X_{j-1}(k) - X_j(k) F_{j-1}^{\mathrm{K}}\right) (F_{j-1}^{\mathrm{K}})^{\mathrm{T}} \\
&= A_j^{\mathrm{T}} \left(C_j G_j - A_j X_j(k) - X_j F_j^{\mathrm{K}}\right) \\
&\quad + \left(C_{j-1} G_{j-1} - A_{j-1} X_{j-1} - X_j(k) F_{j-1}^{\mathrm{K}}\right) (F_{j-1}^{\mathrm{K}})^{\mathrm{T}} \\
&\quad -\alpha(k) A_j^{\mathrm{T}} \left(A_j P_j(k) + P_{j+1}(k) F_j^{\mathrm{K}}\right) \\
&\quad -\alpha(k) \left(A_{j-1} P_{j-1}(k) + P_j(k) F_{j-1}^{\mathrm{K}}\right) (F_{j-1}^{\mathrm{K}})^{\mathrm{T}} \\
&= R_j(k) - \alpha(k) \left[A_j^{\mathrm{T}} \left(A_j P_j + P_{j+1}(k) F_j^{\mathrm{K}}\right) + \left(A_{j-1} P_{j-1}(k) + P_j(k) F_{j-1}^{\mathrm{K}}\right) (F_{j-1}^{\mathrm{K}})^{\mathrm{T}}\right]
\end{aligned}$$

Noticing the formula of $\alpha(k)$ in step 3 of Algorithm 1, we can obtain that

$$\begin{aligned}
\sum_{j=0}^{T-1} \operatorname{tr}\left[R_j^{\mathrm{T}}(k+1) P_j(k)\right] &= \sum_{j=0}^{T-1} \operatorname{tr}\left[R_j^{T}(k) P_j(k)\right] - \alpha(k) \sum_{j=0}^{T-1} \left[\left(A_j P_j(k) + P_{j+1}(k) F_j^{\mathrm{K}}\right)^{\mathrm{T}} A_j P_j(k)\right] \\
&\quad +\alpha(k) \sum_{j=0}^{T-1} \left[\left(A_{j-1} P_{j-1}(k) + P_j(k) F_{j-1}^{\mathrm{K}}\right)^{\mathrm{T}} P_j(k) F_{j-1}^{\mathrm{K}}\right] \\
&= \sum_{j=0}^{T-1} \operatorname{tr}\left[R_j^{T}(k) P_j(k)\right] \\
&\quad -\alpha(k) \sum_{j=0}^{T-1} \left[\left(A_j P_j(k) + P_{j+1}(k) F_j^{\mathrm{K}}\right)^{\mathrm{T}} \left(A_j P_j(k) + P_{j+1}(k) F_j^{\mathrm{K}}\right)\right] \\
&= \sum_{j=0}^{T-1} \operatorname{tr}\left[R_j^{T}(k) P_j(k)\right] - \alpha(k) \sum_{j=0}^{T-1} \left\|A_j P_j(k) + P_{j+1}(k) F_j^{\mathrm{K}}\right\| \\
&= 0
\end{aligned}$$

The second equation in (14) can be verified by similar deduction.

It is easily to check that equation (15) holds for $k = 0$. Then, according to the expression of $P_j(k+1)$ and Equation (14), the following deduction holds.

$$\begin{aligned}
\sum_{j=0}^{T-1} \operatorname{tr}\left[R_j^{\mathrm{T}}(k+1) P_j(k+1)\right] &= -\sum_{j=0}^{T-1} \operatorname{tr}\left[R_j^{\mathrm{T}}(k+1) R_j(k+1)\right] + \frac{\sum_{j=0}^{T-1} \|R_j(k+1)\|^2}{\sum_{j=0}^{T-1} \|R_j(k)\|^2} \sum_{j=0}^{T-1} \operatorname{tr}\left[R_j^{\mathrm{T}}(k+1) P_j(k)\right] \\
&= -\sum_{j=0}^{T-1} \|R_j(k+1)\|^2
\end{aligned}$$

That's to say Equation (15) holds. Applying Kronecker product, we get

$$\begin{aligned}
&\sum_{j=0}^{T-1}\left\|A_jP_j(k)+P_{j+1}(k)F_j^{\mathrm{K}}\right\|^2=\sum_{j=0}^{T-1}\left\|(E\otimes A_j)\operatorname{vec}(P_j(k))+\left((F_j^{\mathrm{K}})^{\mathrm{T}}\otimes E\right)\operatorname{vec}(P_{j+1}(k))\right\|^2\\
=&\left\|\begin{array}{c}(E\otimes A_0)\operatorname{vec}(P_0(k))+\left((F_0^K)^{\mathrm{T}}\otimes E\right)\operatorname{vec}(P_1(k))\\(E\otimes A_1)\operatorname{vec}(P_1(k))+\left((F_1^K)^{\mathrm{T}}\otimes E\right)\operatorname{vec}(P_2(k))\\\vdots\\(E\otimes A_{T-1})\operatorname{vec}(P_{T-1}(k))+\left((F_{T-1}^K)^{\mathrm{T}}\otimes E\right)\operatorname{vec}(P_0(k))\end{array}\right\|^2\\
=&\left\|\begin{bmatrix}E\otimes A_0 & (F_0^K)^{\mathrm{T}}\otimes E & & & \\ & E\otimes A_1 & (F_1^K)^{\mathrm{T}}\otimes E & & \\ & & E\otimes A_2 & & \\ & & & \ddots & (F_{T-2}^K)^{\mathrm{T}}\otimes E\\ (F_{T-1}^K)^{\mathrm{T}}\otimes E & & & & E\otimes A_{T-1}\end{bmatrix}\begin{bmatrix}\operatorname{vec}(P_0(k))\\\operatorname{vec}(P_1(k))\\\operatorname{vec}(P_2(k))\\\vdots\\\operatorname{vec}(P_{T-1}(k))\end{bmatrix}\right\|^2\\
\leq&\ \Pi\sum_{j=0}^{T-1}\left\|P_j(k)\right\|^2,
\end{aligned}\tag{17}$$

where,

$$\Pi=\left\|\begin{bmatrix}E\otimes A_0 & (F_0^K)^{\mathrm{T}}\otimes E & & & \\ & E\otimes A_1 & (F_1^K)^{\mathrm{T}}\otimes E & & \\ & & E\otimes A_2 & & \\ & & & \ddots & (F_{T-2}^K)^{\mathrm{T}}\otimes E\\ (F_{T-1}^K)^{\mathrm{T}}\otimes E & & & & E\otimes A_{T-1}\end{bmatrix}\right\|^2.$$

Define the following function:

$$J_1(k)=\frac{1}{2}\sum_{j=0}^{T-1}\left\|B_jG_j-A_jX_j(t+1)-X_{j+1}(t+1)F_j^{\mathrm{K}}\right\|^2,\tag{18}$$

$$J_2(k)=\frac{1}{2}\sum_{j=0}^{T-1}\left\|C_j^{\mathrm{T}}D_j-A_j^TY_j(t+1)-Y_{j+1}(t+1)F_j^{\mathrm{L}}\right\|^2,\tag{19}$$

By using the expression of $\alpha(k)$, $\beta(k)$, the following relations hold for $k\geq 0$:

$$J_1(k+1)=J_1(k)-\frac{1}{2}\alpha(k)\sum_{j=0}^{T-1}\operatorname{tr}\left[P_j^{\mathrm{T}}(k)R_j(k)\right]\ J_2(k+1)=J_2(k)-\frac{1}{2}\alpha(k)\sum_{j=0}^{T-1}\operatorname{tr}\left[H_j^{\mathrm{T}}(k)N_j(k)\right]$$

Then, one has

$$\begin{aligned}
&J_1(k+1)-J(k)\\
=&\ -\frac{1}{2}\frac{\left(\sum_{j=0}^{T-1}\operatorname{tr}\left[P_j^{\mathrm{T}}(k)R_j(k)\right]\right)^2}{\sum_{j=0}^{T-1}\left\|A_j^{\mathrm{T}}P_j(k)+P_{j+1}(k)F_j^K\right\|^2}\\
\leq&\ 0,
\end{aligned}\tag{20}$$

which means that $\{J(k)\}$ is a descent sequence, so that

$$J_1(k+1)\leq J(0)$$

holds for all $k\geq 0$. Then

$$\begin{aligned}
&\sum_{k=0}^{\infty}\left[J_1(k)-J_1(k+1)\right]=J_1(0)-\lim_{k\to\infty}J(k)\\
<&\ \infty.
\end{aligned}\tag{21}$$

In view of Equation (15), (17) and (21), the following deduction holds:

$$\begin{aligned}
\sum_{k\geq 0}\frac{\left(\sum_{j=0}^{T-1}\|R_j(k)\|^2\right)^2}{\sum_{j=0}^{T-1}\|P_j(k)\|^2} &= \sum_{k\geq 0}\frac{\left(\sum_{j=0}^{T-1}\operatorname{tr}\left[R_j^{\mathrm{T}}(k)P_j(k)\right]\right)^2}{\sum_{j=0}^{T-1}\|P_j(k)\|^2}\\
&\leq \pi\sum_{k\geq 0}\frac{\left(\sum_{j=0}^{T-1}\operatorname{tr}\left[R_j^{\mathrm{T}}(k)P_j(k)\right]\right)^2}{\sum_{j=0}^{T-1}\left\|A_jP_j(k)+P_{j+1}(k)F_j^K\right\|^2} = 2\pi(J(0)-\lim_{k\to\infty}J(k))\\
&< \infty.
\end{aligned}$$

Similar argument on $J_2(k)$ gives the conclusion

$$\sum_{k\geq 0}\frac{\left(\sum_{j=0}^{T-1}\|N_j(k)\|^2\right)^2}{\sum_{j=0}^{T-1}\|H_j(k)\|^2} < \infty.$$

To summarize, the Lemma 1 has been proved. ■

Based on the above lemma, the following conclusion could be drawn as:

Theorem 2 *Consider the completely observable and completely reachable periodic discrete-time linear system (1), the T-periodic matrix $L_j, j\in\overline{0,T-1}$, $K_j, j\in\overline{0,T-1}$, derived from Algorithm 1 is a solution of Problem 1.*

Proof. Let us first prove the convergence of matrix sequence $\{R_j(k)\}, j\in\overline{0,T-1}$ generated from Algorithm 1.

By Lemma 1 and the expressions of $P_j(k+1)$ in Algorithm 1, we have

$$\begin{aligned}
\sum_{j=0}^{T-1}\|P_j(k+1)\|^2 &= \sum_{j=0}^{T-1}\left\|-R_j(k+1)+\frac{\sum_{j=0}^{T-1}\|R_j(k+1)\|^2}{\sum_{j=0}^{T-1}\|R_j(k)\|^2}P_j(k)\right\|^2\\
&= \left(\frac{\sum_{j=0}^{T-1}\|R_j(k+1)\|^2}{\sum_{j=0}^{T-1}\|R_j(k)\|^2}\right)^2\sum_{j=0}^{T-1}\|P_j(k)\|^2+\sum_{j=0}^{T-1}\|R_j(k+1)\|^2. \qquad (22)
\end{aligned}$$

Equation (22) can be written as

$$t(k+1) = t(k)+\frac{1}{\sum_{j=0}^{T-1}\|R_j(k+1)\|^2} \qquad (23)$$

equivalently, where

$$t(k) = \frac{\sum_{j=0}^{T-1}\|P_j(k)\|^2}{\left(\sum_{j=0}^{T-1}\|R_j(k)\|^2\right)^2}.$$

Assume that

$$\lim_{k\to\infty}\sum_{j=0}^{T-1}\|R_j(k)\|^2\neq 0, \qquad (24)$$

which implies that there exists a constant $\delta>0$ such that

$$\sum_{j=0}^{T-1}\|R_j(k)\|^2\geq\delta$$

for all $k\geq 0$. It follows from (23) and (24) that

$$t(k+1)\leq t(k)+\frac{1}{\delta}\leq\cdots\leq t(0)+\frac{k+1}{\delta},$$

which means

$$\frac{1}{t(k+1)} \geq \frac{\delta}{\delta t(0)+k+1}.$$

So we have

$$\sum_{k=1}^{\infty} \frac{1}{t(k)} \geq \sum_{k=1}^{\infty} \frac{\delta}{\delta t(0)+k+1} = \infty.$$

However, according to Equation (16) that

$$\sum_{j=1}^{\infty} \frac{1}{t(j)} < \infty.$$

This gives a contradiction. Thus, there holds

$$\lim_{k\to\infty} \sum_{j=0}^{T-1} \|R_j(k)\|^2 = 0,$$

Similarity, we have

$$\lim_{k\to\infty} \sum_{j=0}^{T-1} \|R_j(k)\|^2 = 0,$$

which indicates that the matrix sequence $\{X_j(k)\}$, $\{Y_j(k)\}$, $j \in \overline{0, T-1}$, generated by Algorithm 1 are convergent to matrices $\{X_j\}$, $\{Y_j\}$, $j \in \overline{0, T-1}$, which are respectively the solutions to the two periodic Sylvester equations (12) and (13). According to the poles assignment theory as previously mentioned, matrix L_j, K_j derived from Algorithm 1 are solutions to Problem 1. ■

3.1 Minimum norm and robust consideration

In this section, we will consider robust poles assignment problem raised in problem 2. In previous work, we have discussed the sensitivity of the closed-loop LDP systems with respect to parameter uncertainties. Here, we revisit it in the following lemma.

Lemma 2 *[17] Let $\Psi = A(T-1)A(T-2)\cdots A(0) \in \mathbb{R}^{n\times n}$ be diagonalizable and $Q \in \mathbb{C}^{n\times n}$ be a nonsingular matrix such that $\Psi = Q^{-1}\Lambda Q \in \mathbb{R}^{n\times n}$, where $\Lambda = \operatorname{diag}\{\lambda_1, \lambda_2, \cdots, \lambda_n\}$ is the Jordan canonical form of matrix Ψ. For a real scalar $\varepsilon > 0$, $\Delta_i(\varepsilon) \in \mathbb{R}^{n\times n}, i \in \overline{0, T-1}$, are matrix functions of ε satisfying*

$$\lim_{\varepsilon\to 0^+} \frac{\Delta_i(\varepsilon)}{\varepsilon} = \Delta_i,$$

where $\Delta_i \in \mathbb{R}^{n\times n}, i \in \overline{0, T-1}$ are constant matrices. Then for any eigenvalue λ of matrix

$$\Psi(\varepsilon) = \left(A(T-1) + \Delta_{T-1}(\varepsilon)\right)\left(A(T-2) + \Delta_{T-2}(\varepsilon)\right)\cdots\left(A(0) + \Delta_0(\varepsilon)\right),$$

the following relation holds:

$$\min_i\{|\lambda_i - \lambda|\} \leq \varepsilon n \kappa_{\mathrm{F}}(Q) \left(\sum_{i=0}^{T-1} \|A(i)\|_{\mathrm{F}}^{T-1}\right) \max_i\{\|\Delta_i\|_{\mathrm{F}}\} + \mathrm{O}(\varepsilon^2). \tag{25}$$

According to Lemma 2, combining the Algorithm 1, one could take the robust performance index of problem 2 as

$$J(G_j, D_j) = \kappa_{\mathrm{F}}(X_0) \sum_{j=0}^{T-1} \|A_j + B_j K_j\|_{\mathrm{F}}^{T-1} + \kappa_{\mathrm{F}}(Y_0) \sum_{j=0}^{T-1} \left\|A_j^{\mathrm{T}} + C_j^{\mathrm{T}} L_j^{\mathrm{T}}\right\|_{\mathrm{F}}^{T-1} \tag{26}$$

Based on the above discussion, the algorithm for robust stabilization based on observer design for LDP systems can be presented as follows.

Algorithm 2 *(Robust stabilization based on periodic observer)*

1. *Perform the operations of step 1-4 of Algorithm 1.*
2. *Based on gradient-based search methods and the index (26), solve the optimization problem*

$$\text{Minimize } J(G_j, D_j),$$

and denote the optimal decision matrix by $G_j^{\text{opt}}, D_j^{\text{opt}}, j \in \overline{0, T-1}$.

3. *Substituting* $G_j^{\text{opt}}, D_j^{\text{opt}}$ *into steps 2-4 of algorithm 1 gives optimization matrices* $X_j^{\text{opt}}, Y_j^{\text{opt}}$.
4. *The robust controller and observer gains can be obtained as*

$$K_j^{\text{opt}} = G_j^{\text{opt}}(X_j^{\text{opt}})^{-1},\ L_j^{\text{opt}} = \left(D_j^{\text{opt}}(Y_j^{\text{opt}})^{-1}\right)^{\mathrm{T}}, j \in \overline{0, T-1}.$$

4 A Numerical Example

Consider LDP system (1) with parameters as follows:

$$\begin{aligned}
A(0) &= \begin{bmatrix} 1 & 2 \\ -2 & 3 \end{bmatrix},\ A(1) = \begin{bmatrix} -1 & 2 \\ 3 & -1 \end{bmatrix},\ A(2) = \begin{bmatrix} -2 & 1 \\ 1 & 3 \end{bmatrix} \\
B(0) &= \begin{bmatrix} -1 \\ 1 \end{bmatrix},\ B(1) = \begin{bmatrix} -1 \\ 1 \end{bmatrix},\ B(2) = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \\
C(0) &= \begin{bmatrix} 2 \\ -1 \end{bmatrix},\ C(1) = \begin{bmatrix} -1 \\ -1 \end{bmatrix},\ C(2) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}
\end{aligned}$$

This is a diverging system and it is easy to prove that the system is completely reachable and completely observable. Hence, we can claim that it can be stabilized by a periodic state feedback law based on a full-dimensional state observer. Without loss of generality, let the pole set the system (3) and (5) be $\Gamma_1 = \{-0.3, 0.3\}$ and $\Gamma_2 = \{-0.4, 0.4\}$, respectively.

According to algorithm 1, by choosing parameter matrices G and D randomly, we obtain a group of solutions as follows:

$$\left\{\begin{aligned} K_0^{\text{rand}} &= \begin{bmatrix} 1.7699 & -1.8268 \end{bmatrix} \\ K_1^{\text{rand}} &= \begin{bmatrix} -1.9615 & -2.3782 \end{bmatrix} \\ K_2^{\text{rand}} &= \begin{bmatrix} -1.1669 & -0.8084 \end{bmatrix} \end{aligned}\right.,\ \left\{\begin{aligned} L_0^{\text{rand}} &= \begin{bmatrix} -2.5762 & -1.4737 \end{bmatrix}^{\mathrm{T}} \\ L_1^{\text{rand}} &= \begin{bmatrix} 0.1217 & 2.0509 \end{bmatrix}^{\mathrm{T}} \\ L_2^{\text{rand}} &= \begin{bmatrix} -1.1765 & -1.6305 \end{bmatrix}^{\mathrm{T}} \end{aligned}\right.$$

Furthermore, employing the robust stabilization algorithm 2, we obtain a group of solution as follows:

$$\left\{\begin{aligned} K_0^{\text{robu}} &= \begin{bmatrix} 1.8432 & -3.5251 \end{bmatrix} \\ K_1^{\text{robu}} &= \begin{bmatrix} -3.1085 & 1.4631 \end{bmatrix} \\ K_2^{\text{robu}} &= \begin{bmatrix} -1.1128 & -2.4826 \end{bmatrix} \end{aligned}\right.,\ \left\{\begin{aligned} L_0^{\text{robu}} &= \begin{bmatrix} -0.6456 & 0.9869 \end{bmatrix}^{\mathrm{T}} \\ L_1^{\text{robu}} &= \begin{bmatrix} 0.3933 & 1.3929 \end{bmatrix}^{\mathrm{T}} \\ L_2^{\text{robu}} &= \begin{bmatrix} -1.0894 & -1.7176 \end{bmatrix}^{\mathrm{T}} \end{aligned}\right.$$

Let discrete reference input $v(t) = 0.1\sin(\frac{\pi}{2} + t)$ and the initial values of state and the observer state be $x_0 = \begin{bmatrix} -1 & 1 \end{bmatrix}^{\mathrm{T}}$, $\hat{x}_0 = \begin{bmatrix} 0 & 0 \end{bmatrix}^{\mathrm{T}}$. We depict the trajectory of state variable x for the original system (1), state variable x and its estimated state of system (6) under $(K_{\text{rand}}, L_{\text{rand}})$, state variable x and its estimated state of system (6) under $(K_{\text{robu}}, L_{\text{robu}})$ in Fig.1 respectively, where the red line denote the histories of xL and the green line denote the histories the observed state $\hat{x}$. From the simulation results, we can see the good performance of the controller and observer generated by the proposed algorithm.

5 Conclusion

A stabilizing controller design method for LDP systems based on periodic full-dimensional state observer is introduced in this paper. As similar with linear time variant systems, the periodic state feedback controller

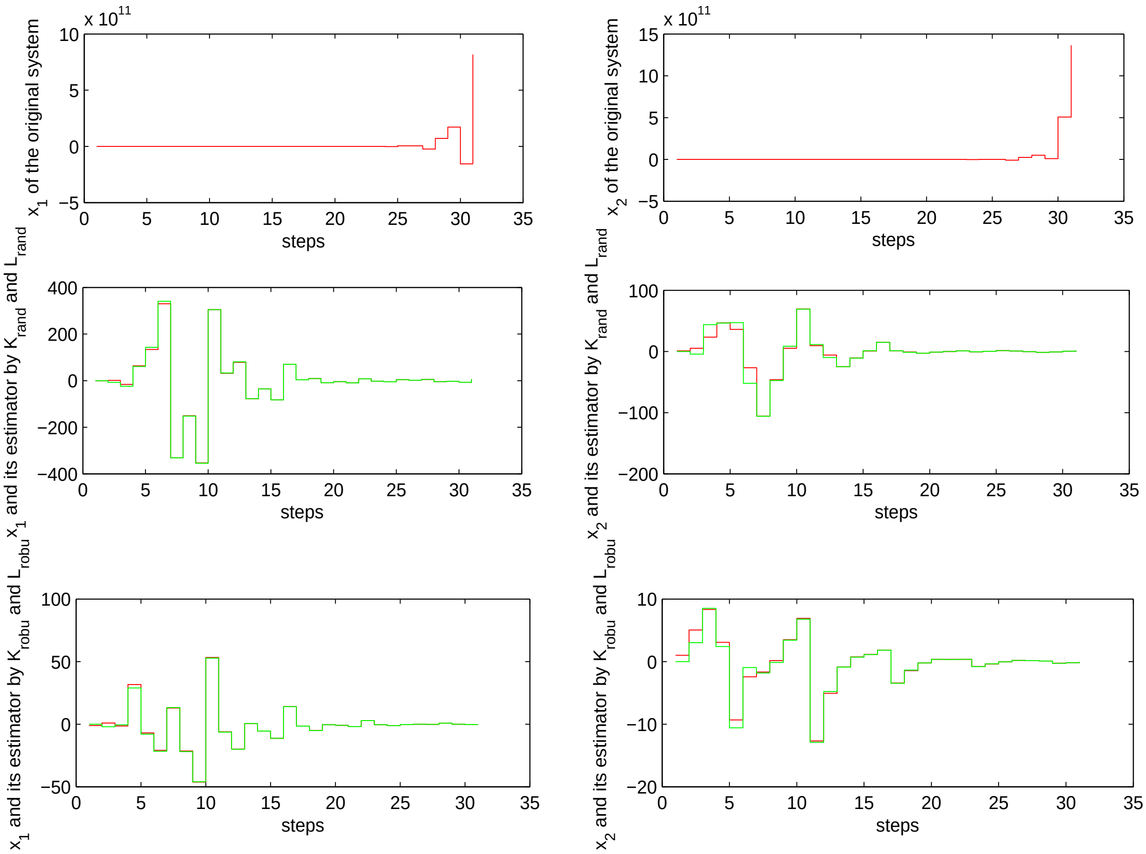


Figure 1: Comparison of state and the observed state under different cases

and periodic observer are designed separately, based on the periodic poles assignment technique. An iterative algorithm is presented to generate periodic observer gains and periodic controller gains simultaneously. In addition, robust stabilization problem is also discussed in this paper, and the corresponding algorithm is derived. The effectiveness of the proposed algorithms are shown by simulation results on an illustrate example.

References

[1] Wang L Y, Xu G, Yin G. State reconstruction for linear time-invariant systems with binary-valued output observations[J]. Systems & Control Letters, 2008, 57(11): 958-963.

[2] Wang L Y, Li C, Yin G G, et al. State observability and observers of linear-time-invariant systems under irregular sampling and sensor limitations[J]. IEEE Transactions on Automatic Control, 2011, 56(11): 2639-2654.

[3] S. P. Xu, S. R. Wang, X. H. Su, The Research of Direct Torque Control Based on Full-Dimensional State Observer[J]. Advanced Materials Research, 2012, 433-440:1576-1581.

[4] Wen G L, Xu D. Observer-based control for full-state projective synchronization of a general class of chaotic maps in any dimension[J]. Physics Letters A, 2004, 333(5-6): 420-425.

[5] Wenli L, Leiting Z, Kan D. Performance analysis of re-adhesion optimization control based on full-dimension state observer[J]. Procedia Engineering, 2011, 23: 531-536.

[6] Angulo M T, Moreno J A, Fridman L. On functional observers for linear systems with unknown inputs and hosm differentiators[J]. Journal of the Franklin Institute, 2014, 351(4): 1982-1994.

[7] Zhang P, Ding S X, Wang G Z, et al. Fault detection of linear discrete-time periodic systems[J]. IEEE Transactions on Automatic Control, 2005, 50(2): 239-244.

[8] Tadeo F, Rami M A. Selection of time-after-injection in bone scanning using compartmental observers[C]//World Congress on Engineering. 2010.

[9] Arzelier D, Peaucelle D, Farges C, et al. Robust analysis and synthesis of linear polytopic discrete-time periodic systems via LMIs[C]//Decision and Control, 2005 and 2005 European Control Conference. CDC-ECC'05. 44th IEEE Conference on. IEEE, 2005: 5734-5739.

[10] Lavaei J, Sojoudi S, Aghdam A G. Pole assignment with improved control performance by means of periodic feedback[J]. IEEE Transactions on Automatic Control, 2010, 55(1): 248-252.

[11] Varga A. Robust and minimum norm pole assignment with periodic state feedback[J]. IEEE Transactions on Automatic Control, 2000, 45(5): 1017-1022.

[12] Zhou B, Zheng W X, Duan G R. Stability and stabilization of discrete-time periodic linear systems with actuator saturation[J]. Automatica, 2011, 47(8): 1813-1820.

[13] Zhou B, Duan G R. Periodic Lyapunov equation based approaches to the stabilization of continuous-time periodic linear systems[J]. IEEE Transactions on Automatic Control, 2012, 57(8): 2139-2146.

[14] De Souza C E, Trofino A. An LMI approach to stabilization of linear discrete-time periodic systems[J]. International Journal of Control, 2000, 73(8): 696-703.

[15] Colaneri P. Output stabilization via pole placement of discrete-time linear periodic systems[J]. IEEE Transactions on Automatic Control, 1991, 36(6): 739-742.

[16] Lv L L, Zhang L. Robust Stabilization Based on Periodic Observers for LDP Systems[J]. Journal of Computational Analysis & Applications, 2016, 20(3): 487-498.

[17] Lv L, Duan G, Zhou B. Parametric pole assignment and robust pole assignment for discrete-time linear periodic systems[J]. SIAM Journal on Control and Optimization, 2010, 48(6): 3975-3996.

SUBORDINATION AND SUPERORDINATION PROPERTIES FOR CERTAIN FAMILY OF INTEGRAL OPERATORS ASSOCIATED WITH MULTIVALENT FUNCTIONS

M. K. AOUF, H. M. ZAYED, AND N. E. CHO

Abstract. The object of the present paper is to obtain subordination, superordination and sandwich-type results related to a certain family of integral operators defined on the space of multivalent functions in the open unit disk. Also we point out relevant connections of the results presented here with those obtained in earlier.

Keywords and phrases: $p-$valent function, differential subordination, superordination, subordination chain, integral operator.

2010 Mathematics Subject Classification: 30C45, 30C50.

1. Introduction

Let $\mathcal{H}=\mathcal{H}(\mathbb{U})$ be the class of functions analytic in $\mathbb{U}=\{z\in\mathbb{C}:|z|<1\}$ and $\mathcal{H}[a,n]$ be the subclass of $\mathcal{H}(\mathbb{U})$ consisting of functions of the form $f(z)=a+a_nz^n+a_{n+1}z^{n+1}+\ldots$ and denote $\mathcal{H}_0:=\mathcal{H}[0,1]$ and $\mathcal{H}:=\mathcal{H}[1,1]$.

Let $\mathcal{P}$ denote the class of functions

$$\mathcal{P}=\{h\in\mathcal{H}[0,1]:h(z)h'(z)\neq 0,\ z\in\mathbb{U}^*:=\mathbb{U}\setminus\{0\}\},\tag{1}$$

and $\mathcal{A}(p)$ be the class of all functions of the form

$$f(z)=z^p+\sum_{k=1}^{\infty}a_{k+p}z^{k+p}\ (p\in\mathbb{N}=\{1,2,...\}),\tag{2}$$

which are analytic in $\mathbb{U}$. We note that $\mathcal{A}(1)=\mathcal{A}$.

For $f,g\in\mathcal{H}(\mathbb{U})$, the function $f(z)$ is said to be subordinate to $g(z)$ or $g(z)$ is superordinate to $f(z)$, if there exists a function $\omega(z)$ analytic in $\mathbb{U}$ with $\omega(0)=0$ and $|\omega(z)|<1$ $(z\in\mathbb{U})$, such that $f(z)=g(\omega(z))$. In such a case we write $f(z)\prec g(z)$. If g is univalent, then $f(z)\prec g(z)$ if and only if $f(0)=g(0)$ and $f(\mathbb{U})\subset g(\mathbb{U})$ (see [14, 15]).

Let $\phi:\mathbb{C}^2\times\mathbb{U}\to\mathbb{C}$ and $h(z)$ be univalent in $\mathbb{U}$. If $\mathfrak{p}(z)$ is analytic in $\mathbb{U}$ and satisfies the first order differential subordination:

$$\phi(\mathfrak{p}(z),z\mathfrak{p}'(z);z)\prec h(z),\tag{3}$$

then $\mathfrak{p}(z)$ is a solution of the differential subordination (3). The univalent function $\mathfrak{q}(z)$ is called a dominant of the solutions of the differential subordination (3) if $\mathfrak{p}(z)\prec\mathfrak{q}(z)$

for all $\mathfrak{p}(z)$ satisfying (3). A univalent dominant $\tilde{\mathfrak{q}}$ that satisfies $\tilde{\mathfrak{q}} \prec \mathfrak{q}$ for all dominants of (3) is called the best dominant. If $\mathfrak{p}(z)$ and $\phi(\mathfrak{p}(z), z\mathfrak{p}'(z); z)$ are univalent in $\mathbb{U}$ and if $\mathfrak{p}(z)$ satisfies the first order differential superordination:

$$h(z) \prec \phi(\mathfrak{p}(z), z\mathfrak{p}'(z); z), \tag{4}$$

then $\mathfrak{p}(z)$ is a solution of the differential superordination (4). An analytic function $\mathfrak{q}(z)$ is called a subordinant of the solutions of the differential superordination (4) if $\mathfrak{q}(z) \prec \mathfrak{p}(z)$ for all $\mathfrak{p}(z)$ satisfying (4). A univalent subordinant $\tilde{\mathfrak{q}}$ that satisfies $\mathfrak{q} \prec \tilde{\mathfrak{q}}$ for all subordinants of (4) is called the best subordinant (see [14, 15]).

For the functions $f_i(z) \in \mathcal{A}(p)$ $(p \in \mathbb{N},\ i = 2, 3, ..., m)$, $h(z) \in \mathcal{P}$ and the parameters $\beta,\ \alpha_1,\ \alpha_2, ..., \alpha_m \in \mathbb{C}$ with $\beta \neq 0$, we introduce the integral operator $I^{p,m}_{h;\alpha_1,\alpha_i,\beta} : \mathcal{A}(p) \to \mathcal{A}(p)$ as follows:

$$I^{p,m}_{h;\alpha_1,\alpha_i,\beta}[f_i](z) = \left[\frac{\alpha_1 + p\sum\limits_{i=2}^{m} \alpha_i}{z^{\alpha_1 - p\beta + p\sum\limits_{i=2}^{m}\alpha_i}} \int\limits_0^z \left(\prod_{i=2}^{m} f_i^{\alpha_i}(t) \right) h^{\alpha_1 - 1}(t) h'(t) dt \right]^{\frac{1}{\beta}}. \tag{5}$$

(All powers are principal ones).

We note the next special cases of the above defined integral operator:

(i) For $p = 1,\ m = 2,\ \alpha_1 = \gamma,\ \alpha_2 = \beta$ and $f_2(t) = f(t)$, we obtain

$$I_{h;\beta,\gamma}(f)(z) = \left(\frac{\beta + \gamma}{z^{\gamma}} \int\limits_0^z f^{\beta}(t) h^{\gamma-1}(t) h'(t) dt \right)^{\frac{1}{\beta}},$$

where the operator $I_{h;\beta,\gamma}$ was introduced and studied by Cho and Bulboacă [6].

(ii) For $p = 1,\ m = 2,\ \alpha_1 = \gamma,\ \alpha_2 = \beta,\ f_2(t) = f(t)$ and $h(t) = t$, we obtain

$$I_{\beta,\gamma}(f)(z) = \left(\frac{\beta + \gamma}{z^{\gamma}} \int\limits_0^z f^{\beta}(t) t^{\gamma-1}(t) dt \right)^{\frac{1}{\beta}},$$

where the operator $I_{\beta,\gamma}$ was introduced by Miller *et al.* [16] and studied by Bulboacă [3–5].

To prove our results, we need the following definitions and lemmas.

Definition 1. *[14] Denote by $\mathcal{Q}$ the set of all functions $q(z)$ that are analytic and injective on $\overline{\mathbb{U}} \backslash E(q)$ where*

$$E(q) = \left\{ \zeta \in \partial\mathbb{U} : \lim_{z \to \zeta} q(z) = \infty \right\}$$

and are such that $q'(\zeta) \neq 0$ for $\zeta \in \partial\mathbb{U} \backslash E(q)$. Further, denote by $\mathcal{Q}(a)$ the subclass of $\mathcal{Q}$ for which $q(0) = a$.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 27, NO.5, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC

Definition 2. *[14] A function $L(z,t)$ $(z \in \mathbb{U},\ t \geq 0)$ is said to be a subordination chain (or Löwner chain) if $L(.,t)$ is analytic and univalent in $\mathbb{U}$ for all $t \geq 0$, $L(z,.)$ is continuously differentiable on $[0,\infty)$ for all $z \in \mathbb{U}$ and $L(z,s) \prec L(z,t)$ for all $0 \leq s \leq t$.*

Lemma 1. *[17] The function $L(z,t) : \mathbb{U} \times [0,\infty) \longrightarrow \mathbb{C}$ of the form*

$$L(z,t) = a_1(t)\, z + a_2(t)\, z^2 + \ldots \ \ (a_1(t) \neq 0;\ t \geq 0)$$

and $\lim\limits_{t\to\infty} |a_1(t)| = \infty$ is a subordination chain if and only if

$$\operatorname{Re}\left\{ \frac{z\dfrac{\partial L(z,t)}{\partial z}}{\dfrac{\partial L(z,t)}{\partial t}} \right\} > 0 \ \ (z \in \mathbb{U};\ t \geq 0),$$

and

$$|L(z,t)| \leq K_0 |a_1(t)| \ \ (|z| < r_0 < 1;\ t \geq 0),$$

for some positive constants K_0 and r_0.

Lemma 2. *[10] Suppose that the function $\mathcal{H} : \mathbb{C}^2 \to \mathbb{C}$ satisfies the condition*

$$\operatorname{Re}\{\mathbf{H}(\mathrm{is};\mathrm{t})\} \leq 0$$

for all real s and for all $t \leq -n(1+s^2)/2$, $n \in \mathbb{N}$. If the function $p(z) = 1 + p_n z^n + p_{n+1} z^{n+1} + \ldots$ is analytic in $\mathbb{U}$ and

$$\operatorname{Re}\{\mathbf{H}(p(z); zp'(z))\} > 0 \ \ (z \in \mathbb{U}),$$

then $\operatorname{Re}\{p(z)\} > 0$ for $z \in \mathbb{U}$.

Lemma 3. *[11] Let $\kappa, \gamma \in \mathbb{C}$ with $\kappa \neq 0$ and let $h \in \mathcal{H}(\mathbb{U})$ with $h(0) = c$. If $\operatorname{Re}\{\kappa h(z) + \gamma\} > 0$ $(z \in \mathbb{U})$, then the solution of the following differential equation:*

$$\mathfrak{q}(z) + \frac{z\mathfrak{q}'(z)}{\kappa \mathfrak{q}(z) + \gamma} = h(z) \ \ (z \in \mathbb{U};\ \mathfrak{q}(0) = c)$$

is analytic in $\mathbb{U}$ and satisfies $\operatorname{Re}\{\kappa \mathfrak{q}(z) + \gamma\} > 0$ for $z \in \mathbb{U}$.

Lemma 4. *[14] Let $\mathfrak{p} \in \mathcal{Q}(a)$ and let $q(z) = a + a_n z^n + a_{n+1} z^{n+1} + \ldots$ be analytic in $\mathbb{U}$ with $q(z) \neq a$ and $n \geq 1$. If q is not subordinate to $\mathfrak{p}$, then there exists two points $z_0 = r_0 e^{i\theta} \in \mathbb{U}$ and $\zeta_0 \in \partial\mathbb{U} \backslash E(q)$ such that*

$$q(\mathbb{U}_{r_0}) \subset \mathfrak{p}(\mathbb{U}),\ q(z_0) = \mathfrak{p}(\zeta_0) \ \text{and}\ z_0 \mathfrak{p}'(z_0) = m\zeta_0 q'(\zeta_0) \ \ (m \geq n).$$

Lemma 5. *[15] Let $q \in \mathcal{H}[a;1]$ and $\varphi : \mathbb{C}^2 \to \mathbb{C}$. Also set $\varphi(q(z), zq'(z)) = h(z)$. If $L(z,t) = \varphi(q(z), tzq'(z))$ is a subordination chain and $q \in H[a,1] \cap \mathcal{Q}(a)$, then*

$$h(z) \prec \varphi(q(z), zq'(z)),$$

implies that $q(z) \prec p(z)$. Furthermore, if $\varphi(q(z), zq'(z)) = h(z)$ has a univalent solution $q \in \mathcal{Q}(a)$, then q is the best subordinant.

Let $c \in \mathbb{C}$ with $\mathrm{Re}\,(c) > 0$ and

$$N = N(c) = \frac{|c|\sqrt{1+2\mathrm{Re}\,(c)} + Im\,(c)}{\mathrm{Re}\,(c)}.$$

If $R = R(z) = \frac{2Nz}{1-z^2}$ is a univalent function and $b = R^{-1}(c)$, then the open door function $R_c(z)$ is defined by

$$R_c(z) = R\left(\frac{z+b}{1+\bar{b}z}\right) \ (z \in \mathbb{U}).$$

The function R_c is univalent in $\mathbb{U}$, $R_c(0) = c$ and $R_c(\mathbb{U}) = R\,(\mathbb{U})$ is the complex plane slit along the half lines $\mathrm{Re}\,(w) = 0,\ \mathrm{Im}\,(\mathrm{w}) \geq \mathrm{N}$ and $\mathrm{Re}\,(w) = 0,\ \mathrm{Im}\,(\mathrm{w}) \leq -\mathrm{N}$.

Lemma 6. *(Integral Existence Theorem [12–14]) Let $\phi, \Phi \in H$ with $\phi(z) \neq 0,\ \Phi(z) \neq 0$ for $z \in \mathbb{U}$. Let $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ with $\beta \neq 0,\ \alpha+\delta = \beta+\gamma$ and $\mathrm{Re}\,(\alpha+\delta) > 0$. If the function $g(z) \in \mathcal{A}$ and*

$$\alpha\frac{zg'(z)}{g(z)} + \frac{z\phi'(z)}{\phi(z)} + \delta \prec R_{\alpha+\delta}(z),$$

then

$$G(z) = \left(\frac{\beta+\gamma}{z^\gamma\Phi(z)}\int_0^z g^\alpha(t)\phi(t)t^{\delta-1}(t)dt\right)^{\frac{1}{\beta}} \in \mathcal{A},$$

$\frac{G(z)}{z} \neq 0 \ \ (z \in \mathbb{U})$ *and*

$$\mathrm{Re}\left(\beta\frac{zG'(z)}{G(z)} + \frac{z\Phi'(z)}{\Phi(z)} + \gamma\right) > 0 \ \ (z \in \mathbb{U})\,.$$

(All powers are principal ones).

Indeed, Lemma 6 is extended for p-valent functions as follows:

Lemma 7. *[18] (see also [1]) Let $p \in \mathbb{N},\ \phi, \Phi \in H$ with $\phi(z) \neq 0,\ \Phi(z) \neq 0$ for $z \in \mathbb{U}$. Let $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ with $\beta \neq 0,\ p\alpha+\delta = p\beta+\gamma$ and $\mathrm{Re}\,(p\alpha+\delta) > 0$. If the function $f(z) \in \mathcal{A}(p)$ and*

$$\mathcal{A}_{p,\alpha,\delta} = \left\{f(z) \in \mathcal{A}(p) : \alpha\frac{zf'(z)}{f(z)} + \frac{z\phi'(z)}{\phi(z)} + \delta \prec R_{p\alpha+\delta}(z)\right\},$$

then

$$F(z) = \left(\frac{p\beta+\gamma}{z^\gamma\Phi(z)}\int_0^z f^\alpha(t)\phi(t)t^{\delta-1}dt\right)^{\frac{1}{\beta}} = z^p + \ldots \in \mathcal{A}(p),$$

$\frac{F(z)}{z^p} \neq 0 \;\; (z \in \mathbb{U})$ *and*

$$\operatorname{Re}\left(\beta \frac{zF'(z)}{F(z)} + \frac{z\Phi'(z)}{\Phi(z)} + \gamma\right) > 0 \;\; (z \in \mathbb{U}).$$

(All powers are principal ones).

2. Main results

Unless otherwise mentioned, we assume throughout this paper that $h \in \mathcal{P}$, $\beta,\ \alpha_1,\ \alpha_2, ..., \alpha_m \in \mathbb{C}$ with $\beta \neq 0$ such that $\operatorname{Re}\left(\alpha_1 + p\sum\limits_{i=2}^{m} \alpha_i\right) > 0$ and all powers are principal ones.

Using similar arguments to Lemma 7, we obtain the following lemma.

Lemma 8. *If $f_i \in \mathcal{A}_{p,h;\alpha_1,\alpha_i}$ $(i = 2, 3, \cdots, m)$, where*

$$\begin{aligned} \mathcal{A}_{p,h;\alpha_1,\alpha_i} \;\; = \;\; & \left\{ f_i(z) \in \mathcal{A}(p) : \sum_{i=2}^{m} \alpha_i \frac{zf_i'(z)}{f_i(z)} + 1 + \frac{zh''(z)}{h'(z)} \right. \\ & \left. + (\alpha_1 - 1)\frac{zh'(z)}{h(z)} \prec R_{\alpha_1 + p\sum\limits_{i=2}^{m}\alpha_i}(z) \right\}, \end{aligned} \tag{6}$$

then $I^{p,m}_{h;\alpha_1,\alpha_i,\beta}[f_i](z) \in \mathcal{A}(p)$, $\frac{I^{p,m}_{h;\alpha_1,\alpha_i,\beta}[f_i](z)}{z^p} \neq 0$ and

$$\operatorname{Re}\left[\beta \frac{z\left(I^{p,m}_{h;\alpha_1,\alpha_i,\beta}[f_i](z)\right)'}{I^{p,m}_{h;\alpha_1,\alpha_i,\beta}[f_i](z)} + \alpha_1 + p\sum_{i=2}^{m}\alpha_i - p\beta\right] > 0 \;\; (z \in \mathbb{U}),$$

where $I^{p,m}_{h;\alpha_1,\alpha_i,\beta}$ is the integral operator defined by (5).

Theorem 1. *Let $f_i, g_i \in \mathcal{A}_{p,h;\alpha_1,\alpha_i}$ $(i = 2, 3, \cdots, m)$ and*

$$\operatorname{Re}\left\{1 + \frac{z\phi''(z)}{\phi'(z)}\right\} > -\delta \tag{7}$$

$$\left(\phi(z) = z\prod_{i=2}^{m}\left(\frac{g_i(z)}{z^p}\right)^{\alpha_i}\left(\frac{h(z)}{z}\right)^{\alpha_1 - 1} h'(z);\;\; z \in \mathbb{U}\right),$$

where δ is given by

$$\delta = \frac{1 + |a|^2 - |1 - a^2|}{4\operatorname{Re}\{a\}} \quad \left(a = \alpha_1 + p\sum_{i=2}^{m}\alpha_i - 1,\ \operatorname{Re}\{a\} > 0\right). \tag{8}$$

Then the subordination condition:

$$z\prod_{i=2}^{m}\left(\frac{f_i(z)}{z^p}\right)^{\alpha_i}\left(\frac{h(z)}{z}\right)^{\alpha_1 - 1} h'(z) \prec \phi(z) \tag{9}$$

implies that

$$z\left(\frac{I^{p,m}_{h;\alpha_1,\alpha_i,\beta}[f_i](z)}{z^p}\right)^{\beta}\prec z\left(\frac{I^{p,m}_{h;\alpha_1,\alpha_i,\beta}[g_i](z)}{z^p}\right)^{\beta} \tag{10}$$

and the function $z\left(\frac{I^{p,m}_{h;\alpha_1,\alpha_i,\beta}[g_i](z)}{z^p}\right)^{\beta}$ *is the best dominant.*

Proof. Define the functions $\Psi(z)$ and $\Phi(z)$ in $\mathbb{U}$ by

$$\Psi(z)=z\left(\frac{I^{p,m}_{h;\alpha_1,\alpha_i,\beta}[f_i](z)}{z^p}\right)^{\beta}\text{ and }\Phi(z)=z\left(\frac{I^{p,m}_{h;\alpha_1,\alpha_i,\beta}[g_i](z)}{z^p}\right)^{\beta}\ \ (z\in\mathbb{U}). \tag{11}$$

From Lemma 8, it follows that these two functions are well defined. We first show that, if

$$q(z)=1+\frac{z\Phi''(z)}{\Phi'(z)}\ \ (z\in\mathbb{U}), \tag{12}$$

then

$$\operatorname{Re}\{q(z)\}>0\ \ (z\in\mathbb{U}).$$

From (5) and the definitions of the functions $\phi(z)$ and $\Phi(z)$, we obtain

$$\left(\alpha_1+p\sum_{i=2}^{m}\alpha_i\right)\phi(z)=z\Phi'(z)+\left(\alpha_1+p\sum_{i=2}^{m}\alpha_i-1\right)\Phi(z). \tag{13}$$

Hence, it follows that

$$1+\frac{z\phi''(z)}{\phi'(z)}=q(z)+\frac{zq'(z)}{q(z)+\alpha_1+p\sum\limits_{i=2}^{m}\alpha_i-1}=h(z)\ \ (z\in\mathbb{U}). \tag{14}$$

It follows from (7) and (14) that

$$\operatorname{Re}\left\{h(z)+\alpha_1+p\sum_{i=2}^{m}\alpha_i-1\right\}>0\ \ (z\in\mathbb{U}). \tag{15}$$

Moreover, by using Lemma 3, we conclude that the differential equation (14) has a solution $q(z)\in H(\mathbb{U})$ with $h(0)=q(0)=1.$ Let

$$\mathbf{H}(u,v)=u+\frac{v}{u+\alpha_1+p\sum\limits_{i=2}^{m}\alpha_i-1}+\delta,$$

where δ is given by (8). From (14) and (15), we obtain $\operatorname{Re}\{\mathbf{H}(q(z);zq'(z))\}>0\ (z\in\mathbb{U})$. To verify the condition

$$\operatorname{Re}\{\mathbf{H}(is;t)\}\leq 0\ \ \left(s\in\mathbb{R};\ t\leq-\frac{1+s^2}{2}\right), \tag{16}$$

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 27, NO.5, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC

we proceed as follows:

$$\operatorname{Re}\{\mathbf{H}(is;t)\} = \operatorname{Re}\left\{is + \frac{t}{is+a} + \delta\right\} = \delta + \frac{t\operatorname{Re}\{a\}}{|is+a|^2} \le -\frac{E_\delta(s)}{2|a+is|^2},$$

where

$$E_\delta(s) = (\operatorname{Re}\{a\} - 2\delta)s^2 - 4\delta(\operatorname{Ima})s + \left(\operatorname{Re}\{a\} - 2\delta|a|^2\right). \tag{17}$$

For δ given by (8), the coefficient of s^2 in the quadratic expression $E_\delta(s)$ given by (17) is positive or equal to zero and $E_\delta(s) \ge 0$. Thus, we see that $\operatorname{Re}\{\mathbf{H}(is;t)\} \le 0$ for all $s \in \mathbb{R}$ and $t \le -\frac{1+s^2}{2}$. Thus, by using Lemma 2, we conclude that

$$\operatorname{Re}\{q(z)\} > 0 \quad (z \in \mathbb{U}),$$

that is, that $\Phi(z)$ defined by (11) is convex (univalent) in $\mathbb{U}$. Next, we prove that the subordination condition (9) implies that

$$\Psi(z) \prec \Phi(z),$$

for $\Psi(z)$ and $\Phi(z)$ defined by (11). Without loss of generality, we assume that $\Phi(z)$ is analytic, univalent on $\overline{\mathbb{U}}$ and

$$\Phi'(\zeta) \neq 0 \quad (|\zeta| = 1).$$

If not, then we replace $\Psi(z)$ and $\Phi(z)$ by $\Psi(\rho z)$ and $\Phi(\rho z)$, respectively, with $0 < \rho < 1$. These new functions have the desired properties on $\overline{\mathbb{U}}$, so we can use them in the proof of our result and the result would follow by letting $\rho \to 1$. Consider the function $L(z,t)$ given by

$$L(z,t) = \left(1 - \frac{1}{\alpha_1 + p\sum_{i=2}^{m}\alpha_i}\right)\Phi(z) + \frac{(1+t)}{\alpha_1 + p\sum_{i=2}^{m}\alpha_i} z\Phi'(z) \quad (0 \le t < \infty;\ z \in \mathbb{U}). \tag{18}$$

We note that

$$\left.\frac{\partial L(z,t)}{\partial z}\right|_{z=0} = \left(1 + \frac{t}{\alpha_1 + p\sum_{i=2}^{m}\alpha_i}\right)\Phi'(0) \neq 0 \quad (0 \le t < \infty;\ z \in \mathbb{U}).$$

This show that the function

$$L(z,t) = a_1(t)z + a_2(t)z^2 + ...,$$

satisfy the conditions $\lim_{t\to\infty}|a_1(t)| = \infty$ and $a_1(t) \neq 0 \ (0 \le t < \infty)$. Further, we have

$$\operatorname{Re}\left\{\frac{z\frac{\partial L(z,t)}{\partial z}}{\frac{\partial L(z,t)}{\partial t}}\right\} = \operatorname{Re}\left\{\alpha_1 + p\sum_{i=2}^{m}\alpha_i - 1 + (1+t)\left(1 + \frac{z\Phi''(z)}{\Phi'(z)}\right)\right\} > 0$$

$$(0 \le t < \infty;\ z \in \mathbb{U}),$$

since $\Phi(z)$ is convex and $\operatorname{Re}\left\{\alpha_1 + p\sum\limits_{i=2}^{m}\alpha_i - 1\right\} > 0$, by using the well-known growth and distortion sharp inequalities for convex functions (see [8]), the second inequality of Lemma 1 is satisfied and so $L(z,t)$ is a subordination chain. It follows from the definition of subordination chain that

$$\phi(z) = \left(1 - \frac{1}{\alpha_1 + p\sum\limits_{i=2}^{m}\alpha_i}\right)\Phi(z) + \frac{1}{\alpha_1 + p\sum\limits_{i=2}^{m}\alpha_i} z\Phi'(z) = L(z,0)$$

and

$$L(z,0) \prec L(z,t) \;\; (0 \le t < \infty),$$

which implies that

$$L(\zeta,t) \notin L(\mathbb{U},0) = \phi(\mathbb{U}) \;\; (0 \le t < \infty;\; \zeta \in \partial\mathbb{U}). \tag{19}$$

If $\Psi(z)$ is not subordinate to $\Phi(z)$, by using Lemma 4, we know that there exist two points $z_0 \in \mathbb{U}$ and $\zeta_0 \in \partial\mathbb{U}$ such that

$$\Psi(z_0) = \Phi(\zeta_0) \;\text{ and }\; z_0\Psi'(z_0) = (1+t)\zeta_0\Phi'(\zeta_0) \;\; (0 \le t < \infty). \tag{20}$$

Hence, by using (10), (18), (20) and (8), we have

$$\begin{aligned} L(\zeta_0,t) &= \left(1 - \frac{1}{\alpha_1 + p\sum\limits_{i=2}^{m}\alpha_i}\right)\Phi(\zeta_0) + \frac{(1+t)}{\alpha_1 + p\sum\limits_{i=2}^{m}\alpha_i}\zeta_0\Phi'(\zeta_0) \\ &= \left(1 - \frac{1}{\alpha_1 + p\sum\limits_{i=2}^{m}\alpha_i}\right)\Psi(z_0) + \frac{1}{\alpha_1 + p\sum\limits_{i=2}^{m}\alpha_i} z_0\Psi'(z_0) \\ &= z_0\prod_{i=2}^{m}\left(\frac{f_i(z_0)}{z_0^p}\right)^{\alpha_i}\left(\frac{h(z_0)}{z_0}\right)^{\alpha_1-1} h'(z_0) \in \phi(\mathbb{U}). \end{aligned}$$

This contradicts (19). Thus, we deduce that $\Psi \prec \Phi$. Considering $\Psi = \Phi$, we see that the function Φ is the best dominant. This completes the proof of Theorem 1. ∎

We now derive the following superordination result.

Theorem 2. *Let* $f_i, g_i \in \mathcal{A}_{p,h;\alpha_1,\alpha_i}$ $(i = 2,3,\cdots,m)$ *and*

$$\operatorname{Re}\left\{1 + \frac{z\phi''(z)}{\phi'(z)}\right\} > -\delta$$

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 27, NO.5, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC

$$\left(\phi\left(z\right)=z\prod_{i=2}^{m}\left(\frac{g_i(z)}{z^p}\right)^{\alpha_i}\left(\frac{h(z)}{z}\right)^{\alpha_1-1}h'(z);\ z\in\mathbb{U}\right),$$

where δ is given by (8). If the function

$$z\prod_{i=2}^{m}\left(\frac{f_i(z)}{z^p}\right)^{\alpha_i}\left(\frac{h(z)}{z}\right)^{\alpha_1-1}h'(z)$$

is univalent in $\mathbb{U}$ and $z\left(\frac{I^{p,m}_{h;\alpha_1,\alpha_i,\beta}[f_i](z)}{z^p}\right)^{\beta}\in\mathcal{H}[0,1]\cap\mathcal{Q}$. Then the superordination condition

$$\phi\left(z\right)\prec z\prod_{i=2}^{m}\left(\frac{f_i(z)}{z^p}\right)^{\alpha_i}\left(\frac{h(z)}{z}\right)^{\alpha_1-1}h'(z) \tag{21}$$

implies that

$$z\left(\frac{I^{p,m}_{h;\alpha_1,\alpha_i,\beta}[g_i](z)}{z^p}\right)^{\beta}\prec z\left(\frac{I^{p,m}_{h;\alpha_1,\alpha_i,\beta}[f_i](z)}{z^p}\right)^{\beta} \tag{22}$$

and the function $z\left(\frac{I^{p,m}_{h;\alpha_1,\alpha_i,\beta}[g_i](z)}{z^p}\right)^{\beta}$ is the best subordinant.

Proof. Suppose that the functions $\Psi(z)$, $\Phi(z)$ and $q(z)$ are defined by (11) and (12), respectively. We will use similar method as in the proof of Theorem 1. As in Theorem 1, we have

$$\phi\left(z\right)=\left(1-\frac{1}{\alpha_1+p\sum\limits_{i=2}^{m}\alpha_i}\right)\Phi\left(z\right)+\frac{1}{\alpha_1+p\sum\limits_{i=2}^{m}\alpha_i}z\Phi'\left(z\right)=\varphi\left(G\left(z\right),zG'\left(z\right)\right)$$

and we obtain

$$\operatorname{Re}\left\{q\left(z\right)\right\}>0\ \ \left(z\in\mathbb{U}\right).$$

Next, to obtain the desired result, we show that $\Phi(z)\prec\Psi(z)$. For this, we suppose that the function

$$L\left(z,t\right)=\left(1-\frac{1}{\alpha_1+p\sum\limits_{i=2}^{m}\alpha_i}\right)\Phi\left(z\right)+\frac{t}{\alpha_1+p\sum\limits_{i=2}^{m}\alpha_i}z\Phi'\left(z\right)\ \ \left(0\leq t<\infty;\ z\in\mathbb{U}\right).$$

We note that

$$\left.\frac{\partial L\left(z,t\right)}{\partial z}\right|_{z=0}=\left(1-\frac{1}{\alpha_1+p\sum\limits_{i=2}^{m}\alpha_i}\right)\Phi'\left(0\right)\neq0\ \ \left(0\leq t<\infty;\ z\in\mathbb{U}\right).$$

This show that the function

$$L(z,t) = a_1(t)\,z + a_2(t)z^2 + ...$$

satisfy the conditions $\lim_{t\to\infty}|a_1(t)| = \infty$ and $a_1(t) \neq 0 \;(0 \leq t < \infty)$. Further, we have

$$\operatorname{Re}\left\{\frac{z\dfrac{\partial L(z,t)}{\partial z}}{\dfrac{\partial L(z,t)}{\partial t}}\right\} = \operatorname{Re}\left\{\alpha_1 + p\sum_{i=2}^{m}\alpha_i - 1 + t\left(1+\frac{z\Phi''(z)}{\Phi'(z)}\right)\right\} > 0$$

$$(0 \leq t < \infty;\; z \in \mathbb{U}),$$

since $\Phi(z)$ is convex and $\operatorname{Re}\left\{\alpha_1 + p\sum_{i=2}^{m}\alpha_i - 1\right\} > 0$. By using the well-known growth and distortion sharp inequalities for convex functions (see [8]), the second inequality of Lemma 1 is satisfied and so $L(z,t)$ is a subordination chain. Therefore, by using Lemma 5, we conclude that the superordination condition (21) must imply the superordination given by (22). Moreover, since the differential equation has a univalent solution Φ, it is the best subordinant. This completes the proof of Theorem 2. ∎

Combining Theorems 1 and 2, the following sandwich-type results are derived.

Theorem 3. *Let* $f, g_j \in \mathcal{A}_{p,h;\alpha_1,\alpha_i}$, $(i = 2,3,\cdots,m;\; j = 1,\,2)$ *and*

$$Re\left\{1+\frac{z\phi_j''(z)}{\phi_j'(z)}\right\} > -\delta$$

$$\left(\phi_j(z) = z\prod_{i=2}^{m}\left(\frac{g_{i,j}(z)}{z^p}\right)^{\alpha_i}\left(\frac{h(z)}{z}\right)^{\alpha_1-1}h'(z);\; z\in\mathbb{U}\right),$$

where δ *is given by (8). If the function*

$$z\prod_{i=2}^{m}\left(\frac{f_i(z)}{z^p}\right)^{\alpha_i}\left(\frac{h(z)}{z}\right)^{\alpha_1-1}h'(z)$$

is univalent in $\mathbb{U}$ *and* $z\left(\frac{I^{p,m}_{h;\alpha_1,\alpha_i,\beta}[f_i](z)}{z^p}\right)^{\beta} \in \mathcal{H}[0,1]\cap\mathcal{Q}$. *Then*

$$\phi_1(z) \prec z\prod_{i=2}^{m}\left(\frac{f_i(z)}{z^p}\right)^{\alpha_i}\left(\frac{h(z)}{z}\right)^{\alpha_1-1}h'(z) \prec \phi_2(z)$$

implies that

$$z\left(\frac{I^{p,m}_{h;\alpha_1,\alpha_i,\beta}[g_{i,1}](z)}{z^p}\right)^{\beta} \prec z\left(\frac{I^{p,m}_{h;\alpha_1,\alpha_i,\beta}[f_i](z)}{z^p}\right)^{\beta} \prec z\left(\frac{I^{p,m}_{h;\alpha_1,\alpha_i,\beta}[g_{i,2}](z)}{z^p}\right)^{\beta}.$$

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 27, NO.5, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC

Moreover, the functions $z\left(\frac{I^{p,m}_{h;\alpha_1,\alpha_i,\beta}[g_{i,1}](z)}{z^p}\right)^\beta$ *and* $z\left(\frac{I^{p,m}_{h;\alpha_1,\alpha_i,\beta}[g_{i,2}](z)}{z^p}\right)^\beta$ *are, respectively, the best subordinant and the best dominant.*

We note that the assumption of Theorem 3 that the functions

$$z\prod_{i=2}^{m}\left(\frac{f_i(z)}{z^p}\right)^{\alpha_i}\left(\frac{h(z)}{z}\right)^{\alpha_1-1}h'(z) \text{ and } z\left(\frac{I^{p,m}_{h;\alpha_1,\alpha_i,\beta}[f_i](z)}{z^p}\right)^\beta$$

need to be univalent in $\mathbb{U}$, may be replaced as in the following corollary.

Corollary 1. *Let* $f, g_j \in \mathcal{A}_{p,h;\alpha_1,\alpha_i}$, $(i = 2, 3, \cdot, m;\ j = 1,\ 2)$ *and*

$$Re\left\{1+\frac{z\phi_j''(z)}{\phi_j'(z)}\right\} > -\delta$$

$$\left(\phi_j(z) = z\prod_{i=2}^{m}\left(\frac{g_{i,j}(z)}{z^p}\right)^{\alpha_i}\left(\frac{h(z)}{z}\right)^{\alpha_1-1}h'(z);\ z\in\mathbb{U}\right)$$

and

$$\text{Re}\left\{1+\frac{z\Theta''(z)}{\Theta'(z)}\right\} > -\delta \tag{23}$$

$$\left(\Theta(z) = z\prod_{i=2}^{m}\left(\frac{f_i(z)}{z^p}\right)^{\alpha_i}\left(\frac{h(z)}{z}\right)^{\alpha_1-1}h'(z);\ z\in\mathbb{U}\right),$$

where δ *is given by (8). Then*

$$\phi_1(z) \prec z\prod_{i=2}^{m}\left(\frac{f_i(z)}{z^p}\right)^{\alpha_i}\left(\frac{h(z)}{z}\right)^{\alpha_1-1}h'(z) \prec \phi_2(z)$$

implies that

$$z\left(\frac{I^{p,m}_{h;\alpha_1,\alpha_i,\beta}[g_{i,1}](z)}{z^p}\right)^\beta \prec z\left(\frac{I^{p,m}_{h;\alpha_1,\alpha_i,\beta}[f_i](z)}{z^p}\right)^\beta \prec z\left(\frac{I^{p,m}_{h;\alpha_1,\alpha_i,\beta}[g_{i,2}](z)}{z^p}\right)^\beta.$$

Proof. To prove Corollary 1, we have to show that condition (23) implies the univalence of $\Theta(z)$ and $\Psi(z) = z\left(\frac{I^{p,m}_{h;\alpha_1,\alpha_i,\beta}[f_i](z)}{z^p}\right)^\beta$. Since $0 \le \delta < \frac{1}{2}$, it follows that $\Theta(z)$ is close to convex function in $\mathbb{U}$ (see [9]) and hence $\Theta(z)$ is univalent in $\mathbb{U}$. Also, by using the same techniques as in the proof of Theorem 1, we can prove that Ψ is convex (univalent) in $\mathbb{U}$, and so the details may be omitted. Therefore, by applying Theorem 3, we obtain the desired result. ∎

Remark 1. *(i) Putting $p = 1$, $m = 2$, $\alpha_1 = \gamma$, $\alpha_2 = \beta$ and $f_2(t) = f(t)$ in Theorem* $1, 2$, 3 *and Corollary 1, we obtain the results by Cho and Bulboacă [6] and the results by Al-Kharsani et al. [2];*

(ii) If we take $\alpha_1 = 0$ in the results mentioned above, then we have those by Aouf et. al [1]. Moreover, putting $p = 1$, $m = 2$, $\alpha_1 = 0$, $\alpha_2 = \beta$ and $f_2(t) = f(t)$ in Theorem $1, 2$, 3 *and Corollary 1, we obtain the results by Cho and Kim [7].*

Acknowledgements

The third author was supported by the Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education, Science and Technology (No. 2016R1D1A1A09916450).

References

[1] M. K. Aouf, A. O. Mostafa and H. M. Zayed, Certain family of integral operators associated with multivalent functions preserving subordination and superordination, to appear in Filomat.

[2] H. A. Al-Kharsani, N. M. Al-Areefi and J. Sokol, A class of Integral operators preserving subordination and superordination for analytic functions, ISRN Math. Anal., 2012 (2012), 1-17.

[3] T. Bulboacă, Integral operators that preserve the subordination, Bull. Korean Math. Soc., 32 (1997), 627–636.

[4] T. Bulboacă, On a class of integral operators that preserve the subordination, Pure Math. Appl., 13 (2002), 87–96.

[5] T. Bulboacă, A class of superordination-preserving integral operators. Indag. Math. (N. S.), 13 (2002), 301–311.

[6] N. E. Cho and T. Bulboacă, Subordination and superordination properties for a class of integral operators, Acta Math. Sin. (Engl. Ser.), 26 (2010), no. 3, 515–522.

[7] N. E. Cho and I. H. Kim, A class of Integral operators preserving subordination and superordination, J. Inequal. Appl., 2008 (2008), 1-14.

[8] D. J. Hallenbeck and T. H. MacGregor, Linear Problems and Convexity Techniques in Geometric Function Theory, Pitman, London, 1984.

[9] W. Kaplan, Close-to-convex schlicht functions, Michigan Math. J., 2 (1952), 169–185.

[10] S. S. Miller and P. T. Mocanu, Differential subordinations and univalent functions, Michigan Math. J., 28 (1981), no. 2, 157–172.

[11] S. S. Miller and P. T. Mocanu, Univalent solutions of Briot-Bouquet differential equations, J. Differential Equations, 56 (1985), no. 3, 297-309.

[12] S. S. Miller and P. T. Mocanu, Integral Operators on Certain Classes of Analytic Functions, Univalent Functions, Fractional Calculus and their Applications, New York: Halstead Press, J Wiley & Sons, 1989: 153–166.

[13] S. S. Miller and P. T. Mocanu, Classes of univalent integral operators, J. Math. Anal. Appl., 157 (1991), no. 1, 147–165.

[14] S. S. Miller and P. T. Mocanu, Differential Subordinations: Theory and Applications, Series on Monographs and Textbooks in Pure and Applied Mathematics, Vol. 225, Marcel Dekker, New York and Basel, 2000.

[15] S. S. Miller and P. T. Mocanu, Subordinants of differential superordinations, Complex Variables Theory Appl., 48 (2003), no. 10, 815–826.

[16] S. S. Miller and P. T. Mocanu and M. O. Reade, Starlike integral operators, Pacific J. Math., 79 (1978), 157-168.
[17] Ch. Pommerenke, Univalent Functions, Vandenhoeck and Ruprecht, Göttingen, 1975.
[18] H. M. Srivastava, M. K. Aouf, A. O. Mostafa and H. M. Zayed, Certain subordination-preserving family of integral operators associated with $p-$valent functions, Appl. Math. Inf. Sci. 11 (4) (2017) 951-960.

(M. K. Aouf) Department of Mathematics, Faculty of Science, Mansoura University, Mansoura 35516, Egypt

E-mail address: mkaouf127@yahoo.com

(H. M. Zayed) Department of Mathematics, Faculty of Science, Menofia University, Shebin Elkom 32511, Egypt

E-mail address: hanaa_zayed42@yahoo.com

(N. E. Cho) Corresponding Author, Department of Applied Mathematics, Pukyong National University, Busan 48513, Korea

E-mail address: necho@pknu.ac.kr

ADDITIVE s-FUNCTIONAL INEQUALITIES AND DERIVATIONS ON BANACH ALGEBRAS

TAEKSEUNG KIM, YOUNGHUN JO*, JUNHA PARK, JAEMIN KIM, CHOONKIL PARK*, AND JUNG RYE LEE

Abstract. In this paper, we introduce the following new additive s-functional inequalities

$$\|f(x-y)+f(y)+f(-x)\| \leq \|s\left(f(x+y)-f(x)-f(y)\right)\|, \tag{0.1}$$

$$\|f(x+y)-f(x)-f(y)\| \leq \|s\left(f(x-y)+f(y)+f(-x)\right)\|, \tag{0.2}$$

where s is a fixed complex number with $|s|<1$, and prove the Hyers-Ulam stability of linear derivations on Banach algebras associated to the additive s-functional inequalities (0.1) and (0.2).

1. Introduction and preliminaries

The stability problem of functional equations originated from a question of Ulam [20] concerning the stability of group homomorphisms. Hyers [6] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [1] for additive mappings and by Rassias [15] for linear mappings by considering an unbounded Cauchy difference. A generalization of the Rassias theorem was obtained by Găvruta [3] by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassias' approach. The stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [8, 9, 13, 14, 17, 18, 19]).

Gilányi [4] showed that if f satisfies the functional inequality

$$\|2f(x)+2f(y)-f(x-y)\| \leq \|f(x+y)\| \tag{1.1}$$

then f satisfies the Jordan-von Neumann functional equation

$$2f(x)+2f(y)=f(x+y)+f(x-y).$$

See also [16]. Fechner [2] and Gilányi [5] proved the Hyers-Ulam stability of the functional inequality (1.1). Park, Cho and Han [12] investigated the Cauchy additive functional inequality

$$\|f(x)+f(y)+f(z)\| \leq \|f(x+y+z)\| \tag{1.2}$$

and the Cauchy-Jensen additive functional inequality

$$\|f(x)+f(y)+2f(z)\| \leq \left\|2f\left(\frac{x+y}{2}+z\right)\right\| \tag{1.3}$$

2010 *Mathematics Subject Classification.* Primary 39B52, 39B62.

Key words and phrases. derivation on Banach algebra; additive s-functional inequality; direct method; Hyers-Ulam stability.

*Corresponding authors.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 27, NO.5, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC

and proved the Hyers-Ulam stability of the functional inequalities (1.2) and (1.3) in Banach spaces.

Park [10, 11] defined additive ρ-functional inequalities and proved the Hyers-Ulam stability of the additive ρ-functional inequalities in Banach spaces and non-Archimedean Banach spaces.

This paper is organized as follows: In Section 2, we prove the Hyers-Ulam stability of linear derivations on Banach algebras associated to the additive s-functional inequality (0.1).

In Section 3, we prove the Hyers-Ulam stability of linear derivations on Banach algebras associated to the additive s-functional inequality (0.2).

Throughout this paper, assume that s is a fixed complex number with $|s| < 1$.

2. Stability of linear derivations on Banach algebras associated to the functional inequality (0.1)

In this section, we prove the Hyers-Ulam stability of linear derivations on Banach algebras associated to the additive s-functional inequality (0.1).

Theorem 2.1. *Let $\theta \geq 0$ and p be real numbers with $p > 2$. Let $f : \mathcal{B} \to \mathcal{B}$ be a mapping satisfying*

$$\|f(\lambda(x-y)) + \lambda f(y) + \lambda f(-x)\| \leq \|s\left(f(x+y) - f(x) - f(y)\right)\| + \theta\left(\|x\|^p + \|y\|^p\right), \tag{2.1}$$

$$\|f(xy) - xf(y) - yf(x)\| \leq \theta\left(\|x\|^p + \|y\|^p\right) \tag{2.2}$$

for all $\lambda \in \mathbb{S}^1 := \{\mu \in \mathbb{C} | |\mu| = 1\}$ and all $x, y \in \mathcal{B}$. Then there exists a unique $\mathbb{C}$-linear derivation $D : \mathcal{B} \to \mathcal{B}$ such that

$$\|f(x) - D(x)\| \leq \frac{2\theta}{(2^p - 2)(1 - |s|)}\|x\|^p \tag{2.3}$$

for all $x \in \mathcal{B}$.

Proof. Letting $x = y = 0$ and $\lambda = -1 \in \mathbb{S}^1$ in (2.1), we get $\|f(0)\| \leq \|sf(0)\|$ and so we get $f(0) = 0$.

Replacing y by x and letting $\lambda = 1$ in (2.1), we get

$$\|f(x) + f(-x)\| \leq \|s\left(f(2x) - 2f(x)\right)\| + 2\theta\|x\|^p \tag{2.4}$$

for all $x \in \mathcal{B}$.

Replacing x by $-x$ and y by x and letting $\lambda = -1$ in (2.1), we get

$$\|f(2x) - 2f(x)\| \leq \|s\left(f(x) + f(-x)\right)\| + 2\theta\|x\|^p \tag{2.5}$$

for all $x \in \mathcal{B}$.

From (2.4) and (2.5), we get

$$\|f(2x) - 2f(x)\| \leq |s|^2\|f(2x) - 2f(x)\| + 2(1 + |s|)\theta\|x\|^p$$

and so

$$\|f(2x) - 2f(x)\| \leq \frac{2\theta}{1 - |s|}\|x\|^p \tag{2.6}$$

for all $x \in \mathcal{B}$. So one can obtain that

$$\left\|f(x) - 2f\left(\frac{x}{2}\right)\right\| \leq \frac{2\theta}{2^p(1 - |s|)}\|x\|^p$$

and hence

$$\left\|2^n f\left(\frac{x}{2^n}\right) - 2^{n+1} f\left(\frac{x}{2^{n+1}}\right)\right\| \le \frac{2 \cdot 2^{n(1-p)}\theta}{2^p(1-|s|)}\|x\|^p$$

for all $x \in \mathcal{B}$. So we get

$$\left\|f(x) - 2^n f\left(\frac{x}{2^n}\right)\right\| \le \sum_{l=0}^{n-1} \frac{2 \cdot 2^{l(1-p)}\theta}{2^p(1-|s|)}\|x\|^p \tag{2.7}$$

for all $x \in \mathcal{B}$.

For positive integers n and m with $n > m$,

$$\left\|2^n f\left(\frac{x}{2^n}\right) - 2^m f\left(\frac{x}{2^m}\right)\right\| \le \sum_{l=m}^{n-1} \frac{2 \cdot 2^{l(1-p)}\theta}{2^p(1-|s|)}\|x\|^p,$$

which tends to zero as $m \to \infty$. So $\{2^n f(\frac{x}{2^n})\}$ is a Cauchy sequence for all $x \in \mathcal{B}$. Since $\mathcal{B}$ is complete, the sequence $\{2^n f(\frac{x}{2^n})\}$ converges for all $x \in \mathcal{B}$. We can define a mapping $D : \mathcal{B} \to \mathcal{B}$ by

$$D(x) = \lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right) \tag{2.8}$$

for all $x \in \mathcal{B}$.

Letting $x = 0$ in (2.1), we get

$$\|f(\lambda x) + \lambda f(-x)\| \le \theta\|x\|^p$$

for all $\lambda \in \mathbb{S}^1$ and all $x \in \mathcal{B}$.

By (2.8), we get

$$\|D(\lambda x) + \lambda D(-x)\| = \lim_{n\to\infty}\left\|2^n f\left(\frac{\lambda x}{2^n}\right) + 2^n \lambda f\left(-\frac{x}{2^n}\right)\right\| \le \lim_{n\to\infty} \frac{2^n}{2^{pn}}\theta\|x\|^p = 0$$

for all $x \in \mathcal{B}$ and all $\lambda \in \mathbb{S}^1$. Hence

$$D(\lambda x) + \lambda D(-x) = 0 \tag{2.9}$$

for all $x \in \mathcal{B}$ and all $\lambda \in \mathbb{S}^1$.

Letting $\lambda = 1$ in (2.9), we get

$$D(x) + D(-x) = 0 \tag{2.10}$$

for all $x \in \mathcal{B}$. Hence

$$D(\lambda x) = \lambda D(x) \tag{2.11}$$

for all $x \in \mathcal{B}$ and all $\lambda \in \mathbb{S}^1$.

Let $\lambda = 1$ in (2.1). By (2.1), (2.8) and (2.10), we get

$$\begin{aligned}
&\|D(x-y) - D(x) + D(y)\| = \lim_{n\to\infty}\left\|2^n f\left(\frac{x-y}{2^n}\right) + 2^n f\left(\frac{y}{2^n}\right) + 2^n f\left(-\frac{x}{2^n}\right)\right\| \\
&\le \lim_{n\to\infty}\left(\left\|s\left(2^n f\left(\frac{x+y}{2^n}\right) - 2^n f\left(\frac{x}{2^n}\right) - 2^n f\left(\frac{y}{2^n}\right)\right)\right\| + 2^{n(1-p)}\theta(\|x\|^p + \|y\|^p)\right) \\
&= \|s\left(D(x+y) - D(x) - D(y)\right)\|
\end{aligned}$$

for all $x, y \in \mathcal{B}$. Hence

$$\|D(x-y) - D(x) + D(y)\| \le \|s(D(x+y) - D(x) - D(y))\| \tag{2.12}$$

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 27, NO.5, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC

for all $x, y \in \mathcal{B}$.

Replacing y by $-y$ in (2.12), we get

$$\|D(x+y) - D(x) - D(y)\| \le \|s(D(x-y) - D(x) + D(y))\| \tag{2.13}$$

for all $x, y \in \mathcal{B}$.

It follows from (2.12) and (2.13) that

$$\|D(x+y) - D(x) - D(y)\| \le \|s^2(D(x+y) - D(x) - D(y))\|$$

for all $x, y \in \mathcal{B}$. Since $|s| < 1$, we get

$$\|D(x+y) - D(x) - D(y)\| = 0$$

for all $x, y \in \mathcal{B}$. So one can obtain that D is additive. Moreover, by passing to the limit in (2.7) as $n \to \infty$, we get the inequality (2.3).

Now let $S : \mathcal{B} \to \mathcal{B}$ be another additive mapping satisfying

$$\|f(x) - S(x)\| \le \frac{2^p}{2^p - 2}\theta\|x\|^p$$

for all $x \in \mathcal{B}$.

$$\begin{aligned} \|D(x) - S(x)\| &= 2^l \left\| D\left(\frac{x}{2^l}\right) - S\left(\frac{x}{2^l}\right) \right\| \\ &\le 2^l \left\| D\left(\frac{x}{2^l}\right) - f\left(\frac{x}{2^l}\right) \right\| + 2^l \left\| f\left(\frac{x}{2^l}\right) - S\left(\frac{x}{2^l}\right) \right\| \\ &\le \frac{2^{l+1}}{2^{lp}} \times \frac{2^p}{2^p - 2}\theta\|x\|^p, \end{aligned}$$

which tends to zero as $l \to \infty$. Thus $D(x) = S(x)$ for all $x \in \mathcal{B}$. This proves the uniqueness of D.

Now let $\mu \in \mathbb{C}$ $(\mu \ne 0)$ and M an integer greater than $4|\lambda|$. Then $|\frac{\lambda}{M}| < \frac{1}{4} < 1 - \frac{2}{3} = \frac{1}{3}$. By [7, Theorem 1], there exist three elements $\mu_1, \mu_2, \mu_3 \in \mathbb{S}^1$ such that $3\frac{\lambda}{M} = \mu_1 + \mu_2 + \mu_3$. By (2.11),

$$\begin{aligned} D(\lambda x) &= D\left(\frac{M}{3} \cdot 3\frac{\lambda}{M}x\right) = M \cdot D\left(\frac{1}{3} \cdot 3\frac{\lambda}{M}x\right) = \frac{M}{3}D\left(3\frac{\lambda}{M}x\right) \\ &= \frac{M}{3}D(\mu_1 x + \mu_2 x + \mu_3 x) = \frac{M}{3}(D(\mu_1 x) + D(\mu_2 x) + D(\mu_3 x)) \\ &= \frac{M}{3}(\mu_1 + \mu_2 + \mu_3)D(x) = \frac{M}{3} \cdot 3\frac{\lambda}{M}D(x) \\ &= \lambda D(x) \end{aligned}$$

for all $x \in \mathcal{B}$. Hence

$$D(\alpha x + \beta y) = D(\alpha x) + D(\beta y) = \alpha D(x) + \beta D(y)$$

for all $\alpha, \beta \in \mathbb{C}(\alpha, \beta \ne 0)$ and all $x, y \in \mathcal{B}$. And $D(0x) = 0 = 0D(x)$ for all $x \in \mathcal{B}$. So the unique additive mapping $D : \mathcal{B} \to \mathcal{B}$ is a $\mathbb{C}$-linear mapping.

It follows from (2.2) and (2.8) that

$$\begin{aligned} \|D(xy) - xD(y) - yD(x)\| &= \lim_{n\to\infty} \left\| 2^{2n} f\left(\frac{x}{2^n}\frac{y}{2^n}\right) - 2^n x f\left(\frac{y}{2^n}\right) - 2^n y f\left(\frac{x}{2^n}\right) \right\| \\ &\le \lim_{n\to\infty} 2^{n(2-p)}\theta(\|x\|^p + \|y\|^p) = 0 \end{aligned}$$

for all $x, y \in \mathcal{B}$. Hence

$$D(xy) = xD(y) + yD(x)$$

for all $x, y \in \mathcal{B}$. Hence the mapping $D : \mathcal{B} \to \mathcal{B}$ is a $\mathbb{C}$-linear derivation satisfying (2.3). □

Theorem 2.2. *Let $\theta \geq 0$ and p be real numbers with $0 < p < 1$. Let $f : \mathcal{B} \to \mathcal{B}$ be a mapping satisfying* (2.1) *and* (2.2)*. Then there exists a unique $\mathbb{C}$-linear derivation $D : \mathcal{B} \to \mathcal{B}$ such that*

$$\|f(x) - D(x)\| \leq \frac{2\theta}{(2-2^p)(1-|s|)}\|x\|^p \tag{2.14}$$

for all $x \in \mathcal{B}$.

Proof. It follows from (2.6) that $\left\|f(x) - \frac{1}{2}f(2x)\right\| \leq \frac{\theta}{1-|s|}\|x\|^p$ and hence

$$\left\|\frac{1}{2^n}f(2^n x) - \frac{1}{2^{n+1}}f(2^{n+1}x)\right\| \leq \frac{2^{pn}\theta}{2^n(1-|s|)}\|x\|^p$$

for all $x \in \mathcal{B}$.

For positive integers n and m with $n > m$,

$$\left\|\frac{1}{2^n}f(2^n x) - \frac{1}{2^m}f(2^m x)\right\| \leq \sum_{l=m}^{n-1}\frac{2^{pl}\theta}{2^l(1-|s|)}\|x\|^p, \tag{2.15}$$

which tends to zero as $m \to \infty$. So $\{\frac{1}{2^n}f(2^n x)\}$ is a Cauchy sequence for all $x \in \mathcal{B}$. Since $\mathcal{B}$ is complete, the sequence $\{\frac{1}{2^n}f(2^n x)\}$ converges for all $x \in \mathcal{B}$. We can define a mapping $D : \mathcal{B} \to \mathcal{B}$ by $D(x) = \lim_{n\to\infty}\frac{1}{2^n}f(2^n x)$ for all $x \in \mathcal{B}$.

Moreover, by letting $m = 0$ and passing to the limit in (2.15) as $n \to \infty$, we get (2.14).

The rest of the proof is similar to the proof of Theorem 2.1. □

3. Stability of linear derivations on Banach algebras associated to the functional inequality (0.2)

In this section, we prove the Hyers-Ulam stability of linear derivations on Banach algebras associated to the additive s-functional inequality (0.2).

Theorem 3.1. *Let $\theta \geq 0$ and p be real numbers with $p > 2$. Let $f : \mathcal{B} \to \mathcal{B}$ be a mapping satisfying* (2.2) *and*

$$\|f(\lambda(x+y)) - \lambda f(x) - \lambda f(y)\| \leq \|s\left(f(x-y) + f(y) + f(-x)\right)\| + \theta\left(\|x\|^p + \|y\|^p\right) \tag{3.1}$$

for all $\lambda \in \mathbb{S}^1$ and all $x, y \in \mathcal{B}$. Then there exists a unique $\mathbb{C}$-linear derivation $D : \mathcal{B} \to \mathcal{B}$ such that

$$\|f(x) - D(x)\| \leq \frac{(2-|s|)\theta}{(2^p-2)(1-|s|)}\|x\|^p \tag{3.2}$$

for all $x \in \mathcal{B}$.

Proof. Letting $x = y = 0$ and $\lambda = -1 \in \mathbb{S}^1$ in (3.1), we get

$$\|3f(0)\| \leq \|3sf(0)\|$$

and so we get $f(0) = 0$.

Letting $y = 0$ and $\lambda = -1$ in (3.1), we get

$$\|f(-x) + f(x)\| \leq \|s\left(f(x) + f(-x)\right)\| + \theta\|x\|^p$$

and so

$$\|f(-x)+f(x)\| \leq \frac{1}{1-|s|}\theta\|x\|^p$$

for all $x \in \mathcal{B}$.

Letting $y = x$ and $\lambda = 1$ in (3.1), we get

$$\begin{aligned} \|f(2x)-2f(x)\| &\leq \|s\left(f(x)+f(-x)\right)\| + 2\theta\|x\|^p \\ &\leq \frac{|s|}{1-|s|}\theta\|x\|^p + 2\theta\|x\|^p = \frac{2-|s|}{1-|s|}\theta\|x\|^p \end{aligned} \tag{3.3}$$

for all $x \in \mathcal{B}$. So one can obtain that

$$\left\|f(x) - 2f\left(\frac{x}{2}\right)\right\| \leq \frac{(2-|s|)\theta}{2^p(1-|s|)}\|x\|^p$$

and hence

$$\left\|2^n f\left(\frac{x}{2^n}\right) - 2^{n+1} f\left(\frac{x}{2^{n+1}}\right)\right\| \leq \frac{(2-|s|)2^{n(1-p)}\theta}{2^p(1-|s|)}\|x\|^p$$

for all $x \in \mathcal{B}$. So we get

$$\left\|f(x) - 2^n f\left(\frac{x}{2^n}\right)\right\| \leq \sum_{l=0}^{n-1} \frac{(2-|s|)2^{l(1-p)}\theta}{2^p(1-|s|)}\|x\|^p \tag{3.4}$$

for all $x \in \mathcal{B}$.

For positive integers n and m with $n > m$,

$$\left\|2^n f\left(\frac{x}{2^n}\right) - 2^m f\left(\frac{x}{2^m}\right)\right\| \leq \sum_{l=m}^{n-1} \frac{(2-|s|)2^{l(1-p)}\theta}{2^p(1-|s|)}\|x\|^p,$$

which tends to zero as $m \to \infty$. So $\{2^n f(\frac{x}{2^n})\}$ is a Cauchy sequence for all $x \in \mathcal{B}$. Since $\mathcal{B}$ is complete, the sequence $\{2^n f(\frac{x}{2^n})\}$ converges for all $x \in \mathcal{B}$. We can define a mapping $D : \mathcal{B} \to \mathcal{B}$ by

$$D(x) = \lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right) \tag{3.5}$$

for all $x \in \mathcal{B}$.

It follows from (3.1) and (3.5) that

$$\begin{aligned} &\|D(\lambda(x+y)) - \lambda D(x) - \lambda D(y)\| = \lim_{n\to\infty} \left\|2^n f\left(\lambda\frac{x+y}{2^n}\right) - 2^n\lambda f\left(\frac{x}{2^n}\right) - 2^n\lambda f\left(\frac{y}{2^n}\right)\right\| \\ &\leq \lim_{n\to\infty}\left(\left\|s\left(2^n f\left(\frac{x-y}{2^n}\right) + 2^n f\left(\frac{y}{2^n}\right) + 2^n f\left(\frac{-x}{2^n}\right)\right)\right\| + 2^{n(1-p)}\theta(\|x\|^p + \|y\|^p)\right) \\ &= \|s\left(D(x-y)+D(y)+D(-x)\right)\| \end{aligned}$$

for all $\lambda \in \mathbb{S}^1$ and all $x, y \in \mathcal{B}$. Hence

$$\|D(\lambda(x+y)) - \lambda D(x) - \lambda D(y)\| = \|s\left(D(x-y)+D(y)+D(-x)\right)\| \tag{3.6}$$

for all $\lambda \in \mathbb{S}^1$ and all $x, y \in \mathcal{B}$.

Letting $\lambda = -1$ and $x = y = 0$ in (3.6), we get $\|3D(0)\| \leq \|3sD(0)\|$ and so $D(0) = 0$.

Replacing x by $-x$ and letting $y = -x$ and $\lambda = -1$ in (3.6), we get

$$\|D(2x) + 2D(-x)\| \leq \|s(D(-x)+D(x))\|$$

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 27, NO.5, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC

for all $x \in \mathcal{B}$.

Letting $y = -x$ and $\lambda = 1$ in (3.6), we get

$$\|D(x) + D(-x)\| \leq \|s(D(2x) + 2D(-x))\| \leq |s|^2 \|D(x) + D(-x)\|$$

and so $D(-x) = -D(x)$ for all $x \in \mathcal{B}$.

Replacing y by $-y$ and letting $\lambda = 1$ in (3.6), we get

$$\|D(x-y) - D(x) + D(y)\| \leq \|s\left(D(x+y) - D(y) - D(x)\right)\|$$

for all $x, y \in \mathcal{B}$.

Letting $\lambda = 1$ in (3.6), we get

$$\begin{aligned} \|D(x+y) - D(x) - D(y)\| &\leq \|s\left(D(x-y) + D(y) + D(-x)\right)\| \\ &\leq |s|^2 \|D(x+y) - D(x) - D(y)\| \end{aligned}$$

for all $x, y \in \mathcal{B}$. Thus $D(x+y) = D(x) + D(y)$ for all $x, y \in \mathcal{B}$.

Letting $y = 0$ in (3.6), we get

$$\|D(\lambda x) - \lambda D(x)\| \leq 0$$

and so $D(\lambda x) = \lambda D(x)$ for all $\lambda \in \mathbb{S}^1$ and $x \in \mathcal{B}$.

The rest of the proof is similar to the proof of Theorem 2.1. □

Theorem 3.2. *Let $\theta \geq 0$ and p be real numbers with $0 < p < 1$. Let $f : \mathcal{B} \to \mathcal{B}$ be a mapping satisfying* (3.1) *and* (2.2). *Then there exists a unique $\mathbb{C}$-linear derivation $D : \mathcal{B} \to \mathcal{B}$ such that*

$$\|f(x) - D(x)\| \leq \frac{(2-|s|)\theta}{(2-2^p)(1-|s|)} \|x\|^p \tag{3.7}$$

for all $x \in \mathcal{B}$.

Proof. It follows from (3.3) that

$$\left\| f(x) - \frac{1}{2} f(2x) \right\| \leq \frac{(2-|s|)\theta}{2(1-|s|)} \|x\|^p$$

and hence

$$\left\| \frac{1}{2^n} f(2^n x) - \frac{1}{2^{n+1}} f(2^{n+1} x) \right\| \leq \frac{(2-|s|)2^{pn}\theta}{2^{n+1}(1-|s|)} \|x\|^p$$

for all $x \in \mathcal{B}$.

For positive integers n and m with $n > m$,

$$\left\| \frac{1}{2^n} f(2^n x) - \frac{1}{2^m} f(2^m x) \right\| \leq \sum_{l=m}^{n-1} \frac{(2-|s|)2^{pl}\theta}{2^{l+1}(1-|s|)} \|x\|^p, \tag{3.8}$$

which tends to zero as $m \to \infty$. So $\{\frac{1}{2^n} f(2^n x)\}$ is a Cauchy sequence for all $x \in \mathcal{B}$. Since $\mathcal{B}$ is complete, the sequence $\{\frac{1}{2^n} f(2^n x)\}$ converges for all $x \in \mathcal{B}$. We can define a mapping $D : \mathcal{B} \to \mathcal{B}$ by $D(x) = \lim_{n\to\infty} \frac{1}{2^n} f(2^n x)$ for all $x \in \mathcal{B}$.

Moreover, by letting $m = 0$ and passing to the limit in (3.8) as $n \to \infty$, we get (3.7).

The rest of the proof is similar to the proofs of Theorems 2.1 and 3.1. □

Acknowledgments

This work was supported by the Seoul Science High School *R&E* Program in 2017. C. Park was supported by Basic Science Research Program through the National Research Foundation of Korea funded by the Ministry of Education, Science and Technology (NRF-2017R1D1A1B04032937).

References

[1] T. Aoki, *On the stability of the linear transformation in Banach spaces*, J. Math. Soc. Japan **2** (1950), 64–66.

[2] W. Fechner, *Stability of a functional inequalities associated with the Jordan-von Neumann functional equation*, Aequationes Math. **71** (2006), 149–161.

[3] P. Găvruta, *A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings*, J. Math. Anal. Appl. **184** (1994), 431–436.

[4] A. Gilányi, *Eine zur Parallelogrammgleichung äquivalente Ungleichung*, Aequationes Math. **62** (2001), 303–309.

[5] A. Gilányi, *On a problem by K. Nikodem*, Math. Inequal. Appl. **5** (2002), 707–710.

[6] D.H. Hyers, *On the stability of the linear functional equation*, Proc. Nat. Acad. Sci. U.S.A. **27** (1941), 222–224.

[7] R.V. Kadison and G. Pedersen, *Means and convex combinations of unitary operators*, Math. Scand. **57** (1985), 249–266.

[8] H. Kim, M. Eshaghi Gordji, A. Javadian and I. Chang, *Homomorphisms and derivations on unital C^*-algebras related to Cauchy-Jensen functional inequality,* J. Math. Inequal. **6** (2012), 557–565.

[9] J. Lee, C. Park and D. Shin, *An AQCQ-functional equation in matrix normed spaces*, Results Math. **27** (2013), 305–318.

[10] C. Park, *Additive ρ-functional inequalities and equations,* J. Math. Inequal. **9** (2015), 17–26.

[11] C. Park, *Additive ρ-functional inequalities in non-Archimedean normed spaces,* J. Math. Inequal. **9** (2015), 397–407.

[12] C. Park, Y. Cho and M. Han, *Stability of functional inequalities associated with Jordan-von Neumann type additive functional equations*, J. Inequal. Appl. **2007**, Art. ID 41820 (2007).

[13] C. Park, K. Ghasemi, S. G. Ghaleh and S. Jang, *Approximate n-Jordan $*$-homomorphisms in C^*-algebras,* J. Comput. Anal. Appl. **15** (2013), 365-368.

[14] C. Park, A. Najati and S. Jang, *Fixed points and fuzzy stability of an additive-quadratic functional equation,* J. Comput. Anal. Appl. **15** (2013), 452–462.

[15] Th.M. Rassias, *On the stability of the linear mapping in Banach spaces*, Proc. Amer. Math. Soc. **72** (1978), 297–300.

[16] J. Rätz, *On inequalities associated with the Jordan-von Neumann functional equation*, Aequationes Math. **66** (2003), 191–200.

[17] S. Shagholi, M. Eshaghi Gordji and M. Bavand Savadkouhi, *Stability of ternary quadratic derivation on ternary Banach algebras,* J. Comput. Anal. Appl. **13** (2011), 1097–1105.

[18] D. Shin, C. Park and Sh. Farhadabadi, *On the superstability of ternary Jordan C^*-homomorphisms,* J. Comput. Anal. Appl. **16** (2014), 964–973.

[19] D. Shin, C. Park and Sh. Farhadabadi, *Stability and superstability of J^*-homomorphisms and J^*-derivations for a generalized Cauchy-Jensen equation,* J. Comput. Anal. Appl. **17** (2014), 125–134.

[20] S. M. Ulam, *A Collection of the Mathematical Problems*, Interscience Publ. New York, 1960.

Taekseung Kim, Younghun Jo, Junha Park, Jaemin Kim
Mathematics Branch, Seoul Science High School, Seoul 03066, Korea
E-mail address: andyk0202@naver.com; stop0422@naver.com; mathmario@naver.com; jm1997a@naver.com

Choonkil Park
Research Institute for Natural Sciences, Hanyang University, Seoul 04763, Korea
E-mail address: baak@hanyang.ac.kr

Jung Rye Lee
Depatment of Mathematics, Daejin University, Kyunggi 11159, Korea
E-mail address: jrlee@daejin.ac.kr

On modulus of convexity of quasi-Banach spaces

Shin Min Kang[1,2], Hussain Minhaj Uddin Ahmad Qadri[3],
Waqas Nazeer[4,*] and Absar Ul Haq[5]

[1]Department of Mathematics and RINS, Gyeongsang National University, Jinju 52828, Korea
e-mail: smkang@gnu.ac.kr

[2]Center for General Education, China Medical University, Taichung 40402, Taiwan

[3]Aitchison College, Lahore 54000, Pakistan
e-mail: minhaj_h@hotmail.com

[4]Division of Science and Technology, University of Education, Lahore 54000, Pakistan
e-mail: nazeer.waqas@ue.edu.pk

[5]Department of Mathematics, University of Management and Technology, Sialkot Campus, Lahore 51410, Pakistan
e-mail: absarulhaq@hotmail.com

Abstract

The aim of this report is to study modulus of convexity $\delta_{\mathcal{B}}$ of a quasi-Banach space $\mathcal{B}$. We prove that $\delta_{\mathcal{B}}$ is convex, continuous, nondecreasing and for arbitrary uniformly convex quasi-Banach space $\mathcal{B}$, $\delta_{\mathcal{B}}(\epsilon) = 1 - \frac{1}{C}\sqrt{1 - \frac{\epsilon^2 C^2}{4}}$. We also prove that a quasi-Banach space $\mathcal{B}$ is uniformly convex if and only if $\delta_{\mathcal{B}}(\epsilon) \geq 0$. Moreover we prove that a non-trivial quasi-Banach space $\mathcal{B}$ is uniformly non-square if and only if $\delta_{\mathcal{B}}(\epsilon) > 0$.

2010 Mathematics Subject Classification: 47H05, 46B20, 46E30

Key words and phrases: modulus of convexity, uniformly convex, uniformly non-square, quasi Banach space.

1 Introduction

Many of the geometric constants for Banach spaces have been investigated so far. These constants play an important role in the description of various geometric structures of Banach spaces. In 1899 Jung [10] was the first who introduced a geometric constant for Banach spaces. In 1936 and 1937, Clarkson [4,5] introduced classical modulus of convexity to define a uniformly convex space. A great number of such moduli have been defined and introduced since then. The theory of the geometry of a Banach space has evolved very rapidly over the past fifty years. By contrast the study of a quasi-Banach space has lagged far behind, even though the first research papers in the subject appeared in the early 1940's [2,4–6]. There are very sound reasons to develop the understanding of these space, but the absence of one of the fundamental tools of functional analysis, the Hahn-Banach

* Corresponding author

theorem, has proved a very significant stumbling block. However, there has been some progress in the non-convex theory and arguably it has contributed to our appreciation of Banach space theory. A systematic study of a quasi-Banach space only really started in the late 1950's and early 1960's with the work of several authors. The efforts of these researchers tended to go in rather separate directions. The subject was given great impetus by the paper of Duren et al. [7] in 1969 which demonstrated both the possibilities for using quasi-Banach spaces in classical function theory and also high-lighted some key problems related to the Hahn-Banach theorem. This opened up many new directions of research. The 1970's and 1980's saw a significant increase in activity with a number of authors contributing to the development of a coherent theory. An important breakthrough was the work of Roberts [13, 14] who showed that the Krein-Milman Theorem fails in general quasi-Banach spaces by developing powerful new techniques. Quasi-Banach spaces (H^p-space when $p < 1$) were also used significantly in Alexandrov's solution of the inner function problem in 1982 [1]. During this period three books on the subject appeared by Turpin [16], Rolewicz [15] (actually an expanded version of a book first published in 1972 and the author, Roberts [14]. In the 1990's it seems to the author that while more and more analysts find that quasi-Banach spaces have uses in their research, paradoxically the interest in developing a general theory has subsided somewhat. The strictly convex Banach spaces were introduced in 1936 by Clarkson, [4], who also studied the concept of uniform convexity. The uniform convexity of L^p spaces, $1 < p < \infty$, was established by Clarkson [4]. The concept of duality map was introduced in 1962 by Beurling and Livingston [3] and was further developed by many others and, De Figueiredo [8]. General properties of the duality map can be found in De Figueiredo [8].

In this paper we aim study modulus of convexity in the setting of quasi Banach spaces.

2 Preliminaries

Throughout this paper $S_{\mathcal{B}}$ is a closed unit ball in a quasi Banach space.

Definition 2.1. A uniformly convex space is a normed vector space so that, for every $0 < \epsilon \leq 2$ there is some $\delta > 0$ so that for any two vectors with $\|x\| = 1$ and $\|y\| = 1$, the condition $\|x - y\| \geq \varepsilon$ implies that $\left\|\frac{x+y}{2}\right\| \leq 1 - \delta$. Intuitively, the center of a line segment inside the unit ball must lie deep inside the unit ball unless the segment is short.

Definition 2.2. A quasi-Banach space $\mathcal{B}$ is said to be uniformly non-square if there exists a positive number $\delta < 2$ such that for any $x_1, x_2 \in S_{\mathcal{B}}$, we have

$$\min\left(\left\|\frac{x_1 + x_2}{C}\right\|, \left\|\frac{x_1 + x_2}{C}\right\|\right) \leq \delta.$$

Definition 2.3. Let $\epsilon \in [0, 2]$ and $C \geq 1$. For a quasi-Banach space $\mathcal{B}$, the modulus of convexity is a function $\delta_{\mathcal{B}} : (0, 2] \longrightarrow [0, 1]$ defined as

$$\delta_{\mathcal{B}}(\epsilon) = \inf\left\{1 - \frac{\|x_1 + x_2\|}{2C} : x_1, x_2 \in S_{\mathcal{B}};\ \frac{\|x_1 - x_2\|}{C} \geq \epsilon\right\}. \tag{2.1}$$

A characteristic or related coefficient of this modulus is

$$\delta_0(\mathcal{B}) = \sup\left\{\epsilon \in [0, 2] : \delta_{\mathcal{B}}(\epsilon) = 0\right\}. \tag{2.2}$$

3 Main results

Lemma 3.1. ([9]) *Every convex function f with convex domain in $\mathbb{R}$ is continuous.*

Lemma 3.2. *Let $\mathcal{B}$ be a quasi-Banach space, and $x_1, x_2 \in S_{\mathcal{B}}$. Then*

$$\frac{\|x_1 + x_2\|}{2C} \leq 1 - \delta_{\mathcal{B}}\left(\frac{\|x_1 - x_2\|}{C}\right). \tag{3.1}$$

Proof. Let $\dim(\mathcal{B}) < \infty$. Let $\epsilon \in [0, 2]$ and choose $u, v \in S_{\mathcal{B}}$ such that $\frac{\|u-v\|}{C}$ is maximal subject to $\frac{\|u-v\|}{C} = \epsilon$. So, here this is enough to prove that $\|u\| = \|v\| = 1$.

The case $\epsilon = 0$ is trivial.

Assume that $\epsilon \neq 0$. Let $x^* \in X^*$ satisfying $\|x^*\| = 1$ and $x^*(u + v) = \frac{\|u+v\|}{2C}$. It would be suffices to prove that if, say, $\|v\| < 1$, then $x^*(v - u) = \epsilon$ and $\|u\| < 1$. Indeed an analogous reasoning would then yields, $x^*(u - v) = \epsilon$ and hence $\epsilon = -\epsilon$, which is a contradiction.

To this end, let $\mathcal{A} = \{w \in \mathcal{B} : \frac{\|w-u\|}{C} = \epsilon\}$. If $w \in \mathcal{A} \cap S_{\mathcal{B}}$, then by maximality of $\frac{\|u+v\|}{2C}$ we get

$$x^*(u + w) \leq \frac{\|u + w\|}{2C} \leq \frac{\|u + v\|}{2C} \leq x^*(u + v).$$

Hence, if we had $\|v\| < 1$, then x^* would attain at v local maximum on $\mathcal{A}$. Consequently, x^* would norm the vector $v - u$, that is, $x^*(v - u) = \frac{\|v-u\|}{C} = \epsilon$. And also

$$\begin{aligned} \|u\| &< \frac{1}{2C}\Big(\|u + v\| + \|u - v\|\Big) \\ &< \frac{1}{2C}\Big[x^*(u + v) + x^*(u - v)\Big] \\ &= x^*(v) < 1 \end{aligned}$$

as permitted. This completes the proof. □

Lemma 3.3. *Let $\mathcal{B}$ be a quasi-Banach space. and $\epsilon \in (0, 2]$. Then the following statements holds:*

(*a*) *$\delta_{\mathcal{B}}(\epsilon)$ is convex and continuous function.*

(*b*) *$\delta_{\mathcal{B}}(\epsilon)$ is a non-decreasing function.*

(*c*) *$\delta_{\mathcal{B}}(\epsilon)/\epsilon$ is a non-decreasing function.*

Proof. (*a*) Consider any two vectors $u, v \in \mathcal{B}$, we denote by $N(u, v)$ the set of all pairs $x, y \in \mathcal{B}$ with $x, y \in S_{\mathcal{B}}(0)$ such that for some real scalars α_1, β_1 we have $x - y = \alpha u$ and $x + y = \beta v$, that is, $N(u, v) = \big\{(x, y) : x - y = \alpha u, x + y = \beta v\big\}$. For $r \in (0, 2)$ define

$$\delta(u, v, r) = \inf\left\{1 - \frac{\|x + y\|}{2C} : \ x, y \in N(u, v), \ \frac{\|x - y\|}{C} \geq r\right\}. \tag{3.2}$$

It is easy to check $\delta(u, v, r) = 0$ for (3.2) as $\|x\| = 1, \forall x \in N(u, v)$. Moreover, for r, for any given $\lambda_1, \lambda_2 \in (0, 2)$ and $\epsilon > 0$ we can choose $x_k, y_k \in N(u, v)$ such that (for $k = 1, 2$)

$$x_k + y_k \geq \lambda_k \quad \text{and} \quad \delta(u, v, \lambda_k) + \frac{\epsilon}{2} \geq 1 - \frac{\|x_k + y_k\|}{2C}.$$

The choice of (x_k, y_k) is possible because of the definition of $\delta(u, v, r)$ in (3.2) as infimum. Now, for $\lambda \in (0, 1)$ we assume

$$x_3 = \lambda x_1 + (1-\lambda)x_2, \quad y_3 = \lambda y_1 + (1-\lambda)y_2. \tag{3.3}$$

$\|x_3\| = \lambda\|x_1\| + (1-\lambda)\|x_2\|$ because $x_1, x_2 \in \overline{S_{\mathcal{B}}(0)}$. Similarly, $(x_k, y_k) \in N(u, v)$ implies that there exist constants such that (for $k = 1, 2$)

$$x_k - y_k = \alpha_k u, x_k - y_k = \beta_k v. \tag{3.4}$$

From equation (3.3) we have

$$\begin{aligned}
x_3 - y_3 &= \lambda x_1 + (1-\lambda)x_2 - \lambda y_1 - (1-\lambda)y_2 \\
&= \lambda[x_1 - y_1] + (1-\lambda)[x_2 - y_2] \\
&= \lambda[\alpha_1 u] + (1-\lambda)[\alpha_2 u] \\
&= [\lambda\alpha_1 + (1-\lambda)\alpha_2]u.
\end{aligned}$$

Similarly,

$$\begin{aligned}
x_3 - y_3 &= \lambda x_1 + (1-\lambda)x_2 - \lambda y_1 + (1-\lambda)y_2 \\
&= \lambda[x_1 - y_1] + (1-\lambda)[x_2 - y_2] \\
&= \lambda[\beta_1 v] + (1-\lambda)[\beta_2 v] \\
&= [\lambda\beta_1 + (1-\lambda)\beta_2]v.
\end{aligned}$$

Now we have

$$\|x_3 - y_3\| = [\lambda\alpha_1 + (1-\lambda)\alpha_2]\|u\|. \tag{3.5}$$

Similarly,

$$\|x_3 - y_3\| = [\lambda\beta_1 + (1-\lambda)\beta_2]\|v\|. \tag{3.6}$$

Therefore, using (3.5) and (3.6), generally, we get,

$$\|x_3 - y_3\| = \lambda\epsilon_1 + (1-\lambda)\epsilon_2. \tag{3.7}$$

Now we have

$$\begin{aligned}
\delta(u, v, [\lambda(\epsilon_1) + (1-\lambda)\epsilon_2]) &\leq 1 - \frac{\|x_3 + y_3\|}{2C} \\
&= 1 - \frac{\lambda\|x_1 + y_1\| + (1-\lambda)\|x_2 + y_2\|}{2C} \\
&\leq \lambda\left[1 - \frac{\|x_1 + y_1\|}{2C}\right] + (1-\lambda)\left[1 - \frac{\|x_2 + y_2\|}{2C}\right] \\
&= \lambda[\delta(u, v, \epsilon_1] + (1-\lambda)[\delta(u, v, \epsilon_2].
\end{aligned}$$

Belonging to some $N(u, v)$ since $\delta_{\mathcal{B}}(u, v, \epsilon)$ is convex, which shows that $\delta_{\mathcal{B}}(\epsilon)$ is convex. Since $\delta_{\mathcal{B}}(\epsilon)$ is convex, so is continuous by Lemma 3.1.

(b) Let $0 \leq \epsilon_1 \leq \epsilon_2 \leq 2$ and $x_1, x_2 \in S_{\mathcal{B}}$ satisfying $\frac{\|x_1 - x_2\|}{C} \leq \epsilon_2$ and $\frac{\|x_1 + x_2\|}{2C} \leq 1 - \delta_{\mathcal{B}}(\epsilon_2)$. Then letting $\mathcal{E} = \frac{\epsilon_2 - \epsilon_1}{2\epsilon_2}$ and $x = x_1 + \mathcal{E}(x_2 - x_1)$ and $y = x_2 - \mathcal{E}(x_2 - x_1)$ we have $x, y \in S_{\mathcal{B}}$ and $\frac{\|x - y\|}{C} \leq \epsilon_1$.

Now applying Lemma 3.2 we get

$$\delta_{\mathcal{B}}(\epsilon_1) \leq 1 - \frac{x+y}{2C} \leq 1 - \frac{x_1+x_2}{2C} \leq \delta_{\mathcal{B}}(\epsilon_2),$$

which shows that $\delta_{\mathcal{B}}(\epsilon)$ is a non-decreasing function.

(c) Fix $\eta \in (0,2]$ with $\eta < \epsilon$. Let $x_1, x_2 \in \mathcal{B}$ such that $\|x_1\| = \|x_2\| = 1$ and $\frac{\|x_1-x_2\|}{C} = \epsilon$. Here, it will be suffices to show that

$$\frac{\delta_{\mathcal{B}}(\eta)}{\eta} \leq \frac{\delta_{\mathcal{B}}(\epsilon)}{\epsilon}.$$

Consider

$$\begin{aligned} u_1 &= \frac{\eta}{\epsilon}x_1 + \left(1 - \frac{\eta}{\epsilon}\right)\left[\frac{x_1+x_2}{\|x_1+x_2\|}\right], \\ u_2 &= \frac{\eta}{\epsilon}x_2 + \left(1 - \frac{\eta}{\epsilon}\right)\left[\frac{x_1+x_2}{\|x_1+x_2\|}\right], \end{aligned}$$

then

$$u_1 - u_2 = \frac{\eta}{\epsilon}(x_1 - x_2) \quad \Longrightarrow \quad \frac{\|u_1-u_2\|}{\eta} = C.$$

And

$$\frac{u_1+u_2}{2} = \left[\frac{x_1+x_2}{\|x_1+x_2\|}\right]\left(1 - \frac{\eta}{\epsilon} + \frac{\eta\|x_1+x_2\|}{2\epsilon}\right),$$

thus

$$\frac{u_1+u_2}{2C} = \left[\frac{x_1+x_2}{\|x_1+x_2\|}\right]\left(\frac{1}{C} - \frac{\eta}{\epsilon C} + \frac{\eta\|x_1+x_2\|}{2\epsilon C}\right).$$

This implies that

$$\begin{aligned} \left\|\frac{x_1+x_2}{\|x_1+x_2\|} - \frac{u_1+u_2}{2C}\right\| &= 1 - \left(\frac{1}{C} - \frac{\eta}{\epsilon C} + \frac{\eta\|x_1+x_2\|}{2\epsilon C}\right) \\ &= 1 - \frac{\|u_1+u_2\|}{2C}. \end{aligned}$$

Here note that

$$\begin{aligned} \left\|\frac{x_1+x_2}{\|x_1+x_2\|} - \frac{x_1+x_2}{2C}\right\| &= \|x_1+x_2\|\left(\frac{1}{\|x_1+x_2\|} - \frac{1}{2C}\right) \\ &= 1 - \frac{\|x_1+x_2\|}{2C}. \end{aligned}$$

Now we have

$$\begin{aligned} \frac{\left\|\frac{x_1+x_2}{\|x_1+x_2\|} - \frac{u_1+u_2}{2C}\right\|}{\|u_1-u_2\|} &= \frac{C}{\eta}\left(\frac{\eta}{\epsilon C} - \frac{\eta\|x_1+x_2\|}{2\epsilon C}\right) \\ &= \frac{1}{\epsilon}\left(1 - \frac{\|x_1+x_2\|}{2C}\right) \\ &= \frac{\left\|\frac{x_1+x_2}{\|x_1+x_2\|} - \frac{x_1+x_2}{2C}\right\|}{\|x_1-x_2\|}, \end{aligned}$$

then

$$\begin{aligned}
\frac{\delta(\eta)}{\eta} = \frac{1-\frac{\|u_1+u_2\|}{2C}}{\|u_1-u_2\|} &= \frac{\left\|\frac{x_1+x_2}{\|x_1+x_2\|}-\frac{u_1+u_2}{2c}\right\|}{\|u_1-u_2\|} \\
&= \frac{\left\|\frac{x_1+x_2}{\|x_1+x_2\|}-\frac{x_1+x_2}{2C}\right\|}{\|x_1-x_2\|} \\
&= \frac{1-\frac{\|x_1+x_2\|}{2C}}{\|x_1-x_2\|} \leq \frac{1}{\epsilon}\left[1-\frac{\|x_1+x_2\|}{2C}\right] \\
&= \frac{1}{\epsilon}\big(\delta_{\mathcal{B}}(\epsilon)\big).
\end{aligned}$$

This completes the proof. □

Proposition 3.4. *Let $\mathcal{B}$ be a quasi-Banach space. Then $\forall 0 \leq \epsilon_1 \leq \epsilon_2 \leq 2$ we have*

$$\frac{\delta_{\mathcal{B}}(\epsilon_2)-\delta_{\mathcal{B}}(\epsilon_1)}{\epsilon_2-\epsilon_1} \leq \frac{1-\delta_{\mathcal{B}}(\epsilon_1)}{2-\epsilon_1}. \tag{3.8}$$

Proof. Let

$$\epsilon_2 = 2\left(\frac{\epsilon_2-\epsilon_1}{2-\epsilon_1}\right)+\epsilon_1\left(1-\frac{\epsilon_2-\epsilon_1}{2-\epsilon_1}\right).$$

Then we have

$$\begin{aligned}
\delta_{\mathcal{B}}(\epsilon_2) &= \delta_{\mathcal{B}}\left[2\left(\frac{\epsilon_2-\epsilon_1}{2-\epsilon_1}\right)+\epsilon_1\left(1-\frac{\epsilon_2-\epsilon_1}{2-\epsilon_1}\right)\right] \\
&\leq \delta_{\mathcal{B}}(2)\left(\frac{\epsilon_2-\epsilon_1}{2-\epsilon_1}\right)+\delta_{\mathcal{B}}(\epsilon_1)\left(1-\frac{\epsilon_2-\epsilon_1}{2-\epsilon_1}\right) \\
&\leq \delta_{\mathcal{B}}(2)\left(\frac{\epsilon_2-\epsilon_1}{2-\epsilon_1}\right)+\delta_{\mathcal{B}}(\epsilon_1)-\delta_{\mathcal{B}}(\epsilon_1)\left(\frac{\epsilon_2-\epsilon_1}{2-\epsilon_1}\right) \\
&= \left(\frac{\epsilon_2-\epsilon_1}{2-\epsilon_1}\right)[\delta_{\mathcal{B}}(2)-\delta_{\mathcal{B}}(\epsilon_1)]+\delta_{\mathcal{B}}(\epsilon_1).
\end{aligned}$$

Now

$$\delta_{\mathcal{B}}(\epsilon_2)-\delta_{\mathcal{B}}(\epsilon_1) \leq \left(\frac{\epsilon_2-\epsilon_1}{2-\epsilon_1}\right)[1-\delta_{\mathcal{B}}(\epsilon_1)].$$

Hence

$$\frac{\delta_{\mathcal{B}}(\epsilon_2)-\delta_{\mathcal{B}}(\epsilon_1)}{\epsilon_2-\epsilon_1} \leq \frac{1-\delta_{\mathcal{B}}(\epsilon_1)}{2-\epsilon_1}.$$

This completes the proof. □

Theorem 3.5. *Let $\mathcal{B}$ be a uniformly convex space. Then for every $d>0$, $\varepsilon>0$, and for arbitrary vectors, $x_1, x_2 \in \mathcal{B}$ with $\|x_1\| \leq d$, $\|x_2\| \leq d$ and $\frac{\|x_1-x_2\|}{C} \geq \varepsilon$, there exists $\delta>0$ such that*

$$\frac{\|x_1+x_2\|}{2C} \leq \left[1-\delta\left(\frac{\varepsilon}{d}\right)\right]d.$$

Proof. For any arbitrary $x_1, x_2 \in \mathcal{B}$ we assume that

$$z_1 = \frac{x_1}{d}, \quad z_2 = \frac{x_2}{d}, \quad \text{and set } \epsilon = \frac{\varepsilon}{d}.$$

Obviously $\epsilon > 0$. Moreover, with $\|x_1\| \leq 1$ and $\|x_2\| \leq 1$ we have

$$\|z_1 - z_2\| = \frac{1}{d}\|x_1 - x_2\| \geq \frac{\varepsilon}{d} = \epsilon.$$

Now, for uniform convexity, we have

$$\delta = \delta\left(\frac{\varepsilon}{d}\right) > 0 \quad \text{and} \quad \frac{\|z_1 + z_2\|}{2C} \leq 1 - \delta(\epsilon),$$

which implies that

$$\frac{\|x_1 + x_2\|}{2dC} \leq 1 - \delta\left(\frac{\varepsilon}{d}\right),$$

thus

$$\frac{\|x_1 + x_2\|}{2C} \leq \left[1 - \delta\left(\frac{\varepsilon}{d}\right)\right] d.$$

This completes the proof. □

Theorem 3.6. *A quasi-Banach space $\mathcal{B}$ is uniformly convex iff $\delta_{\mathcal{B}}(\epsilon) \geq 0$.*

Proof. If X is uniformly convex, then, for given $\epsilon > 0$ there exists $\delta > 0$ such that $\forall x_1, x_2 \in \mathcal{B}$ with $\|x_1\| = 1$, $\|x_2\| = 1$ and $\frac{\|x_1 - x_2\|}{C} \geq \epsilon$

$$1 - \frac{\|x_1 + x_2\|}{2C} \geq \delta \quad \Longrightarrow \quad \delta_{\mathcal{B}}(\epsilon) > 0.$$

Conversely, assume that $\delta_{\mathcal{B}}(\epsilon) > 0$ for every $\epsilon \in (0, 2]$. Let fix $\epsilon \in (0, 2]$ and then take $x_1, x_2 \in \mathcal{B}$ with $\|x_1\| = 1$, $\|x_2\| = 1$ and $\frac{\|x_1 - x_2\|}{C} \geq \epsilon$. Then

$$0 < \delta_{\mathcal{B}}(\epsilon) \leq 1 - \frac{\|x_1 + x_2\|}{2C}.$$

This implies that $1 - \frac{\|x_1 + x_2\|}{2C} \leq 1 - \delta$ with $\delta = \delta_{\mathcal{B}}(\epsilon)$, which does not depends upon either x_1 or x_2. This completes the proof. □

Theorem 3.7. *For arbitrary uniformly convex quasi-Banach space $\mathcal{B}$,*

$$\delta_{\mathcal{B}}(\epsilon) = 1 - \frac{1}{C}\sqrt{1 - \frac{\epsilon^2 C^2}{4}}.$$

Proof. Let $x_1, x_2 \in \mathcal{B}$ with $\|x_1\| = 1$, $\|x_2\| = 1$ and $\frac{\|x_1 - x_2\|}{C} = \epsilon$. Then using the parallelogram identity,

$$\|x_1 + x_2\|^2 + \|x_1 - x_2\|^2 = 2\big(\|x_1\|^2 + \|x_2\|^2\big),$$

thus

$$\begin{aligned} \|x_1 + x_2\|^2 &= 2\big(\|x_1\|^2 + \|x_2\|^2\big) - \|x_1 - x_2\|^2 \\ &= 2(1^2 + 1^2) - \|x_1 - x_2\|^2 \\ &= 2(2) - (\epsilon C)^2 \\ &= 4 - \epsilon^2 C^2, \end{aligned}$$

hence

$$\|x_1 + x_2\| = \sqrt{4 - \epsilon^2 C^2},$$

thus we have

$$1 - \frac{\|x_1 + x_2\|}{2C} = 1 - \frac{\sqrt{4 - \epsilon^2 C^2}}{2C},$$

which implies that

$$\inf \left\{ 1 - \frac{\|x_1 + x_2\|}{2C} \right\} = 1 - \frac{1}{C}\sqrt{1 - \frac{\epsilon^2 C^2}{4}}.$$

Hence, we get

$$\delta_{\mathcal{B}}(\epsilon) = 1 - \frac{1}{C}\sqrt{1 - \frac{\epsilon^2 C^2}{4}}.$$

This completes the proof. □

Theorem 3.8. *A non-trivial quasi-Banach space $\mathcal{B}$ is uniformly non-square if and only if $\delta_{\mathcal{B}}(\epsilon) > 0$.*

Proof. Let $\mathcal{B}$ be uniformly non-square. Set $\epsilon = 2 - 2\delta$, $\epsilon \in (0, 2)$. Then

$$\delta_{\mathcal{B}}(\epsilon) \geq 1 - \frac{\epsilon}{2} > 0.$$

Conversely, let there is $\epsilon_0 \in (0, 2)$ such that $\delta_{\mathcal{B}}(\epsilon) > 0$, that is,

$$\delta_{\mathcal{B}}(\epsilon) \geq \eta_0 > 0 \quad \text{for some } \eta_0 \in (0, 1).$$

Let $2 - 2\delta = \epsilon \in [\epsilon_0, 2)$. Then

$$\delta \in (0, 1 - \epsilon_0/2] \quad \text{and} \quad \delta_{\mathcal{B}}(2 - 2\delta) = \delta_{\mathcal{B}}(\epsilon) \geq \eta_0 > 0.$$

This indicates that for any $x, y \in S_{\mathcal{B}}$, if

$$\frac{\|x - y\|}{C} \geq 2 - 2\delta,$$

then

$$1 - \frac{\|x + y\|}{2C} \geq \eta_0.$$

Let $\delta' = \min\{\delta, \eta_0\}$. Then of course $\delta' \in (0, 1)$.

Now we just need to show that either $\frac{\|x-y\|}{2C} \leq 1 - \delta'$ or $\frac{\|x+y\|}{2C} \leq 1 - \delta'$. If

$$\frac{\|x + y\|}{2C} \leq 1 - \delta',$$

then we are done:

Let we consider

$$\frac{\|x + y\|}{2C} > 1 - \delta'.$$

Then

$$\|x - y\| > 2C(1 - \delta') \geq 2C(1 - \delta).$$

By this assumption we get

$$1 - \frac{\|x+y\|}{2C} \geq \eta_0,$$

which implies that

$$\frac{\|x+y\|}{2C} \leq 1 - \eta_0 \leq (1 - \delta'),$$

which shows that $\mathcal{B}$ is uniformly non-square. This completes the proof. □

Proposition 3.9. *Let $\mathcal{B}$ be a quasi-Banach space and H be a Hilbert space. Then*

$$\delta_{\mathcal{B}}(\epsilon) \leq \delta_H(\epsilon), \quad \forall \epsilon \in [0,2]. \tag{3.9}$$

Proof. From Theorem 3.7 we can easily prove the result. □

References

[1] A. B. Alexandrov, The existence of inner functions in a ball, *Mat. Sb. (N.S.)*, **118(160)** (1982), 147–163.

[2] T. Aoki, Locally bounded linear topological spaces, *Proc. Imp. Acad. Tokyo*, **18** (1942), 588–594.

[3] A. Beurling and A. E. Livingston, A theorem on duality mappings in Banach spaces, *Ark. Mat.*, **4** (1962), 405–411.

[4] J. A. Clarkson, Uniformly convex spaces, *Trans. Amer. Math. Soc.*, **40** (1936), 396–414.

[5] J. A. Clarkson, The von Neumann-Jordan constant for the Lebesgue spaces, *Ann. of Math.*, **38** (1937), 114–115.

[6] M. M. Day, The spaces L^p with $0 < p < 1$, *Bull. Amer. Math. Soc.*, **46** (1940), 816–823.

[7] P. L. Duren, B. W. Romberg, and A. L. Shields, Linear functionals on H^p spaces with $0 < p < 1$, *J. Reine Angew. Math.*, **238** (1969), 32–60.

[8] D. G. de Figueiredo, Topics in Nonlinear Functional Analysis (Vol. 48), University of Maryland, Institute for Fluid Dynamics and Applied Mathematics, 1967.

[9] H. M. U. A. Qadri and Q. Mehmood, On moduli and constants of quasi-Banach space, *Open J. Math. Sci.*, (in press)

[10] H. W. E. Jung, Über die kleinste kugel die eine räumliche figur einschliesst, University of California Libraries, 1899.

[11] Y. C. Kwun, Q. Mehmood, W. Nazeer, A. U. Haq and S. M. Kang, Relations between generalized von Neumann-Jordan and James constants for quasi-Banach spaces, *J. Inequal. Appl.*, **2016** (2016), Article ID 171, 10 pages.

[12] W. Nazeer, Q. Mehmood, S. M. Kang, and A. U. Haq, Generalized von Neumann-Jordan and James constants for quasi-Banach spaces, *J. Comput. Anal. Appl.*, **25** (2018), 1043–1052.

[13] J. W. Roberts, Pathological compact convex sets in $l_p[0,1]$, $0 \leq p < 1$. In The Altgeld Book, University of Illinois Functional Analysis Seminar, 1975-76.

[14] J. W. Roberts, A nonlocally convex F-space with the Hahn-Banach approximation property, Banach spaces of analytic functions (Proc. Pelczynski Conf., Kent State Univ., Kent, Ohio, 1976). Lecture Notes in Math. 604, Springer Berlin, 1977, pp. 76–81.

[15] S. Rolewicz, On a certain class of linear metric spaces, *Bull. Acad. Polon. Sci.*, **5** (1957), 471–473.

[16] P. Turpin, Convexités dans les espaces vectoriels topologiques généraux, *Dissertationes Math.* (*Rozprawy Mat.*), **131** (1976), 221 pages.

TABLE OF CONTENTS, JOURNAL OF COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 27, NO. 5, 2019

TABLE OF CONTENTS, JOURNAL OF COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 27, NO. 5, 2019

(continued)

Volume 27, Number 6 November 15, 2019
ISSN:1521-1398 PRINT,1572-9206 ONLINE

Journal of Computational Analysis and Applications

EUDOXUS PRESS,LLC

Journal of Computational Analysis and Applications
ISSNno.'s:1521-1398 PRINT,1572-9206 ONLINE

SCOPE OF THE JOURNAL

An international publication of Eudoxus Press, LLC
(fifteen times annually)

Editor in Chief: George Anastassiou
Department of Mathematical Sciences,
University of Memphis, Memphis, TN 38152-3240, U.S.A
ganastss@memphis.edu
http://www.msci.memphis.edu/~ganastss/jocaaa

The main purpose of "J.Computational Analysis and Applications" is to publish high quality research articles from all subareas of Computational Mathematical Analysis and its many potential applications and connections to other areas of Mathematical Sciences. Any paper whose approach and proofs are computational,using methods from Mathematical Analysis in the broadest sense is suitable and welcome for consideration in our journal, except from Applied Numerical Analysis articles. Also plain word articles without formulas and proofs are excluded. The list of possibly connected mathematical areas with this publication includes, but is not restricted to: Applied Analysis, Applied Functional Analysis, Approximation Theory, Asymptotic Analysis, Difference Equations, Differential Equations, Partial Differential Equations, Fourier Analysis, Fractals, Fuzzy Sets, Harmonic Analysis, Inequalities, Integral Equations, Measure Theory, Moment Theory, Neural Networks, Numerical Functional Analysis, Potential Theory, Probability Theory, Real and Complex Analysis, Signal Analysis, Special Functions, Splines, Stochastic Analysis, Stochastic Processes, Summability, Tomography, Wavelets, any combination of the above, e.t.c.

"J.Computational Analysis and Applications" is a peer-reviewed Journal. See the instructions for preparation and submission of articles to JoCAAA. Assistant to the Editor: Dr.Razvan Mezei,mezei_razvan@yahoo.com, Madison,WI,USA.

Journal of Computational Analysis and Applications(JoCAAA) is published by **EUDOXUS PRESS,LLC**,1424 Beaver Trail Drive,Cordova,TN38016,USA,anastassioug@yahoo.com http://www.eudoxuspress.com. **Annual Subscription Prices**:For USA and Canada,Institutional:Print $800, Electronic OPEN ACCESS. Individual:Print $400. For any other part of the world add $160 more(handling and postages) to the above prices for Print. No credit card payments.

JoCAAA is reviewed and abstracted by AMS Mathematical Reviews,MATHSCI,and Zentralblaat MATH.

Editorial Board
Associate Editors of Journal of Computational Analysis and Applications

Dartmouth College
8000 Cummings Hall,
Hanover, NH 03755-8000
603-646-3843 (X 3546 Secr.)
e-mail:george.cybenko@dartmouth.edu
Approximation Theory and Neural Networks

Sever S. Dragomir
School of Computer Science and Mathematics, Victoria University,
PO Box 14428,
Melbourne City,
MC 8001, AUSTRALIA
Tel. +61 3 9688 4437
Fax +61 3 9688 4050
sever.dragomir@vu.edu.au
Inequalities, Functional Analysis, Numerical Analysis, Approximations, Information Theory, Stochastics.

Oktay Duman
TOBB University of Economics and Technology,
Department of Mathematics, TR-06530,
Ankara, Turkey,
oduman@etu.edu.tr
Classical Approximation Theory, Summability Theory, Statistical Convergence and its Applications

Saber N. Elaydi
Department Of Mathematics
Trinity University
715 Stadium Dr.
San Antonio, TX 78212-7200
210-736-8246
e-mail: selaydi@trinity.edu
Ordinary Differential Equations, Difference Equations

J .A. Goldstein
Department of Mathematical Sciences
The University of Memphis
Memphis, TN 38152
901-678-3130
jgoldste@memphis.edu
Partial Differential Equations, Semigroups of Operators

H. H. Gonska
Department of Mathematics
University of Duisburg
Duisburg, D-47048
Germany
011-49-203-379-3542
e-mail: heiner.gonska@uni-due.de
Approximation Theory, Computer Aided Geometric Design

John R. Graef
Department of Mathematics
University of Tennessee at Chattanooga
Chattanooga, TN 37304 USA
John-Graef@utc.edu
Ordinary and functional differential equations, difference equations, impulsive systems, differential inclusions, dynamic equations on time scales, control theory and their applications

Weimin Han
Department of Mathematics
University of Iowa
Iowa City, IA 52242-1419
319-335-0770
e-mail: whan@math.uiowa.edu
Numerical analysis, Finite element method, Numerical PDE, Variational inequalities, Computational mechanics

Tian-Xiao He
Department of Mathematics and Computer Science
P.O. Box 2900, Illinois Wesleyan University
Bloomington, IL 61702-2900, USA
Tel (309)556-3089
Fax (309)556-3864
the@iwu.edu
Approximations, Wavelet, Integration Theory, Numerical Analysis, Analytic Combinatorics

Margareta Heilmann
Faculty of Mathematics and Natural Sciences, University of Wuppertal
Gaußstraße 20
D-42119 Wuppertal, Germany,
heilmann@math.uni-wuppertal.de
Approximation Theory (Positive Linear Operators)

Xing-Biao Hu
Institute of Computational Mathematics
AMSS, Chinese Academy of Sciences
Beijing, 100190, CHINA
hxb@lsec.cc.ac.cn

Instructions to Contributors
Journal of Computational Analysis and Applications

An international publication of Eudoxus Press, LLC, of TN.

Editor in Chief: George Anastassiou

Department of Mathematical Sciences
University of Memphis
Memphis, TN 38152-3240, U.S.A.

1. Manuscripts files in Latex and PDF and in English, should be submitted via email to the Editor-in-Chief:

Prof.George A. Anastassiou
Department of Mathematical Sciences
The University of Memphis
Memphis,TN 38152, USA.
Tel. 901.678.3144
e-mail: ganastss@memphis.edu

Authors may want to recommend an associate editor the most related to the submission to possibly handle it.

Also authors may want to submit a list of six possible referees, to be used in case we cannot find related referees by ourselves.

2. Manuscripts should be typed using any of TEX,LaTEX,AMS-TEX,or AMS-LaTEX and according to EUDOXUS PRESS, LLC. LATEX STYLE FILE. (Click HERE to save a copy of the style file.)They should be carefully prepared in all respects. Submitted articles should be brightly typed (not dot-matrix), double spaced, in ten point type size and in 8(1/2)x11 inch area per page. Manuscripts should have generous margins on all sides and should not exceed 24 pages.

3. Submission is a representation that the manuscript has not been published previously in this or any other similar form and is not currently under consideration for publication elsewhere. A statement transferring from the authors(or their employers,if they hold the copyright) to Eudoxus Press, LLC, will be required before the manuscript can be accepted for publication.The Editor-in-Chief will supply the necessary forms for this transfer.Such a written transfer of copyright,which previously was assumed to be implicit in the act of submitting a manuscript,is necessary under the U.S.Copyright Law in order for the publisher to carry through the dissemination of research results and reviews as widely and effective as possible.

4. The paper starts with the title of the article, author's name(s) (no titles or degrees), author's affiliation(s) and e-mail addresses. The affiliation should comprise the department, institution (usually university or company), city, state (and/or nation) and mail code.

The following items, 5 and 6, should be on page no. 1 of the paper.

5. An abstract is to be provided, preferably no longer than 150 words.

6. A list of 5 key words is to be provided directly below the abstract. Key words should express the precise content of the manuscript, as they are used for indexing purposes.

The main body of the paper should begin on page no. 1, if possible.

7. All sections should be numbered with Arabic numerals (such as: 1. INTRODUCTION) .
Subsections should be identified with section and subsection numbers (such as 6.1. Second-Value Subheading).
If applicable, an independent single-number system (one for each category) should be used to label all theorems, lemmas, propositions, corollaries, definitions, remarks, examples, etc. The label (such as Lemma 7) should be typed with paragraph indentation, followed by a period and the lemma itself.

8. Mathematical notation must be typeset. Equations should be numbered consecutively with Arabic numerals in parentheses placed flush right, and should be thusly referred to in the text [such as Eqs.(2) and (5)]. The running title must be placed at the top of even numbered pages and the first author's name, et al., must be placed at the top of the odd numbed pages.

9. Illustrations (photographs, drawings, diagrams, and charts) are to be numbered in one consecutive series of Arabic numerals. The captions for illustrations should be typed double space. All illustrations, charts, tables, etc., must be embedded in the body of the manuscript in proper, final, print position. In particular, manuscript, source, and PDF file version must be at camera ready stage for publication or they cannot be considered.

Tables are to be numbered (with Roman numerals) and referred to by number in the text. Center the title above the table, and type explanatory footnotes (indicated by superscript lowercase letters) below the table.

10. List references alphabetically at the end of the paper and number them consecutively. Each must be cited in the text by the appropriate Arabic numeral in square brackets on the baseline.
References should include (in the following order):
initials of first and middle name, last name of author(s)
title of article,

name of publication, volume number, inclusive pages, and year of publication.

Authors should follow these examples:

<u>Journal Article</u>

1. H.H.Gonska,Degree of simultaneous approximation of bivariate functions by Gordon operators, (journal name in italics) *J. Approx. Theory*, 62,170-191(1990).

<u>Book</u>

2. G.G.Lorentz, (title of book in italics) *Bernstein Polynomials* (2nd ed.), Chelsea,New York,1986.

<u>Contribution to a Book</u>

3. M.K.Khan, Approximation properties of beta operators,in(title of book in italics) *Progress in Approximation Theory* (P.Nevai and A.Pinkus,eds.), Academic Press, New York,1991,pp.483-495.

11. All acknowledgements (including those for a grant and financial support) should occur in one paragraph that directly precedes the References section.

12. Footnotes should be avoided. When their use is absolutely necessary, footnotes should be numbered consecutively using Arabic numerals and should be typed at the bottom of the page to which they refer. Place a line above the footnote, so that it is set off from the text. Use the appropriate superscript numeral for citation in the text.

13. After each revision is made please again submit via email Latex and PDF files of the revised manuscript, including the final one.

14. Effective 1 Nov. 2009 for current journal page charges, contact the Editor in Chief. Upon acceptance of the paper an invoice will be sent to the contact author. The fee payment will be due one month from the invoice date. The article will proceed to publication only after the fee is paid. The charges are to be sent, by money order or certified check, in US dollars, payable to Eudoxus Press, LLC, to the address shown on the Eudoxus homepage.

No galleys will be sent and the contact author will receive one (1) electronic copy of the journal issue in which the article appears.

15. This journal will consider for publication only papers that contain proofs for their listed results.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 27, NO.6, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC

Asymptotic behavior of equilibrium point for a system of fourth-order rational difference equations

Ping Liu [1], Changyou Wang [1, 2*], Yonghong Li [1*], Rui Li [3]

1. College of Science, Chongqing University of Posts and Telecommunications, Chongqing, 400065, People's Republic of China
2. College of Applied Mathematics, Chengdu University of Information Technology, Chengdu, Sichuan 610225, People's Republic of China
3. College of Automation, Chongqing University of Posts and Telecommunications, Chongqing 400065, People's Republic of China

Abstract

Our aim in this paper is to investigate the dynamics of a system of fourth-order rational difference equations

$$x_{n+1} = \frac{x_{n\text{-}3} - y_{n\text{-}1}}{A + x_{n\text{-}3}y_{n\text{-}1}}, \quad y_{n+1} = \frac{y_{n\text{-}3} - x_{n\text{-}1}}{A + y_{n\text{-}3}x_{n\text{-}1}}, \quad n = 0, 1, \cdots,$$

where the parameter A is arbitrary positive real number and the initial conditions $x_{-3}, x_{-2}, x_{-1}, x_0, y_{-3}, y_{-2}, y_{-1}, y_0$ are arbitrary nonnegative real numbers. By using new iteration method for the more general nonlinear difference equations and inequality skills, we establish some sufficient conditions which guarantee the existence, unstability and global asymptotic stability of the equilibriums for this nonlinear system. Numerical examples to the difference system are given to verify our theoretical results.

Keywords: difference system; equilibrium point; asymptotical stability; unstability

1. Introduction

Because of the necessity for some techniques that can be used in mathematical models describing real situations, nonlinear difference equations have been studied in the fields of population biology, economics, probability theory, genetics, psychology etc (see, e.g., [1-4] and the references therein). In recent years, with the dramatically development of

---

*Corresponding authors at: College of Science, Chongqing University of Posts and Telecommunications, Chongqing, 400065, People's Republic of China
Email addresses: wangcy@cuit.edu.cn (C.Y. Wang), liyh@cqupt.edu.cn (Y.H. Li).

computer-based computational techniques, difference equations are found to be much appropriate mathematical representations for computer simulation and experiment (see, e.g., [5-8]). However, it is more interesting to investigate the behavior of solutions of a system of higher-order rational difference equations and to discuss the asymptotic stability of their equilibrium points (for example, see [9-19]).

Recently, Bajo and Liz [20] described the asymptotic behavior and the stability properties of the solution to the following nonlinear second-order difference equation

$$x_{n+1}=\frac{x_{n-1}}{a+bx_nx_{n-1}},\quad n=0,1,\cdots. \tag{1.1}$$

for all values of the real parameters $a,b,$ and any initial condition $(x_{-1},x_0)\in R^2$.

In [21], Kurbanli, Cinar, and Yalcinkaya investigated the positive solutions of the following difference equations

$$x_{n+1}=\frac{x_{n-1}}{y_nx_{n-1}+1},\quad y_{n+1}=\frac{y_{n-1}}{x_ny_{n-1}+1},\qquad n=0,1,\cdots, \tag{1.2}$$

where $(x_k,y_k)\in[0,\infty)$ for $k=-1,0$.

Moreover, Touafek and Elsayed [22] deal with the periodic nature and the form of the solutions of the following systems of rational difference equations.

$$x_{n+1}=\frac{x_{n-3}}{\pm1\pm x_{n-3}y_{n-1}},\quad y_{n+1}=\frac{y_{n-3}}{\pm1\pm y_{n-3}x_{n-1}},\quad n=0,1,\cdots, \tag{1.3}$$

with a nonzero real number's initial conditions.

As an extension of (1.3), Elsayed [23] continuously dealt with the existence of solutions and the periodicity character of the following systems of rational difference equations

$$x_{n+1}=\frac{x_ny_{n-1}}{\mathrm{y}_n(x_ny_{n-1}\pm1)},\quad y_{n+1}=\frac{y_nx_{n-1}}{x_n(y_nx_{n-1}\pm1)},\quad n=0,1,\cdots, \tag{1.4}$$

where the initial conditions x_{-1},x_0,y_{-1} and y_0 are nonzero real numbers.

More recently, Khan and Qureshi [24] study the equilibrium points, local asymptotic stability of equilibrium point, unstability of equilibrium points, global character of equilibrium point, periodicity behavior of positive solutions and rate of convergence of positive solutions of the following systems

$$x_{n+1}=\frac{\alpha x_{n-1}}{\beta-\gamma y_ny_{n-1}},\quad y_{n+1}=\frac{\alpha_1y_{n-1}}{\beta_1-\gamma_1x_nx_{n-1}},\quad n=0,1,\cdots, \tag{1.5}$$

and

$$x_{n+1} = \frac{ay_{n-1}}{b - cx_n x_{n-1}}, \quad y_{n+1} = \frac{a_1 x_{n-1}}{b_1 - c_1 y_n y_{n-1}}, \quad n = 0, 1, \cdots. \tag{1.6}$$

Especially, Yalçınkaya [25] investigated the sufficient condition for the global asymptotic stability of the following systems of difference equations

$$x_{n+1} = \frac{x_n + y_{n-1}}{x_n y_{n-1} - 1}, \quad y_{n+1} = \frac{y_n + x_{n-1}}{y_n x_{n-1} - 1}, \qquad n = 0, 1, \cdots. \tag{1.7}$$

where the initial conditions x_{-1}, x_0, y_{-1} and y_0 are nonzero real numbers.

Motivated by works [20-25], our aim in this paper is to investigate the dynamics of a system of fourth-order rational difference equations

$$x_{n+1} = \frac{x_{n-3} - y_{n-1}}{A + x_{n-3} y_{n-1}}, \quad y_{n+1} = \frac{y_{n-3} - x_{n-1}}{A + y_{n-3} x_{n-1}}, \quad n = 0, 1, \cdots, \tag{1.8}$$

where $A \in (0, \infty)$ and $(x_n, y_n) \in [0, \infty) \times [0, \infty)$ for $n = -3, -2, -1, 0, \cdots$.

For more related work, one can refer to [26-35] and references therein.

2. Some preliminary results

To prove the main results in this paper we first give some definitions and preliminary results [36-38] which are basically used throughout this paper.

Lemma 2.1 Let I_x, I_y be some intervals of real numbers and let $f : I_x^{k+1} \times I_y^{l+1} \to I_x$, $g : I_x^{k+1} \times I_y^{l+1} \to I_y$ be continuously differentiable functions. Then for every set of initial conditions $(x_i, y_j) \in I_x \times I_y$, $(i = -k, -k+1, \cdots, 0,\ j = -l, -l+1, \cdots, 0)$, the following system of difference equations

$$\begin{cases} x_{n+1} = f(x_n, x_{n-1}, \cdots, x_{n-k}, y_n, y_{n-1}, \cdots, y_{n-l}), \\ y_{n+1} = g(x_n, x_{n-1}, \cdots, x_{n-k}, y_n, y_{n-1}, \cdots, y_{n-l}), \end{cases} \quad n = 0, 1, 2, \cdots, \tag{2.1}$$

has a unique solution $\{(x_i, y_j)\}_{i=-k, j=-l}^{+\infty, +\infty}$.

Definition 2.1 A point $(\bar{x}, \bar{y}) \in I_x \times I_y$ is called an equilibrium point of system (2.1) if

$$\bar{x} = f(\bar{x}, \bar{x}, \cdots, \bar{x}, \bar{y}, \bar{y}, \cdots, \bar{y}), \quad \bar{y} = g\left(\bar{x}.\bar{x}, \cdots, \bar{x}, \bar{y}, \bar{y}, \cdots, \bar{y}\right).$$

That is, $(x_n, y_n) = (\bar{x}, \bar{y})$ for $n \ge 0$ is the solution of difference system (2.1), or equivalently, $(\bar{x}, \bar{y})$ is a fixed point of the vector map (f, g).

Definition 2.2 Assume that $(\bar{x}, \bar{y})$ be an equilibrium point of the system (2.1). Then, we have

(i) An equilibrium point $(\bar{x}, \bar{y})$ is called locally stable if for every $\varepsilon > 0$, there exits $\delta > 0$ such that for any initial conditions $(x_i, y_i) \in I_x \times I_y$ $(i = -k, \cdots, 0, j = -l, \cdots, 0)$, with $\sum_{i=-k}^{0} |x_i - \bar{x}| < \delta$, $\sum_{j=-l}^{0} |y_j - \bar{y}| < \delta$, we have $|x_n - \bar{x}| < \varepsilon, |y_n - \bar{y}| < \varepsilon$ for any $n > 0$.

(ii) An equilibrium point $(\bar{x}, \bar{y})$ is called attractor if $lim_{n\to\infty} x_n = \bar{x}$, $lim_{n\to\infty} y_n = \bar{y}$ for any initial conditions $(x_i, y_i) \in I_x \times I_y$ $(i = -k, \cdots, 0, j = -l, \cdots, 0)$.

(iii) An equilibrium point $(\bar{x}, \bar{y})$ is called asymptotically stable if it is stable, and $(\bar{x}, \bar{y})$ is also attractor.

(iv) An equilibrium point $(\bar{x}, \bar{y})$ is called unstable if it is not locally stable.

Definition 2.3 Let $(\bar{x}, \bar{y})$ be an equilibrium point of the vector map $F = (f, x_n, \cdots, x_{n-k}, g, y_n, \cdots, y_{n-l})$, where f and g are continuously differentiable functions at $(\bar{x}, \bar{y})$. The linearized system of (1.8) about the equilibrium point $(\bar{x}, \bar{y})$ is $X_{n+1} = F(X_n) = F_j \cdot X_n$, where F_J is the Jacobian matrix of the system (1.8) about $(\bar{x}, \bar{y})$ and $X_n = (x_n, \cdots, x_{n-k}, y_n, \cdots, y_{n-l})^T$.

Definition 2.4 let p, q, s, t be four nonnegative integers such that $p + q = n, s + t = m$. Splitting $x = (x_1, x_2, \cdots, x_n)$ into $x = ([x]_p, [x]_q)$ and $y = (y_1, y_2, \cdots, y_m)$ into $y = ([y]_s, [y]_t)$, where $[x]_\sigma$ denotes a vector with σ-components of x. We say that the function $f(x_1, x_2, \cdots, x_n, y_1, y_2, \cdots, y_m)$ possesses a mixed monotone property in subsets $I_x^n \times I_y^m$ of $R^n \times R^m$ if $f([x]_p, [x]_q, [y]_s, [y]_t)$ is monotone non-decreasing in each component of $([x]_p, [y]_s)$, and is monotone non-increasing in each component of $([x]_q, [y]_t)$ for $(x, y) \in I_x^n \times I_y^m$. In particular, if $q = 0, t = 0$, then it is said to be monotone non-decreasing in $I_x^n \times I_y^m$.

Lemma 2.2 Assume that $X(n+1) = F(X(n))$, $n = 0, 1, \cdots$, is a system of difference equations and $\bar{X}$ is the equilibrium point of this system i.e., $F(\bar{X}) = \bar{X}$. Then we have

(i) If all eigenvalues of the Jacobian matrix J_F about $\bar{X}$ lie inside the open unit disk

$|\lambda|<1$, then $\bar{X}$ is locally asymptotically stable.

(ii) If one of eigenvalues of the Jacobian matrix J_F about $\bar{X}$ has norm greater than one, then $\bar{X}$ is unstable.

Lemma 2.3 Assume that $X(n+1)=F(X(n))$, $n=0,1,\cdots$, is a system of difference equations and $\bar{X}$ is the equilibrium point of this system, the characteristic polynomial of this system about the equilibrium point $\bar{X}$ is $P(\lambda)=a_0\lambda^n+a_1\lambda^{n-1}+\cdots+a_{n-1}\lambda+a_n=0$, with the real coefficients and $a_0>0$. Then all roots of the polynomial $P(\lambda)$ lie inside the open unit disk $|\lambda|<1$ if and only if

$$\Delta_k>0 \quad \text{for} \quad k=1,2,\cdots,n, \tag{2.2}$$

where Δ_k is the principal minor of order k of the $n\times n$ matrix

$$\Delta_n=\begin{bmatrix} a_1 & a_3 & a_5 & \cdots & 0 \\ a_0 & a_2 & a_4 & \cdots & 0 \\ 0 & a_1 & a_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_n \end{bmatrix}.$$

3. Main results and their proofs

In this section, we shall investigate the qualitative behavior of the system (1.8). Let $(\bar{x},\bar{y})$ be an equilibrium point of system (1.8), then the system (1.8) has a unique equilibrium $(0,0)$ when $0<A\le 2$, and the system (1.8) has following three equilibrium points $P_0=(0,0)$, $P_1=(\sqrt{A-2},-\sqrt{A-2})$, and $P_2=(-\sqrt{A-2},\sqrt{A-2})$ if $A>2$.

To construct corresponding linearlized form of the nonlinear system (1.8), we consider the transformation

$$(x_n,x_{n-1},x_{n-2},x_{n-3},y_n,y_{n-1},y_{n-2},y_{n-3})\mapsto(f,f_1,f_2,f_3,g,g_1,g_2,g_3)\ , \tag{3.1}$$

where

$$f=\frac{x_{n-3}-y_{n-1}}{A+x_{n-3}y_{n-1}},f_i=x_{n-i+1},g=\frac{y_{n-3}-x_{n-1}}{A+y_{n-3}x_{n-1}},\ g_i=y_{n-i+1},\ i=1,2,3.$$

The Jacobian matrix about the equilibrium point $(\overline{x},\overline{y})$ under the transformation (3.1) is given by

$$F_J(\overline{x},\overline{y})=\begin{bmatrix} 0 & 0 & 0 & \delta_1 & 0 & \delta_2 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \delta_3 & 0 & 0 & 0 & 0 & 0 & \delta_4 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix},$$

where $\delta_1=\dfrac{A+\overline{y}^2}{(A+\overline{x}\,\overline{y})^2}$, $\delta_2=\dfrac{-A-\overline{x}^2}{(A+\overline{x}\,\overline{y})^2}$, $\delta_2=\dfrac{-A-\overline{y}^2}{(A+\overline{x}\,\overline{y})^2}$, $\delta_4=\dfrac{A+\overline{x}^2}{(A+\overline{x}\,\overline{y})^2}$.

Theorem 3.1 If $A>1$, then the equilibrium point $(0,0)$ of the system (1.8) is locally asymptotically stable.

Proof: We can easily obtain that the linearlized system of (1.8) about the equilibrium point $(0,0)$ is

$$\varphi_{n+1}=D\varphi_n \tag{3.2}$$

where

$$\varphi_n=\begin{bmatrix} x_n \\ x_{n-1} \\ x_{n-2} \\ x_{n-3} \\ y_n \\ y_{n-1} \\ y_{n-2} \\ y_{n-3} \end{bmatrix}, \qquad D=\begin{bmatrix} 0 & 0 & 0 & \frac{1}{A} & 0 & -\frac{1}{A} & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{A} & 0 & 0 & 0 & 0 & 0 & \frac{1}{A} \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

The characteristic equation of (3.2) is

$$f(\lambda)=(\lambda^4-\frac{1}{A})^2=0\,. \tag{3.3}$$

In view of $A>1$, it is clear that all roots of characteristic equation (3.3) lie inside unit disk. Hence the equilibrium $(0,0)$ is locally asymptotically stable by Lemma 2.1.

Theorem 3.2 Let I_x, I_y be some intervals of real numbers and assume that $f: I_x^{k+1} \times I_y^{l+1} \to I_x$ and $g: I_x^{k+1} \times I_y^{l+1} \to I_y$ be continuously differentiable functions satisfying mixed monotone property. If there exits

$$\begin{cases} m_0 \le \min\{x_{-k}, \cdots, x_0, y_{-l}, \cdots, y_0\} \le \max\{x_{-k}, \cdots, x_0, y_{-l}, \cdots, y_0\} \le M_0, \\ n_0 \le \min\{x_{-k}, \cdots, x_0, y_{-l}, \cdots, y_0\} \le \max\{x_{-k}, \cdots, x_0, y_{-l}, \cdots, y_0\} \le N_0, \end{cases} \tag{3.4}$$

such that

$$\begin{cases} m_0 \le f([m_0]_p, [M_0]_q, [n_0]_s, [N_0]_t) \le f([M_0]_p, [m_0]_q, [N_0]_s, [n_0]_t) \le M_0, \\ n_0 \le g([m_0]_{p_1}, [M_0]_{q_1}, [n_0]_{s_1}, [N_0]_{t_1}) \le g([M_0]_{p_1}, [m_0]_{q_1}, [N_0]_{s_1}, [n_0]_{t_1}) \le N_0, \end{cases} \tag{3.5}$$

then there exit $(m, M) \in [m_0, M_0]^2$ and $(n, N) \in [n_0, N_0]^2$ satisfying

$$\begin{cases} M = f([M]_p, [m]_q, [N]_s, [n]_t), \quad m = f([m]_p, [M]_q, [n]_s, [N]_t), \\ N = g([M]_{p_1}, [m]_{q_1}, [N]_{s_1}, [n]_{t_1}), \quad n = g([m]_{p_1}, [M]_{q_1}, [n]_{s_1}, [N]_{t_1}). \end{cases} \tag{3.6}$$

Moreover, if $m = M, n = N$, then equation (2.1) has a unique equilibrium point $(\bar{x}, \bar{y}) \in [m_0, M_0] \times [n_0, N_0]$ and every solution of (2.1) converges to $(\bar{x}, \bar{y})$.

Proof. Using m_0, M_0, n_0 and N_0 as two couples of initial iterations, we construct four sequences $\{m_i\}, \{M_i\}, \{n_i\}$, and $\{N_i\}$ $(i = 1, 2, \cdots)$ from the following equations

$$\begin{cases} m_i = f([m_{i-1}]_p, [M_{i-1}]_q, [n_{i-1}]_s, [N_{i-1}]_t), \quad M_i = f([M_{i-1}]_p, [m_{i-1}]_q, [N_{i-1}]_s, [n_{i-1}]_t), \\ n_i = g([m_{i-1}]_{p_1}, [M_{i-1}]_{q_1}, [n_{i-1}]_{s_1}, [N_{i-1}]_{t_1}), \quad N_i = g([M_{i-1}]_{p_1}, [m_{i-1}]_{q_1}, [N_{i-1}]_{s_1}, [n_{i-1}]_{t_1}). \end{cases} \tag{3.7}$$

It is obvious from the mixed monotone property of f and g that the sequences $\{m_i\}, \{M_i\}$, $\{n_i\}$ and $\{N_i\}$ possess the following monotone property

$$\begin{cases} m_0 \le m_1 \le \cdots \le m_i \le \cdots \le M_i \le \cdots \le M_1 \le M_0, \\ n_0 \le n_1 \le \cdots \le n_i \le \cdots \le N_i \le \cdots \le N_1 \le N_0, \end{cases} \tag{3.8}$$

where i=0, 1, 2, $\cdots$, and

$$m_i \le x_u \le M_i, n_i \le y_v \le N_i, \text{ for } u \ge (k+1)i + 1, v \ge (l+1)i + 1, i = 0, 1, 2, \cdots. \tag{3.9}$$

Set

$$m = \lim_{i \to \infty} m_i, M = \lim_{i \to \infty} M_i, \quad n = \lim_{i \to \infty} n_i, N = \lim_{i \to \infty} N_i. \tag{3.10}$$

Then

$$m \le \liminf_{i\to\infty} x_i \le \limsup_{i\to\infty} x_i \le M, \quad n \le \liminf_{i\to\infty} y_i \le \limsup_{i\to\infty} y_i \le N. \tag{3.11}$$

By the continuity of f and g, one has

$$\begin{cases} M = f([M]_p,[m]_q,[N]_s,[n]_t), m = f([m]_p,[M]_q,[n]_s,[N]_t), \\ N = g([M]_{p_1},[m]_{q_1},[N]_{s_1},[n]_{t_1}), n = g([m]_{p_1},[M]_{q_1},[n]_{s_1},[N]_{t_1}). \end{cases} \tag{3.12}$$

Moreover, if $m = M, n = N$, then $m = M = \lim_{i\to\infty} x_i = \bar{x}, n = N = \lim_{i\to\infty} y_i = \bar{y}$, and then the proof is complete.

Theorem 3.3 If $1 < A$, then the equilibrium point $(0,0)$ of the system (1.8) is global attractor for any condition $(x_{-3}, x_{-2}x_{-1}, x_0, y_{-3}, y_{-2}, y_{-1}, y_0) \in (0,\infty)^8$.

Proof: Let $(f,g):(0,\infty)^4 \times (0,\infty)^4 \to (0,\infty)\times(0,\infty)$ be a function defined by

$$f(x_n, x_{n-1}, x_{n-2}, x_{n-3}, y_n, y_{n-1}, y_{n-2}, y_{n-3}) = \frac{x_{n-3} - y_{n-1}}{A + x_{n-3}y_{n-1}}, \tag{3.13}$$

and

$$g(x_n, x_{n-1}, x_{n-2}, x_{n-3}, y_n, y_{n-1}, y_{n-2}, y_{n-3}) = \frac{y_{n-3} - x_{n-1}}{A + x_{n-1}y_{n-3}}. \tag{3.14}$$

Set

$$f = \frac{x-y}{A+xy}, \quad g = \frac{y-x}{A+xy}, \tag{3.15}$$

we can obtain that

$$\begin{aligned} f_x &= \frac{A+y^2}{(A+xy)^2} > 0, \quad g_x = \frac{-A-y^2}{(A+xy)^2} < 0, \\ f_y &= \frac{-A-x^2}{(A+xy)^2} < 0, \quad g_x = \frac{A+x^2}{(A+xy)^2} > 0, \end{aligned} \tag{3.16}$$

which implies that f and g possess a mixed monotone property.

Let $M_0 = N_0 = \max\{x_{-3}, x_{-2}x_{-1}, x_0, y_{-3}, y_{-2}, y_{-1}, y_0\}$ and $-A/M_0 < m_0 = n_0 < M_0/(1-A)$. Thus, we have

$$m_0 < \frac{m_0 - N_0}{A + m_0N_0} < \frac{M_0 - n_0}{A + M_0n_0} < M_0, n_0 < \frac{n_0 - M_0}{A + n_0M_0} < \frac{N_0 - m_0}{A + N_0m_0} < N_0. \tag{3.17}$$

Moreover, from (1.8) and Theorem 3.2, one can derive that there exists $m, M \in [m_0, M_0]$,

$n, N \in [n_0, N_0]$ satisfying

$$m = \frac{m-N}{A+mN}, n = \frac{n-M}{A+nM}, M = \frac{M-n}{A+nM}, N = \frac{N-m}{A+Nm}. \tag{3.18}$$

Hence, we have

$$M = N = m = n = 0.$$

According to Lemma 2.2 and Theorem 3.2, If $1 < A$, the unique equilibrium $(0,0)$ is not only locally asymptotically stable, but also a global attractor. The proof is complete.

Theorem 3.4 If $A < 1$, then the equilibrium point $(0,0)$ is unstable.

Proof: It is easy to see that there exist roots of characteristic equation (3.3) lie outside unit disk when $A < 1$. According to Lemma 2.2, the equilibrium point (0, 0) is unstable.

Theorem 3.5 The equilibrium points $p_1 = (\sqrt{A-2}, -\sqrt{A-2})$, and $p_2 = (-\sqrt{A-2}, \sqrt{A-2})$ of the system (1.8) are locally asymptotically stable when $2 < A < 3$. And the equilibrium points p_1 and p_2 of the system (1.8) are unstable when $A > 3$.

Proof: We can easily obtain that the linear equations of the system (1.8) about the equilibrium point $p_1 = (\sqrt{A-2}, -\sqrt{A-2})$ is

$$\varphi_{n+1} = D^* \varphi_n,$$

where

$$\varphi_n = \begin{bmatrix} x_n \\ x_{n-1} \\ x_{n-2} \\ x_{n-3} \\ y_n \\ y_{n-1} \\ y_{n-2} \\ y_{n-3} \end{bmatrix}, \quad D = \begin{bmatrix} 0 & 0 & 0 & \frac{A-1}{2} & 0 & \frac{1-A}{2} & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1-A}{2} & 0 & 0 & 0 & 0 & 0 & \frac{A-1}{2} \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}. \tag{3.19}$$

The characteristic equation of (3.18) is

$$f(\lambda) = (\lambda^4 - \frac{A-1}{2})^2 = 0. \tag{3.20}$$

Hence, we have that the equilibrium point p_1 of the system (1.8) is locally asymptotically stable when $2 < A < 3$, and the equilibrium point p_1 of the system (1.8) is unstable when

$A > 3$. The stability and unstability of the equilibrium point p_2 can be proved similarly.

4. Numerical simulations

In this section some numerical examples are given in order to confirm the results of the previous sections and to support our theoretical discussions. These examples represent different types of qualitative behavior of solutions of the system (1.8). As examples, we consider the following difference equations

$$x_{n+1} = \frac{x_{n-3} - y_{n-1}}{3 + x_{n-3}y_{n-1}}, \quad y_{n+1} = \frac{y_{n-3} - x_{n-1}}{3 + y_{n-3}x_{n-1}}, \quad n = 0, 1, \cdots, \tag{4.1}$$

$$x_{n+1} = \frac{x_{n-3} - y_{n-1}}{5 + x_{n-3}y_{n-1}}, \quad y_{n+1} = \frac{y_{n-3} - x_{n-1}}{5 + y_{n-3}x_{n-1}}, \quad n = 0, 1, \cdots, \tag{4.2}$$

and

$$x_{n+1} = \frac{x_{n-3} - y_{n-1}}{0.5 + x_{n-3}y_{n-1}}, \quad y_{n+1} = \frac{y_{n-3} - x_{n-1}}{0.5 + y_{n-3}x_{n-1}}, \quad n = 0, 1, \cdots. \tag{4.3}$$

By employing the software package MATLAB7.0, we can solve the numerical solutions of the system (4.1), (4.2) and (4.3) which are shown respectively in Figures 4.1-Figure 4.4. More precisely, it is obvious that the equations (4.1) satisfy the conditions of Theorems 3.1 and Figure 4.1 shows that the solution of the difference system (1.8) is local stability if $A = 3$ and the initial conditions $x_{-3} = 6, x_{-2} = 4, x_{-1} = 2, x_0 = 8, y_{-3} = 1, y_{-2} = 4$, $y_{-1} = 2$ and $y_0 = 3$. We can also see that the equations (4.1) satisfy the conditions of Theorems 3.2 and Theorem 3.3, and Figure 4.2 shows that the solution of the difference system (1.8) is globally asymptotically stable where $A = 3$, $n_0 = m_0 = -0.9, N_0 = M_0 = 0.9$ and the initial conditions $x_{-3} = 0.01, x_{-2} = 0.02, x_{-1} = 0.01, x_0 = 0.03, y_{-3} = 0.2, y_{-2} = 0.4$, $y_{-1} = 0.8$ and $y_0 = 0.7$. It can be noticed that the equations (4.2) satisfy the conditions of Theorems 3.1, Theorems 3.2 and Theorem 3.3, and Figure 4.3 shows that the solution of the difference system (1.8) is globally asymptotically stable where $A = 5$, $n_0 = m_0 = -0.3, N_0 = M_0 = 0.5$ and the initial conditions $x_{-3} = 0.2, x_{-2} = 0.06, x_{-1} = 0.4, x_0 = 0.08, y_{-3} = 0.02, y_{-2} = 0.04$, $y_{-1} = 0.01$ and $y_0 = 0.1$. It is clear that the equations (4.3) satisfy the conditions of Theorem 3.4, and Figure 4.4 shows that the solution of the difference system (1.8) is unstable where $A = 0.5$, and the initial conditions $x_{-3} = 1.6, x_{-2} = 1, x_{-1} = 1.5, x_0 = 1.8, y_{-3} = 1, y_{-2} = 4$, $y_{-1} = 2$ and $y_0 = 3$.

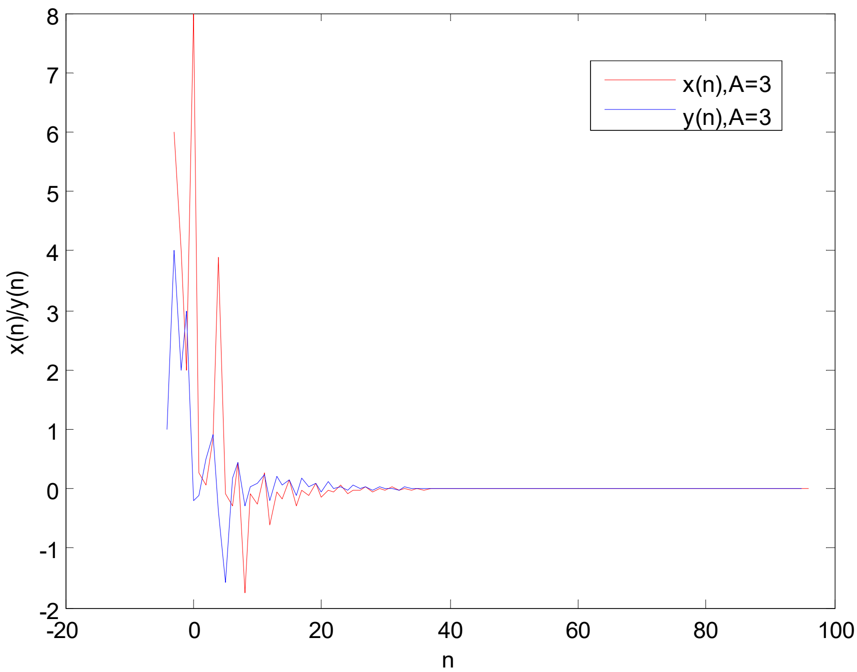


Figure 4.1. Solutions of (4.1) with A=3 and the initial conditions

$x_{-3}=6, x_{-2}=4, x_{-1}=2, x_0=8, y_{-3}=1, y_{-2}=4$, $y_{-1}=2$ and $y_0=3$

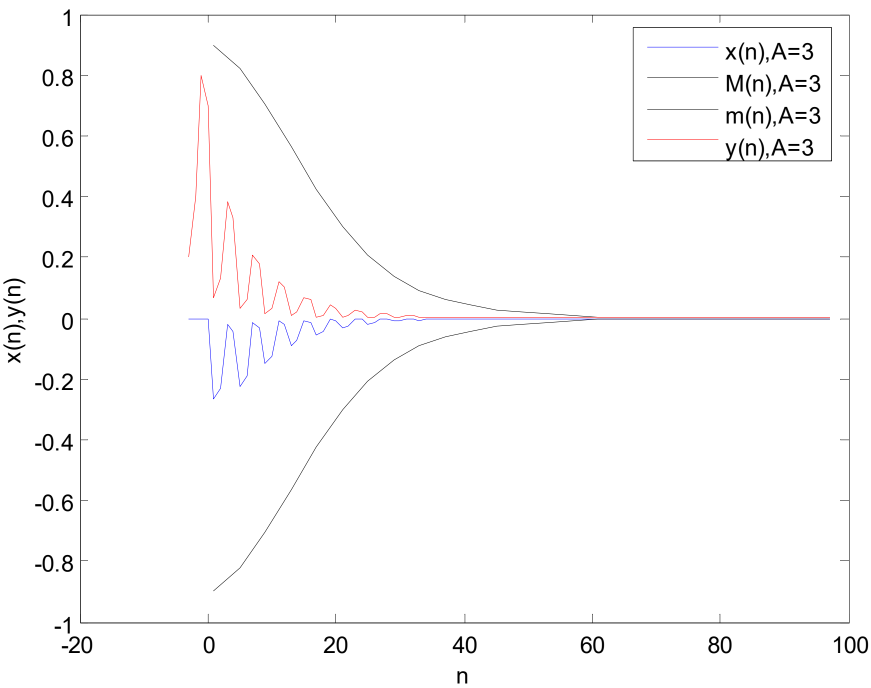


Figure 4.2. Solutions of (4.1) with $A=3$, $n_0=m_0=-0.9, N_0=M_0=0.9$ and the initial conditions

$x_{-3}=0.01, x_{-2}=0.02, x_{-1}=0.01, x_0=0.03, y_{-3}=0.2, y_{-2}=0.4$, $y_{-1}=0.8$ and $y_0=0.7$

11

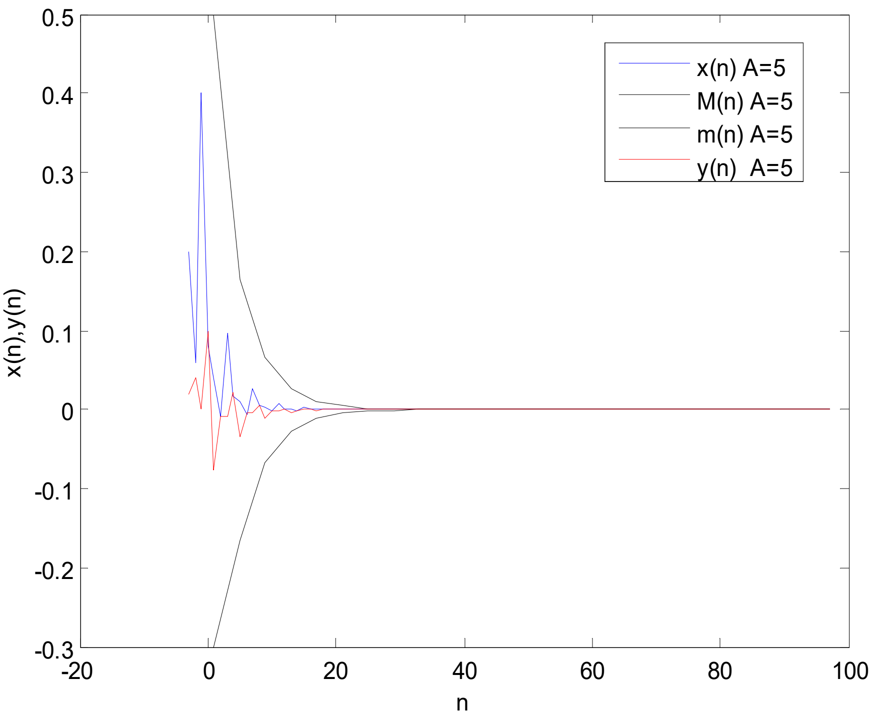


Figure 4.3. Solutions of (4.2) with $A=5$, $n_0=m_0=-0.3, N_0=M_0=0.5$ and the initial conditions

$x_{-3}=0.2, x_{-2}=0.06, x_{-1}=0.4, x_0=0.08, y_{-3}=0.02, y_{-2}=0.04$, $y_{-1}=0.01$ and $y_0=0.1$

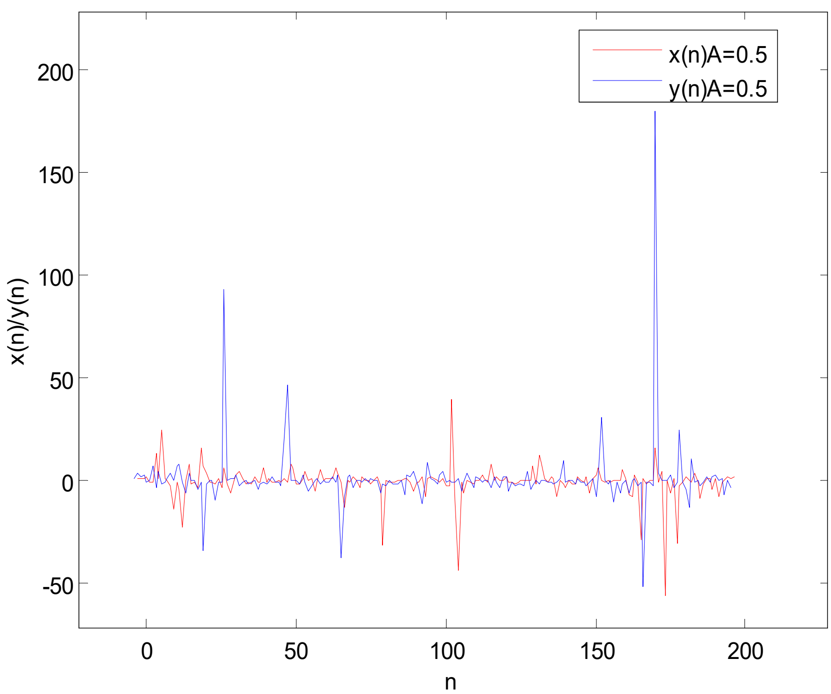


Figure 4.4. Solutions of (4.3) with $A=0.5$, and the initial conditions

$x_{-3}=1.6, x_{-2}=1, x_{-1}=1.5, x_0=1.8, y_{-3}=1, y_{-2}=4$, $y_{-1}=2$ and $y_0=3$

12

5. Conclusions

This paper presents the use of a variational iteration method for systems of nonlinear difference equations. This technique is a powerful tool for solving various difference equations and can also be applied to other nonlinear differential equations in mathematical physics. The numerical simulations show that this method is an effective and convenient one. The variational iteration method provides an efficient method to handle the nonlinear structure. Computations are performed using the software package MATLAB7.0.

We have dealt with the problem of global asymptotic stability analysis for a class of nonlinear high order difference equations. The general sufficient conditions have been obtained to ensure the existence, unstability and global asymptotic stability of the equilibrium point for the nonlinear difference equations. These criteria generalize and improve some known results. In particular, some illustrate examples are given to show the effectiveness of the obtained results. In addition, the sufficient conditions that we obtained are very simple, which provide flexibility for the application and analysis of nonlinear difference equation.

Acknowledgements

This work is supported Science Fund for Distinguished Young Scholars (cstc2014jcyjjq40004) of China, the National Nature Science Fund (Project nos.11372366 and 61503053) of China, the Natural Science Foundation Project of CQ CSTC (Grant nos. cstc2015jcyjA00034, cstc2015jcyjBX0135 and cstc2015jjA20016) of China.

References

[1] E.C. Pielou, Population and Community Ecology, Gordon and Breach, London, 1975.

[2] E.P. Popov, Automatic Regulation and Control, Nauka, Moscow, 1966 (in Russian).

[3] S. Stević, Behaviour of the positive solutions of the generalized Beddington-Holt equation, PanAm. Math. J. 10 (4) (2000) 77-85.

[4] E.M. Elsayed, On the solutions and periodic nature of some systems of difference equations, Int. J. Biomath. 7 (6) (2014), Article ID:1450067.

[5] S. Stević, Global stability and asymptotics of some classes of rational difference equations, J. Math. Anal. Appl. 316 (1) (2006) 60-68.

[6] S. Stević, Asymptotics of some classes of higher-order difference equations, Discrete Dyn. Nat. Soc. 2007 (2007), Article ID: 56813.

[7] D.B. Iricanin, S. Stević, Some systems of nonlinear difference equations of higher order with periodic solutions, Dyn. Contin. Discret. Impuls. Syst. Ser. A-Math Anal. 13 (3) (2006) 499-507.

[8] E.M. Elsayed, Behavior and expression of the solutions of some rational difference equations, J. Comput. Anal. Appl. 15 (1) (2013) 73-81.

[9] L.X. Hu, W.S. He, H.M. Xia, Global asymptotic behavior of a rational difference equation, Appl. Math. Comput. 218 (15) (2012) 7818-7828.

[10] G. Papaschinopoulos, M. Radin, C. J. Schinas, Study of the asymptotic behavior of the solutions of three systems of difference equations of exponential form, Appl. Math. Comput. 218 (9) (2012) 5310-5318.

[11] Y. Muroya, E. Ishiwata, N. Guglielmi, Global stability for nonlinear difference equations with variable coefficients, J. Math. Anal. Appl. 334 (1) (2007) 232-247.

[12] M. Galewski, A note on the existence of a bounded solution for a nonlinear system of difference equations, J. Differ. Equ. Appl. 16 (1) (2010) 121-124.

[13] E.A. Grove, G. Ladas, Periodicities in Nonlinear Difference Equations, Chapman and Hall/CRC Press, Boca Raton, 2004.

[14] C.Y. Wang, S. Wang, Z.W. Wang, F. Gong, R. Wang, Asymptotic stability for a class of nonlinear difference equation, Discrete Dyn. Nat. Soc. 2010 (2010), Article ID 791610.

[15] C.Y. Wang, Q.H. Shi, S. Wang, Asymptotic behavior of equilibrium point for a family of rational difference equation, Adv. Differ. Equ. 2010 (2010), Article ID 505906.

[16] C.Y. Wang, S. Wang, L.R. LI, Q.H. Shi, Asymptotic behavior of equilibrium point for a class of nonlinear difference equation, Adv. Differ. Equ. 2009 (2009), Article ID 214309.

[17] C.Y. Wang, S. Wang, W. Wang, Global asymptotic stability of equilibrium point for a family of rational difference equations, Appl. Math. Lett. 24 (5) (2011) 714-718.

[18] M.M. El-Dessoky, E.M. Elsayed, E.O. Alzahrani, The form of solutions and periodic nature for some rational difference equations systems, J. Nonlinear Sci. Appl. 9 (10) (2016) 5629-5647.

[19] E.M. Elsayed, Solutions of rational difference system of order two, Math. Comput. Model. 55 (2012) 378-384.

[20] I. Bajo, E. Liz, Global behaviour of a second-order nonlinear difference equation, J. Differ. Equ. Appl. 17 (10) (2011) 1471-1486.

[21] A.S. Kurbanli, C. Cinar, I. Yalcinkaya, On the behavior of positive solutions of the system of rational difference equations $x_{n+1} = \frac{x_{n-1}}{y_n x_{n-1} + 1}, y_{n+1} = \frac{y_{n-1}}{x_n y_{n-1} + 1}$, Math. Comput. Model. 53 (2011) 1261-1267.

[22] N. Touafek, E.M. Elsayed, On the solutions of systems of rational difference equations, Math. Comput. Model. 55 (2012) 1987-1997

[23] E.M. Elsayed, Solution for systems of difference equations of rational form of order two, Comput. Appl. Math. 33 (3) (2014) 751-765.

[24] A.Q. Khan, M.N. Qureshi, Global dynamics of some systems of rational difference

equations, J. Egypt. Math. Soc. 24 (1) (2016) 30-36.

[25] I. Yalçınkaya, On the global asymptotic behavior of a system of two nonlinear difference equations, ARS Comb. 95 (2010) 151-159.

[26] H. Sedaghat, Reduction of order, periodicity and boundedness in a class of nonlinear, higher order difference equations, Comput. Math. Appl. 66 (11) (2013) 2231-2238.

[27] T.H. Thai, V.V. Khuong, Global asymptotic stability of a second-order system of difference equations, Indian J. Pure Appl. Math. 45 (2) (2014) 185-198.

[28] A. Khaliq, F. Alzahranib, E.M. Elsayed, Global attractivity of a rational difference equation of order ten, J. Nonlinear Sci. Appl. 9 (6) (2016) 4465-4477.

[29] M.M. El-Dessoky, E.M. Elsayed, E.O. Alzahrani, The form of solutions and periodic nature for some rational difference equations systems, J. Nonlinear Sci. Appl. 9 (10) (2016) 5629-5647.

[30] M.M. El-Dessoky, On the dynamics of a higher order rational difference equations, J. Egypt. Math. Soc. 25 (1) (2017) 28-36.

[31] R. Abo-Zeid, On the oscillation of a third order rational difference equation, J. Egypt. Math. Soc. 23 (1) (2015) 62-66.

[32] M. Saleh, N. Alkoumi, Aseel Farhat, On the dynamics of a rational difference equation $x_{n+1}=\dfrac{\alpha+\beta x_n+\gamma x_{n-k}}{Bx_n+Cx_{n-k}}$, Chaos Solitons Fractals 96 (1) (2017) 76-84.

[33] A. Khaliq, F. Alzahranib, E. M. Elsayed, Global attractivity of a rational difference equation of order ten, J. Nonlinear Sci. Appl. 9 (6) (2016) 4465-4477.

[34] C.Y. Wang, X.J. Fang, R. Li, On the solution for a system of two rational difference equations, J. Comput. Anal. Appl. 20 (1) (2016) 175-186

[35] C.Y. Wang, X.J. Fang, R. Li, On the dynamics of a certain four-order fractional difference equations, J. Comput. Anal. Appl. 22 (5) (2017) 968-976.

[36] V.L. Kocic, G. Ladas, Global Behavior of Nonlinear Difference Equations of Higher Order with Applications, Kluwer Academic, Dordrecht, 1993.

[37] H. Sedaghat, Nonlinear Difference Equations: Theory with Applications to Social Science Models, Kluwer Academic Publishers, Dordrecht, 2003.

[38] E. Camouzis, G. Ladas, Dynamics of Third-order Rational Difference Equations: With Open Problems and Conjectures, Chapman and Hall/HRC, Boca Raton, 2007.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 27, NO.6, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC

A version of the Hadamard inequality for Caputo fractional derivatives and related results

Shin Min Kang[1,2], Ghulam Farid[3], Waqas Nazeer[4,*] and Saira Naqvi[5]

[1]Department of Mathematics and RINS, Gyeongsang National University, Jinju 52828, Korea
e-mail: smkang@gnu.ac.kr

[2]Center for General Education, China Medical University, Taichung 40402, Taiwan

[3]Department of Mathematics, Comsats Institute of Information Technology, Attock 43600, Pakistan
e-mail: ghlmfarid@ciit-attock.edu.pk

[4]Division of Science and Technology, University of Education, Lahore 54000, Pakistan
e-mail: nazeer.waqas@ue.edu.pk

[5]Department of Mathematics, Comsats Institute of Information Technology, Attock 43600, Pakistan
e-mail: naqvisaira2013@gmail.com

Abstract

In this paper we are interested to give the Hadamard inequality for n-times differentiable convex functions via Caputo fractional derivatives. We also find bounds of a difference of this inequality.

2010 Mathematics Subject Classification: 26A51, 26D10, 26D15

Key words and phrases: convex functions, Hadamard inequality, Caputo Fractional derivatives

1 introduction

Fractional calculus was mainly a study kept for the finest minds in mathematics. The history of fractional calculus is as old as the history of differential calculus. It does indeed provide several potentially useful tools for solving differential and integral equations, and various other problems involving special functions of mathematical physics as well as their extensions and generalizations in one and more variables. Fourier, Eulern and Laplace are among those mathematicians who showed a casual interest by fractional calculus and

* Corresponding author

mathematical consequences. A lot of them established definitions by means of their own notion and style. Most renowned of these definitions are the Grunwald-Letnikov and Riemann-Liouville definitions [6–8].

In the following we give the definition of Caputo fractional derivatives [6].

Definition 1.1. Let $\alpha > 0$ and $\alpha \notin \{1, 2, 3, ...\}$, $n = [\alpha] + 1$, $f \in AC^n[a, b]$, the space of functions having nth derivatives absolutely continuous. The right-sided and left-sided Caputo fractional derivatives of order α are defined as follows:

$$({}^C D^{\alpha}_{a+} f)(x) = \frac{1}{\Gamma(n-\alpha)} \int_a^x \frac{f^{(n)}(t)}{(x-t)^{\alpha-n+1}} dt, x > a \tag{1.1}$$

and

$$({}^C D^{\alpha}_{b-} f)(x) = \frac{(-1)^n}{\Gamma(n-\alpha)} \int_x^b \frac{f^{(n)}(t)}{(t-x)^{\alpha-n+1}} dt, x < b. \tag{1.2}$$

If $\alpha = n \in \{1, 2, 3, ...\}$ and usual derivative $f^{(n)}(x)$ of order n exists, then Caputo fractional derivative $({}^C D^{\alpha}_{a+} f)(x)$ coincides with $f^{(n)}(x)$, whereas $({}^C D^{\alpha}_{b-} f)(x)$ coincides with $f^{(n)}(x)$ with exactness to a constant multiplier $(-1)^n$. In particular we have

$$({}^C D^0_{a+} f)(x) = ({}^C D^0_{b-} f)(x) = f(x)$$

where $n = 1$ and $\alpha = 0$.

Definition 1.2. ([7]) Let $f \in L[a, b]$. Then Riemann-Liouville fractional integrals of f of order α are defined as follows

$$I^{\alpha}_{a+} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t) dt, \; x > a$$

and

$$I^{\alpha}_{b-} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t) dt, \; x < b.$$

In [10], Sarikaya et al. proved following Hadamard-type inequalities for Riemann-Liouville fractional integrals:

Theorem 1.3. *Let $f : [a, b] \to \mathbb{R}$ be a positive function with $0 \le a < b$ and $f \in L_1[a, b]$. If f is a convex function on [a,b], then the following inequalities for fractional integrals hold*

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\le \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}} \left[I^{\alpha}_{(\frac{a+b}{2})+} f(b) + I^{\alpha}_{(\frac{a+b}{2})-} f(a) \right] \\ &\le \frac{f(a)+f(b)}{2} \end{aligned} \tag{1.3}$$

with $\alpha > 0$.

Theorem 1.4. *Let $f : [a, b] \to \mathbb{R}$ be a differentiable mapping on (a, b) with $a < b$. If $|f'|^q$ is convex on $[a, b]$ for $q \geq 1$, then the following inequality for fractional integrals holds*

$$
\begin{aligned}
&\left| \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha} [I^\alpha_{(\frac{a+b}{2})+} f(b) + I^\alpha_{(\frac{a+b}{2})-} f(a)] - f\left(\frac{a+b}{2}\right) \right| \\
&\leq \frac{b-a}{4(\alpha+1)} \left(\frac{1}{2(\alpha+2)}\right)^{\frac{1}{q}} \left[\left((\alpha+1)|f'(a)|^q + (\alpha+3)|f'(b)|^q\right)^{\frac{1}{q}} \right. \qquad (1.4) \\
&\left. + \left((\alpha+3)|f'(a)|^q + (\alpha+1)|f'(b)|^q\right)^{\frac{1}{q}} \right].
\end{aligned}
$$

with $\alpha > 0$.

Theorem 1.5. *Let $f : [a, b] \to \mathbb{R}$ be a differentiable mapping on (a, b) with $a < b$. If $|f'|^q$ is convex on $[a, b]$ for $q > 1$, then the following inequality for fractional integral holds*

$$
\begin{aligned}
&\left| \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha} [I^\alpha_{(\frac{a+b}{2})+} f(b) + I^\alpha_{(\frac{a+b}{2})-} f(a)] - f\left(\frac{a+b}{2}\right) \right| \\
&\leq \frac{b-a}{4} \left(\frac{1}{\alpha p+1}\right)^{\frac{1}{p}} \left(\left(\frac{|f'(a)|^q + 3|f'(b)|q}{4}\right)^{\frac{1}{q}} + \left(\frac{3|f'(a)|^q + |f'(b)|^q}{4}\right)^{\frac{1}{q}} \right) \qquad (1.5) \\
&\leq \frac{b-a}{4} \left(\frac{4}{\alpha p+1}\right)^{\frac{1}{p}} [|f'(a)| + |f'(b)|],
\end{aligned}
$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

In recent days many researchers have focused their attention in establishing inequalities of Hadamard type via utilization of fractional integral operators, (see, [1–5, 9]) and references therein. In this paper we are interested to give versions of inequalities (1.3), (1.4) and (1.5) for n-times differentiable convex functions via Caputo fractional derivatives.

In the whole paper $C^n[a, b]$ denotes the space of n-times differentiable functions such that $f^{(n)}$ are continuous on $[a, b]$.

2 Hadamard-type inequalities for Caputo fractional derivatives

In this section we give a version of the Hadamard inequality via Caputo fractional derivatives. First we prove the following lemma.

Lemma 2.1. *Let $g : [a, b] \to \mathbb{R}$ be a function such that $g \in C^n[a, b]$, also let $g^{(n)}$ is integrable and symmetric to $\frac{a+b}{2}$. Then we have*

$$
\begin{aligned}
({}^C D^\alpha_{a+} g)(b) &= (-1)^n ({}^C D^\alpha_{b-} g)(a) \\
&= \frac{1}{2}[({}^C D^\alpha_{a+} g)(b) + (-1)^n ({}^C D^\alpha_{b-} g)(a)].
\end{aligned}
$$

Proof. By symmetricity of $g^{(n)}$ we have $g^{(n)}(a+b-x) = g^{(n)}(x)$, where $x \in [a,b]$. Replacing x with $a+b-x$ in the following integral we have

$$\begin{aligned}({}^C D^{\alpha}_{a+}g)(b) &= \frac{1}{\Gamma(n-\alpha)}\int_a^b \frac{g^{(n)}(x)}{(b-x)^{\alpha-n+1}}dx\\ &= \frac{1}{\Gamma(n-\alpha)}\int_a^b \frac{g^{(n)}(a+b-x)}{(x-a)^{\alpha-n+1}}dx\\ &= \frac{1}{\Gamma(n-\alpha)}\int_a^b \frac{g^{(n)}(x)}{(x-a)^{\alpha-n+1}}dx\\ &= (-1)^n({}^C D^{\alpha}_{b-}g)(a).\end{aligned}$$

□

Theorem 2.2. *Let $f : [a,b] \to \mathbb{R}$ be a positive function with $0 \le a < b$ and $f \in C^n[a,b]$. If $f^{(n)}$ is a convex function on $[a,b]$, then the following inequalities for Caputo fractional derivatives hold*

$$\begin{aligned}&f^{(n)}\left(\frac{a+b}{2}\right)\\ &\le \frac{2^{n-\alpha-1}\Gamma(n-\alpha+1)}{(b-a)^{n-\alpha}}\left[({}^C D^{\alpha}_{(\frac{a+b}{2})+}f)(b) + (-1)^n({}^C D^{\alpha}_{(\frac{a+b}{2})-}f)(a)\right] \qquad (2.1)\\ &\le \frac{f^{(n)}(a)+f^{(n)}(b)}{2}.\end{aligned}$$

Proof. From convexity of $f^{(n)}$ we have

$$f^{(n)}\left(\frac{x+y}{2}\right) \le \frac{f^{(n)}(x)+f^{(n)}(y)}{2}.$$

Setting $x = \frac{t}{2}a + \frac{(2-t)}{2}b, y = \frac{(2-t)}{2}a + \frac{t}{2}b$ for $t \in [0,1]$. Then $x, y \in [a,b]$ and above inequality gives

$$2f^{(n)}\left(\frac{a+b}{2}\right) \le f^{(n)}\left(\frac{t}{2}a + \frac{2-t}{2}b\right) + f^{(n)}\left(\frac{2-t}{2}a + \frac{t}{2}b\right),$$

multiplying both sides of above inequality with $t^{n-\alpha-1}$ and integrating over $[0,1]$ we have

$$\begin{aligned}&2f^{(n)}\left(\frac{a+b}{2}\right)\int_0^1 t^{n-\alpha-1}dt\\ &\le \int_0^1 t^{n-\alpha-1}f^{(n)}\left(\frac{t}{2}a+\frac{2-t}{2}b\right)dt + \int_0^1 t^{n-\alpha-1}f^{(n)}\left(\frac{2-t}{2}a+\frac{t}{2}b\right)dt\\ &= \frac{2^{n-\alpha}\Gamma(\alpha)}{(b-a)^{n-\alpha}}\left[({}^C D^{\alpha}_{(\frac{a+b}{2})+}f)(b) + (-1)^n({}^C D^{\alpha}_{(\frac{a+b}{2})-}f)(a)\right],\end{aligned}$$

from which one can have

$$\begin{aligned}&f^{(n)}\left(\frac{a+b}{2}\right)\\ &\le \frac{2^{n-\alpha-1}\Gamma(n-\alpha+1)}{(b-a)^{n-\alpha}}\left[({}^C D^{\alpha}_{(\frac{a+b}{2})+}f)(b) + (-1)^n({}^C D^{\alpha}_{(\frac{a+b}{2})-}f)(a)\right]. \qquad (2.2)\end{aligned}$$

On the other hand convexity of $f^{(n)}$ gives

$$\begin{aligned}
&f^{(n)}\left(\frac{t}{2}a+\frac{2-t}{2}b\right)+f^{(n)}\left(\frac{2-t}{2}a+\frac{t}{2}b\right)\\
&\leq \frac{t}{2}f^{(n)}(a)+\frac{2-t}{2}f^{(n)}(b)+\frac{2-t}{2}f^{(n)}(a)+\frac{t}{2}f^{(n)}(b),
\end{aligned}$$

multiplying both sides of above inequality with $t^{n-\alpha-1}$ and integrating over $[0,1]$ we have

$$\begin{aligned}
&\int_0^1 t^{n-\alpha-1}f^{(n)}\left(\frac{t}{2}a+\frac{2-t}{2}b\right)dt+\int_0^1 t^{n-\alpha-1}f^{(n)}\left(\frac{2-t}{2}a+\frac{t}{2}b\right)dt\\
&\leq \left[f^{(n)}(a)+f^{(n)}(b)\right]\int_0^1 t^{n-\alpha-1}dt,
\end{aligned}$$

from which one can have

$$\begin{aligned}
&\frac{2^{n-\alpha-1}\Gamma(n-\alpha+1)}{(b-a)^{n-\alpha}}\left[({}^CD^{\alpha}_{(\frac{a+b}{2})+}f)(b)+(-1)^n({}^CD^{\alpha}_{(\frac{a+b}{2})-}f)(a)\right]\\
&\leq \frac{f^{(n)}(a)+f^{(n)}(b)}{2}.
\end{aligned} \tag{2.3}$$

Combining inequality (2.2) and inequality (2.3) we get inequality (2.1). □

3 Caputo fractional inequalities related to the Hadamard inequality

In this section we give the bounds of a difference of the Hadamard inequality proved in previous section. For our results we use the following lemma.

Lemma 3.1. *Let $f:[a,b]\to\mathbb{R}$ be a differentiable mapping on (a,b) with $a<b$. If $f\in C^{n+1}[a,b]$, then the following equality for Caputo fractional derivatives holds*

$$\begin{aligned}
&\frac{2^{n-\alpha-1}\Gamma(n-\alpha+1)}{(b-a)^{n-\alpha}}\left[({}^CD^{\alpha}_{(\frac{a+b}{2})+}f)(b)+(-1)^n({}^CD^{\alpha}_{(\frac{a+b}{2})-}f)(a)\right]\\
&\quad -f^{(n)}\left(\frac{a+b}{2}\right)\\
&=\frac{b-a}{4}\left[\int_0^1 t^{n-\alpha}f^{(n+1)}\left(\frac{t}{2}a+\frac{2-t}{2}b\right)dt\right.\\
&\quad \left.-\int_0^1 t^{n-\alpha}f^{(n+1)}\left(\frac{2-t}{2}a+\frac{t}{2}b\right)dt\right].
\end{aligned} \tag{3.1}$$

Proof. One can note that

$$\begin{aligned}
&\frac{b-a}{4}\left[\int_0^1 t^{n-\alpha} f^{(n+1)}\left(\frac{t}{2}a+\frac{2-t}{2}b\right)dt\right] \\
&=\frac{b-a}{4}\left[t^{n-\alpha}\frac{2}{a-b}f^{(n)}\left(\frac{t}{2}a+\frac{2-t}{2}b\right)|_0^1\right. \\
&\left.-\int_0^1 \alpha t^{n-\alpha-1}\frac{2}{a-b}f^{(n)}\left(\frac{t}{2}a+\frac{2-t}{2}b\right)\right] \\
&=\frac{b-a}{4}\left[-\frac{2}{b-a}f^{(n)}\left(\frac{a+b}{2}\right)\right. \\
&\left.-\frac{2\alpha}{(a-b)}\int_b^{\frac{a+b}{2}}\left(\frac{2}{b-a}(b-x)\right)^{n-\alpha-1}\frac{2}{a-b}f^{(n)}(x)dx\right] \\
&=\frac{b-a}{4}\left[-\frac{2}{b-a}f^{(n)}\left(\frac{a+b}{2}\right)+\frac{2^{n-\alpha+1}\Gamma(n-\alpha+1)}{(b-a)^{n-\alpha+1}}(-1)^n({}^C D^{\alpha}_{(\frac{a+b}{2})-}f)(b)\right].
\end{aligned} \tag{3.2}$$

Similarly

$$\begin{aligned}
&-\frac{b-a}{4}\left[\int_0^1 t^{n-\alpha} f^{(n+1)}\left(\frac{2-t}{2}a+\frac{t}{2}b\right)dt\right] \\
&=-\frac{b-a}{4}\left[\frac{2}{b-a}f^{(n)}\left(\frac{a+b}{2}\right)-\frac{2^{n-\alpha+1}\Gamma(n-\alpha+1)}{(b-a)^{n-\alpha+1}}({}^C D^{\alpha}_{(\frac{a+b}{2})+}f)(a)\right].
\end{aligned} \tag{3.3}$$

Combining (3.2) and (3.3) one can have (3.1). ☐

Using the above lemma we give the following Caputo fractional Hadamard-type inequality.

Theorem 3.2. *Let $f:[a,b]\to\mathbb{R}$ be a differentiable mapping on (a,b) with $a<b$ and $f\in C^{n+1}[a,b]$. If $|f^{(n+1)}|^q$ is convex on $[a,b]$ for $q\geq 1$, then the following inequality for Caputo fractional derivatives holds*

$$\begin{aligned}
&\left|\frac{2^{n-\alpha-1}\Gamma(n-\alpha+1)}{(b-a)^{n-\alpha}}\left[({}^C D^{\alpha}_{(\frac{a+b}{2})+}f)(b)+(-1)^n({}^C D^{\alpha}_{(\frac{a+b}{2})-}f)(a)\right]\right. \\
&\left.\quad -f^{(n)}\left(\frac{a+b}{2}\right)\right| \\
&\leq\frac{b-a}{4(n-\alpha+1)}\left(\frac{1}{2(n-\alpha+2)}\right)^{\frac{1}{q}}\left[\left[(n-\alpha+1)\left|f^{(n+1)}(a)\right|^q\right.\right. \\
&\quad\left.+(n-\alpha+3)\left|f^{(n+1)}(b)\right|^q\right]^{\frac{1}{q}}+\left[(n-\alpha+3)\left|f^{(n+1)}(a)\right|^q\right. \\
&\quad\left.\left.+(n-\alpha+1)\left|f^{(n+1)}(b)\right|^q\right]^{\frac{1}{q}}\right].
\end{aligned}$$

Proof. From Lemma 3.1 and convexity of $|f^{(n+1)}|$ and for $q=1$ we have

$$\begin{aligned}
&\left|\frac{2^{n-\alpha-1}\Gamma(n-\alpha+1)}{(b-a)^{n-\alpha}}\left[({}^C D^{\alpha}_{(\frac{a+b}{2})+}f)(b)+({}^C D^{\alpha}_{(\frac{a+b}{2})-}f)(a)\right]\right.\\
&\quad\left.-f^{(n)}\left(\frac{a+b}{2}\right)\right|\\
&\leq \frac{b-a}{4}\int_0^1 t^{n-\alpha}\left(\left|f^{(n+1)}\left(\frac{t}{2}a+\frac{2-t}{2}b\right)\right|dt+\left|f^{(n+1)}\left(\frac{2-t}{2}a+\frac{t}{2}b\right)\right|\right)dt.\\
&=\frac{b-a}{4\left(n-\alpha+1\right)}\left[|f^{(n+1)}(a)|+|f^{(n+1)}(b)|\right].
\end{aligned}$$

For $q>1$ using Lemma 3.1 we have

$$\begin{aligned}
&\left|\frac{2^{n-\alpha-1}\Gamma(n-\alpha+1)}{(b-a)^{n-\alpha}}[({}^C D^{\alpha}_{(\frac{a+b}{2})+}f)(b)+(-1)^n({}^C D^{\alpha}_{(\frac{a+b}{2})-}f)(a)]\right.\\
&\quad\left.-f\left(\frac{a+b}{2}\right)\right|\\
&\leq \frac{b-a}{4}\left[\int_0^1 t^{n-\alpha}\left|f^{(n+1)}\left(\frac{t}{2}a+\frac{2-t}{2}b\right)\right|dt\right.\\
&\left.+\int_0^1 t^{n-\alpha}\left|f^{(n+1)}\left(\frac{2-t}{2}a+\frac{t}{2}b\right)\right|dt\right].
\end{aligned}$$

Using power mean inequality we get

$$\begin{aligned}
&\left|\frac{2^{n-\alpha-1}\Gamma(n-\alpha+1)}{(b-a)^{n-\alpha}}\left[({}^c D^{\alpha}_{(\frac{a+b}{2})+}f)(b)+(-1)^n({}^C D^{\alpha}_{(\frac{a+b}{2})-}f)(a)\right]\right.\\
&\quad\left.-f^{(n)}\left(\frac{a+b}{2}\right)\right|\\
&\leq \frac{b-a}{4}\left(\frac{1}{n-\alpha+1}\right)^{\frac{1}{p}}\left[\left[\int_0^1 t^{n-\alpha}\left|f^{(n+1)}\left(\frac{t}{2}a+\frac{2-t}{2}b\right)\right|^q dt\right]^{\frac{1}{q}}\right.\\
&\quad\left.+\left[\int_0^1 t^{n-\alpha}\left|f^{(n+1)}\left(\frac{2-t}{2}a+\frac{t}{2}b\right)\right|^q dt\right]^{\frac{1}{q}}\right].
\end{aligned}$$

Convexity of $|f^{(n+1)}|^q$ gives

$$\begin{aligned}
&\Bigg|\frac{2^{n-\alpha-1}\Gamma(n-\alpha+1)}{(b-a)^{n-\alpha}}\left[({}^CD^{\alpha}_{(\frac{a+b}{2})+}f)(b)+(-1)^n({}^CD^{\alpha}_{(\frac{a+b}{2})-}f)(a)\right]\\
&\quad -f^{(n)}\left(\frac{a+b}{2}\right)\Bigg|\\
&\leq \frac{b-a}{4}\left(\frac{1}{n-\alpha+1}\right)^{\frac{1}{p}}\left[\left[\int_0^1 t^{n-\alpha}\left(\frac{t}{2}|f^{(n+1)}(a)|^q+\frac{2-t}{2}|f^{(n+1)}(b)|^q\right)dt\right]^{\frac{1}{q}}\right.\\
&\quad \left.+\left[\int_0^1 t^{n-\alpha}\left(\frac{2-t}{2}|f^{(n+1)}(a)|^q+\frac{t}{2}|f^{(n+1)}(b)|^q\right)dt\right]^{\frac{1}{q}}\right]\\
&= \frac{b-a}{4}\left(\frac{1}{n-\alpha+1}\right)^{\frac{1}{p}}\left[\left[\frac{|f^{(n+1)}(a)|^q}{2(n-\alpha+2)}+\frac{|f^{(n+1)}(b)|^q}{n-\frac{\alpha}{k}+1}-\frac{|f^{(n+1)}(b)|^q}{2(n-\alpha+2)}\right]^{\frac{1}{q}}\right.\\
&\quad \left.+\left[\frac{|f^{(n+1)}(a)|^q}{n-\alpha+1}-\frac{|f^{(n+1)}(a)|^q}{2(n-\alpha+2)}+\frac{|f^{(n+1)}(b)|^q}{2(n-\alpha+2)}\right]^{\frac{1}{q}}\right],
\end{aligned}$$

which after a little computation gives the required result. □

Theorem 3.3. *Let $f:[a,b]\to\mathbb{R}$ be a function such that $f\in C^{n+1}[a,b]$, $a<b$. If $|f^{(n+1)}|^q$ is convex on $[a,b]$ for $q>1$, then the following inequality for Caputo fractional derivatives holds*

$$\begin{aligned}
&\Bigg|\frac{2^{n-\alpha-1}\Gamma(n-\alpha+1)}{(b-a)^{n-\alpha}}\left[({}^CD^{\alpha}_{(\frac{a+b}{2})+}f)(b)+(-1)^n({}^CD^{\alpha}_{(\frac{a+b}{2})-}f)(a)\right]\\
&\quad -f^{(n)}\left(\frac{a+b}{2}\right)\Bigg|\\
&\leq \frac{b-a}{4}\left(\frac{1}{np-\alpha p+1}\right)^{\frac{1}{p}}\left[\left(\frac{|f^{(n+1)}(a)|^q+3|f^{(n+1)}(b)|^q}{4}\right)^{\frac{1}{q}}\right.\\
&\quad \left.+\left(\frac{3|f^{(n+1)}(a)|^q+|f^{(n+1)}(b)|^q}{4}\right)^{\frac{1}{q}}\right]\\
&\leq \frac{b-a}{4}\left(\frac{4}{3(np-\alpha p+1)}\right)^{\frac{1}{p}}[|f^{(n+1)}(a)|+|f^{(n+1)}(b)|],
\end{aligned} \tag{3.4}$$

with $\frac{1}{p}+\frac{1}{q}=1$.

Proof. From Lemma 3.1 we have

$$\begin{aligned}
&\left|\frac{2^{n-\alpha-1}\Gamma(n-\alpha+1)}{(b-a)^{n-\alpha}}\left[({}^{C}D^{\alpha}_{(\frac{a+b}{2})+}f)(b)+(-1)^{n}({}^{C}D^{\alpha}_{(\frac{a+b}{2})-}f)(a)\right]\right.\\
&\quad\left.-f^{(n)}\left(\frac{a+b}{2}\right)\right|\\
&\leq\frac{b-a}{4}\left[\int_0^1 t^{n-\alpha}\left|f^{(n+1)}\left(\frac{t}{2}a+\frac{2-t}{2}b\right)\right|dt\right.\\
&\quad\left.+\int_0^1 t^{n-\alpha}\left|f^{(n+1)}\left(\frac{2-t}{2}a+\frac{t}{2}b\right)\right|dt\right].
\end{aligned}$$

From Hölder's inequality we get

$$\begin{aligned}
&\left|\frac{2^{n-\alpha-1}\Gamma(n-\alpha+1)}{(b-a)^{n-\alpha}}\left[({}^{C}D^{\alpha}_{(\frac{a+b}{2})+}f)(b)+(-1)^{n}({}^{C}D^{\alpha}_{(\frac{a+b}{2})-}f)(a)\right]\right.\\
&\quad\left.-f^{(n)}\left(\frac{a+b}{2}\right)\right|\\
&\leq\frac{b-a}{4}\left[\left[\int_0^1 t^{np-\alpha p}dt\right]^{\frac{1}{p}}\left[\int_0^1\left|f^{(n+1)}\left(\frac{t}{2}a+\frac{2-t}{2}b\right)\right|^q dt\right]^{\frac{1}{q}}\right.\\
&\quad\left.+\left[\int_0^1 t^{np-\alpha p}dt\right]^{\frac{1}{p}}\left[\int_0^1\left|f^{(n+1)}\left(\frac{2-t}{2}a+\frac{t}{2}b\right)\right|^q dt\right]^{\frac{1}{q}}\right].
\end{aligned}$$

Convexity of $|f^{(n+1)}|^q$ gives

$$\begin{aligned}
&\left|\frac{2^{n-\alpha-1}\Gamma(n-\alpha+1)}{(b-a)^{n-\alpha}}\left[({}^{C}D^{\alpha}_{(\frac{a+b}{2})+}f)(b)+(-1)^{n}({}^{C}D^{\alpha}_{(\frac{a+b}{2})-}f)(a)\right]\right.\\
&\quad\left.-f^{(n)}\left(\frac{a+b}{2}\right)\right|\\
&\leq\frac{b-a}{4}\left(\frac{1}{np-\alpha p+1}\right)^{\frac{1}{p}}\left[\left[\int_0^1\left(\frac{t}{2}|f^{(n+1)}(a)|^q+\frac{2-t}{2}|f^{(n+1)}(b)|^q\right)dt\right]^{\frac{1}{q}}\right.\\
&\quad\left.+\left[\int_0^1\left(\frac{2-t}{2}|f^{(n+1)}(a)|^q+\frac{t}{2}|f^{(n+1)}(b)|^q\right)dt\right]^{\frac{1}{q}}\right]\\
&=\frac{b-a}{4}\left(\frac{1}{np-\alpha p+1}\right)^{\frac{1}{p}}\left[\left[\frac{|f^{(n+1)}(a)|^q+3|f^{(n+1)}(b)|^q}{4}\right]^{\frac{1}{q}}\right.\\
&\quad\left.+\left[\frac{3|f^{(n+1)}(a)|^q+|f^{(n+1)}(b)|^q}{4}\right]^{\frac{1}{q}}\right].
\end{aligned}$$

For second inequality of (3.4) we use Minkowski's inequality as follows

$$\begin{aligned}
&\left|\frac{2^{n-\alpha-1}\Gamma(n-\alpha+1)}{(b-a)^{n-\alpha}}\left[({}^{C}D^{\alpha}_{(\frac{a+b}{2})+}f)(b)+(-1)^{n}({}^{C}D^{\alpha}_{(\frac{a+b}{2})-}f)(a)\right]\right.\\
&\quad\left.-f^{(n)}\left(\frac{a+b}{2}\right)\right|\\
&\leq \frac{b-a}{16}\left(\frac{4}{np-\alpha p+1}\right)^{\frac{1}{p}}\left[\left[|f^{(n+1)}(a)|^{q}+3|f^{(n+1)}(b)|^{q}\right]^{\frac{1}{q}}\right.\\
&\quad\left.+\left[3|f^{(n+1)}(a)|^{q}+|f^{(n+1)}(b)|^{q}\right]^{\frac{1}{q}}\right]\\
&\leq \frac{b-a}{16}\left(\frac{4}{np-\alpha p+1}\right)^{\frac{1}{p}}(3^{\frac{1}{q}}+1)(|f^{(n+1)}(a)|+|f^{(n+1)}(b)|)\\
&\leq \frac{b-a}{4}\left(\frac{4}{3(np-\alpha p+1)}\right)^{\frac{1}{p}}(|f^{(n+1)}(a)|+|f^{(n+1)}(b)|).
\end{aligned}$$

□

Acknowledgement

The research work of author Ghulam Farid is supported by Higher Education Commission of Pakistan under NRPU 2016, Project No. 5421.

References

[1] G. Abbas and G. Farid, Some integral inequalities for m-convex functions via generalized fractional integral operator containing generalized Mittag-Leffler function, *Cogent Math.*, **3** (2016), Article ID 1269589, 12 pages.

[2] G. Abbas, K. A. Khan, G. Farid and A. Ur Rehman, Generalizations of some fractional integral inequalities via generalized Mittag-Leffler function, *J. Inequal. Appl.*, **2017** 2017, Paper No. 121, 10 pages.

[3] G. Farid, A. Ur Rehman and B. Tariq, On Hadamard-type inequalities for m-convex functions via Riemann-Liouville fractional integrals, *Stud. Univ. Babeş-Bolyai Math.*, **62** (2017), 141–150.

[4] G. Farid, A. Ur Rehman and M. Zahara, On Hadamard inequalities for k-fractional integrals, *Nonlinear Funct. Anal. Appl.*, **21** (2016), 463–478.

[5] G. Farid, U. N. Katugampola and M. Usman, Ostrowski type fractional integral inequalities for S-Godunova-Levin functions via Katugampola fractional integrals, *Open J. Math. Sci.*, **1** (2017), 97–110.

[6] A. A. Kilbas, H. M. Srivastava and J. J. Trujillo, *Theory and applications of fractional differential equations*, Elsevier, Amsterdam, 2006.

[7] A. Loverro, *Fractional Calculus*: History, Definitions and Applications for the Engineer, Department of Aerospace and Mechanical Engineering, University of Notre Dame, 2004.

[8] S. Miller and B. Ross, *An introduction to the fractional calculus and fractional differential equations*, John Wiley and Sons, Inc., New York, 1993.

[9] M. Z. Sarikaya, E. Set, H. Yaldiz and N. Basak, Hermite-Hadamard's inequalities for fractional integrals and related fractional inequalities, *Math. Comput. Model.*, **57** (2013), 2403–2407.

[10] M. Z. Sarikaya and H. Yildirim, On Hermite-Hadamard type inequalities for Riemann-Liouville fractional integrals, *Miskolc Math. Notes*, **17** (2016), 1049–1059.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 27, NO.6, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC

A hesitant fuzzy ordered information system

Haidong Zhang[1]*, Yanping He[2]

1. School of Mathematics and Computer Science, Northwest MinZu University
Lanzhou, Gansu, 730030, P. R. China
2. School of Electrical Engineering, Northwest MinZu University
Lanzhou, Gansu, 730030, P. R. China

Abstract

Hesitant fuzzy information systems are generalized types of traditional information systems. First, a dominance relation is defined by the score function of hesitant fuzzy value in hesitant fuzzy information systems. By introducing the dominance relation to hesitant fuzzy information systems, we then establish a dominance-based rough set model by replacing the indiscernibility relation in classic rough set theory with the dominance relation, and develop a ranking approach for all objects based on dominance classes. Furthermore, to simplify the knowledge representation, we provide an attribute reduction approach to eliminate the redundant information. And an example is provided to illustrate the validity of this approach.

Key words: Dominance relation; Dominance-based rough set; Hesitant fuzzy information systems; Reduction

1 Introduction

As a mathematical approach to handle imprecision, vagueness and uncertainty in data analysis, rough set theory introduced by Pawlak [22, 23] is a valid means of granular computing [24]. In Pawlak's rough set model, the equivalence relation is a key tool and can represent information systems or decision tables. However, the equivalence relation is a very stringent condition that may limit the application of rough sets in practical problems. Therefore many researchers have generalized the notion of Pawlak's rough set by replacing the equivalence relation with other binary relations. It may be a fuzzy, intuitionistic fuzzy, interval-valued fuzzy, hesitant fuzzy or other indiscernibility one within the generalized rough sets [1, 3, 4, 15, 21, 27, 31, 34, 39, 40, 42–51, 54, 55, 59].

The aforementioned rough sets, such as fuzzy rough set [1,3,21,27,34,39], intuitionistic fuzzy rough set [15, 54, 55], hesitant fuzzy rough set [4, 40, 46], and so on, do not consider

*Corresponding author. Address: School of Mathematics and Computer Science Northwest MinZu University, LanZhou, Gansu, 730030, P.R.China. E-mail:lingdianstar@163.com

attributes with preference-ordered domains. However, in many real-life situations, we are always faced with some problems in which the ordering of properties of the considered attributes plays a key role. In such case, to take into consideration the ordering properties of criteria, Greco et al. [8–11] generalized the notion of Pawlak's rough set and initiated the dominance-based rough sets approach (DRSA) by replacing the indiscernibility relation with a dominance relation. In DRSA, the knowledge approximated is a collection of upward and downward unions of classes and the dominance classes are sets of objects defined by a dominance relation in which condition attributes are the criteria and classes are preference ordered. Up to now, many fruitful results in DRSA have been achieved [5, 12, 13, 25, 29, 30, 52].

Hesitant fuzzy (HF) set theory, initiated by Torra and Narukawa [32] and Torra [33] as one of the extensions of Zadeh's fuzzy set [56], permits the membership degree of an element to a set having several possible different values. Because HF set can express the hesitant information more comprehensively than other extensions of fuzzy set, it has been applied in dealing with lots of decision making problems successfully [2, 6, 17, 18, 28, 35–38, 57]. Although rough sets and HF sets both capture particular facets of the same notion-imprecision, studies on the combination of rough set theory and HF set theory are rare. In [40], Yang et al. proposed the concept of HF rough sets by integrating HF sets with rough sets. However, Zhang et al. [46] pointed out that hesitant fuzzy subset based on the hesitant fuzzy rough sets is not necessarily antisymmetric. To remedy this defect, they introduced an HF rough set over two universes and give a new decision making approach in uncertainty environment using the model. Subsequently, Zhang et al. [47] extended the rough set into interval-valued hesitant fuzzy environment and introduced the concept of interval-valued hesitant fuzzy rough sets. In typical hesitant fuzzy background, Zhang and Yang [53] studied the constructive approach to rough set approximation operators and proposed a typical hesitant fuzzy rough set. By combining the hesitant fuzzy linguistic term set and rough set, Zhang et al. [41] developed a general framework for the study of hesitant fuzzy linguistic rough sets over two universes.

On the one hand, hybrid models integrating an HF set with a rough set are rarely developed despite the above mentioned research efforts. Knowledge reduction is also an important task in classic and generalized rough set theory. However, the issue has rarely been discussed under the hesitant fuzzy environment. On the other hand, it is well known that the rough set data analysis starts from information systems which contain data about objects of interest, characterized by a finite set of attributes. As an important type of data tables, information systems on decision problems have been widely studied [7, 14, 16, 19, 20, 25, 26]. However, in general, we may not have enough expertise or possess a sufficient level of knowledge to precisely express our preferences over the objects by using a value or a single term, and then, we may usually have a certain hesitancy between a few different values. In such a case, the traditional information system can not express our preferences or assessments by only a single term or value. Considering the facts, it is natural for

us to investigate information systems in the context of hesitant fuzzy settings which is called hesitant fuzzy information systems. So how to make a decision by a dominance relation is an urgent need in hesitant fuzzy information systems. The aim of this paper is to introduce a dominance relation to hesitant fuzzy information systems and establish a rough set approach by replacing the indiscernibility relation with the dominance relation. Then we develop a reduction approach in hesitant fuzzy ordered information systems for eliminating redundant information from the perspective of the ordering of objects.

The rest of the paper is organized as follows. In Section 2, by reviewing some basic concepts, a dominance relation is introduced to hesitant fuzzy information systems and some properties are discussed. Section 3 establishes a dominance-based rough set approach in hesitant fuzzy ordered information systems by replacing the indiscernibility relation with a dominance relation. In Section 4, a ranking approach is established through the notions of dominance degree and whole dominance degree. Section 5 proposes a reduction approach in hesitant fuzzy ordered information system for eliminating redundant information from the perspective of the ordering of objects. Finally, we conclude the paper in Section 6.

2 Dominance relation in hesitant fuzzy information systems

In [32, 33], Torra and Narukawa introduced the notions related to HF sets.

Definition 2.1 ([32, 33]) *Let U be a fixed set, a hesitant fuzzy set $\mathbb{A}$ on U is in terms of a function $h_{\mathbb{A}}(x)$ that when applied to U returns a subset of [0,1].*

To be easily understood, Xia and Xu [35] denoted the HF set by a mathematical symbol:

$\mathbb{A} = \{< x, h_{\mathbb{A}}(x) > | x \in U\}$,

where $h_{\mathbb{A}}(x)$ is a set of some different values in [0,1], standing for the possible membership degrees of the element $x \in U$ to $\mathbb{A}$.

For convenience, Xia and Xu [35] called $h_{\mathbb{A}}(x)$ an HF element, and denoted the set of all HF sets on U by $HF(U)$.

To compare the HF elements, Xia and Xu [35] defined the following comparison laws:

Definition 2.2 ([35]) *For an HF element h, $s(h) = \frac{1}{\#h} \sum_{\gamma \in h} \gamma$ is called the score function of h, where $\#h$ is the number of the elements in h. For two HF elements h_1 and h_2, if $s(h_1) > s(h_2)$, then $h_1 \succ h_2$; if $s(h_1) = s(h_2)$, then $h_1 = h_2$.*

An HF information system is a quadruple $\mathcal{I} = (U, AT, V, f)$, where

- U is a non-empty finite set of objects called the universe;
- AT is a non-empty finite set of attributes;
- V is the domain of all attributes, i.e., $V = V_{AT} = \bigcup_{a \in AT} V_a$;
- $f : U \times AT \longrightarrow V$ is a total function such that $f(x, a) \in V_a$ for every $a \in AT, x \in U$, called

Table 1: An HF information system

U	a_1	a_2	a_3	a_4	a_5
x_1	{0.4,0.6,0.7}	{0.4,0.5,0.6}	{0.3,0.4,0.6}	{0.1,0.3,0.4}	{0.4,0.5,0.8}
x_2	{0.4,0.5,0.6}	{0.0,0.4,0.5}	{0.5,0.6,0.7}	{0.4,0.6,0.7}	{0.2,0.3,0.4}
x_3	{0.5,0.6,0.7}	{0.5,0.7,0.8}	{0.6,0.8,0.9}	{0.5,0.7,0.8}	{0.6,0.8,0.9}
x_4	{0.4,0.7,0.8}	{0.4,0.6,0.7}	{0.5,0.7,0.8}	{0.8,0.9,1.0}	{0.7,0.8,0.9}
x_5	{0.4,0.6,0.8}	{0.4,0.5,0.8}	{0.4,0.6,0.7}	{0.5,0.7,0.8}	{0.4,0.6,0.8}
x_6	{0.2,0.3,0.4}	{0.2,0.3,0.6}	{0.3,0.4,0.5}	{0.4,0.6,0.9}	{0.3,0.6,0.7}
x_7	{0.1,0.4,0.5}	{0.5,0.6,0.7}	{0.4,0.6,0.7}	{0.3,0.7,0.8}	{0.6,0.8,0.9}
x_8	{0.2,0.5,0.7}	{0.2,0.6,0.8}	{0.3,0.4,0.5}	{0.4,0.6,0.8}	{0.4,0.5,0.8}

an information function, where V_a is a set of HF elements. Denote as $f(x,a) = h_a(x)$, then we call it the HF value of x under the attribute a. In particular, if the information function $f(x,a)$ contains only one real number, the HF information system degenerates into a traditional information system [29].

Example 2.3 *An HF information system is given in Table 1, where* $U = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8\}$, $AT = \{a_1, a_2, a_3, a_4, a_5\}$.

In practical decision analysis, we always consider a dominance relation between objects that are possibly dominant in terms of values of an attributes set in an HF information system. Generally, an increasing preference and a decreasing preference can be considered by a decision maker. If the domain of an attribute is ordered by a decreasing or increasing preference, then the attribute is a criterion.

Definition 2.4 *An HF information system is called an HF ordered information system (HFOIS) if all attributes are criterions.*

On the basis of Definition 2.2, we develop an approach to rank two objects whose attribute characters are described by HF values.

Definition 2.5 *Let* $\mathcal{I} = (U, AT, V, f)$ *be an HFOIS. For* $x, y \in U$, *denote as*

$$x \succeq_A y \Longleftrightarrow \forall a \in A, f(x,a) \succeq f(y,a) \Longleftrightarrow \forall a \in A, f(x,a) \succ f(y,a) \vee f(x,a) = f(y,a),$$

then we say that x dominates y with respect to $A \subseteq AT$ if $x \succeq_A y$, denoted by $x\mathbb{R}_A^{\succeq}y$. Where $\mathbb{R}_A^{\succeq} = \{(y,x) \in U \times U | y \succeq_A x\}$ is called a dominance relation in HFOIS. Analogously, we call the relation $\mathbb{R}_A^{\preceq}$ a dominated relation in HFOIS, which can be defined as follows:

$$\mathbb{R}_A^{\preceq} = \{(y,x) \in U \times U | x \succeq_A y\}.$$

From Definitions 2.5 and 2.2, we can easily obtain the following theorem.

Theorem 2.6 *Let $\mathcal{I} = (U, AT, V, f)$ be an HFOIS and $A \subseteq AT$, then*

(1) $\mathbb{R}_A^{\succeq}$ and $\mathbb{R}_A^{\preceq}$ are reflexive, transitive and unsymmetric;

(2) $\mathbb{R}_A^{\succeq} = \bigcap_{a\in A} \mathbb{R}_{\{a\}}^{\succeq}$, $\mathbb{R}_A^{\preceq} = \bigcap_{a\in A} \mathbb{R}_{\{a\}}^{\preceq}$.

The dominance class induced by the dominance relation $\mathbb{R}_A^{\succeq}$ is the set of objects dominating x, i.e.,

$$\begin{aligned}[x]_A^{\succeq} &= \{y \in U | f(y,a) \succ f(x,a) \vee f(y,a) = f(x,a)(\forall a \in A)\} \\ &= \{y \in U | (y,x) \in \mathbb{R}_A^{\succeq}\},\end{aligned}$$

where $[x]_A^{\succeq}$ describes the set of objects that may dominate x and is called the A-dominating set with respect to $x \in U$.

Similarly, the dominance class induced by the dominated relation $\mathbb{R}_A^{\preceq}$ is the set of objects dominated by x, i.e.,

$$\begin{aligned}[x]_A^{\preceq} &= \{y \in U | f(x,a) \succ f(y,a) \vee f(x,a) = f(y,a)(\forall a \in A)\} \\ &= \{y \in U | (y,x) \in \mathbb{R}_A^{\preceq}\},\end{aligned}$$

where $[x]_A^{\preceq}$ describes the set of objects that may be dominated by x and is called the A-dominated set with respect to $x \in U$.

Let $U/\mathbb{R}_A^{\succeq}$ denote classification on the universe, which is the family set $\{[x]_A^{\succeq} | x \in U\}$. Any element from $U/\mathbb{R}_A^{\succeq}$ is called a dominance class with respect to A. Dominance classes in $U/\mathbb{R}_A^{\succeq}$ do not constitute a partition of U, but constitute a covering of U.

In the text that follows, without loss of generality, we adopt the dominance relation $\mathbb{R}_A^{\succeq}$ for investigating HFOIS and consider attributes with increasing preference.

Theorem 2.7 *Let $\mathcal{I} = (U, AT, V, f)$ be an HFOIS and $A, B \subseteq AT$.*

(1) If $B \subseteq A \subseteq AT$, then $\mathbb{R}_B^{\succeq} \supseteq \mathbb{R}_A^{\succeq} \supseteq \mathbb{R}_{AT}^{\succeq}$.

(2) If $B \subseteq A \subseteq AT$, then $[x]_B^{\succeq} \supseteq [x]_A^{\succeq} \supseteq [x]_{AT}^{\succeq}$.

(3) If $x_j \in [x_i]_A^{\succeq}$, then $[x_j]_A^{\succeq} \subseteq [x_i]_A^{\succeq}$ and $[x_i]_A^{\succeq} = \bigcup\{[x_j]_A^{\succeq} : x_j \in [x_i]_A^{\succeq}\}$.

(4) $[x_i]_A^{\succeq} = [x_j]_A^{\succeq}$ iff $f(x_i, a) = f(x_j, a)(\forall a \in A)$.

Proof. (1) and (2) are straightforward.

(3) If $x_j \in [x_i]_A^{\succeq}$, by Definition 2.5, then $f(x_j, a) \succeq f(x_i, a)$ for all $a \in A$. Similarly, for all $x \in [x_j]_A^{\succeq}$, we have $f(x, a) \succeq f(x_j, a)$. According to the transitivity of the dominance relation $\mathbb{R}_A^{\succeq}$, then $f(x, a) \succeq f(x_i, a)$, i.e. $x \in [x_i]_A^{\succeq}$. Thus $[x_j]_A^{\succeq} \subseteq [x_i]_A^{\succeq}$. Consequently, $[x_i]_A^{\succeq} = \bigcup\{[x_j]_A^{\succeq} : x_j \in [x_i]_A^{\succeq}\}$.

(4) "$\Rightarrow$" Assume that $[x_i]_A^{\succeq} = [x_j]_A^{\succeq}$, then $[x_i]_A^{\succeq} \subseteq [x_j]_A^{\succeq}$. Based on the result (3), for all $a \in A$, we have $f(x_i, a) \succeq f(x_j, a)$. Similarly, we can conclude that $f(x_j, a) \succeq f(x_i, a)$. Consequently, $f(x_i, a) = f(x_j, a)(\forall a \in A)$.

"$\Leftarrow$" It can be directly derived from the definition of the set of objects dominating x.
□

Example 2.8 (Continued from Example 2.3). Compute the classification induced by the dominance relation $\mathbb{R}^{\succeq}_{AT}$ in Table 1.

From Table 1, we have

$U/\mathbb{R}^{\succeq}_{AT} = \{[x_1]^{\succeq}_{AT}, [x_2]^{\succeq}_{AT}, \ldots, [x_8]^{\succeq}_{AT}\}$,

where

$[x_1]^{\succeq}_{AT} = \{x_1, x_3, x_4, x_5\}$, $[x_2]^{\succeq}_{AT} = \{x_2, x_3, x_4\}$, $[x_3]^{\succeq}_{AT} = \{x_3\}$, $[x_4]^{\succeq}_{AT} = \{x_4\}$,
$[x_5]^{\succeq}_{AT} = \{x_3, x_4, x_5\}$, $[x_6]^{\succeq}_{AT} = \{x_3, x_4, x_5, x_6\}$, $[x_7]^{\succeq}_{AT} = \{x_3, x_7\}$, $[x_8]^{\succeq}_{AT} = \{x_3, x_4, x_5, x_8\}$.

From Example 2.8, it is evident that dominance classes in $U/\mathbb{R}^{\succeq}_{AT}$ do not constitute a partition of U, but constitute a covering of U.

3 Rough set approach to HFOIS

In this section, we shall investigate the problems of set approximation and roughness measure with respect to the dominance relation $\mathbb{R}^{\succeq}_{A}$ in HFOIS.

Definition 3.1 *Let $\mathcal{I} = (U, AT, V, f)$ be an HFOIS. For any $X \subseteq U$ and $A \subseteq AT$, the lower and upper approximations of the set X with respect to the dominance relation $\mathbb{R}^{\succeq}_{A}$ are defined as follows:*

$\underline{\mathbb{R}^{\succeq}_{A}}(X) = \{x \in U | [x]^{\succeq}_{A} \subseteq X\}$,
$\overline{\mathbb{R}^{\succeq}_{A}}(X) = \{x \in U | [x]^{\succeq}_{A} \cap X \neq \emptyset\}$.

From Definition 3.1, we can easily obtain the following theorem.

Theorem 3.2 *Let $\mathcal{I} = (U, AT, V, f)$ be an HFOIS and $X, Y \subseteq U$, then*

(1) $\underline{\mathbb{R}^{\succeq}_{A}}(X) = \sim \overline{\mathbb{R}^{\succeq}_{A}}(\sim X)$, $\overline{\mathbb{R}^{\succeq}_{A}}(X) = \sim \underline{\mathbb{R}^{\succeq}_{A}}(\sim X)$;

(2) $\underline{\mathbb{R}^{\succeq}_{A}}(X) \subseteq X \subseteq \overline{\mathbb{R}^{\succeq}_{A}}(X)$;

(3) $A \subseteq AT \Longrightarrow \underline{\mathbb{R}^{\succeq}_{A}}(X) \subseteq \underline{\mathbb{R}^{\succeq}_{AT}}(X)$, $\overline{\mathbb{R}^{\succeq}_{A}}(X) \supseteq \overline{\mathbb{R}^{\succeq}_{AT}}(X)$;

(4) $X \subseteq Y \Longrightarrow \underline{\mathbb{R}^{\succeq}_{A}}(X) \subseteq \underline{\mathbb{R}^{\succeq}_{A}}(Y)$, $\overline{\mathbb{R}^{\succeq}_{A}}(X) \subseteq \overline{\mathbb{R}^{\succeq}_{A}}(Y)$;

(5) $\underline{\mathbb{R}^{\succeq}_{A}}(X \cap Y) = \underline{\mathbb{R}^{\succeq}_{A}}(X) \cap \underline{\mathbb{R}^{\succeq}_{A}}(Y)$, $\overline{\mathbb{R}^{\succeq}_{A}}(X \cup Y) = \overline{\mathbb{R}^{\succeq}_{A}}(X) \cup \overline{\mathbb{R}^{\succeq}_{A}}(Y)$;

(6) $\underline{\mathbb{R}^{\succeq}_{A}}(X \cup Y) \supseteq \underline{\mathbb{R}^{\succeq}_{A}}(X) \cup \underline{\mathbb{R}^{\succeq}_{A}}(Y)$, $\overline{\mathbb{R}^{\succeq}_{A}}(X \cap Y) \subseteq \overline{\mathbb{R}^{\succeq}_{A}}(X) \cap \overline{\mathbb{R}^{\succeq}_{A}}(Y)$;

(7) $\underline{\mathbb{R}^{\succeq}_{A}}(\emptyset) = \overline{\mathbb{R}^{\succeq}_{A}}(\emptyset) = \emptyset$, $\underline{\mathbb{R}^{\succeq}_{A}}(U) = \overline{\mathbb{R}^{\succeq}_{A}}(U) = U$;

(8) $\underline{\mathbb{R}^{\succeq}_{A}}(\underline{\mathbb{R}^{\succeq}_{A}}(X)) = \underline{\mathbb{R}^{\succeq}_{A}}(X)$, $\overline{\mathbb{R}^{\succeq}_{A}}(\overline{\mathbb{R}^{\succeq}_{A}}(X)) = \overline{\mathbb{R}^{\succeq}_{A}}(X)$.

Theorem 3.3 *Let $\mathcal{I} = (U, AT, V, f)$ be an HFOIS and $A \subseteq AT$. If $\mathbb{R}^{\succeq}_{A} = \mathbb{R}^{\succeq}_{AT}$, then $\underline{\mathbb{R}^{\succeq}_{A}}(X) = \underline{\mathbb{R}^{\succeq}_{AT}}(X)$ and $\overline{\mathbb{R}^{\succeq}_{A}}(X) = \overline{\mathbb{R}^{\succeq}_{AT}}(X)$.*

Proof. It is directly derived from Definitions 2.5 and 3.1. $\square$

Generally speaking, the uncertainty of a set is due to the existence of the borderline region. The wider the borderline region of a set is, the lower the accuracy of the set is. To express the idea precisely, some basic measures (accuracy and roughness) are defined to depict the quality of the rough approximation of a set. In the following, we introduce the concepts of roughness measure and accuracy measure to measure the imprecision of rough sets induced by dominance relation $\mathbb{R}_A^{\succeq}$ in HFOIS.

Definition 3.4 *Let $\mathcal{I} = (U, AT, V, f)$ be an HFOIS, $X \subseteq U$ and $A \subseteq AT$. Then the roughness measure $\rho_X^{\mathbb{R}_A^{\succeq}}$ of the set X with respect to the dominance relation $\mathbb{R}_A^{\succeq}$ is defined as follows:*

$$\rho_X^{\mathbb{R}_A^{\succeq}} = 1 - \frac{|\underline{\mathbb{R}_A^{\succeq}}(X)|}{|\overline{\mathbb{R}_A^{\succeq}}(X)|},$$

where $|\cdot|$ denotes the cardinality of a set. If $\overline{\mathbb{R}_A^{\succeq}}(X) = \emptyset$, we define $\rho_X^{\mathbb{R}_A^{\succeq}} = 0$. $\eta_X^{\mathbb{R}_A^{\succeq}} = \frac{|\underline{\mathbb{R}_A^{\succeq}}(X)|}{|\overline{\mathbb{R}_A^{\succeq}}(X)|}$ is referred to as the accuracy measure of X with respect to the dominance relation $\mathbb{R}_A^{\succeq}$.

According to Definition 3.4 and Theorem 3.2(2), we observe that $0 \leq \rho_X^{\mathbb{R}_A^{\succeq}} \leq 1$ and $0 \leq \eta_X^{\mathbb{R}_A^{\succeq}} \leq 1$.

Obviously, by Theorem 3.3 and Definition 3.4, we can draw the following conclusion.

Theorem 3.5 *Let $\mathcal{I} = (U, AT, V, f)$ be an HFOIS and $A \subseteq AT$. If $\mathbb{R}_A^{\succeq} = \mathbb{R}_{AT}^{\succeq}$, then $\rho_X^{\mathbb{R}_A^{\succeq}} = \rho_X^{\mathbb{R}_{AT}^{\succeq}}$ and $\eta_X^{\mathbb{R}_A^{\succeq}} = \eta_X^{\mathbb{R}_{AT}^{\succeq}}$.*

Theorem 3.6 *Let $\mathcal{I} = (U, AT, V, f)$ be an HFOIS, $X \subseteq U$ and $A \subseteq AT$, then the following holds:*

(1) $\rho_X^{\mathbb{R}_{AT}^{\succeq}} \leq \rho_X^{\mathbb{R}_A^{\succeq}}$,

(2) $\eta_X^{\mathbb{R}_{AT}^{\succeq}} \geq \eta_X^{\mathbb{R}_A^{\succeq}}$.

Proof. (1) Since $A \subseteq AT$, by Theorem 3.2(3) we have $\underline{\mathbb{R}_A^{\succeq}}(X) \subseteq \underline{\mathbb{R}_{AT}^{\succeq}}(X)$ and $\overline{\mathbb{R}_A^{\succeq}}(X) \supseteq \overline{\mathbb{R}_{AT}^{\succeq}}(X)$. It implies that $\frac{|\underline{\mathbb{R}_A^{\succeq}}(X)|}{|\overline{\mathbb{R}_A^{\succeq}}(X)|} \leq \frac{|\underline{\mathbb{R}_{AT}^{\succeq}}(X)|}{|\overline{\mathbb{R}_{AT}^{\succeq}}(X)|}$. According to Definition 3.4, then $\rho_X^{\mathbb{R}_A^{\succeq}} = 1 - \frac{|\underline{\mathbb{R}_A^{\succeq}}(X)|}{|\overline{\mathbb{R}_A^{\succeq}}(X)|} \geq 1 - \frac{|\underline{\mathbb{R}_{AT}^{\succeq}}(X)|}{|\overline{\mathbb{R}_{AT}^{\succeq}}(X)|} = \rho_X^{\mathbb{R}_{AT}^{\succeq}}$, i.e., $\rho_X^{\mathbb{R}_{AT}^{\succeq}} \leq \rho_X^{\mathbb{R}_A^{\succeq}}$.

(2) It is directly derived from the result (1) and Definition 3.4. $\square$

Example 3.7 *Consider HFOIS in Table 1. Let $A = \{a_1, a_4, a_5\} \subseteq AT$ and $X = \{x_2, x_3, x_5, x_7\}$. Now we compute the rough sets of X induced by $U/\mathbb{R}_{AT}^{\succeq}$ and $U/\mathbb{R}_A^{\succeq}$, respectively.*

By Definition 3.1 and Example 2.8, the rough set $(\underline{\mathbb{R}^{\succeq}_{AT}}(X), \overline{\mathbb{R}^{\succeq}_{AT}}(X))$ *can be obtained as follows:*

$\underline{\mathbb{R}^{\succeq}_{AT}}(X) = \{x_3, x_7\}$, $\overline{\mathbb{R}^{\succeq}_{AT}}(X)) = \{x_1, x_2, x_3, x_5, x_6, x_7, x_8\}$.

Then we compute the classification set induced by the dominance relation $U/\mathbb{R}^{\succeq}_{A}$*. By Table 1, we have*

$U/\mathbb{R}^{\succeq}_{A} = \{[x_1]^{\succeq}_{A}, [x_2]^{\succeq}_{A}, \ldots, [x_8]^{\succeq}_{A}\}$,

where

$[x_1]^{\succeq}_{A} = \{x_1, x_3, x_4, x_5\}$, $[x_2]^{\succeq}_{A} = \{x_2, x_3, x_4, x_5\}$, $[x_3]^{\succeq}_{A} = \{x_3, x_4\}$, $[x_4]^{\succeq}_{A} = \{x_4\}$,
$[x_5]^{\succeq}_{A} = \{x_3, x_4, x_5\}$, $[x_6]^{\succeq}_{A} = \{x_3, x_4, x_5, x_6\}$, $[x_7]^{\succeq}_{A} = \{x_3, x_4, x_7\}$, $[x_8]^{\succeq}_{A} = \{x_3, x_4, x_5, x_8\}$.

Similarly, by Definition 3.1, we calculate the rough set $(\underline{\mathbb{R}^{\succeq}_{A}}(X), \overline{\mathbb{R}^{\succeq}_{A}}(X))$ *as follows:*

$\underline{\mathbb{R}^{\succeq}_{A}}(X) = \emptyset$, $\overline{\mathbb{R}^{\succeq}_{A}}(X)) = \{x_1, x_2, x_3, x_5, x_6, x_7, x_8\}$.

Therefore, we have

$$\rho_X^{\mathbb{R}^{\succeq}_{A}} = 1 - \frac{|\underline{\mathbb{R}^{\succeq}_{A}}(X)|}{|\overline{\mathbb{R}^{\succeq}_{A}}(X)|} = 1, \quad \rho_X^{\mathbb{R}^{\succeq}_{AT}} = 1 - \frac{|\underline{\mathbb{R}^{\succeq}_{AT}}(X)|}{|\overline{\mathbb{R}^{\succeq}_{AT}}(X)|} = 1 - \frac{2}{7} = \frac{5}{7},$$

$$\eta_X^{\mathbb{R}^{\succeq}_{A}} = \frac{|\underline{\mathbb{R}^{\succeq}_{A}}(X)|}{|\overline{\mathbb{R}^{\succeq}_{A}}(X)|} = 0, \quad \eta_X^{\mathbb{R}^{\succeq}_{AT}} = \frac{|\underline{\mathbb{R}^{\succeq}_{AT}}(X)|}{|\overline{\mathbb{R}^{\succeq}_{AT}}(X)|} = \frac{2}{7}.$$

Thus, $\rho_X^{\mathbb{R}^{\succeq}_{AT}} \le \rho_X^{\mathbb{R}^{\succeq}_{A}}$ *and* $\eta_X^{\mathbb{R}^{\succeq}_{AT}} \ge \eta_X^{\mathbb{R}^{\succeq}_{A}}$.

4 Ranking for all objects in HFOIS

In [58], Zhang et al. defined the concept of dominance degrees for ranking all objects in classical ordered information systems. Inspired by the idea, we introduce a dominance degree between two objects in HFOIS as follows:

Definition 4.1 *Let* $\mathcal{I} = (U, AT, V, f)$ *be an HFOIS and* $A \subseteq AT$*. Dominance degree between two objects with respect to the dominance relation* $\mathbb{R}^{\succeq}_{A}$ *is defined as*

$$\mathbb{D}_A(x_i, x_j) = \frac{|\sim [x_i]^{\succeq}_{A} \cup [x_j]^{\succeq}_{A}|}{|U|},$$

where $|\cdot|$ *denotes the cardinality of a set,* $x_i, x_j \in U$.

Theorem 4.2 *Dominance degree* $\mathbb{D}_A(x_i, x_j)$ *satisfies the following properties:*

(1) $\frac{1}{|U|} \le \mathbb{D}_A(x_i, x_j) \le 1$;

(2) if $(x_j, x_k) \in \mathbb{R}^{\succeq}_{A}$*, then* $\mathbb{D}_A(x_i, x_j) \le \mathbb{D}_A(x_i, x_k)$ *and* $\mathbb{D}_A(x_j, x_i) \ge \mathbb{D}_A(x_k, x_i)$.

Proof. (1) It is straightforward.

(2) Assume that $(x_j, x_k) \in \mathbb{R}_A^{\succeq}$. By Theorem 2.7, then $[x_j]_A^{\succeq} \subseteq [x_k]_A^{\succeq}$. Therefore, we have

$$\begin{aligned}
\mathbb{D}_A(x_i, x_j) - \mathbb{D}_A(x_i, x_k) &= \frac{1}{|U|}(| \sim [x_i]_A^{\succeq} \cup [x_j]_A^{\succeq}| - | \sim [x_i]_A^{\succeq} \cup [x_k]_A^{\succeq}|) \\
&\leq \frac{1}{|U|}(| \sim [x_i]_A^{\succeq} \cup [x_k]_A^{\succeq}| - | \sim [x_i]_A^{\succeq} \cup [x_k]_A^{\succeq}|) \\
&= 0,
\end{aligned}$$

$$\begin{aligned}
\mathbb{D}_A(x_j, x_i) - \mathbb{D}_A(x_k, x_i) &= \frac{1}{|U|}(| \sim [x_j]_A^{\succeq} \cup [x_i]_A^{\succeq}| - | \sim [x_k]_A^{\succeq} \cup [x_i]_A^{\succeq}|) \\
&\geq \frac{1}{|U|}(| \sim [x_k]_A^{\succeq} \cup [x_i]_A^{\succeq}| - | \sim [x_k]_A^{\succeq} \cup [x_i]_A^{\succeq}|) \\
&= 0.
\end{aligned}$$

That is, $\mathbb{D}_A(x_i, x_j) \leq \mathbb{D}_A(x_i, x_k)$ and $\mathbb{D}_A(x_j, x_i) \geq \mathbb{D}_A(x_k, x_i)$. □

According to Definition 4.1, we may construct a dominance relation matrix with respect to A induced by the dominance relation $\mathbb{R}_A^{\succeq}$. Based on the dominance relation matrix, the whole dominance degree of each object can be calculated by the following formula

$$\mathbb{D}_A(x_i) = \frac{1}{|U|-1} \sum_{j \neq i} \mathbb{D}_A(x_i, x_j), \ x_i, x_j \in U. \tag{1}$$

Obviously, by the concepts of dominance degree and whole dominance degree, the following theorem holds.

Theorem 4.3 *Let $\mathcal{I} = (U, AT, V, f)$ be an HFOIS and $A \subseteq AT$. If $\mathbb{R}_A^{\succeq} = \mathbb{R}_{AT}^{\succeq}$, then $\mathbb{D}_A(x_i, x_j) = \mathbb{D}_{AT}(x_i, x_j)$ and $\mathbb{D}_A(x_i) = \mathbb{D}_{AT}(x_i)$.*

By employing the whole dominance degree of each object on the universe, we may rank all objects by the values of $\mathbb{D}_A(x_i)$. The following example is given to demonstrate the application of this method.

Example 4.4 *(Continued from Example 2.8). Rank all objects in U based on the dominance relation $\mathbb{R}_{AT}^{\succeq}$. By the concept of dominance degree, we obtain the dominance relation matrix as follows*

$$\begin{pmatrix}
1 & 0.75 & 0.625 & 0.625 & 0.875 & 0.875 & 0.625 & 0.875 \\
0.875 & 1 & 0.75 & 0.75 & 0.875 & 0.875 & 0.75 & 0.875 \\
1 & 1 & 1 & 0.875 & 1 & 1 & 1 & 1 \\
1 & 1 & 0.875 & 1 & 1 & 1 & 0.875 & 1 \\
1 & 0.875 & 0.75 & 0.75 & 1 & 1 & 0.75 & 1 \\
0.875 & 0.75 & 0.625 & 0.625 & 0.875 & 1 & 0.625 & 0.875 \\
0.875 & 0.875 & 0.875 & 0.75 & 0.875 & 0.875 & 1 & 0.875 \\
0.875 & 0.875 & 0.625 & 0.625 & 0.875 & 0.875 & 0.625 & 1
\end{pmatrix}.$$

Therefore, by Equation 1, the whole dominance degree of each object x_i can be calculated as follows:

$\mathbb{D}_{AT}(x_1) = 0.75$, $\mathbb{D}_{AT}(x_2) = 0.82$, $\mathbb{D}_{AT}(x_3) = 0.98$, $\mathbb{D}_{AT}(x_4) = 0.96$,

$\mathbb{D}_{AT}(x_5) = 0.875$, $\mathbb{D}_{AT}(x_6) = 0.75$, $\mathbb{D}_{AT}(x_7) = 0.857$, $\mathbb{D}_{AT}(x_8) = 0.768$.

An object with larger value implies a better object. Therefore, based on the values of $\mathbb{D}_{AT}(x_i)$, we can rank all objects as follows:

$$x_3 \succeq x_4 \succeq x_5 \succeq x_7 \succeq x_8 \succeq \begin{pmatrix} x_1 \\ x_6 \end{pmatrix}.$$

5 Attribute reduction in HFOIS

In order to simplify knowledge representation in HFOIS, it is necessary for us to reduce some dispensable attributes in the context of dominance relations. In this section, we will develop an approach to attribute reduction in a given HFOIS.

Definition 5.1 *Let $\mathcal{I} = (U, AT, V, f)$ be an HFOIS and $A \subseteq AT$. For any $B \subset A$, if $\mathbb{R}^{\succeq}_{A} = \mathbb{R}^{\succeq}_{AT}$ and $\mathbb{R}^{\succeq}_{B} \neq \mathbb{R}^{\succeq}_{AT}$, then we call A an attribute reduction of $\mathcal{I}$.*

By Definition 5.1, we can easily verify the following conclusion holds.

Theorem 5.2 *Let $\mathcal{I} = (U, AT, V, f)$ be an HFOIS and $A \subseteq AT$. If A is an attribute reduction of $\mathcal{I}$, then $\mathbb{D}_A(x_i, x_j) = \mathbb{D}_{AT}(x_i, x_j)$, $x_i, x_j \in U$.*

In what follows we define several special attributes in HFOIS as follows:

Definition 5.3 *Let $\mathcal{I} = (U, AT, V, f)$ be an HFOIS. If $\mathbb{R}^{\succeq}_{AT} = \mathbb{R}^{\succeq}_{(AT-\{a\})}$, an attribute $a \in AT$ is called dispensable with respect to the dominance relation $\mathbb{R}^{\succeq}_{AT}$; otherwise, a is called indispensable. The set of all indispensable attributes is called a core with respect to the dominance relation $\mathbb{R}^{\succeq}_{AT}$.*

Definition 5.4 *Let $\mathcal{I} = (U, AT, V, f)$ be an HFOIS and $A \subseteq AT$. Denote by $Dis(x, y) = \{a \in A | (x, y) \notin \mathbb{R}^{\succeq}_{\{a\}}\}$, then we call $Dis(x, y)$ a discernibility attribute set between x and y, and $DIS = (Dis(x, y) : x, y \in U)$ a discernibility matrix of the HFOIS.*

Theorem 5.5 *Let $\mathcal{I} = (U, AT, V, f)$ be an HFOIS and $A \subseteq AT$. Suppose that $Dis(x, y)$ is the discernibility attribute set of $\mathcal{I}$; then $\mathbb{R}^{\succeq}_{AT} = \mathbb{R}^{\succeq}_{A}$ iff $A \cap Dis(x, y) \neq \emptyset$ $(Dis(x, y) \neq \emptyset)$.*

Proof. "$\Longrightarrow$" Assume that $\mathbb{R}^{\succeq}_{AT} = \mathbb{R}^{\succeq}_{A}$, for any $y \in U$ then $[y]^{\succeq}_{AT} = [y]^{\succeq}_{A}$. If some $x \notin [y]^{\succeq}_{AT}$, then $x \notin [y]^{\succeq}_{A}$. Therefore, there exists $a \in A$ such that $(x, y) \notin \mathbb{R}^{\succeq}_{\{a\}}$. Thus, $a \in Dis(x, y)$. Consequently, if $Dis(x, y) \neq \emptyset$, we have $A \cap Dis(x, y) \neq \emptyset$.

"$\Longleftarrow$" Based on Definition 5.4, we can observe that if $(x, y) \notin \mathbb{R}^{\succeq}_{AT}$ for any $(x, y) \in U \times U$, then $Dis(x, y) \neq \emptyset$. Since $A \cap Dis(x, y) \neq \emptyset$, there exists $a \in A$ such that $a \in Dis(x, y)$, i.e., $(x, y) \notin \mathbb{R}^{\succeq}_{\{a\}}$. Thus $(x, y) \notin \mathbb{R}^{\succeq}_{A}$. Consequently, $\mathbb{R}^{\succeq}_{AT} \supseteq \mathbb{R}^{\succeq}_{A}$. On the other hand, note that $A \subseteq AT$, then we have $\mathbb{R}^{\succeq}_{AT} \subseteq \mathbb{R}^{\succeq}_{A}$. Hence, $\mathbb{R}^{\succeq}_{AT} = \mathbb{R}^{\succeq}_{A}$. $\square$

Table 2: The discernibility matrix of Table 1

U	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8
x_1	$\emptyset$	a_3a_4	$a_1a_2a_3a_4a_5$	$a_1a_2a_3a_4a_5$	$a_1a_2a_3a_4a_5$	a_4	$a_2a_3a_4a_5$	a_2a_4
x_2	$a_1a_2a_5$	$\emptyset$	$a_1a_2a_3a_4a_5$	$a_1a_2a_3a_4a_5$	$a_1a_2a_4a_5$	$a_2a_4a_5$	$a_2a_4a_5$	$a_2a_4a_5$
x_3	$\emptyset$	$\emptyset$	$\emptyset$	$a_1a_4a_5$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
x_4	$\emptyset$	$\emptyset$	a_2a_3	$\emptyset$	$\emptyset$	$\emptyset$	a_2	$\emptyset$
x_5	$\emptyset$	a_3	$a_2a_3a_5$	$a_1a_3a_4a_5$	$\emptyset$	$\emptyset$	a_2a_5	$\emptyset$
x_6	$a_1a_2a_3a_5$	a_1a_3	$a_1a_2a_3a_4a_5$	$a_1a_2a_3a_4a_5$	$a_1a_2a_3a_4a_5$	$\emptyset$	$a_1a_2a_3a_5$	$a_1a_2a_5$
x_7	a_1	a_1a_3	$a_1a_2a_3a_4$	$a_1a_3a_4a_5$	a_1a_4	a_4	$\emptyset$	a_1
x_8	a_1a_3	a_1a_3	$a_1a_2a_3a_4a_5$	$a_1a_2a_3a_4a_5$	$a_1a_2a_3a_4a_5$	a_4	$a_2a_3a_5$	$\emptyset$

Definition 5.6 *Let $\mathcal{I} = (U, AT, V, f)$ be an HFOIS, $A \subseteq AT$ and $Dis(x, y)$ the discernibility attributes set of $\mathcal{I}$ with respect to $\mathbb{R}^{\succeq}_{AT}$. Denote as*

$$\mathcal{M} = \bigwedge \left\{ \bigvee \{a : a \in Dis(x, y) | x, y \in U\} \right\},$$

then we call $\mathcal{M}$ a discernibility function.

From the definition of minimal disjunctive normal form of the discernibility function and Theorem 5.5, we can easily verify the following conclusion.

Theorem 5.7 *Let $\mathcal{I} = (U, AT, V, f)$ be an HFOIS. The minimal disjunctive normal form of $\mathcal{M}$ is*

$$\mathcal{M} = \bigvee_{k=1}^{t} \left(\bigwedge_{s=1}^{q_k} a_{i_s} \right).$$

Denoted by $\mathcal{B}_k = \{a_{i_s} : s = 1, 2, \ldots, q_k\}$, then $\{\mathcal{B}_k : k = 1, 2, \ldots, t\}$ are the family of all attribute reductions of $\mathcal{I}$.

By Theorem 5.7, a practical approach to attribute reductions of HFOIS is provided. In the following, we shall illustrate how to obtain attribute reductions of an HFOIS by an example.

Example 5.8 *(Continued from Example 2.3). According to Definition 5.4, we obtain the discernibility matrix of Table 1 (see Table 2). Thus, we have*

$$\begin{aligned}
\mathcal{M} = {} & (a_1 \vee a_2 \vee a_5) \wedge (a_1 \vee a_2 \vee a_3 \vee a_5) \wedge a_1 \wedge (a_1 \vee a_3) \wedge (a_3 \vee a_4) \wedge a_3 \\
& \wedge (a_1 \vee a_2 \vee a_3 \vee a_4 \vee a_5) \wedge (a_2 \vee a_3) \wedge (a_2 \vee a_3 \vee a_5) \wedge (a_1 \vee a_2 \vee a_3 \vee a_4) \\
& \wedge (a_1 \vee a_4 \vee a_5) \wedge (a_1 \vee a_3 \vee a_4 \vee a_5) \wedge (a_1 \vee a_2 \vee a_4 \vee a_5) \wedge (a_1 \vee a_4) \wedge a_4 \\
& \wedge (a_2 \vee a_4 \vee a_5) \wedge (a_2 \vee a_3 \vee a_4 \vee a_5) \wedge a_2 \wedge (a_2 \vee a_5) \wedge (a_2 \vee a_4) \\
= {} & a_1 \wedge a_2 \wedge a_3 \wedge a_4
\end{aligned}$$

Therefore, there is only one attribute reduction for the HFOIS, which is $\{a_1, a_2, a_3, a_4\}$*. From the perspective of the ordering of objects, the attributes* a_1, a_2, a_3 *and* a_4 *are indispensable in Table 1.*

6 Conclusions

Although the conventional rough set theory is a powerful and useful mathematical tool to deal with uncertainty information, it can not deal with ordering objects instead of classifying objects. In this situation, we have investigated information systems in the context of hesitant fuzzy settings, which is called hesitant fuzzy information systems. The hesitant fuzzy information system is an important type of data tables, which is generalized from the traditional information systems. First, based on the score function of hesitant fuzzy value, a dominance relation has been introduced to hesitant fuzzy information systems. Then we have established a rough set approach in HFOIS by replacing the indiscernibility relation with the dominance relation, and given a ranking approach to all objects by employing the whole dominance degree of each object. Finally, from the perspective of the ordering of objects, we have also developed a reduction approach in HFOIS for eliminating redundant information.

Acknowledgements

The authors would like to thank the anonymous referees for their valuable comments and suggestions. This work is supported by the Natural Science Foundation of Gansu Province (No. 1606RJZA003), the Research Project Funds for Higher Education Institutions of Gansu Province (No. 2016B-005), the Fundamental Research Funds for the Central Universities of Northwest MinZu University (No. 31920170010) and the first-class discipline program of Northwest Minzu University.

References

[1] M. D. Cock, C. Cornelis, E.E. Kerre, Fuzzy rough sets: the forgotten step, IEEE Transactions on Fuzzy Systems 15 (1) (2007) 121-130.

[2] N. Chen, Z. S. Xu, M. M. Xia, Correlation coefficients of hesitant fuzzy sets and their applications to clustering analysis, Applied Mathematical Modelling 37 (2013) 2197-2211.

[3] D. Dubois, H. Prade, Rough fuzzy sets and fuzzy rough sets, International Journal of General Systems 17 (1990) 191-209.

[4] D. Deepak, S. J. John, Hesitant fuzzy rough sets through hesitant fuzzy relations, Annals of Fuzzy Mathematics and Informatics 8 (1) (2014) 33-46.

[5] K. Dembczynski, R. Pindur, R. Susmaga, Dominance-based rough set classifier without induction of decision rules, Electronic Notes in Theoretical Computer Science 82 (4) (2003) 84-95.

[6] B. Farhadinia, Information measures for hesitant fuzzy sets and interval-valued hesitant fuzzy sets, Information Sciences 240 (2013) 129-144.

[7] J.W. Guan, D.A. Bell, Rough computational methods for information systems, Artificial Intelligence 105 (1998) 77-103.

[8] S. Greco, B. Matarazzo, R. Slowinski, A new rough set approach to multicriteria and multiattribute classification, Lecture Notes in Artificial Intelligence 1424 (1998) 60-67.

[9] S. Greco, B. Matarazzo, R. Slowinski, Rough sets theory for multicriteria decision analysis, European Journal of Operational Research 129 (2001) 1-47.

[10] S. Greco, B. Matarazzo, R. Slowinski, Rough sets methodology for sorting problems in presence of multiple attributes and criteria, European Journal of Operational Research 138 (2002) 247-259.

[11] S. Greco, B. Matarazzo, R. Slowinski, J. Stefanowski, An algorithm for induction of decision rules consistent with the dominance principle, Lecture Notes in Artificial Intelligence 2005 (2001) 304-313.

[12] B. Huang, H.X. Li, D.K. Wei, Dominance-based rough set model in intuitionistic fuzzy information systems, Knowledge-Based Systems 28 (2012) 115-123.

[13] B. Huang, Y.L. Zhuang, H.X. Li, D.K. Wei, A dominance intuitionistic fuzzy-rough set approach and its applications, Applied Mathematical Modelling 37 (2013) 7128-7141.

[14] G. Jeon, D. Kim, J. Jeong, Rough sets attributes reduction based expert system in interlaced video sequences, IEEE Transactions on Consumer Electronics 52 (4) (2006) 1348-1355.

[15] S.P. Jena, S.K. Ghosh, Intuitionistic fuzzy rough sets, Notes on Intuitionistic Fuzzy Sets 8 (2002) 1-18.

[16] M. Kryszkiewicz, Rough set approach to incomplete information systems, Information Sciences 112 (1998) 39-49.

[17] D.C. Liang, Liu D, A novel risk decision-making based on decision-theoretic rough sets under hesitant fuzzy information, IEEE Transactions on Fuzzy Systems 23 (2) (2015) 237-247.

[18] H.C. Liao, Z.S Xu, A VIKOR-based method for hesitant fuzzy multi-criteria decision making, Fuzzy Optimization Decision Making 12 (2013) 373-392.

[19] J.Y. Liang, D.Y. Li, Uncertainty and Knowledge Acquisition in Information Systems, Science Press, Beijing, China, 2005.

[20] J.Y. Liang, Y.H. Qian, Axiomatic approach of knowledge granulation in information systems, Lecture Notes in Artificial Intelligence 4304 (2006) 1074-1078.

[21] S. Nanda, S. Majumda, Fuzzy rough sets, Fuzzy Sets and Systems 45 (1992) 157-160.

[22] Z. Pawlak, Rough sets, International Journal of Computer Information Sciences 11 (1982) 145-172.

[23] Z. Pawlak, Rough Sets-Theoretical Aspects to Reasoning about Data, Kluwer Academic Publisher, Boston, 1991.

[24] W. Pedrycz, Granular Computing: Analysis and Design of Intelligent Systems, CRC Press/Francis Taylor, Boca Raton, 2013.

[25] Y.H. Qian, J.Y. Liang, C.Y. Dang, Interval ordered information systems, Computers and Mathematics with Applications 56 (2008) 1994-2009.

[26] Y.H. Qian, J.Y. Liang, C.Y. Dang, Converse approximation and rule extraction from decision tables in rough set theory, Computers and Mathmatics with Applications 55 (2008) 1754-1765.

[27] A.M. Radzikowska, E.E. Kerre, A comparative study of fuzzy rough sets, Fuzzy Sets and Systems 126 (2002) 137-155.

[28] R. M. Rodrguez, L. Martnez, F.Herrera, Hesitant fuzzy linguistic term sets for decision making, IEEE Transactions on Fuzzy Systems 20 (1) (2012) 109-119.

[29] M.W. Shao, W.X. Zhang, Dominance relation and rules in an incomplete ordered information system, International Journal of Intelligent Systems 20 (2005) 13-27.

[30] Y. Sai, Y.Y. Yao, N. Zhong, Data analysis and mining in ordered information tables, in: Proceedings of 2001 IEEE International Conference on Data Mining, IEEE Computer Society Press, 2001, pp. 497-504.

[31] S.P. Tiwari, Arun K. Srivastava, Fuzzy rough sets, fuzzy preorders and fuzzy topologies, Fuzzy Sets and Systems 210 (2013) 63-68.

[32] V. Torra, Y. Narukawa, On hesitant fuzzy sets and decision, The 18th IEEE International Conference on Fuzzy Systems, Korea, 2009, pp. 1378-1382.

[33] V. Torra, Hesitant fuzzy sets, International Journal of Intelligent Systems 25 (2010) 529-539.

[34] W.Z. Wu, W.X. Zhang, Constructive and axiomatic approaches of fuzzy approximation operators, Information Sciences 159 (2004) 233-254.

[35] M.M. Xia, Z.S. Xu, Hesitant fuzzy information aggregation in decision making, International Journal of Approximate Reasoning 52 (2011) 395-407.

[36] Z.S. Xu, M. M. Xia, Distance and similarity measures for hesitant fuzzy sets, Information Sciences 181 (2011) 2128-2138.

[37] Z.S. Xu, M. M. Xia, On distance and correlation measures of hesitant fuzzy information, International Journal of Intelligent Systems 26 (2011) 410-425.

[38] Z.S. Xu, X.L. Zhang, Hesitant fuzzy multi-attribute decision making based on TOPSIS with incomplete weight information, Knowledge-Based Systems 52 (2013) 53-64.

[39] D.S. Yeung, D.G. Chen, E.C.C. Tsang, J.W.T. Lee, X.Z. Wang, On the generalization of fuzzy rough sets, IEEE Transactions on Fuzzy Systems 13 (2005) 343-361.

[40] X.B. Yang, X.N. Song, Y.S. Qi, J.Y. Yang, Constructive and axiomatic approaches to hesitant fuzzy rough set, Soft Computing 18 (2014) 1067-1077.

[41] C. Zhang, D.Y.Li, J.Y. Liang, Hesitant fuzzy linguistic rough set over two universes model and its applications, International Journal of Machine Learning and Cybernetics (2016), doi:10.1007/s13042-016-0541-z.

[42] J.M. Zhan, Q. Liu, T. Herawan, A novel soft rough set: soft rough hemirings and its multicriteria group decision making, Applied Soft Computing 54 (2017) 393-402.

[43] J.M. Zhan, M. I. Ali, N. Mehmood, On a novel uncertain soft set model: Z-soft fuzzy rough set model and corresponding decision making methods, Applied Soft Computing 56 (2017) 446-457.

[44] J.M. Zhan, K.Y. Zhu, A novel soft rough fuzzy set: Z-soft rough fuzzy ideals of hemirings and corresponding decision making, Soft Computing 21 (2017) 1923-1936.

[45] J.M. Zhan, Q. Liu, W. Zhu, Another approach to rough soft hemirings and corresponding decision making, Soft Computing 21 (2017) 3769-3780.

[46] H.D. Zhang, L. Shu, S.L. Liao, Hesitant fuzzy rough set over two universes and its application in decision making, Soft Computing 21 (2017) 1803-1816.

[47] H.D. Zhang, L. Shu, S.L. Liao, On interval-valued hesitant fuzzy rough approximation operators, Soft Computing 20 (2016) 189-209.

[48] H.D. Zhang, L. Shu, S.L. Liao, Topological structures of interval-valued hesitant fuzzy rough set and its application, Journal of Intelligent and Fuzzy Systems 30 (2)(2016) 1029-1043.

[49] H.D. Zhang, L. Shu, Generalized interval-valued fuzzy rough set and its application in decision making, International Journal of Fuzzy Systems 17 (2) (2015) 279-291.

[50] H.D. Zhang, L. Shu, S.L. Liao, C.R. Xiawu, Dual hesitant fuzzy rough set and its application, Soft Computing 21 (2017) 3287-3305.

[51] H.D. Zhang, Y.P. He, L.L. Xiong, Multi-granulation dual hesitant fuzzy rough sets, Journal of Intelligent and Fuzzy Systems 30 (2016) 623-637.

[52] H.Y. Zhang, Y. Leung, L. Zhou, Variable-precision-dominance-based rough set approach to interval-valued information systems, Information Sciences 244 (2013) 75-91.

[53] H.Y. Zhang, S.Y. Yang, Representations of typical hesitant fuzzy rough sets, Journal of Intelligent and Fuzzy Systems 31 (2016) 457-468.

[54] L. Zhou, W.Z. Wu, On genernalized intuitionistic fuzzy approximation operators, Information Sciences 178 (2008) 2448-2465.

[55] L. Zhou, W.Z. Wu, On characterization of intuitonistic fuzzy rough sets based on intuitionistic fuzzy implicators, Information Sciences 179 (2009) 883-898.

[56] L.A. Zadeh, Fuzzy sets, Information and Control 8 (1965) 378-352.

[57] N. Zhang, G. Wei, Extension of VIKOR method for decision making problem based on hesitant fuzzy set, Applied Mathematical Modelling 37 (7) (2013) 4938-4947.

[58] W.X. Zhang, G.F. Qiu, Uncertain Decision Making Based on Rough Sets, Science Press, Beijing, China, 2005.

[59] X.H. Zhang, B. Zhou, P. Li, A general frame for intuitionistic fuzzy rough sets, Information Sciences 216 (2012) 34-49.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 27, NO.6, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC

THE STABILITY OF CUBIC FUNCTIONAL EQUATIONS WITH INVOLUTION IN MODULAR SPACES

CHANGIL KIM AND GILJUN HAN*

Abstract. In this paper, we prove the generalized Hyers-Ulam stability for the following cubic functional equation with involution

$$f(2x+y)+f(2x+\sigma(y))-2f(x+y)-2f(x+\sigma(y))-12f(x)=0$$

in modular spaces by using a fixed point theorem.

1. Introduction and preliminaries

In 1940, Ulam proposed the following stability problem (cf. [21]):

"Let G_1 be a group and G_2 a metric group with the metric d. Given a constant $\delta > 0$, does there exist a constant $c > 0$ such that if a mapping $f : G_1 \longrightarrow G_2$ satisfies $d(f(xy), f(x)f(y)) < c$ for all $x, y \in G_1$, then there exists a unique homomorphism $h : G_1 \longrightarrow G_2$ with $d(f(x), h(x)) < \delta$ for all $x \in G_1$?"

In the next year, Hyers [6] gave the first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' theorem was generalized by Aoki [2] for additive mappings and by Rassias [17] for linear mappings by considering an unbounded Cauchy difference, the latter of which has influenced many developments in the stability theory. This area is then referred to as the generalized Hyers-Ulam stability. A generalization of the Rassias' theorem was obtained by Găvruta [5] by replasing the unbounded Cauchy difference by a general control function in the spirit of Rassias' approach.

A problem that mathematicians has dealt with is "how to generalize the classical function space L^p". A first attempt was made by Birnhaum and Orlicz in 1931. This generalization found many applications in differential and intergral equations with kernls of nonpower types. The more abstract generalization was given by Nakano [14] in 1950 based on replacing the particular integral form of the functional by an abstract one that satisfies some good properties. This functional was called *modular*. Since then, these have been thoroughly developed by several mathematicians, for example, Amemiya [1], Koshi and Shimogaki [9], Yamamuro [23], Orlicz [15], Mazur [11], Musielak [12], Luxemburg [10], Turpin [20]. This idea was refined and generalized by Musielak and Orlicz [13] in 1959.

Recently, Sadeghi [18] presented a fixed point method to prove the generalized Hyers-Ulam stability of functional equations in modular spaces with the $\triangle_2$-condition, Wongkum, Chaipunya, and Kumam [22] proved the fixed point theorem and the generalized Hyers-Ulam stability for quadratic mappings in a modular

2010 *Mathematics Subject Classification.* 39B52, 39B72, 47H09.

Key words and phrases. Fixed point theorem, Hyers-Ulam stability, cubic functional equations, modular spaces.

* Corresponding author.

space whose modular is convex, lower semi-continuous but do not satisfy the $\triangle_2$-condition, and Park, Bodaghi, and Kim [16] proved the generalized Hyers-Ulam stability for additive mappings in a modular space with $\triangle_2$-conditions.

Let X and Y be real vector spaces. For an additive mapping $\sigma : X \longrightarrow X$ with $\sigma(\sigma(x)) = x$ for all $x \in X$, σ is called *an involution* of X. For a given involution $\sigma : X \longrightarrow X$, the functional equation

$$f(x+y) + f(x+\sigma(y)) = 2f(x) \tag{1.1}$$

is called *an additive functional equation with involution* and a solution of (1.1) is called *an additive mapping with involution*. For a given involution $\sigma : X \longrightarrow X$, the functional equation

$$f(x+y) + f(x+\sigma(y)) = 2f(x) + 2f(y) \tag{1.2}$$

is called *the quadratic functional equation with involution* and a solution of (1.2) is called *a quadratic mapping with involution*. The functional equation (1.2) has been studied by Stetkær [19] and the generalized Hyers-Ulam stability for (1.2) has been obtained by Bouikhalene et al. [3, 4, 7].

In this paper, we prove the generalized Hyers-Ulam stability for the following cubic functional equation with involution

$$f(2x+y) + f(2x+\sigma(y)) - 2f(x+y) - 2f(x+\sigma(y)) - 12f(x) = 0 \tag{1.3}$$

in modular spaces without the $\triangle_2$-condition and the convexity by using a fixed point theorem. Unlike Banach spaces and F-spaces, due to the triangle inequlity in modular spaces, we need subtle calculation in the proofs of Theorem 2.1 and Theorem 2.2

Definition 1.1. Let X be a vector space over a field $\mathbb{K}(\mathbb{R}, \mathbb{C}, \text{or } \mathbb{N})$.
(1) A generalized functional $\rho : X \longrightarrow [0, \infty]$ is called *a modular* if
(M1) $\rho(x) = 0$ if and only if $x = 0$,
(M2) $\rho(\alpha x) = \rho(x)$ for every scalar α with $|\alpha| = 1$, and
(M3) $\rho(z) \le \rho(x) + \rho(y)$ whenever z is a convex combination of x and y.
(2) If (M3) is replaced by
(M4) $\rho(\alpha x + \beta y) \le \alpha\rho(x) + \beta\rho(y)$
for all $x, y \in V$ and for all nonnegative real numbers α, β with $\alpha + \beta = 1$, then we say that ρ is *convex*.

For any modular ρ on X, the modular space X_ρ is defined by

$$X_\rho = \{x \in X \mid \rho(\lambda x) \to 0 \ as \ \lambda \to 0\}$$

and the modular space X_ρ can be equipped with a norm called the Luxemburg norm, defined by

$$\|x\|_\rho = \inf\Big\{\lambda > 0 \mid \rho\Big(\frac{x}{\lambda}\Big) \le 1\Big\}.$$

Let X_ρ be a modular space and $\{x_n\}$ a sequence in X_ρ. Then (i) $\{x_n\}$ is called *ρ-Cauchy* if for any $\epsilon > 0$, one has $\rho(x_n - x_m) < \epsilon$ for sufficiently large $m, n \in \mathbb{N}$, (ii) $\{x_n\}$ is called *ρ-convergent* to a point $x \in X_\rho$ if $\rho(x_n - x) \to 0$ as $n \to \infty$, and (iii) a subset K of X_ρ is called *ρ-complete* if each ρ-Cauchy sequence is ρ-convergent to a point in K.

Another unnatural behavior one usually encounter is that the convergence of a sequence $\{x_n\}$ to x does not imply that $\{cx_n\}$ converges to cx for some $c \in \mathbb{K}$.

Thus, many mathematicians imposed some additional conditions for a modular to meet in order to make the multiples of $\{x_n\}$ converge naturally. Such preferences are referred to mostly under the term related to $\triangle_2$-condition.

A modular space X_ρ is said to *satisfy the* $\triangle_2$*-condition* if there exists $k \geq 2$ such that $X_\rho(2x) \leq kX_\rho(x)$ for all $x \in X$. Some authors varied the notion so that only $k > 0$ is required and called it *the* $\triangle_2$*-type condition*. In fact, one may see that these two notions coincide. There are still a number of equivalent notions related to the $\triangle_2$-condition. In [8], Khamsi proved a series of fixed point theorems in modular spaces where the modulars do not satisfy $\triangle_2$-conditions. His results exploit one unifying hypothesis in which the boundedness of an orbit is assumed.

Example 1.2. A convex function ζ defined on the interval $[0, \infty)$, nondecreasing and continuous, such that $\zeta(0) = 0, \zeta(\alpha) > 0$ for $\alpha > 0$, $\zeta(\alpha) \to \infty$ as $\alpha \to \infty$, is called an Orlicz function. Let (Ω, Σ, μ) be a measure space and $L^0(\mu)$ the set of all measurable real valued (or complex valued) functions on Ω. Deine the Orlicz modular ρ_ζ on $L^0(\mu)$ by the formula

$$\rho_\zeta(f) = \int_\Omega \zeta(|f|)d\mu.$$

The associated modular space with respect to this modular is called an Orlicz space, and will be denoted by (L^ζ, Ω, μ) or briefly L^ζ. In other words,

$$L^\zeta = \{f \in L^0(\mu) \mid \rho_\zeta(\lambda f) < \infty \ \text{ for some } \ \lambda > 0\}.$$

It is known that the Orlicz space L^ζ is ρ_ζ-complete. Moreover, $(L^\zeta, \|\cdot\|_{\rho_\zeta})$ is a Banach space, where the Luxemburg norm $\|\cdot\|_{\rho_\zeta}$ is defined as follows

$$\|f\|_{\rho_\zeta} = \inf\Big\{\lambda > 0 \ \Big| \ \int_\Omega \zeta\Big(\frac{|f|}{\lambda}\Big)d\mu \leq 1\Big\}.$$

Further, if μ is the Lebesgue measure on $\mathbb{R}$ and $\zeta(t) = e^t - 1$, then ρ_ζ does not satisfy the $\triangle_2$-condition.

For a modular space X_ρ, a nonempty subset C of X_ρ, and a mapping $T : C \longrightarrow C$, *the orbit of* T *at* $x \in C$ is the set

$$\mathbb{O}(x) = \{x, Tx, T^2x, \cdots\}.$$

The quantity $\delta_\rho(x) = \sup\{\rho(u - v) \mid u, v \in \mathbb{O}(x)\}$ is called *the orbital diameter of* T *at* x and if $\delta_\rho(x) < \infty$, then one says that T has *a bounded orbit at* x.

Khamsi [8] proved a series of fixed point theorems in modular spaces where the modulars do not satisfy $\triangle_2$-conditions. His results exploit one unifying hypothesis in which the boundedness of an orbit is assumed.

Lemma 1.3. [8] *Let X_ρ be a modular space whose induced modular is lower semi-continuous and let $C \subseteq X_\rho$ be a ρ-complete subset. If $T : C \longrightarrow C$ is a ρ-contraction, that is, there is a constant $L \in [0, 1)$ such that*

$$\rho(Tx - Ty) \leq L\rho(x - y), \ \forall x, y \in C$$

and T has a bounded orbit at a point $x_0 \in C$, then the sequence $\{T^n x_0\}$ is ρ-convergent to a point $w \in C$.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 27, NO.6, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC

For any modular ρ on X and any linear space V, we define a set $\mathbb{M}$

$$\mathbb{M} := \{g : V \longrightarrow X_\rho \mid g(0) = 0\}$$

and the generalized function $\widetilde{\rho}$ on $\mathbb{M}$ by for each $g \in \mathbb{M}$,

$$\widetilde{\rho}(g) := \inf\{c > 0 \mid \rho(g(x)) \le c\psi(x, 0),\ \forall x \in V\},$$

where $\psi : V^2 \longrightarrow [0, \infty)$ is a mapping. The proof of the following lemma is similar to the proof of Lemma 10 in [22].

Lemma 1.4. *Let V be a linear space, X_ρ a ρ-complete modular space where ρ is lower semi-continuous and $f : V \longrightarrow X_\rho$ a mapping with $f(0) = 0$. Let $\psi : V^2 \longrightarrow [0, \infty)$ be a mapping such that*

$$\lim_{n\to\infty} \frac{\psi(2^n x, 2^n y)}{8^n} = 0,\ \ \psi(2x, 2x) \le 8L\psi(x, x) \tag{1.4}$$

for all $x, y \in V$ and some L with $0 \le L < 1$. Then we have the following :
(1) $\mathbb{M}$ is a linear space,
(2) $\widetilde{\rho}$ is a modular on $\mathbb{M}$,
(3) if ρ is convex, then $\widetilde{\rho}$ is convex,
(4) $\mathbb{M}_{\widetilde{\rho}} = \mathbb{M}$ and $\mathbb{M}_{\widetilde{\rho}}$ is $\widetilde{\rho}$-complete, and
(5) $\widetilde{\rho}$ is lower semi-continuous.

Proof. (1), (2), and (3) are trivial.

(4) By the definition of $\mathbb{M}_{\widetilde{\rho}}$, $\mathbb{M}_{\widetilde{\rho}} = \mathbb{M}$. Let $\epsilon > 0$. Take any $\widetilde{\rho}$-Cauchy sequence $\{g_n\}$ in $\mathbb{M}_{\widetilde{\rho}}$. Then there is an $l \in \mathbb{N}$ such that for $n, m \in \mathbb{N}$ with $n, m \ge l$,

$$\rho(g_n(x) - g_m(x)) \le \epsilon\psi(x, 0) \tag{1.5}$$

for all $x \in V$. Hence $\{g_n(x)\}$ is a ρ-Cauchy sequence in X_ρ for all $x \in V$. Since X_ρ is a ρ-complete modular space, there is a mapping $g : V \longrightarrow X_\rho$ such that $\rho(g_n(x) - g(x)) \longrightarrow 0$ as $n \to \infty$ for all $x \in V$. Since each $g_n \in \mathbb{M}$, there is an $m \in \mathbb{N}$ such that

$$\rho(g_m(0) - g(0)) = \rho(g(0)) \le \epsilon$$

and hence $g \in \mathbb{M}_{\widetilde{\rho}}$. Since ρ is lower semi-continuous, by (1.5), we have

$$\rho(g_n(x) - g(x)) \le \liminf_{m\to\infty} \rho(g_n(x) - g_m(x)) \le \epsilon\psi(x, 0)$$

for all $x \in V$. Hence $\mathbb{M}_{\widetilde{\rho}}$ is $\widetilde{\rho}$-complete.

(5) Suppose that $\{g_n\}$ is a sequence in $\mathbb{M}_{\widetilde{\rho}}$ which is $\widetilde{\rho}$-convergent to $g \in \mathbb{M}_{\widetilde{\rho}}$. Let $\epsilon > 0$. Then for any $n \in \mathbb{N}$, there is a positive real number c_n such that

$$\widetilde{\rho}(g_n) \le c_n \le \widetilde{\rho}(g_n) + \epsilon$$

and so

$$\begin{aligned} \rho(g(x)) &\le \liminf_{n\to\infty} \rho(g_n(x)) \\ &\le \liminf_{n\to\infty} c_n\psi(x, 0) \le \Big(\liminf_{n\to\infty} \rho(g_n(x)) + \epsilon\Big)\psi(x, 0) \end{aligned} \tag{1.6}$$

for all $x \in V$. Hence $\widetilde{\rho}$ is lower semi-continuous. □

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 27, NO.6, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC

2. The generalized Hyers-Ulam stability for (1.3) in modular spaces

Throughout this section, we assume that every modular is lower semi-continuous. In this section, we prove the generalized Hyers-Ulam stability for (1.3).

For any $f : V \longrightarrow X_\rho$ and any involution $\sigma : V \longrightarrow V$, let

$$Df(x, y) = f(2x + y) + f(2x + \sigma(y)) - 2f(x + y) - 2f(x + \sigma(y)) - 12f(x).$$

Theorem 2.1. *Let V be a linear space, X_ρ a ρ-complete modular space and $f : V \longrightarrow X_\rho$ a mapping with $f(0) = 0$. Let $\phi : V^2 \longrightarrow [0, \infty)$ be a mapping such that*

$$\phi(2x, 2y) \le 8L\phi(x, y), \ \ \phi(x + \sigma(x), y + \sigma(y)) \le 8L\phi(x, y) \tag{2.1}$$

for all $x, y \in V$ and some L with $0 < L < \frac{1}{16}$ and

$$\rho(Df(x, y)) \le \phi(x, y) \tag{2.2}$$

for all $x, y \in V$. Then there exists a unique cubic mapping $F : V \longrightarrow X_\rho$ with involution such that

$$\rho\Big(F(x) - \frac{1}{4}f(x)\Big) \le \frac{2}{1 - 8L}\phi(x, 0) \tag{2.3}$$

for all $x \in V$.

Proof. Let $\psi(x, y) = \phi(x, y) + \phi(y, x)$ for all $x, y \in V$. Then ψ satisfies (1.4) and hence, by Lemma 1.4, $\widetilde{\rho}$ is a lower semi-continuous convex modular on $\mathbb{M}_{\widetilde{\rho}}$, $\mathbb{M}_{\widetilde{\rho}} = \mathbb{M}$, and $\mathbb{M}_{\widetilde{\rho}}$ is $\widetilde{\rho}$-complete. Define $T : \mathbb{M}_{\widetilde{\rho}} \longrightarrow \mathbb{M}_{\widetilde{\rho}}$ by

$$Tg(x) = \frac{1}{8}\Big(g(2x) + g(x + \sigma(x))\Big)$$

for all $g \in \mathbb{M}_{\widetilde{\rho}}$ and all $x \in V$. Let $g, h \in \mathbb{M}_{\widetilde{\rho}}$. Suppose that $\widetilde{\rho}(g - h) \le c$ for some nonnegative real number c. Then by (2.1), we have

$$\begin{aligned}\rho(Tg(x) - Th(x)) &\le \rho\Big(\frac{1}{4}[g(2x) - h(2x)]\Big) + \rho\Big(\frac{1}{4}[g(x + \sigma(x)) - h(x + \sigma(x))]\Big)\\ &\le 16Lc\psi(x, 0)\end{aligned}$$

for all $x \in V$ and so $\widetilde{\rho}(Tg - Th) \le 16L\widetilde{\rho}(g - h)$. Hence T is a $\widetilde{\rho}$-contraction. By (2.2), we get

$$\rho(f(x) + f(\sigma(x))) \le \phi(0, x) \tag{2.4}$$

and

$$\rho\Big(f(2x) - 8f(x)\Big) \le \rho(2f(2x) - 16f(x)) \le \phi(x, 0) \tag{2.5}$$

for all $x \in X$. Letting $x = x + \sigma(x)$ in (2.4), by (M3), we have

$$\rho(f(x + \sigma(x))) \le \rho(2f(x + \sigma(x))) \le \phi(0, x + \sigma(x)) \le 8L\phi(0, x), \tag{2.6}$$

for all $x \in X$ and by (2.5) and (M3), we get

$$\rho\Big(\frac{1}{2^3}f(2x) - f(x)\Big) \le \rho(f(2x) - 8f(x)) \le \phi(x, 0) \tag{2.7}$$

for all $x \in X$.

Now, we claim that T has a bounded orbit at $\frac{1}{4}f$. By the definition of T, we have

$$T^n f(x + \sigma(x)) = \frac{1}{2^{2n}}f(2^n(x + \sigma(x))) \tag{2.8}$$

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 27, NO.6, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC

for all $x \in V$ and for all $n \in \mathbb{N}$. Hence by (2.1), (2.6), and (2.8), we have

$$\rho(T^n f(x+\sigma(x))) \le \rho(f(2^n(x+\sigma(x)))) \le (8L)^{n+1}\phi(0,x)$$

for all $x \in V$ and for all $n \in \mathbb{N}$. By (2.7), for any nonnegative integer n, we obtain

$$\begin{aligned}
&\rho(\frac{1}{2}T^n f(x) - \frac{1}{2}f(x)) \\
&\le \rho\Big(T^n f(x) - \frac{1}{2^3}f(2x)\Big) + \rho(\frac{1}{2^3}f(2x) - f(x)) \\
&\le \rho\Big(\frac{1}{2}T^{n-1}f(2x) - \frac{1}{2}f(2x)\Big) + \rho\Big(T^{n-1}f(x+\sigma(x))\Big) + \phi(x,0) \\
&\le \rho\Big(\frac{1}{2}T^{n-1}f(2x) - \frac{1}{2}f(2x)\Big) + (8L)^n\phi(0,x) + \phi(x,0)
\end{aligned}$$

for all $x \in V$ and by induction, we have

$$\begin{aligned}
\rho\Big(\frac{1}{2}T^n f(x) - \frac{1}{2}f(x)\Big) &\le \sum_{i=0}^{n-1}(8L)^{n-i}\phi(0,2^i x) + \sum_{i=0}^{n-1}\phi(2^i x, 0) \\
&\le n(8L)^n\phi(0,x) + \frac{1}{1-8L}\phi(x,0)
\end{aligned} \tag{2.9}$$

for all $x \in V$ and all $n \in \mathbb{N}$. Hence by (2.9), we get

$$\rho\Big(\frac{1}{4}T^n f(x) - \frac{1}{4}T^m f(x)\Big) \le \frac{2}{1-8L}\phi(x,0) + [n(8L)^n + m(8L)^m]\phi(0,x) \tag{2.10}$$

for all $x \in V$ and all all nonnegative integers n, m and since $0 < L < \frac{1}{16}$, by (2.10), we have

$$\rho\Big(\frac{1}{4}T^n f(x) - \frac{1}{4}T^m f(x)\Big) \le 4\phi(x,0) + \phi(0,x) \le 4\psi(x,0)$$

for all $x \in V$ and all nonnegative integers n, m. Hence we have

$$\widetilde{\rho}\Big(T^n\frac{1}{4}f - T^m\frac{1}{4}f\Big) \le 4$$

all nonnegative integers n, m and thus T has a bounded orbit at $\frac{1}{4}f$.

By Lemma 1.3, there is an $F \in \mathbb{M}_{\widetilde{\rho}}$ such that $\{T^n\frac{1}{4}f\}$ is $\widetilde{\rho}$-convergent to F. Since $\widetilde{\rho}$ is lower semi-continuous, we get

$$0 \le \widetilde{\rho}(TF - F) \le \liminf_{n\to\infty}\widetilde{\rho}\Big(TF - T^{n+1}\frac{1}{4}f\Big) \le \liminf_{n\to\infty} 16L\widetilde{\rho}\Big(F - T^n\frac{1}{4}f\Big) = 0$$

and hence F is a fixed point of T in $\mathbb{M}_{\widetilde{\rho}}$. By induction, we can easily show that

$$\begin{aligned}
T^n f(x) &= \frac{1}{2^{3n}}f(2^n x) + \frac{1}{2^{3n}}\sum_{i=0}^{n-1}2^i f(2^{n-1}(x+\sigma(x))) \\
&= \frac{1}{2^{3n}}f(2^n x) + \frac{2^n-1}{2^{3n}}f(2^{n-1}(x+\sigma(x)))
\end{aligned}$$

for all $x \in X$ and $n \in \mathbb{N}$. Moreover, we have

$$\rho\Big(\frac{1}{2^8}DF(x,y)\Big) \le \rho\Big(\frac{1}{2^7}\Big[DF(x,y) - T^n\frac{1}{4}Df(x,y)\Big]\Big) + \rho\Big(\frac{1}{2^7}T^n\frac{1}{4}Df(x,y)\Big) \tag{2.11}$$

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 27, NO.6, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC

for all $x, y \in V$ and all $n \in \mathbb{N}$. Note that

$$\begin{aligned}
&\rho\Big(\frac{1}{2^7}\Big[DF(x,y) - T^n\frac{1}{4}Df(x,y)\Big]\Big)\\
&\le \rho\Big(\frac{1}{2^6}\Big[F(2x+y) - T^n\frac{1}{4}f(2x+y)\Big]\Big) + \rho\Big(\frac{1}{2^4}\Big[F(2x+\sigma(y)) - T^n\frac{1}{4}f(2x+\sigma(y))\Big]\Big)\\
&+ \rho\Big(\frac{1}{2^4}\Big[2F(x+y) - T^n\frac{1}{4}2f(x+y)\Big]\Big) + \rho\Big(\frac{1}{2^4}\Big[2F(x+\sigma(y)) - T^n\frac{1}{4}2f(x+\sigma(y))\Big]\Big)\\
&+ \rho\Big(\frac{1}{2^4}\Big[12F(x) - T^n\frac{1}{4}12f(x)\Big]\Big)
\end{aligned}$$

for all $x, y \in V$ and all $n \in \mathbb{N}$. Since $\{T^n\frac{1}{4}f\}$ is $\widetilde{\rho}$-convergent to F, we get

$$\lim_{n\to\infty}\rho\Big(\frac{1}{2^7}\Big[DF(x,y) - T^n\frac{1}{4}Df(x,y)\Big]\Big) = 0 \tag{2.12}$$

for all $x, y \in V$. Further, we have

$$\begin{aligned}
&\rho\Big(\frac{1}{2^7}T^n\frac{1}{4}Df(x,y)\Big) = \rho\Big(\frac{1}{2^9}T^nDf(x,y)\Big)\\
&\le \rho\Big(\frac{1}{2^{3n+8}}Df(2^nx,2^ny)\Big) + \rho\Big(\frac{2^n-1}{2^{3n+8}}Df(2^{n-1}(x+\sigma(x)),2^{n-1}(y+\sigma(y)))\Big)\\
&\le \phi(2^nx,2^ny) + \phi(2^{n-1}(x+\sigma(x)),2^{n-1}(y+\sigma(y)))\\
&\le 2(8L)^n\phi(x,y)
\end{aligned}$$

for all $x, y \in V$ and all $n \in \mathbb{N}$. Letting $n \to \infty$ in the last inequality, we get

$$\lim_{n\to\infty}\rho\Big(\frac{1}{2^7}T^n\frac{1}{4}Df(x,y)\Big) = 0 \tag{2.13}$$

for all $x, y \in V$. By (2.11), (2.12), (2.13), and (M1), we obtain

$$DF(x,y) = 0$$

for all $x, y \in V$ and hence F is a cubic mapping with involution. Moreover, since ρ is lower semi-continuous, by (2.10), we get

$$\rho\Big(F(x) - \frac{1}{4}f(x)\Big) \le \frac{2}{1-8L}\phi(x,0)$$

for all $x \in X$. □

If ρ is covex, then Theorem 2.1 can replaced by the following theorem.

Theorem 2.2. *All conditions of Theorem 2.1 are assumed. Further, suppose that ρ is a convex modular and $0 < L < \frac{1}{2}$. Then there exists a unique cubic mapping $F : V \longrightarrow X_\rho$ with involution such that*

$$\rho\Big(F(x) - \frac{1}{4}f(x)\Big) \le \frac{1}{2^4(1-L)}\phi(x,0) \tag{2.14}$$

for all $x \in V$.

Proof. Let $\psi(x,y) = \phi(x,y) + \phi(y,x)$ for all $x, y \in V$. Then ψ satisfies (1.4) and hence, by Lemma 1.4, $\widetilde{\rho}$ is a lower semi-continuous convex modular on $\mathbb{M}_{\widetilde{\rho}}$, $\mathbb{M}_{\widetilde{\rho}} = \mathbb{M}$, and $\mathbb{M}_{\widetilde{\rho}}$ is $\widetilde{\rho}$-complete. Define $T : \mathbb{M}_{\widetilde{\rho}} \longrightarrow \mathbb{M}_{\widetilde{\rho}}$ by

$$Tg(x) = \frac{1}{8}\Big(g(2x) + g(x+\sigma(x))\Big)$$

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 27, NO.6, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC

for all $g \in \mathbb{M}_{\widetilde{\rho}}$ and all $x \in V$. Let $g, h \in \mathbb{M}_{\widetilde{\rho}}$. Suppose that $\widetilde{\rho}(g-h) \le c$ for some nonnegative real number c. Then by (2.1), we have

$$\begin{aligned}\rho(Tg(x)-Th(x)) &\le \frac{1}{2}\rho\Big(\frac{1}{4}[g(2x)-h(2x)]\Big)+\frac{1}{2}\rho\Big(\frac{1}{4}[g(x+\sigma(x))-h(x+\sigma(x))]\Big)\\ &\le 2Lc\psi(x,0)\end{aligned}$$

for all $x \in V$ and so $\widetilde{\rho}(Tg-Th) \le 2L\widetilde{\rho}(g-h)$. Hence T is a $\widetilde{\rho}$-contraction. By (2.2), we get

$$\rho(f(x)+f(\sigma(x))) \le \phi(0,x) \tag{2.15}$$

and

$$\rho\Big(f(2x)-8f(x)\Big) \le \frac{1}{2}\rho(2f(2x)-16f(x)) \le \frac{1}{2}\phi(x,0) \tag{2.16}$$

for all $x \in X$. Letting $x = x+\sigma(x)$ in (2.15), by (M3), we have

$$\rho(f(x+\sigma(x))) \le \frac{1}{2}\rho(2f(x+\sigma(x))) \le \frac{1}{2}\phi(0,x+\sigma(x)) \le 4L\phi(0,x), \tag{2.17}$$

for all $x \in X$ and by (2.16) and (M3), we get

$$\rho\Big(\frac{1}{2^3}f(2x)-f(x)\Big) \le \frac{1}{2^3}\rho(f(2x)-8f(x)) \le \frac{1}{2^4}\phi(x,0) \tag{2.18}$$

for all $x \in X$.

Now, we claim that T has a bounded orbit at $\frac{1}{4}f$. By the definition of T, we have

$$T^n f(x+\sigma(x)) = \frac{1}{2^{2n}}f(2^n(x+\sigma(x))) \tag{2.19}$$

for all $x \in V$ and for all $n \in \mathbb{N}$. Hence by (2.1), (2.15), and (2.19), we have

$$\rho(T^n f(x+\sigma(x))) \le \frac{1}{2^{2n}}\rho(f(2^n(x+\sigma(x)))) \le 2(2L)^{n+1}\phi(0,x)$$

for all $x \in V$ and for all $n \in \mathbb{N}$. By (2.18), for any nonnegative integer n, we obtain

$$\begin{aligned}&\rho\Big(\frac{1}{2}T^n f(x)-\frac{1}{2}f(x)\Big)\\ &\le \frac{1}{2}\rho\Big(T^n f(x)-\frac{1}{2^3}f(2x)\Big)+\frac{1}{2}\rho\Big(\frac{1}{2^3}f(2x)-f(x)\Big)\\ &\le \frac{1}{2^3}\rho\Big(\frac{1}{2}T^{n-1}f(2x)-\frac{1}{2}f(2x)\Big)+\frac{1}{2^4}\rho\Big(T^{n-1}f(x+\sigma(x))\Big)+\frac{1}{2^5}\phi(x,0)\\ &\le \frac{1}{2^3}\rho\Big(\frac{1}{2}T^{n-1}f(2x)-\frac{1}{2}f(2x)\Big)+\frac{(2L)^n}{2^3}\phi(0,x)+\frac{1}{2^5}\phi(x,0)\end{aligned}$$

for all $x \in V$ and by induction, we have

$$\begin{aligned}\rho\Big(\frac{1}{2}T^n f(x)-\frac{1}{2}f(x)\Big) &\le \sum_{i=0}^{n-1}\frac{(2L)^{n-i}}{2^{3(i+1)}}\phi(0,2^i x)+\frac{1}{2^5}\sum_{i=0}^{n-1}\frac{1}{2^{3i}}\phi(2^i x,0)\\ &\le \frac{(2L)^n}{4}\phi(0,x)+\frac{1}{2^5(1-L)}\phi(x,0)\end{aligned} \tag{2.20}$$

for all $x \in V$ and all $n \in \mathbb{N}$. Hence by (2.20), we get

$$\rho\Big(\frac{1}{4}T^n f(x)-\frac{1}{4}T^m f(x)\Big) \le \frac{1}{2^4(1-L)}\phi(x,0)+\frac{1}{4}[(2L)^n+(2L)^m]\phi(0,x) \tag{2.21}$$

for all $x \in V$ and all all nonnegative integers n, m and since $0 < L < \frac{1}{2}$, by (2.21), we have

$$\begin{aligned}\rho\Big(\frac{1}{4}T^n f(x) - \frac{1}{4}T^m f(x)\Big) &\le \frac{1}{2^4(1-L)}\phi(x,0) + \frac{1}{2}\phi(0,x)\\ &\le \frac{1}{8}\phi(x,0) + \frac{1}{2}\phi(0,x)\\ &\le \frac{1}{2}\psi(x,0)\end{aligned}$$

for all $x \in V$ and all nonnegative integers n, m. Hence we have

$$\widetilde{\rho}\Big(T^n\frac{1}{4}f - T^m\frac{1}{4}f\Big) \le \frac{1}{2}$$

all nonnegative integers n, m and thus T has a bounded orbit at $\frac{1}{4}f$.

By Lemma 1.3, there is an $F \in \mathbb{M}_{\widetilde{\rho}}$ such that $\{T^n\frac{1}{4}f\}$ is $\widetilde{\rho}$-convergent to F. Since $\widetilde{\rho}$ is lower semi-continuous, we get

$$0 \le \widetilde{\rho}(TF - F) \le \liminf_{n\to\infty} \widetilde{\rho}\Big(TF - T^{n+1}\frac{1}{4}f\Big) \le \liminf_{n\to\infty} 2L\widetilde{\rho}\Big(F - T^n\frac{1}{4}f\Big) = 0$$

and hence F is a fixed point of T in $\mathbb{M}_{\widetilde{\rho}}$. By induction, we can easily show that

$$\begin{aligned}T^n f(x) &= \frac{1}{2^{3n}}f(2^n x) + \frac{1}{2^{3n}}\sum_{i=0}^{n-1} 2^i f(2^{n-1}(x+\sigma(x)))\\ &= \frac{1}{2^{3n}}f(2^n x) + \frac{2^n - 1}{2^{3n}}f(2^{n-1}(x+\sigma(x)))\end{aligned}$$

for all $x \in X$ and $n \in \mathbb{N}$. Moreover, we have

(2.22)
$$\rho\Big(\frac{1}{2^8}DF(x,y)\Big) \le \frac{1}{2}\rho\Big(\frac{1}{2^7}\Big[DF(x,y) - T^n\frac{1}{4}Df(x,y)\Big]\Big) + \frac{1}{2}\rho\Big(\frac{1}{2^7}T^n\frac{1}{4}Df(x,y)\Big)$$

for all $x, y \in V$ and all $n \in \mathbb{N}$. Note that

$$\begin{aligned}&\rho\Big(\frac{1}{2^7}\Big[DF(x,y) - T^n\frac{1}{4}Df(x,y)\Big]\Big)\\ &\le \frac{1}{2}\rho\Big(\frac{1}{2^6}\Big[F(2x+y) - T^n\frac{1}{4}f(2x+y)\Big]\Big) + \frac{1}{2^3}\rho\Big(\frac{1}{2^4}\Big[F(2x+\sigma(y)) - T^n\frac{1}{4}f(2x+\sigma(y))\Big]\Big)\\ &+ \frac{1}{2^3}\rho\Big(\frac{1}{2^4}\Big[2F(x+y) - T^n\frac{1}{4}2f(x+y)\Big]\Big) + \frac{1}{2^3}\rho\Big(\frac{1}{2^4}\Big[2F(x+\sigma(y)) - T^n\frac{1}{4}2f(x+\sigma(y))\Big]\Big)\\ &+ \frac{1}{2^3}\rho\Big(\frac{1}{2^4}\Big[12F(x) - T^n\frac{1}{4}12f(x)\Big]\Big)\end{aligned}$$

for all $x, y \in V$ and all $n \in \mathbb{N}$. Since $\{T^n\frac{1}{4}f\}$ is $\widetilde{\rho}$-convergent to F, we get

$$\lim_{n\to\infty} \rho\Big(\frac{1}{2^7}\Big[DF(x,y) - T^n\frac{1}{4}Df(x,y)\Big]\Big) = 0 \tag{2.23}$$

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 27, NO.6, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC

for all $x, y \in V$. Further, we have

$$\begin{aligned}
&\rho\Big(\frac{1}{2^7}T^n\frac{1}{4}Df(x,y)\Big) = \rho\Big(\frac{1}{2^9}T^nDf(x,y)\Big)\\
&\le \frac{1}{2}\rho\Big(\frac{1}{2^{3n+8}}Df(2^nx,2^ny)\Big) + \frac{1}{2}\rho\Big(\frac{2^n-1}{2^{3n+8}}Df(2^{n-1}(x+\sigma(x)),2^{n-1}(y+\sigma(y)))\Big)\\
&\le \frac{1}{2^{3n+9}}\phi(2^nx,2^ny) + \frac{2^n-1}{2^{3n+9}}\phi(2^{n-1}(x+\sigma(x)),2^{n-1}(y+\sigma(y)))\\
&\le \frac{(2L)^n}{2^9}\phi(x,y)
\end{aligned}$$

for all $x, y \in V$ and all $n \in \mathbb{N}$. Letting $n \to \infty$ in the last inequality, we get

$$\lim_{n\to\infty}\rho\Big(\frac{1}{2^7}T^n\frac{1}{4}Df(x,y)\Big) = 0 \tag{2.24}$$

for all $x, y \in V$. By (2.22), (2.23), (2.24), and (M1), we obtain

$$DF(x,y) = 0$$

for all $x, y \in V$ and hence F is a cubic mapping with involution. Moreover, since ρ is lower semi-continuous, by (2.21), we get

$$\rho\Big(F(x) - \frac{1}{4}f(x)\Big) \le \frac{1}{2^4(1-L)}\phi(x,0)$$

for all $x \in X$. □

It is well-known that every normed space is a modular space with $\rho(x) = \|x\|$. Using Theorem 2.2, we have the following corollary.

Corollary 2.3. *Let X and Y be normed spaces and ϵ, θ, and p be real numbers with $\epsilon \ge 0$, $\theta \ge 0$, and $0 < p < \frac{3}{2}$. Let $f : X \longrightarrow Y$ be a mapping with involution σ such that $f(0) = 0$ and*

$$\|D_f(x,y)\| \le \epsilon + \theta(\|x\|^{2p} + \|y\|^{2p} + \|x\|^p\|y\|^p)$$

and

$$\|x + \sigma(x)\| \le 2\|x\|$$

for all $x, y \in X$. Then there is a cubic mapping $F : X \longrightarrow Y$ with involution such that

$$\|F(x) - f(x)\| \le \frac{1}{2(8-2^{2p})}(\epsilon + \theta\|x\|^{2p})$$

for all $x \in X$.

Proof. Let $\rho(z) = \|z\|$ for all $y \in Y$ and $\phi(x,y) = \epsilon + \theta(\|x\|^{2p} + \|y\|^{2p} + \|x\|^p\|y\|^p)$ for all $x, y \in V$. Then ρ is a convex modular on a normed space Y, $Y = Y_\rho$, and

$$\phi(2x,2y) \le 2^{2p}\phi(x,y), \;\; \phi(x+\sigma(x), y+\sigma(y)) \le 2^{2p}\phi(x,y)$$

for all $x, y \in V$. By Theorem 2.2, we have the results. □

Using Example 1.3, we get the following example.

Example 2.4. Let ϵ, θ, and p be real numbers with $\epsilon \geq 0$, $\theta \geq 0$, and $0 < p < \frac{3}{2}$. Let ζ be an Orlicz function and L^ζ the Orlicz space. Let $f : V \longrightarrow L^\zeta$ be a mapping with involution σ such that $f(0) = 0$ and

$$\int_\Omega \zeta(|f(2x+y) + f(2x+\sigma(y)) - 2f(x+y) - 2f(x+\sigma(y)) - 12f(x)|)d\mu$$
$$\leq \epsilon + \theta(\|x\|^{2p} + \|y\|^{2p} + \|x\|^p\|y\|^p)$$

and

$$\|x + \sigma(x)\| \leq 2\|x\|$$

for all $x, y \in X$. Then there is a cubic mapping $F : X \longrightarrow Y$ with involution such that

$$\int_\Omega \zeta(|F(x) - \frac{1}{4}f(x)|)d\mu \leq \frac{1}{2(8-2^{2p})}(\epsilon + \theta\|x\|^{2p})$$

for all $x \in X$.

Define a mapping $\rho_2 : \mathbb{R} \longrightarrow \mathbb{R}$ by $\rho_2(x) = |x|^{\frac{1}{2}}$. Then clearly, ρ_2 is a modular on $\mathbb{R}$ and $\mathbb{R}_{\rho_2} = \mathbb{R}$. Note that

$$\Big|\frac{1}{2} \times 2 + \frac{1}{2} \times 4\Big|^{\frac{1}{2}} = \sqrt{3} > \frac{\sqrt{2}}{2} + 1 = \frac{1}{2} \times |2|^{\frac{1}{2}} + \frac{1}{2} \times |4|^{\frac{1}{2}}.$$

Hence ρ_2 is not convex. Moreover, since $(\mathbb{R}, |\cdot|)$ is a complete normed space, we can easily show that $(\mathbb{R}, \rho_2)$ is a complete modular space. Using these and Theorem 2.1, we have the following example.

Example 2.5. Let ϵ, θ, and p be real numbers with $\epsilon \geq 0$, $\theta \geq 0$, and $0 < p < \frac{3}{2}$. Let $f : V \longrightarrow \mathbb{R}$ be a mapping with involution σ such that $f(0) = 0$ and

$$|f(2x+y) + f(2x+\sigma(y)) - 2f(x+y) - 2f(x+\sigma(y)) - 12f(x)|^{\frac{1}{2}}$$
$$\leq \epsilon + \theta(\|x\|^{2p} + \|y\|^{2p} + \|x\|^p\|y\|^p)$$

and

$$\|x + \sigma(x)\| \leq 2\|x\|$$

for all $x, y \in X$. Then there is a cubic mapping $F : X \longrightarrow Y$ with involution such that

$$|F(x) - \frac{1}{4}f(x)|^{\frac{1}{2}} \leq \frac{2}{1-2^{2p}}(\epsilon + \theta\|x\|^{2p})$$

for all $x \in X$.

References

[1] I. Amemiya, On the representation of complemented modular lattices, J. Math. Soc. Japan. **9**(1957), 263-279.

[2] T. Aoki, On the stability of the linear transformation in Banach spaces, J. Math. Soc. Japan **2**(1950), 64-66.

[3] B. Boukhalene, E. Elqorachi, and Th. M. Rassias, *On the generalized Hyers-Ulam stability of the quadratic functional equation with a general involution*, Nonlinear Funct. Anal. Appl. 12. no 2 (2007), 247-262.

[4] ———, *On the Hyers-Ulam stability of approximately pexider mappings*, Math. Ineqal. Appl. 11 (2008), 805-818.

[5] P. Găvruta, A generalization of the Hyer-Ulam-Rassias stability of approximately additive mappings, J. Math. Anal. Appl. **184**(1994), 431-436.

[6] D. H. Hyers, On the stability of the linear functional equation, Proc. Nat. Acad. Sci. **27**(1941), 222-224.

[7] S. M. Jung, Z. H. Lee, *A fixed point approach to the stability of quadratic functional equation with involution*, Fixed Point Theory Appl. 2008.
[8] M. A. Khamsi, Quasicontraction mappings in modular spaces without 2-condition, Fixed Point Theory and Applications, **2008**(2008), 1-6.
[9] S. Koshi and T. Shimogaki, On F-norms of quasi-modular spaces, J. Fac. Sci., Hokkaido Univ., Ser. 1 **15**(1961), 202-218.
[10] W. A. Luxemburg, Banach function spaces. PhD thesis, Delft University of Technology, Delft, The Netherlands 1959.
[11] B. Mazur, Modular curves and the Eisenstein ideal. Publ. Math. IHS **47**(1978), 33-186.
[12] J. Musielak, Orlicz Spaces and Modular Spaces. Lecture Notes in Mathematics, vol. 1034. Springer, Berlin 1983.
[13] J. Musielak and W. Orlicz, On modular spaces, Studia Mathematica, **18**(1959), 591-597.
[14] H. Nakano, Modular semi-ordered spaces, Tokyo, Japan, 1959.
[15] W. Orlicz, Collected Papers, vols. I, II. PWN, Warszawa 1988.
[16] C. Park, A. Bodaghi, and S. O. Kim, A fixed point approach to stability for additive mappings in a modular space with $\triangle_2$-conditions, J. Comput. Anal. Appl. **24** (2018), 1036-1048.
[17] Th. M. Rassias, On the stability of the linear mapping in Banach sapces, Proc. Amer. Math. Sco. **72**(1978), 297-300.
[18] G. Sadeghi, A fixed point approach to stability of functional equations in modular spaces, Bulletin of the Malaysian Mathematical Sciences Society. Second Series, **37**(2014), 333-344.
[19] H. Stetkær, *Functional equations on abelian groups with involution*, Aequationes Math. 54 (1997), 144-172.
[20] P. Turpin, Fubini inequalities and bounded multiplier property in generalized modular spaces, Comment. Math. **1**(1978), 331-353.
[21] S. M. Ulam, Problems in Modern Mathematics, Wiley, New York; 1964.
[22] K. Wongkum, P. Chaipunya, and P. Kumam, On the generalized Ulam-Hyers-Rassias stability of quadratic mappings in modular spaces without $\triangle_2$-conditions, **2015**(2015), 1-6.
[23] S. Yamamuro, On conjugate spaces of Nakano spaces, Trans. Am. Math. Soc. **90**(1959), 291-311.

Department of Mathematics Education, Dankook University, 152, Jukjeon-ro, Suji-gu, Yongin-si, Gyeonggi-do, 448-701, KOREA
E-mail address: kci206@hanmail.net

Department of Mathematics Education, Dankook University, 152, Jukjeon-ro, Suji-gu, Yongin-si, Gyeonggi-do, 448-701, Korea
E-mail address: gilhan@dankook.ac.kr

A nonstandard finite difference method applied to a mathematical cholera model with spatial diffusion

Shu Liao [a] Weiming Yang [a]

[a] School of Mathematics and Statistics
Chongqing Technology and Business University, Chongqing, 400067,China

Abstract

In this paper, we propose a nonstandard finite difference (NSFD) scheme to solve numerically a cholera epidemic model with spatial diffusion. Through constructing discrete Lyapunov functions, we prove the globally asymptotical stabilities of the disease-free equilibrium and the chronic infection equilibrium, which coincide with the corresponding continuous model . Finally, numerical simulations are provided to illustrate the theoretical results.

Cholera, partial differential equation, nonstandard finite difference scheme, Lyapunov function, global stability.

1 Introduction

Cholera is an infection of the intestines caused by the bacterium called Vibrio cholerae and can spread rapidly in areas with inadequate treatment of sewage and drinking water. The World Health Organization (WHO) has warned that there are an estimated 3-5 million infected cases and 28,800-130,000 deaths worldwide due to cholera every year. Since 1817, seven cholera pandemics have spread in many places, with periodic outbreaks such as the latest one in Yemen in October 2016, which is the worst cholera outbreak in the world. The total cases in Yemen have exceeded half a million, with nearly 2,000 deaths reported, since the outbreak began to spread rapidly at the end of April 2017 due to the deteriorating hygiene and sanitation conditions and the disrupted water supply across the country. There have been massive outbreaks of cholera in many developing countries of Africa and South-east Asia, including Congo (2008), Iraq (2008), Zimbabwe (2008-2009), Vietnam (2009), Nigeria (2010), Haiti (2010), Mexico (2013), South Sudan (2014), and Somalia (2017).

In recent years, many epidemic models have been proposed to a better understanding of the transmission of cholera. In 2001, Codeço [1] proposed a $SIRB$ epidemic model to study the transmission of cholera in which B represents the *V. cholerae* concentration in water. Hartley Morris and Smith [2] in 2006 discovered a representative hyperinfectious state of the pathogen, which is the 'explosive' infectivity of freshly shed *V. cholerae* based on the laboratory results. Tien and Earn [3] proposed a water-borne disease model with multiple transmission pathways: both direct human-to-human and indirect water-to-human transmissions, and identified how these transmission routes influence disease dynamics. Mukandavire *et al.* [4] simplified Hartley's model to understand transmission dynamics of cholera outbreak in Zimbabwe. Liao and Wang [5] in 2011 conducted a dynamical analysis of the Hartley's model to study the stability of both the disease-free and endemic equilibria so as to explore the complex epidemic and endemic dynamics of the disease. Bertuzzo *et al.* [6] based on

the Codeco's work and developed a partial differential equation model for cholera epidemics. Their results suggested that cholera outbreaks may be triggered by time scales of disease dynamics. In a recently study, Safi *et al.* [7] designed a new two-strain model to assess the impact of basic control measures and dose-structured mass vaccination on cholera transmission dynamics in a population. More papers in the field of cholera epidemic models are presented in ([8–11]).

Nowadays, more and more researchers consider to discretize the continuous models for practical purposes. One of the reasons is that most numerical methods like traditional Euler, Runge-Kutta and some standard procedures of MATLAB software will fail to solve nonlinear systems generating oscillations, chaos, and unsteady states if the time step size increases to a critical size. The other reason is that the results of the discrete time models are more accurate and convenient to describe infectious diseases and can preserve as much as possible the qualitative properties of the corresponding continuous models. The nonstandard finite difference (NSFD) scheme developed by Mickens ([12–14]) performs well and has been applied to many articles. An NSFD discretization must satisfy one of the following two conditions ([15, 16]): nonlocal approximation is used and discretization of derivative must be a denominator function. Cui *et al.* [17] employed an NSFD scheme to discuss a class of SIR epidemic model with vaccination and treatment. The dynamical properties of their discretized model were analysed to demonstrate that the discretized epidemic model maintains essential properties of the corresponding continuous model, such as positivity property, boundness of solutions, equilibrium points and their local stability properties. Suryanto *et al.* [18] constructed an NSFD scheme to solve a SIR epidemic model with modified saturated incidence rate. From their numerical simulations, the NSFD scheme allowed large time step size to save the computational cost. Qin *et al.* [19] proposed an NSFD method for an epidemic model which described the hepatitis B virus infection with spatial dependence. They have shown that the NSFD method is unconditionally positive by using the M-matrix theory. Moveover, asymptotical stabilities of the steady-state solutions were fully determined by constructing discrete Lyapunov functions independent of the time and space step sizes. Manna and Chakrabarty [20] analysed a spatiotemporal model for HBV infection by using an NSFD scheme, and studied the global stability properties of the discretized model. The simulation results demonstrated the advantages of the usage of NSFD method over the other schemes. For more investigations on NSFD scheme can be found in ([21–24]).

In 2015, Wang and Wang [25] proposed a PDE model to simulate cholera infection with spatial diffusion, taking multiple transmission ways into account among the human host, the pathogen, and the environment. The model in their paper assumes that both the human population and the bacteria undergo a diffusion process and is given by the following system of PDEs:

$$\frac{\partial S}{dt} = \Lambda - \beta_W \frac{W(x,t)S(x,t)}{\kappa + W(x,t)} - \beta_h S(x,t)I(x,t) - \mu S(x,t) + D_1 \triangle S, \tag{1}$$

$$\frac{\partial I}{dt} = \beta_W \frac{W(x,t)S(x,t)}{\kappa + W(x,t)} + \beta_h S(x,t)I(x,t) - (\gamma + \mu + u_1)I(x,t) + D_2 \triangle I, \tag{2}$$

$$\frac{\partial W}{dt} = \xi I(x,t) - \delta W(x,t) + D_3 \triangle W, \tag{3}$$

$$\frac{\partial R}{dt} = \gamma I(x,t) - \mu R(x,t) + D_4 \triangle R, \tag{4}$$

where $S(x,t)$, $I(x,t)$, $R(x,t)$ and $W(x,t)$ denote the susceptible, the infected, the recovered populations and the density of *V. cholerae* at location x and time t, respectively. The parameters β_h and β_W denote the concentrations of the hyperinfectious (HI) and less-infectious (LI) vibrios, respectively. μ represents the natural death rate that is not related to the disease, u_1 defines the rate of disease-related death, κ is the concentration of vibrios in contaminated water, ξ the natural decay rate of *V. cholerae*, δ the bacterial death rate, γ the recovery rate, and D_i $(i = 1,2,3,4)$ are the diffusion coefficients.

Ω is a bounded domain in R^n with smooth boundary $\partial\Omega$, $\triangle$ is the Laplacian operator, that is $\triangle = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$ with n is the number of spatial dimensions of the domain Ω. The Neumann boundary conditions of the model system are:

$$\frac{\partial S}{dt} = \frac{\partial I}{dt} = \frac{\partial W}{dt} = \frac{\partial R}{dt} = 0, x \in \partial\Omega. \tag{5}$$

In the case that the diffusion coefficients D_i are all equal to zero, according to Wang and Wang's [25], we know that the basic reproduction number is given by:

$$R_0 = \frac{\Lambda}{\mu\delta\kappa(\gamma+\mu+u_1)}(\xi\beta_W + \delta\kappa\beta_h). \tag{6}$$

And the disease-free equilibrium $E_0(S_0, I_0, W_0, R_0)$ is $(\frac{\Lambda}{\mu}, 0, 0, 0)$, the endemic equilibrium $E^*(S^*, I^*, W^*, R^*)$ is determined by:

$$S^* = \frac{\Lambda}{\mu} - \frac{(\gamma+\mu+u_1)I^*}{\mu}, I^* = \frac{\beta_e S^*}{\gamma+\mu+u_1-\beta_h S^*} - \frac{\delta\kappa}{\xi}, W^* = \frac{\xi I^*}{\delta}, R^* = \frac{\gamma I^*}{\mu}.$$

Wang and Wang' paper [25] also established the following results:

Theorem 1 *Assume $D_i = 0$, then for model system (1-4), (1) the disease-free equilibrium E_0 is locally and globally asymptotically stable if $R_0 < 1$; and (2) if $R_0 > 1$, the unique chronic infection equilibrium E^* is globally asymptotically stable.*

In this paper, we consider the cholera spatially dependent model proposed in Wang and wang [25] and construct an NSFD scheme for this model. As far as we know, there are few studies on the continuous cholera models designed as discrete equations. The rest of the paper is organized as follows. In the next section, we construct a discretized cholera model with diffusion from the continuous model by using the nonstandard finite difference method. In Section 3 and Section 4, the global asymptotic stability analysis of the equilibria is performed by using discrete Lyapunov functions. In Section 5, we carry out the numerical study of the discrete model, which confirms our theoretical results. Finally, the conclusions are summarized in Section 6.

2 A discretized model

Assume $\Omega = [a,b]$ with $a, b \in R$, let $\triangle t$ be the time step size and $\triangle x = \frac{(b-a)}{N}$ be the space step size, $t_k = k\triangle t$ for $k \in N$ be the time mesh point, where N is the set of all non-negative

integers. The space mesh point is $X_n = n\Delta x$ for $n \in \{0, 1, \cdots, N\}$. At each point, we denote approximations of $S(x_n, t_k)$, $I(x_n, t_k)$, $W(x_n, t_k)$ and $R(x_n, t_k)$ by S_n^k, I_n^k, W_n^k and R_n^k, respectively. For the sake of convenience, a $(N+1)-$ dimensional vector

$$U^k = (U_0^k, U_1^k, \cdots, U_N^k)^T \tag{7}$$

is used to represent all the approximation solutions at the time t_k. The notation $(\cdot)^T$ denotes the transposition of a vector, and all components of a vector U are non-negative.

We construct the following NSFD method for model system (1-4):

$$\frac{S_n^{k+1} - S_n^k}{\triangle t} = \Lambda - \beta_W \frac{S_n^{k+1} W_n^k}{\kappa + W_n^k} - \beta_h S_n^{k+1} I_n^k - \mu S_n^{k+1} + D_1 \frac{S_{n+1}^{k+1} - 2S_n^{k+1} + S_{n-1}^{k+1}}{(\triangle x)^2}, \tag{8}$$

$$\frac{I_n^{k+1} - I_n^k}{\triangle t} = \beta_W \frac{S_n^{k+1} W_n^k}{\kappa + W_n^k} + \beta_h S_n^{k+1} I_n^k - (\gamma + \mu + u_1) I_n^{k+1} + D_2 \frac{I_{n+1}^{k+1} - 2I_n^{k+1} + I_{n-1}^{k+1}}{(\triangle x)^2}, \tag{9}$$

$$\frac{W_n^{k+1} - W_n^k}{\triangle t} = \xi I_n^{k+1} - \delta W_n^{k+1} + D_3 \frac{W_{n+1}^{k+1} - 2W_n^{k+1} + W_{n-1}^{k+1}}{(\triangle x)^2}, \tag{10}$$

$$\frac{R_n^{k+1} - R_n^k}{\triangle t} = \gamma I_n^{k+1} - \mu R_n^{k+1} + D_4 \frac{R_{n+1}^{k+1} - 2R_n^{k+1} + R_{n-1}^{k+1}}{(\triangle x)^2}, \tag{11}$$

with discrete initial value conditions

$$S_n^0 = \psi_1(x_n), I_n^0 = \psi_2(x_n), W_n^0 = \psi_3(x_n), R_n^0 = \psi_4(x_n),$$

for $n \in \{0, 1, \cdots, N\}$, and discrete boundary condition is given as:

$$S_{-1}^k = S_0^k, S_N^k = S_{N+1}^k, I_{-1}^k = I_0^k, I_N^k = I_{N+1}^k, W_{-1}^k = W^k, W_N^k = W_{N+1}^k, R_{-1}^k = R_0^k, R_N^k = R_{N+1}^k.$$

It is easy to check that the solutions of the discrete system (8-11) are positive, and have the disease-free equilibrium E_0 and the chronic infection equilibrium E^*, which are the same as that of the model (1-4).

3 Global stability of the disease-free equilibrium

Since R does not appear in the first three equations of the system (8-11), we only need to study the system (8-10). In this section, we establish the global stability of the disease-free equilibrium of system (8-10) by constructing a discrete Lyapunov function.

Theorem 2 *If $R_0 < 1$, the disease-free equilibrium E_0 of the model system (8-10) is globally asymptotically stable.*

Proof Define a discrete Lyapunov function

$$L^k = \sum_{n=0}^{N} \frac{1}{\Delta t} [S_0 g(\frac{S_n^k}{S_0}) + I_n^k + \frac{(\gamma + \mu + u_1)(1 + \delta \Delta t)}{\xi} W_n^k], \tag{12}$$

where the function $g(x) = x - 1 - lnx$, $x \in R^+$, clearly, $g(x) \geq 0$ with equality only if $x = 1$. Thus we have $L^k \geq 0$ with equality if and only if $S_n^k = S_0$, $I_n^k = 0$ and $W_n^k = 0$ for all $n \in \{0, 1, \dots, N\}$. Then, along the trajectory of (8-10), we have

$$\begin{aligned}
L^{k+1} - L^k &= \sum_{n=0}^{N} \frac{1}{\Delta t}[S_n^{k+1} - S_n^k + S_0 ln \frac{S_n^k}{S_n^{k+1}} + I_n^{k+1} - I_n^k + \frac{\gamma + \mu + u_1}{\xi}(W_n^{k+1} - W_n^k)] \\
&+ \frac{\delta(\gamma + \mu + u_1)}{\xi}(W_n^{k+1} - W_n^k) \\
&= \sum_{n=0}^{N}[2\Lambda - \frac{\beta_W S_n^{k+1} W_n^k}{\kappa + W_n^k} - \beta_h S_n^{k+1} I_n^k - \mu S_n^{k+1} + D_1 \frac{S_{n+1}^{k+1} - 2S_n^{k+1} + S_{n-1}^{k+1}}{(\triangle x)^2} \\
&- \frac{\Lambda^2}{\mu S_n^{k+1}} + \frac{\Lambda \beta_W W_n^k}{\mu(\kappa + W_n^k)} + \frac{\Lambda \beta_h I_n^k}{\mu} - \frac{\Lambda D_1}{\mu S_n^{k+1}} \frac{S_{n+1}^{k+1} - 2S_n^{k+1} + S_{n-1}^{k+1}}{(\triangle x)^2} \\
&+ \frac{\beta_W S_n^{k+1} W_n^k}{\kappa + W_n^k} + \beta_h S_n^{k+1} I_n^k - (\gamma + \mu + u_1) I_n^{k+1} + D_2 \frac{I_{n+1}^{k+1} - 2I_n^{k+1} + I_{n-1}^{k+1}}{(\triangle x)^2} \\
&+ (\gamma + \mu + u_1) I_n^{k+1} - \frac{\delta(\gamma + \mu + u_1)}{\xi} W_n^{k+1} + D_3 \frac{(\gamma + \mu + u_1)}{\xi} \frac{W_{n+1}^{k+1} - 2W_n^{k+1} + W_{n-1}^{k+1}}{(\triangle x)^2}] \\
&+ \frac{\delta(\gamma + \mu + u_1)}{\xi}(W_n^{k+1} - W_n^k) \\
&\leq \sum_{n=0}^{N}[\Lambda(2 - \frac{\Lambda}{\mu S_n^{k+1}} - \frac{\mu S_n^{k+1}}{\Lambda}) + (\gamma + \mu + u_1) I_n^k (R_0 - 1)] \\
&+ D_1 \frac{S_{N+1}^{k+1} - S_N^{k+1}}{(\triangle x)^2} + D_1 \frac{S_0^{k+1} - S_{-1}^{k+1}}{(\triangle x)^2} + D_2 \frac{I_{N+1}^{k+1} - I_N^{k+1}}{(\triangle x)^2} + D_2 \frac{I_0^{k+1} - I_{-1}^{k+1}}{(\triangle x)^2} \\
&+ D_3 \frac{(\gamma + \mu + u_1)}{\xi} \frac{W_{N+1}^{k+1} - W_N^{k+1}}{(\triangle x)^2} + D_3 \frac{(\gamma + \mu + u_1)}{\xi} \frac{W_0^{k+1} - W_{-1}^{k+1}}{(\triangle x)^2} \\
&= \sum_{n=0}^{N}[\Lambda(2 - \frac{\Lambda}{\mu S_n^{k+1}} - \frac{\mu S_n^{k+1}}{\Lambda}) + (\gamma + \mu + u_1) I_n^k (R_0 - 1)].
\end{aligned}$$

Since $2 - \frac{\Lambda}{\mu S_n^{k+1}} - \frac{\mu S_n^{k+1}}{\Lambda} \leq 0$ by the arithmetic-geometric inequality, it then follows that if $R_0 < 1$, $L^{k+1} - L^k < 0$, for all $k \in \mathbb{N}$ and the equality holds if and only if $S_n^{k+1} = \frac{\Lambda}{\mu}$. This yields that $\{L^k\}$ is a monotone decreasing sequence. Thus, there exists a constant L_0 such that $lim_{k \to +\infty}(L^{k+1} - L^k) = 0$. Therefore, we have $lim_{k \to +\infty} S_n^k = \frac{\Lambda}{\mu}$, $lim_{k \to +\infty} I_n^k = 0$, $lim_{k \to +\infty} W_n^k = 0$, for all $n \in \{0, 1, \dots N\}$. Hence, E_0 is globally asymptotically stable when $R_0 < 1$. This completes the proof. ■

4 Global Stability of the chronic infection equilibrium

In this section we concern with the global stability of the chronic infection steady state of system (8-10) when $R_0 > 1$.

Theorem 3 *If $R_0 > 1$, the chronic infection equilibrium E^* of the model system (8-10) is globally asymptotically stable.*

Using the expression for S^* along with system (8-10) and discrete boundary conditions, we first have

$$\begin{aligned}
&\sum_{n=0}^{N}\frac{1}{\Delta t}[g(\frac{S_n^{k+1}}{S^*})-g(\frac{S_n^k}{S^*})]\\
&\leq\sum_{n=0}^{N}\frac{1}{\Delta t}[(S_n^{k+1}-S_n^k)(\frac{S_n^{k+1}-S^*}{S^*S_n^{k+1}})]\\
&=\sum_{n=0}^{N}\frac{1}{S^*}[(\Lambda-\frac{\beta_W S_n^{k+1}W_n^k}{\kappa+W_n^k}+\beta_h S_n^{k+1}I_n^k-\mu S_n^{k+1}+D_1\frac{S_{n+1}^{k+1}-2S_n^{k+1}+S_{n-1}^{k+1}}{(\triangle x)^2})(1-\frac{S^*}{S_n^{k+1}})]\\
&=\sum_{n=0}^{N}\frac{1}{S^*}[(\frac{\beta_W S^*W^*}{\kappa+W^*}+\beta_h S^*I^*+\mu S^*-\frac{\beta_W S_n^{k+1}W_n^k}{\kappa+W_n^k}-\beta_h S_n^{k+1}I_n^k-\mu S_n^{k+1})(1-\frac{S^*}{S_n^{k+1}})]\\
&+\sum_{n=0}^{N}\frac{1}{S^*}[(D_1\frac{S_{n+1}^{k+1}-2S_n^{k+1}+S_{n-1}^{k+1}}{(\triangle x)^2})(1-\frac{S^*}{S_n^{k+1}})]\\
&=\sum_{n=0}^{N}[-\frac{\mu(S_n^{k+1}-S^*)^2}{S^*S_n^{k+1}}+\frac{\beta_W W^*}{\kappa+W^*}(1-\frac{S^*}{S_n^{k+1}})(1-\frac{(\kappa+W^*)S_n^{k+1}W_n^k}{(\kappa+W_n^k)S^*W^*})\\
&+\beta_h I^*(1-\frac{S^*}{S_n^{k+1}})(1-\frac{S_n^{k+1}I_n^k}{S^*I^*})]-D_1\sum_{n=0}^{N-1}\frac{(S_{n+1}^{k+1}-S_n^{k+1})^2}{(\triangle x)^2S_{n+1}^{k+1}S_n^{k+1}}.
\end{aligned}$$

In the same way, we have

$$\begin{aligned}
&\sum_{n=0}^{N}\frac{1}{\Delta t}[g(\frac{I_n^{k+1}}{I^*})-g(\frac{I_n^k}{I^*})]\\
&\leq\sum_{n=0}^{N}\frac{1}{\Delta t}[(I_n^{k+1}-I_n^k)(\frac{I_n^{k+1}-I^*}{I^*I_n^{k+1}})]\\
&=\sum_{n=0}^{N}\frac{1}{I^*}[\frac{\beta_W S_n^{k+1}W_n^k}{\kappa+W_n^k}+\beta_h S_n^{k+1}I_n^k-(\gamma+\mu+u_1)I_n^{k+1}\\
&+D_2(\frac{I_{n+1}^{k+1}-2I_n^{k+1}+I_{n-1}^{k+1}}{(\triangle x)^2})(1-\frac{I^*}{I_n^{k+1}})]\\
&=\sum_{n=0}^{N}[\frac{\beta_W}{I^*}(1-\frac{I^*}{I_n^{k+1}})(\frac{S_n^{k+1}W_n^k}{\kappa+W_n^k}-\frac{I_n^{k+1}S^*W^*}{(\kappa+W^*)I^*})+\beta_h S^*(1-\frac{I^*}{I_n^{k+1}})(\frac{S_n^{k+1}I_n^k}{S^*I^*}-\frac{I_n^{k+1}}{I^*})]\\
&+\frac{1}{I^*}\sum_{n=0}^{N}[(D_2\frac{I_{n+1}^{k+1}-2I_n^{k+1}+I_{n-1}^{k+1}}{(\triangle x)^2})(1-\frac{I^*}{I_n^{k+1}})]\\
&=\sum_{n=0}^{N}[\frac{\beta_W}{I^*}(1-\frac{I^*}{I_n^{k+1}})(\frac{S_n^{k+1}W_n^k}{\kappa+W_n^k}-\frac{I_n^{k+1}S^*W^*}{(\kappa+W^*)I^*})+\beta_h S^*(1-\frac{I^*}{I_n^{k+1}})(\frac{S_n^{k+1}I_n^k}{S^*I^*}-\frac{I_n^{k+1}}{I^*})]
\end{aligned}$$

$$- D_2\sum_{n=0}^{N-1}\frac{(I_{n+1}^{k+1}-I_n^{k+1})^2}{(\triangle x)^2 I_{n+1}^{k+1}I_n^{k+1}}.$$

Similarly, by letting $\xi I^* = \delta W^*$, we obtain:

$$\begin{aligned}
&\sum_{n=0}^{N}\frac{1}{\Delta t}[g(\frac{W_n^{k+1}}{W^*}) - g(\frac{W_n^k}{W^*})] \\
&\le \sum_{n=0}^{N}\frac{1}{\Delta t}[(W_n^{k+1}-W_n^k)(\frac{W_n^{k+1}-W^*}{W^*W_n^{k+1}})] \\
&= \sum_{n=0}^{N}\frac{1}{W^*}[(\xi I_n^{k+1} - \delta W_n^{k+1} + D_3\frac{W_{n+1}^{k+1}-2W_n^{k+1}+W_{n-1}^{k+1}}{(\triangle x)^2})(1-\frac{W^*}{W_n^{k+1}})] \\
&= \sum_{n=0}^{N}[\frac{\delta}{W^*}(1-\frac{W^*}{W_n^{k+1}})(\frac{W^*I_n^{k+1}}{I^*} - W_n^{k+1})] + \sum_{n=0}^{N}\frac{1}{W^*}[(\frac{W_{n+1}^{k+1}-2W_n^{k+1}+W_{n-1}^{k+1}}{(\triangle x)^2})(1-\frac{W^*}{W_n^{k+1}})] \\
&= \sum_{n=0}^{N}[\frac{\delta}{W^*}(1-\frac{W^*}{W_n^{k+1}})(\frac{W^*I_n^{k+1}}{I^*} - W_n^{k+1})] - D_3\sum_{n=0}^{N-1}\frac{(W_{n+1}^{k+1}-W_n^{k+1})^2}{(\triangle x)^2W_{n+1}^{k+1}W_n^{k+1}}.
\end{aligned}$$

We then define the following Lyapunov function:

$$H^k = \sum_{n=0}^{N}\frac{1}{\Delta t}[\frac{1}{\beta_h I^*}g(\frac{S_n^k}{S^*}) + \frac{1}{\beta_h S^*}g(\frac{I_n^k}{I^*}) + \frac{\beta_W}{\beta_h\delta I^*}g(\frac{W_n^k}{W^*})]. \tag{13}$$

Thus, $H^k \ge 0$ for all $k \in \mathbb{N}$, with equality if and only if $S_n^k = S^*$, $I_n^k = I^*$ and $W_n^k = W^*$ for all $n \in \{0, 1, \ldots N\}$. The difference of H^k is:

$$\begin{aligned}
H^{k+1} - H^k &= \sum_{n=0}^{N}[\frac{1}{\beta_h I^*}(\frac{S_n^{k+1}-S_n^k}{S^*} + ln\frac{S_n^k}{S_n^{k+1}}) + \frac{1}{\beta_h S^*}(\frac{I_n^{k+1}-I_n^k}{I^*} + ln\frac{I_n^k}{I_n^{k+1}}) \\
&+ \frac{\beta_W}{\delta\beta_h I^*}(\frac{W_n^{k+1}-W_n^k}{W^*} + ln\frac{W_n^k}{W_n^{k+1}})] - D_1\sum_{n=0}^{N-1}\frac{(S_{n+1}^{k+1}-S_n^{k+1})^2}{(\Delta x)^2S_{n+1}^{k+1}S_n^{k+1}} \\
&- D_2\sum_{n=0}^{N-1}\frac{(I_{n+1}^{k+1}-I_n^{k+1})^2}{(\Delta x)^2I_{n+1}^{k+1}I_n^{k+1}} - D_3\sum_{n=0}^{N-1}\frac{(W_{n+1}^{k+1}-W_n^{k+1})^2}{(\Delta x)^2W_{n+1}^{k+1}W_n^{k+1}} \\
&\le \sum_{n=0}^{N}\{-\frac{\mu(S_n^{k+1}-S^*)^2}{\beta_h S_n^{k+1}S^*I^*} + (2 - \frac{S^*}{S_n^{k+1}} - \frac{I_n^{k+1}}{I^*} - \frac{S_n^{k+1}I_n^k}{I_n^{k+1}S^*} + \frac{I_n^{k+1}}{I^*}) \\
&- \frac{\beta_W W^*}{\beta_h I^*(\kappa+W^*)}[\frac{S^*}{S_n^{k+1}} + \frac{I_n^{k+1}}{I^*} + \frac{S_n^{k+1}W_n^kI^*(\kappa+W^*)}{I_n^{k+1}(\kappa+W_n^k)S^*W^*} - \frac{W_n^k(\kappa+W^*)}{(\kappa+W_n^k)W^*} - 2] \\
&- \frac{\beta_W W^*}{\beta_h I^*(\kappa+W^*)}(\frac{W_n^{k+1}}{W^*} + \frac{I_n^{k+1}W^*}{W_n^{k+1}I^*} - \frac{I_n^{k+1}}{I^*} - 1)\} \\
&- D_1\sum_{n=0}^{N-1}\frac{(S_{n+1}^{k+1}-S_n^{k+1})^2}{(\Delta x)^2S_{n+1}^{k+1}S_n^{k+1}} - D_2\sum_{n=0}^{N-1}\frac{(I_{n+1}^{k+1}-I_n^{k+1})^2}{(\Delta x)^2I_{n+1}^{k+1}I_n^{k+1}} - D_3\sum_{n=0}^{N-1}\frac{(W_{n+1}^{k+1}-W_n^{k+1})^2}{(\Delta x)^2W_{n+1}^{k+1}W_n^{k+1}}
\end{aligned}$$

$$
\begin{aligned}
&\leq \sum_{n=0}^{N}\{-\frac{\mu(S_n^{k+1}-S^*)^2}{\beta_h S_n^{k+1}S^*I^*}-[g(\frac{S^*}{S_n^{k+1}})+g(\frac{S_n^{k+1}I_n^k}{I_n^{k+1}S^*})+\frac{I_n^{k+1}}{I^*}-ln\frac{I_n^{k+1}}{I_n^k}]\\
&-\frac{\beta_W W^*}{\beta_h I^*(\kappa+W^*)}[\frac{S^*}{S_n^{k+1}}+\frac{I_n^{k+1}}{I^*}+\frac{S_n^{k+1}I^*(\kappa+W^*)}{I_n^{k+1}S^*W^*}-\frac{\kappa+W^*}{W^*}-2]\\
&-\frac{\beta_W W^*}{\beta_h I^*(\kappa+W^*)}(\frac{W_n^{k+1}}{W^*}+\frac{I_n^{k+1}W^*}{W_n^{k+1}I^*}-\frac{I_n^{k+1}}{I^*}-1)\}\\
&-D_1\sum_{n=0}^{N-1}\frac{(S_{n+1}^{k+1}-S_n^{k+1})^2}{(\Delta x)^2S_{n+1}^{k+1}S_n^{k+1}}-D_2\sum_{n=0}^{N-1}\frac{(I_{n+1}^{k+1}-I_n^{k+1})^2}{(\Delta x)^2I_{n+1}^{k+1}I_n^{k+1}}-D_3\sum_{n=0}^{N-1}\frac{(W_{n+1}^{k+1}-W_n^{k+1})^2}{(\Delta x)^2W_{n+1}^{k+1}W_n^{k+1}}\\
&\leq \sum_{n=0}^{N}\{-\frac{\mu(S_n^{k+1}-S^*)^2}{\beta_h S_n^{k+1}S^*I^*}-g(\frac{S^*}{S_n^{k+1}})-g(\frac{S_n^{k+1}I_n^k}{I_n^{k+1}S^*})\\
&-\frac{\beta_W W^*}{\beta_h I^*(\kappa+W^*)}[g(\frac{S^*}{S_n^{k+1}})+g(\frac{S_n^{k+1}I_n^{k+1}(\kappa+W^*)}{I_n^{k+1}S^*W^*})+g(\frac{W_n^{k+1}}{W^*})+g(\frac{I_n^{k+1}W^*}{W_n^{k+1}I^*})]\}\\
&-D_1\sum_{n=0}^{N-1}\frac{(S_{n+1}^{k+1}-S_n^{k+1})^2}{(\Delta x)^2S_{n+1}^{k+1}S_n^{k+1}}-D_2\sum_{n=0}^{N-1}\frac{(I_{n+1}^{k+1}-I_n^{k+1})^2}{(\Delta x)^2I_{n+1}^{k+1}I_n^{k+1}}-D_3\sum_{n=0}^{N-1}\frac{(W_{n+1}^{k+1}-W_n^{k+1})^2}{(\Delta x)^2W_{n+1}^{k+1}W_n^{k+1}}.
\end{aligned}
$$

It is easy to see $H^{k+1}-H^k \leq 0$ for all $k \in \mathbb{N}$. Then there exists a constant H^* such that $lim_{k\to+\infty}(H^{k+1}-H^k)=0$, which implies $lim_{k\to+\infty}S_n^k=S^*$. Combined with system (8-10), we have $lim_{k\to+\infty}I_n^k=I^*$ and $lim_{k\to+\infty}W_n^k=W^*$ as well, for all $n\in\{0,1,\dots N\}$. Hence, E^* is globally asymptotically stable when $R_0>1$. This completes the proof.

5 Numerical results

In this section, we propose numerical simulations to verify the stability properties of the NSFD scheme. We use the data regarding the course of the cholera in Zimbabwe during 2008-2009, which is the worst outbreak in Africa in the past 30 years with over 100,000 humans have been infected and more than 4,300 killed. The total population in Zimbabwe is 12,347,240, for mathematical simplicity, we scale down all data numbers by a factor of 1,200. All epidemiological parameter values for cholera in literature are given as: $\Lambda=4.5$, $\mu=0.000442$, $\xi=70$, $\delta=0.2333$, $u_1=0.04$, $\gamma=1.4$, $\kappa=1000000$ ([2, 4, 5, 9]). In addition, the initial values are taken as $I(x,0)=10\times\exp(-x)$, $S(x,0)=1000\times\exp(-x)$, $W(x,0)=10\times\exp(-x)$, and $R(x,0)=10\times\exp(-x)$, where $x\in[0,50]$.

Let the grid sizes used in the simulation are $\Delta x=0.5$ and $\Delta t=0.1$, respectively, and the diffusion coefficients D_i are all fixed as 0.01. The discussions in ([2,4,5,9]) indicate that parameters β_W and β_h are sensitive and vary from place to place, so we first set $\beta_W=0.0001$ and $\beta_h=0.0001$, which renders $R_0=0.7070<1$. Hence, model system has a disease-free equilibrium in this case, the number of infectious decreases quickly and the disease dies out. It can be observed from Figure 1, where the steady state approaches to $E_0=(0.6,0,0)$. For the other case, we choose $\beta_W=0.0001$, $\beta_h=0.000236$ and do not change the other parameter values, which gives $R_0=1.6683>1$, the chronic infection steady state is $E^*=(0.5135,1899.14,8200.46)$ by calculation, the infected steady state is stable as can

be observed numerically in Figure 2. We then examine the case with different sets of initial conditions when $R_0 > 1$, also obtain almost the same patterns.

Figure 3 compares the profile when we choose two different combinations of D_i, as, $(0.01, 0.05, 0.01, 0.05)$ and $(0.05, 0.1, 0.05, 0.1)$ for $R_0 > 1$. Only the distribution of the density of $I(x,t)$ is depicted, similar results for the other two variables $S(x,t)$ and $R(x,t)$ are not presented here. Comparing Fig. 3 and Fig 1.(a), we can find that diffusion coefficients have no effect on the convergence of solutions, but the larger diffusion coefficients will deduce the number of infected population and speed up the arrived time at the chronic infection equilibrium.

In a addition, we perform numerical simulations of a standard finite difference (SFD) scheme to compare the results with NSFD scheme using the same discrete boundary conditions and parameter values in Figure 4. The stronger competitiveness of NSFD scheme has been proved by its succsess in preserving the global stability of equilibrium and the failure of the SFD method.

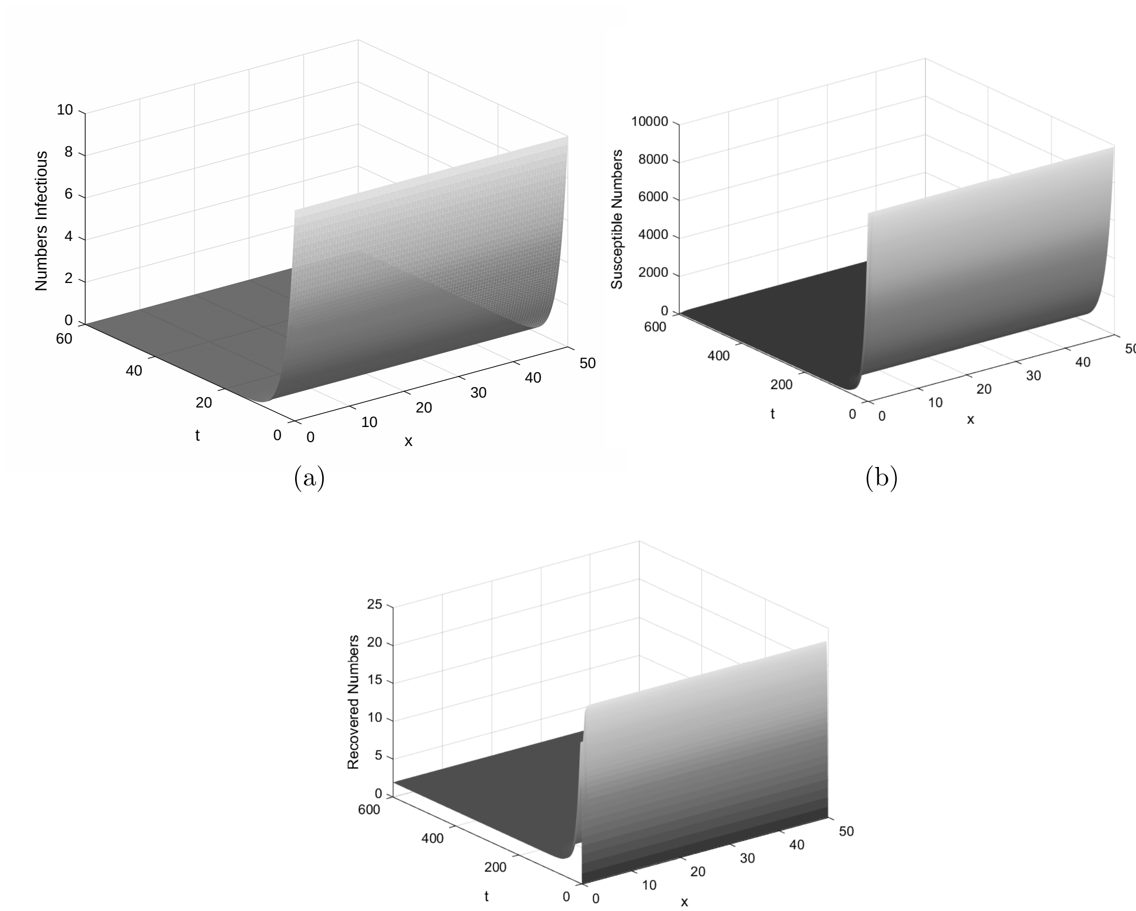
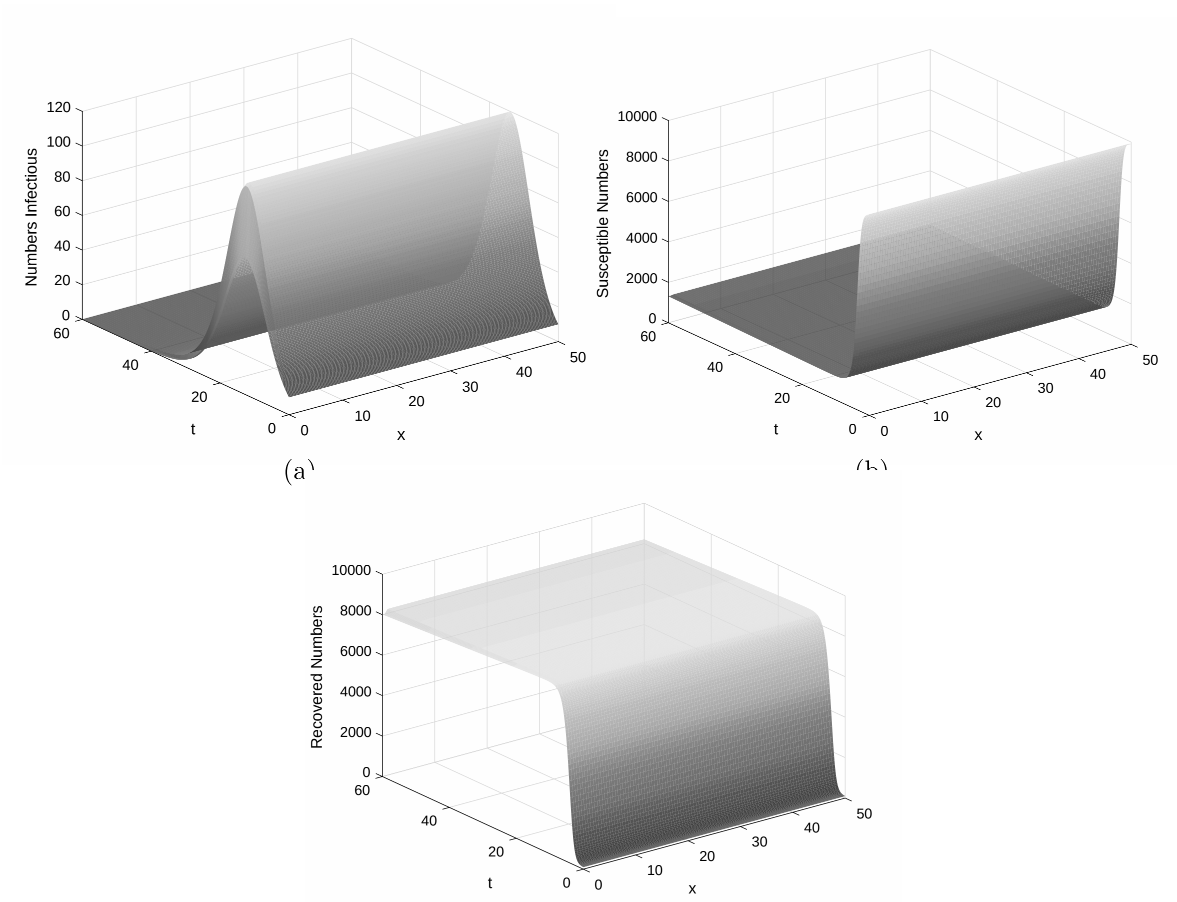


Figure 1: Graphs of the numerical solutions of the NSFD method when $R_0 < 1$.

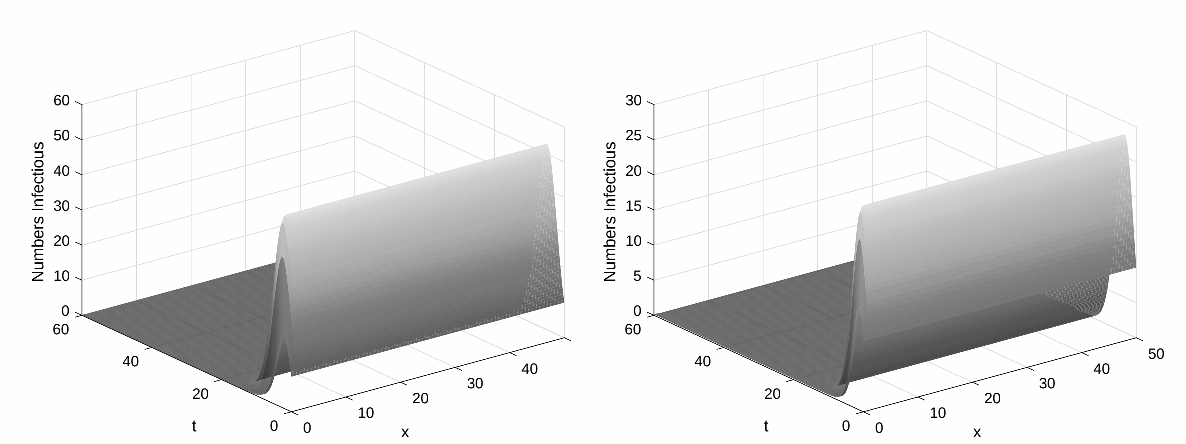
6 Conclusions and discussions

In this article, we derive a discrete cholera infection model with spatial diffusion by using an NSFD method. We show that the disease-free steady state of the discrete model is globally



(c)

Figure 2: Graphs of the numerical solutions of the NSFD method when $R_0 > 1$.



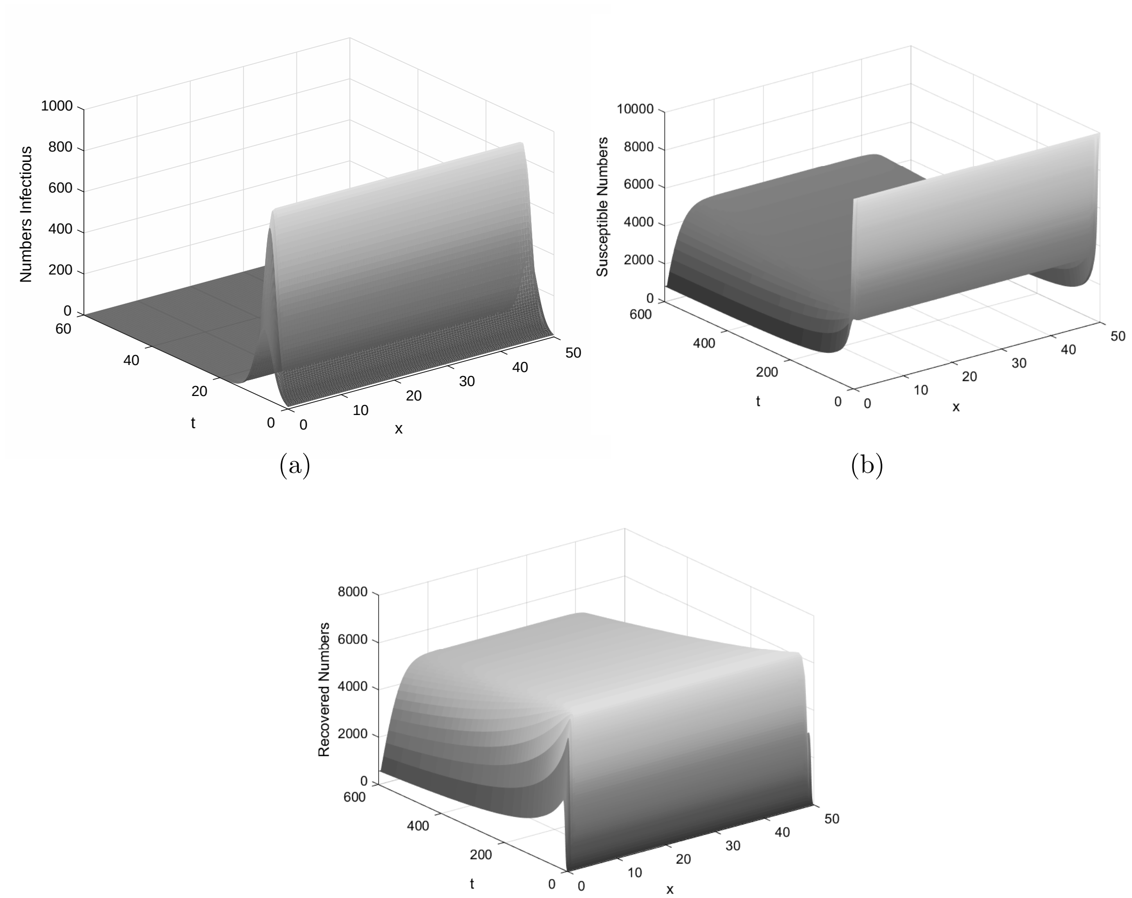
(a) (b)

Figure 3: Dynamics of infected population when $R_0 > 1$ for two different sets of D_i.

asymptotically stable if the basic reproduction number $R_0 < 1$, and the chronic infection equilibrium is globally asymptotically stable when $R_0 > 1$. In a word, our results (Theorem 2 and Theorem 3) imply that the discretization scheme (8-11) is dynamically consistent with the continuous system with respect to the globally asymptotical stability of the steady-state solutions. Our simulation results also conclude that the diffusion coefficients have no relation to the global stability of such cholera epidemic. Finally, numerical results show the advantage of our method in comparison to an SFD method. Application of this method to the general delayed discrete epidemic models is our future work.

Acknowledgments

This work was partially supported by the Natural Science Foundation of China (11401059), the Natural Science Foundation of CQ (cstc2015jcyjA00024, cstc2017jcyjAX0067), Scientific and Technological Research Program of Chongqing Municipal Education Commission (KJ1600610, KJ1706163).



(a) (b)

(c)

Figure 4: Graphs of the numerical solutions of the SFD method.

References

[1] Codeço CT, Endemic and epidemic dynamics of cholera: the role of the aquatic reservoir, *BMC Infectious Diseases*, **1**. 1, 2001.

[2] Hartley DM, Morris JG and Smith DL, Hyperinfectivity: A critical element in the ability of V. cholerae to cause epidemics?, *PLoS Medicine*, **3**(1): 63-69, 2006.

[3] Tien JH and Earn DJD, Multiple transmission pathways and disease dynamics in a waterborne pathogen model, *Bulletin of Mathematical Biology*, **72**(6): 1502-1533, 2010.

[4] Mukandavire Z, Liao S, Wang J, Gaff H, Smith DL and Morris JG, Estimating the reproductive numbers for the 2008–2009 cholera outbreaks in Zimbabwe, *Proceedings of the National Academy of Sciences of the United States of America*, **108**(21): 8767-8772, 2011.

[5] Liao S and Wang J, Stability analysis and application of a mathematical cholera model, *Mathematical Biosciences and Engineering*, **8**(3): 733-752, 2011.

[6] Bertuzzo E, Casagrandi R, Gatto M, Rodriguez-Iturbe I and Rinaldo A, on spatially explicit models of cholera epidemics, *Journal of the Royal Society Interface*, **7**(43): 321-333, 2010.

[7] Safi MA, Melesse DY and Gumel AB, Analysis of a Multi-strain Cholera Model with an Imperfect Vaccine, *Bulletin of Mathematical Biology*, **75**(7): 1104-1137, 2013.

[8] Misra AK and Singh V, A delay mathematical model for the spread and control of water borne diseases, *Journal of Theoretical Biology*, **301**(5): 49-56, 2012.

[9] Liao S and Yang W, On the dynamics of a vaccination model with multiple transmission ways, *International Journal of Applied Mathematics and Computer Science*, **23**(4): 761-772, 2013.

[10] Tuite AR, Tien JH, Eisenberg MC, Earn DJD, Ma J and Fisman DN, Cholera Epidemic in Haiti: Using a Transmission Model to Explain Spatial Spread of Disease and Identify Optimal Control Interventions, *Annals of Internal Medicine*, **154**(2011): 293-302, 2010.

[11] Capone F, De CV and De LR, Influence of diffusion on the stability of equilibria in a reactionCdiffusion system modeling cholera dynamic, *Mathematical Biology*, **71**(5): 1107-1131, 2015.

[12] Mickens RE, Exact solutions to a finite difference model of a nonlinear reaction-advection equation: implications for numerical analysis, *Numerical Methods for Partial Differential Equations*, **5**: 313-325, 1989.

[13] Mickens RE, *Nonstandard Finite difference models of differential equations*, World Scientific, Singapore, 1994.

[14] Mickens RE, Dynamic consistency: a fundamental principle for constructing nonstandard finite difference schemes for differential equations, *Journal of Difference Equations & Applications*, **11**(7): 645-653, 2005.

[15] Mickens RE, Calculation of denominator functions for nonstandard finite difference schemes for differential equations satisfying a positivity condition, *Numerical Methods for Partial Differential Equations*, **23**(3): 672-691, 2006.

[16] Mickens RE, Discretizations of nonlinear differential equations using explicit nonstandard methods, *Journal of Computational and Applied Mathematics*, **110**(1): 181-185, 1999.

[17] Cui Q, Yang X and Zhang Q, An NSFD scheme for a class of SIR epidemic models with vaccination and treatment, *Journal of Difference Equations and Applications*, **20**(3): 416-422, 2014.

[18] Suryanto A, Kusumawinahyu WM, Darti I and Yanti I, Dynamically consistent discrete epidemic model with modified saturated incidence rate, *Applied Mathematics and Computation*, **32**(2): 373-383, 2013.

[19] Qin W, Wang L and Ding X, A non-standard finite difference method for a hepatitis B virus infection model with spatial diffusion, *Journal of Difference Equations and Applications*, **20**(12): 1641-1651, 2014.

[20] Manna K and Chakrabarty SP, Global stability and a non-standard finite difference scheme for a diffusion driven HBV model with capsids, *Journal of Difference Equations and Applications*, **21**(10): 918-933, 2015.

[21] Villanueva R, Arenas A and Gonzalez-Parra G, A nonstandard dynamically consistent numerical scheme applied to obesity dynamics, *Journal of Applied Mathematics*, Article ID 640154, 2008.

[22] Jodar L, Villanueva RJ, Arenas AJ and Gonzalez GC, Nonstandard numerical methods for a mathematical model for influenza disease, *Mathematics and Computers in Simulation*, **79**(3): 622-633, 2008.

[23] Arenas AJ, Gonzalez-Parra G and Chen-Charpentier BM, A nonstandard numerical scheme of predictor-corrector type for epidemic models, *Computers and Mathematics with Applications*, **59**(12): 3740-3749, 2010.

[24] Garba SM, Gumel AB and Lubuma JMS, Dynamically-consistent non-standard finite difference method for an epidemic model, *Mathematical and Computer Modelling*, **53**(1-2): 131-150, 2011.

[25] Wang X and Wang J, Analysis of cholera epidemics with bacterial growth and spatial movement, *Journal of Biological Dynamics*, **9**(sup1): 233-261, 2014.

On the Higher Order Difference Equation $x_{n+1}=\alpha x_n+\beta x_{n-l}+\gamma x_{n-k}+\dfrac{ax_nx_{n-k}}{bx_n+cx_{n-l}+dx_{n-k}}$

M. M. El-Dessoky[1,2] and K. S. Al-Basyouni[1]
[1]King Abdulaziz University, Faculty of Science, Mathematics Department,
P. O. Box 80203, Jeddah 21589, Saudi Arabia.
[2]Department of Mathematics, Faculty of Science, Mansoura University,
Mansoura 35516, Egypt.
E-mail: dessokym@mans.edu.eg; kalbasyouni@kau.edu.sa

ABSTRACT

The main objective of this paper is to investigate the global stability of the solutions, the boundedness and the periodic character of the nonlinear difference equation

$$x_{n+1}=\alpha x_n+\beta x_{n-l}+\gamma x_{n-k}+\frac{ax_nx_{n-k}}{bx_n+cx_{n-l}+dx_{n-k}},\qquad n=0,\ 1,\ ...,$$

where the parameters $\alpha,\ \beta,\ \gamma,\ a,\ b,\ c$ and d are positive real numbers and the initial conditions $x_{-s},\ x_{-s+1},...,\ x_{-1}$, x_0 are positive real numbers where $s=max\{l,\ k\}$. Some numerical examples will be given to explicate our results.

Keywords: Difference equations, Stability, Global stability, Boundedness, Periodic solutions.

Mathematics Subject Classification: 39A10

1. INTRODUCTION

Our goal is to study some qualitative behavior of the solutions of the difference equation

$$x_{n+1}=\alpha x_n+\beta x_{n-l}+\gamma x_{n-k}+\frac{ax_nx_{n-k}}{bx_n+cx_{n-l}+dx_{n-k}},\qquad n=0,\ 1,\ ..., \tag{1}$$

where the parameters $\alpha,\ \beta,\ \gamma,\ a,\ b,\ c$ and d are positive real numbers and the initial conditions $x_{-s},\ x_{-s+1},...,\ x_{-1}$, x_0 are positive real numbers where $s=max\{l,\ k\}$.

Recently there has been a great interest in studying the qualitative properties of rational difference equations. For the systematical studies of rational and nonrational difference equations, one can refer to the papers [1-270] and references therein.

Ibrahim [4] investigated the global attractivity of the positive solutions of the difference equation

$$x_{n+1}=\frac{x_{n-(2k+1)}}{1+x_{n-k}x_{n-(2k+1)}},\qquad n=0,1,....$$

Zayed et al. et al. [5] studied the periodicity, the boundedness and the global stability of the positive solution of the difference equation,

$$x_{n+1}=\frac{\alpha x_n+\beta x_{n-1}+\gamma x_{n-2}+\delta x_{n-3}}{Ax_n+Bx_{n-1}+Cx_{n-2}+Dx_{n-3}},\qquad n=0,1,....$$

In [6] El-Dessoky investigated the global stability character and the periodicity of solutions of the recursive sequence

$$x_{n+1}=\frac{ax_{n-l}+bx_{n-k}}{c+dx_{n-l}x_{n-k}},\quad n=0,1,....$$

Guo-Mei Tang et al. [7] obtained the global behavior of solutions of the following nonlinear difference equation

$$x_{n+1}=\frac{\alpha+x_n}{A+Bx_n+x_{n-k}},\qquad n=0,1,....$$

Papaschinopoulos et al. [8] studied the asymptotic behavior and the periodicity of the positive solutions of the nonautonomous difference equation

$$x_{n+1} = A_n + \tfrac{x_{n-1}^p}{x_n^q}, \qquad n = 0, 1, ...$$

El-Dessoky [9] obtained the global stability, the boundedness and the periodicity of the nonlinear difference equation

$$x_{n+1} = ax_n + bx_{n-k} + cx_{n-l} - \tfrac{dx_{n-s}}{ex_{n-s} - \alpha x_{n-t}}, \qquad n = 0,\ 1,\ ....$$

Nirmaladevi et al. [10] studied the periodicity solution and the global stability of nonlinear difference equation

$$y_{n+1} = Py_n + Qy_{n-k} + Ry_{n-l} + \tfrac{by_{n-k}}{dy_{n-k} - ey_{n-l}}, \qquad n = 0, 1, ...$$

"Let I be some interval of real numbers and let

$$F : I^{s+1} \to I,$$

be a continuously differentiable function. Then for every set of initial conditions $x_{-s}, x_{-s+1}, ..., x_0 \in I$, the difference equation

$$x_{n+1} = F(x_n, x_{n-1}, ..., x_{n-s}), \quad n = 0, 1, ..., \tag{2}$$

has a unique solution $\{x_n\}_{n=-s}^{\infty}$.

Definition 1.1. *(Equilibrium Point)*
A point $\overline{x} \in I$ is called an equilibrium point of the difference equation (2) if

$$\overline{x} = F(\overline{x}, \overline{x}, ..., \overline{x}).$$

That is, $x_n = \overline{x}$ for $n \geq 0$, is a solution of the difference equation (2), or equivalently, $\overline{x}$ is a fixed point of F.

Definition 1.2. *(Stability)*
Let $\overline{x} \in (0,\ \infty)$ be an equilibrium point of the difference equation (2). Then, we have
(i) The equilibrium point $\overline{x}$ of the difference equation (2) is called locally stable if for every $\epsilon > 0$, there exists $\delta > 0$ such that for all $x_{-s}, ..., x_{-1}, x_0 \in I$ with

$$|x_{-t} - \overline{x}| + ... + |x_{-1} - \overline{x}| + |x_0 - \overline{x}| < \delta,$$

we have

$$|x_n - \overline{x}| < \epsilon \qquad \textit{for all} \quad n \geq -t.$$

(ii) The equilibrium point $\overline{x}$ of the difference equation (2) is called locally asymptotically stable if $\overline{x}$ is locally stable solution of equation (2) and there exists $\gamma > 0$, such that for all $x_{-t}, ..., x_{-1},\ x_0 \in I$ with

$$|x_{-s} - \overline{x}| + ... + |x_{-1} - \overline{x}| + |x_0 - \overline{x}| < \gamma,$$

we have

$$\lim_{n\to\infty} x_n = \overline{x}.$$

(iii) The equilibrium point $\overline{x}$ of the difference equation (2) is called global attractor if for all $x_{-s}, ..., x_{-1}, x_0 \in I$, we have

$$\lim_{n\to\infty} x_n = \overline{x}.$$

(iv) The equilibrium point $\overline{x}$ of the difference equation (2) is called globally asymptotically stable if $\overline{x}$ is locally stable, and $\overline{x}$ is also a global attractor of the difference equation (2).

(v) The equilibrium point $\overline{x}$ of the difference equation (2) is called unstable if $\overline{x}$ is not locally stable.

Definition 1.3. *(Periodicity)*
A sequence $\{x_n\}_{n=-s}^{\infty}$ *is said to be periodic with period* p *if* $x_{n+p} = x_n$ *for all* $n \geq -t$. *A sequence* $\{x_n\}_{n=-s}^{\infty}$ *is said to be periodic with prime period* p *if* p *is the smallest positive integer having this property.*

Definition 1.4. *Equation (2) is called permanent and bounded if there exists numbers* M *and* m *with* $0 < m < M < \infty$ *such that for any initial conditions* $x_{-s}, ..., x_{-1}, x_0 \in (0, \infty)$ *there exists a positive integer* N *which depends on these initial conditions such that*

$$m \leq x_n \leq M \qquad \text{for all } n > N.$$

Definition 1.5. *The linearized equation of the difference equation (2) about the equilibrium* $\overline{x}$ *is the linear difference equation*

$$y_{n+1} = \sum_{i=0}^{s} \frac{\partial F(\overline{x}, \overline{x}, ..., \overline{x})}{\partial x_{n-i}} y_{n-i}. \tag{3}$$

Now, assume that the characteristic equation associated with (3) is

$$p(\lambda) = p_0\lambda^s + p_1\lambda^{s-1} + ... + p_{s-1}\lambda + p_s = 0, \tag{4}$$

where

$$p_i = \frac{\partial F(\overline{x}, \overline{x}, ..., \overline{x})}{\partial x_{n-i}}.$$

Theorem 1.6. *[1]: Assume that* $p_i \in R$, $i = 1, 2, ..., s$ *and* s *is non-negative integer. Then*

$$\sum_{i=1}^{s} |p_i| < 1,$$

is a sufficient condition for the asymptotic stability of the difference equation

$$y_{n+s} + p_1 y_{n+s-1} + ... + p_s y_n = 0, \;\; n = 0, 1, ... \,.$$

Theorem 1.7. *[1]: Consider the the difference equation (2) where* $F \in C(I^{t+1}, R)$ *and* I *is an open interval of real numbers. Let* $\overline{x}$ *be an equilibrium point of the difference equation (2). Finally, suppose that* F *satisfies the following two conditions:*

(i) F *is nondecreasing in each of its argements.*

(ii) F *satisfies the negative feedback property*

$$[F(x, \; x, \; ..., \; x) - x]\,(x - \overline{x}) < 0, \quad \text{for all } \; x \in I - \{0\}\,.$$

Then the equilibrium point $\overline{x}$ *isa global attractor of all solutions of the difference equation (2)."*

2. LOCAL STABILITY

In this section, we study the local stability character of the equilibrium point of equation (1).

Equation (1) has equilibrium point and is given by

$$\overline{x} = \alpha\overline{x} + \beta\overline{x} + \gamma\overline{x} + \tfrac{a\overline{x}^2}{b\overline{x} + c\overline{x} + d\overline{x}},$$

$$[(1 - \alpha - \beta - \gamma)\,(b + c + d) - a]\,\overline{x}^2 = 0.$$

If $(1 - \alpha - \beta - \gamma)\,(b + c + d) \neq a$, then the equilibrium point of the difference equation (1) is $\overline{x} = 0$.

Let $f : (0,\ \infty)^3 \longrightarrow (0,\ \infty)$ be a continuous function defined by

$$f(u,\ v,\ w) = \alpha u + \beta v + \gamma w + \tfrac{auw}{bu+cv+dw}.$$

Therefore, it follows that

$$\frac{\partial f(u,\ v,\ w)}{\partial u} = \alpha + \frac{aw(cv+dw)}{(bu+cv+dw)^2},\ \frac{\partial f(u,\ v,\ w)}{\partial v} = \beta - \frac{acuw}{(bu+cv+dw)^2} \text{ and } \frac{\partial f(u,\ v,\ w)}{\partial w} = \gamma + \frac{au(bu+cv)}{(bu+cv+dw)^2}.$$

THEOREM 2.1. *The zero equilibrium $\overline{x}$ of the difference equation (1) is locally asymptotically stable if*

$$\left(\alpha+\beta+\gamma\right)\left(b+c+d\right)+a<1. \tag{5}$$

Proof: So, we can write Eq. (6) at zero equilibrium point $\overline{x}=0$

$$\begin{aligned}\frac{\partial f(\overline{x},\ \overline{x},\ \overline{x})}{\partial u} &= \alpha + \frac{a(c+d)}{(b+c+d)^2} = p_1,\ \frac{\partial f(\overline{x},\ \overline{x},\ \overline{x})}{\partial v} = \beta - \frac{ac}{(b+c+d)^2} = p_2 \\ \text{and } \frac{\partial f(\overline{x},\ \overline{x},\ \overline{x})}{\partial w} &= \gamma + \frac{a(b+c)}{(b+c+d)^2} = p_3.\end{aligned}$$

Then the linearized equation of equation (1) about $\overline{x}$ is

$$y_{n+1} - p_1 y_{n-k} - p_2 y_{n-l} - p_3 y_{n-s} = 0,$$

It follows by Theorem 1 that, equation (1) is asymptotically stable if and only if

$$|p_1| + |p_2| + |p_3| < 1.$$

Thus,

$$\left|\alpha + \frac{a(c+d)}{(b+c+d)^2}\right| + \left|\beta - \frac{ac}{(b+c+d)^2}\right| + \left|\gamma + \frac{a(b+c)}{(b+c+d)^2}\right| < 1,$$

and so

$$\alpha + \frac{a(c+d)}{(b+c+d)^2} + \beta - \frac{ac}{(b+c+d)^2} + \gamma + \frac{a(b+c)}{(b+c+d)^2} < 1,$$

$$\begin{aligned}\alpha+\beta+\gamma+\frac{a(b+c+d)}{(b+c+d)^2} &< 1, \\ \left(\alpha+\beta+\gamma\right)\left(b+c+d\right)+a &< 1.\end{aligned}$$

The proof is complete.

Example 1. Consider $l = 2$, $k = 3$, $\alpha = 0.3$, $\beta = 0.02$, $\gamma = 0.01$, $a = 0.1$, $b = 0.2$, $c = 0.3$ and $d = 0.7$ and the initial conditions $x_{-3} = 0.2$, $x_{-2} = 0.4$, $x_{-1} = 0.6$ and $x_0 = 0.1$, the zero solution of the difference equation (1) is local stability (see Fig. 1).

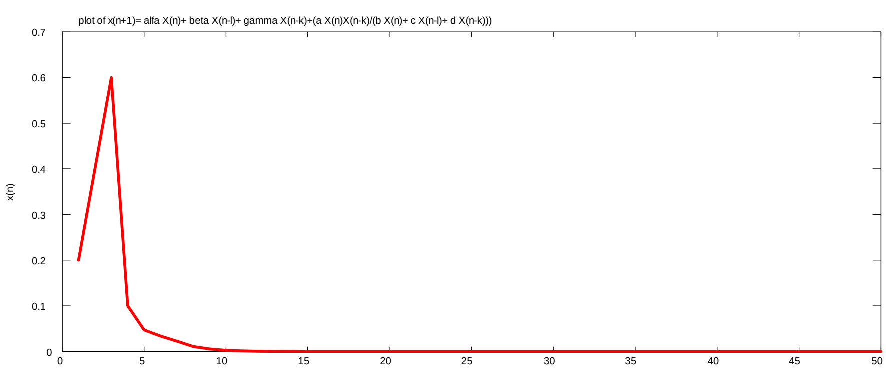


Figure 1. Sketch the behavior of zero solution of equation (1) is local stable.

Example 2. The solution of the difference equation (1) is unstable if $l = 2$, $k = 3$, $\alpha = 0.3$, $\beta = 0.2$, $\gamma = 0.1$, $a = 0.5$, $b = 0.2$, $c = 0.3$ and $d = 0.7$ and the initial conditions $x_{-3} = 0.2$, $x_{-2} = 0.4$, $x_{-1} = 0.6$ and $x_0 = 0.1$. (See Fig. 2).

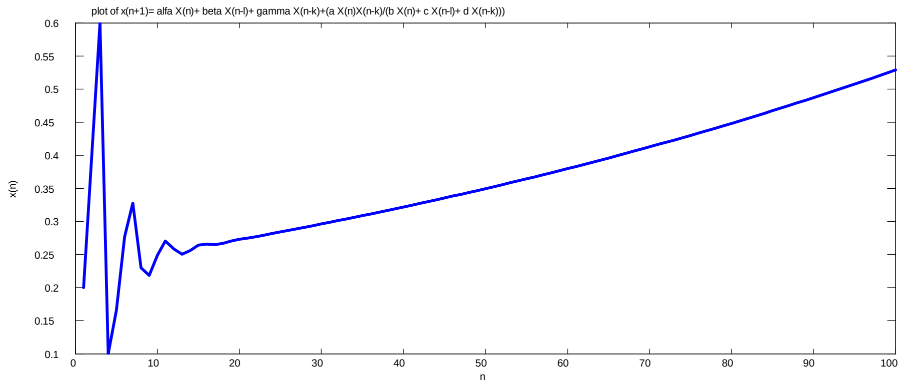


Figure 2. Draw the behavior of the solution of equation (1) is unstable.

3. GLOBAL STABILITY

In this section, the global asymptotic stability of equation (1) is studied.

THEOREM 3.1. *The equilibrium point* $\overline{x}$ *is a global attractor of Eq. (1) if* $\alpha + \beta + \gamma \neq 1$.

Proof: Suppose that ζ and η are real numbers and assume that $F : [\zeta, \eta]^3 \longrightarrow [\zeta, \eta]$ is a function defined by

$$F(x,\ y,\ z) = \alpha x + \beta y + \gamma z + \frac{axz}{bx+cy+dz}.$$

Then

$$\frac{\partial F(x,\ y,\ z)}{\partial x} = \alpha + \frac{az(cy+dz)}{(bx+cy+dz)^2},\ \frac{\partial F(x,\ y,\ z)}{\partial y} = \beta - \frac{acxz}{(bx+cy+dz)^2} \text{ and } \frac{\partial F(x,\ y,\ z)}{\partial z} = \gamma + \frac{ax(bx+cy)}{(bx+cy+dz)^2}.$$

Now, we can see that the function $F(x,\ y,\ z)$ nondecreasing in $x,\ y$ and z. Then

$$\begin{aligned}
[F(x,\ x,\ x) - x]\,(x - \overline{x}) &= \left[\alpha x + \beta x + \gamma x + \frac{ax^2}{bx+cx+dx} - x\right](x - \overline{x}) \\
&= -\left[\left(1 - \alpha - \beta - \gamma - \frac{a}{b+c+d}\right)x\right](x - 0) \\
&= -\left(1 - \alpha - \beta - \gamma - \frac{a}{b+c+d}\right)x^2 < 0
\end{aligned}$$

If $\alpha + \beta + \gamma + \frac{a}{b+c+d} < 1$, then $F(x,\ y,\ z)$ satisfies the negative feedback property

$$[F(x,\ x,\ x) - x]\,(x - \overline{x}_0) < 0, \quad \text{for } \overline{x}_0 = 0.$$

According to Theorem 2, then $\overline{x}$ is a global attractor of Eq. (1). This completes the proof.

Example 3. The solution of the difference equation (1) is global stability when $l = 2$, $k = 3$, $\alpha = 0.03$, $\beta = 0.02$, $\gamma = 0.01$, $a = 0.1$, $b = 0.2$, $c = 0.3$ and $d = 0.7$ and the initial conditions $x_{-3} = 0.2$, $x_{-2} = 0.4$, $x_{-1} = 0.6$ and $x_0 = 0.1$. (See Fig. 3).

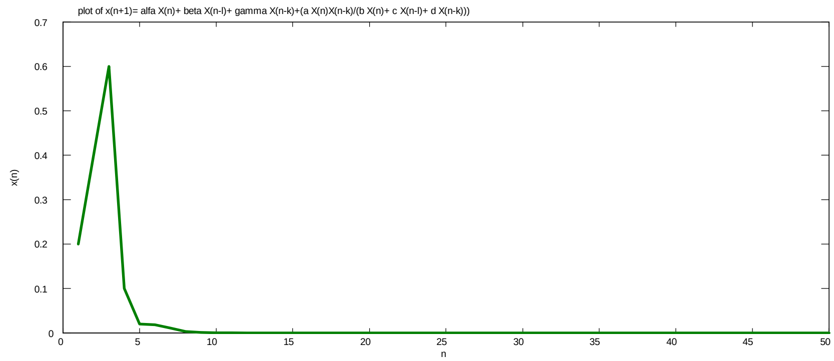


Figure 3. Plot the behavior of the solution of equation (1) is global stability.

4. BOUNDEDNESS OF THE SOLUTIONS

In this section, we investigate the boundedness nature of the positive solutions of equation (1).

THEOREM 4.1. *Every solution of Equation (1) is bounded if one of the following conditions holds:*

$$(i)\ \alpha+\frac{a}{d} < 1,\ \beta<1 \text{ and } \gamma<1. \tag{6}$$

$$(ii)\ \alpha < 1,\ \beta<1 \text{ and } \gamma+\frac{a}{b}<1. \tag{7}$$

Proof: First we prove every solution of Equation (1) is bounded if $\alpha+\frac{a}{d}<1,\ \beta<1$ and $\gamma<1$. Let $\{x_n\}_{n=-s}^{\infty}$ be a solution of Equation (1). It follows from Equation (1) that

$$\begin{aligned} x_{n+1} &= \alpha x_n+\beta x_{n-l}+\gamma x_{n-k}+\tfrac{a x_n x_{n-k}}{b x_n+c x_{n-l}+d x_{n-k}}, \\ &\leqslant \alpha x_n+\beta x_{n-l}+\gamma x_{n-k}+\tfrac{a x_n x_{n-k}}{d x_{n-k}} \\ &= \left(\alpha+\tfrac{a}{d}\right)x_n+\beta x_{n-l}+\gamma x_{n-k} \\ &< x_n+x_{n-l}+x_{n-k}. \end{aligned}$$

Then

$$x_{n+1}<x_n+x_{n-l}+x_{n-k} \quad \text{for all } n\geq 0.$$

So every solution of Eq. (1) is bounded from above by $M=x_0+x_{-l}+x_{-k}$.

Second we prove every solution of Equation (1) is bounded if $\alpha<1,\ \beta<1$ and $\gamma+\frac{a}{b}<1$. Let $\{x_n\}_{n=-s}^{\infty}$ be a solution of Equation (1). It follows from Equation (1) that

$$\begin{aligned} x_{n+1} &= \alpha x_n+\beta x_{n-l}+\gamma x_{n-k}+\tfrac{a x_n x_{n-k}}{b x_n+c x_{n-l}+d x_{n-k}}, \\ &\leqslant \alpha x_n+\beta x_{n-l}+\gamma x_{n-k}+\tfrac{a x_n x_{n-k}}{b x_n} \\ &= \alpha x_n+\beta x_{n-l}+\left(\gamma+\tfrac{a}{b}\right)x_{n-k} \\ &< x_n+x_{n-l}+x_{n-k}. \end{aligned}$$

Then

$$x_{n+1}<x_n+x_{n-l}+x_{n-k} \quad \text{for all } n\geq 0.$$

So every solution of Eq. (1) is bounded from above by $M=x_0+x_{-l}+x_{-k}$.

THEOREM 4.2. *Every solution of Equation (1) is unbounded if $\alpha>1$or $\beta>1$ or $\gamma>1$.*

Proof: Let $\{x_n\}_{n=-s}^{\infty}$ be a solution of Equation (1).Then from Equation (1) we see that

$$x_{n+1} = \alpha x_n + \beta x_{n-l} + \gamma x_{n-k} + \frac{a x_n x_{n-k}}{b x_n + c x_{n-l} + d x_{n-k}} > \alpha x_n \quad \text{for all} \quad n \geq 0.$$

We see that the right hand side can be written as follows

$$z_{n+1} = \alpha z_{n-l}.$$

then

$$z_{ln+i} = \alpha^n z_{l+i} + const., \qquad i = 0, 1, ..., l,$$

and this equation is unstable because $\alpha > 1$, and $\lim_{n\to\infty} z_n = \infty$.Then by using ratio test $\{x_n\}_{n=-s}^{\infty}$ is unbounded from above.

Similarly we can prof that every solution of Eq. (1) is unbounded if $\beta > 1$ or $\gamma > 1$. Thus, the proof is now completed.

Example 4. We assume $l = 2$, $k = 3$, $\alpha = 1.3$, $\beta = 0.2$, $\gamma = 0.1$, $a = 0.1$, $b = 0.2$, $c = 0.3$ and $d = 0.7$ and the initial conditions $x_{-3} = 0.2$, $x_{-2} = 0.4$, $x_{-1} = 0.6$ and $x_0 = 0.1$, the solution of the difference equation (1) is unbounded (see Fig. 4).

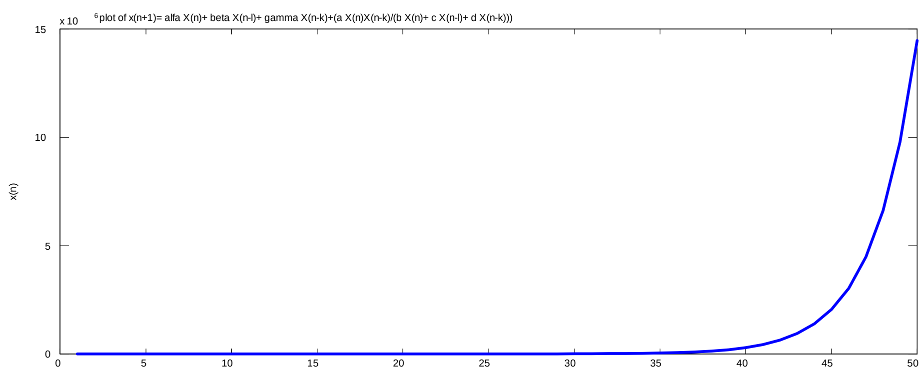


Figure 4. Plot the behavior of the solution of equation (1) is unbounded.

5. EXISTENCE OF PERIODIC SOLUTIONS

THEOREM 5.1. *Suppose that l and k are even positive integers, then equation (1) has no prime period two solutions.*

Proof: First suppose that there exists a prime period two solution

$$...P,\ Q,\ P,\ Q,\ ...,$$

of equation (1). We see from equation (1) when l and k are an even, then $x_n = x_{n-l} = x_{n-k}$. It follows equation (1) that

$$P = \alpha Q + \beta Q + \gamma Q + \frac{aQ^2}{bQ + cQ + dQ},$$

and

$$Q = \alpha P + \beta P + \gamma P + \frac{aP^2}{bP + cP + dP}.$$

Therefore,

$$(b + c + d)\, P = (b + c + d)\, (\alpha + \beta + \gamma) Q + aQ, \tag{8}$$

$$(b + c + d)\, Q = (b + c + d)\, (\alpha + \beta + \gamma) P + aP, \tag{9}$$

Subtracting (9) from (8) gives

$$(b+c+d)(P-Q) = ((b+c+d)(\alpha+\beta+\gamma)+a)(Q-P)$$

$$(P-Q)[(b+c+d)(1+\alpha+\beta+\gamma)+a] = 0$$

Since $(b+c+d)(1+\alpha+\beta+\gamma)+a \neq 0$, then $p=q$. This is a contradiction. Thus, the proof is completed.

THEOREM 5.2. *Let l is even and k is odd positive integers, then equation (1) has no positive prime period two solutions.*

Proof: First suppose that there exists a prime period two solution

$$...P,\ Q,\ P,\ Q, ...,$$

of equation (1). We see from equation (1) when l is an even and k is an odd, then $x_n = x_{n-l}$ and $x_{n+1} = x_{n-k}$. It follows equation (1) that

$$P = \alpha Q + \beta Q + \gamma P + \tfrac{aQP}{bQ+cQ+dP},$$

and

$$Q = \alpha P + \beta P + \gamma Q + \tfrac{aPQ}{bP+cP+dQ}.$$

Therefore,

$$(b+c)(1-\gamma)PQ + dP^2 = (b+c)(\alpha+\beta)Q^2 + d(\alpha+\beta)PQ + aQP, \tag{10}$$

$$(b+c)(1-\gamma)PQ + dQ^2 = (b+c)(\alpha+\beta)P^2 + d(\alpha+\beta)PQ + aPQ, \tag{11}$$

Subtracting (11) from (10) gives

$$d(P^2-Q^2) = (b+c)(\alpha+\beta)(Q^2-P^2)$$

$$(P^2-Q^2)(d+(b+c)(\alpha+\beta)) = 0$$

Then $P = \pm Q$. This is a contradiction. Thus, the proof is completed.

THEOREM 5.3. *Suppose that l is odd and k is even positive integers, then equation (1) has no positive prime period two solutions.*

Proof: First suppose that there exists a prime period two solution

$$...P,\ Q,\ P,\ Q, ...,$$

of equation (1). We see from equation (1) when k is an even and l is an odd, then $x_n = x_{n-k}$ and $x_{n+1} = x_{n-l}$. It follows equation (1) that

$$P = \alpha Q + \beta P + \gamma Q + \tfrac{aQ^2}{bQ+cP+dQ},$$

and

$$Q = \alpha P + \beta Q + \gamma P + \tfrac{aP^2}{bP+cQ+dP}.$$

Therefore,

$$(b+d)(1-\beta)PQ + cP^2 = (b+d)(\alpha+\gamma)Q^2 + c(\alpha+\gamma)PQ + aQ^2, \tag{12}$$

$$(b+d)(1-\beta)PQ + cQ^2 = (b+d)(\alpha+\gamma)P^2 + c(\alpha+\gamma)PQ + aP^2, \tag{13}$$

Subtracting (13) from (12) gives

$$c(P^2-Q^2) = ((b+d)(\alpha+\gamma)+a)(Q^2-P^2)$$

$$\left(P^2-Q^2\right)\left[c\left((b+d)\left(\alpha+\gamma\right)+a\right)\right]=0$$

Then $P=\pm Q$. This is a contradiction. Thus, the proof is completed.

Theorem 5.4. *Let l, k are odd positive integers. If*

$$(1+\alpha-\beta-\gamma)\left(b+c+d\right)-a\neq 0,$$

then Eq. (1) has no prime period two solution.

Proof: First suppose that there exists a prime period two solution

$$...P,\ Q,\ P,\ Q,\ ...,$$

of equation (1). We see from equation (1) when l and k are an odd, then $x_{n+1}=x_{n-l}=x_{n-k}$. It follows equation (1) that

$$P=\alpha Q+\beta P+\gamma P+\tfrac{aP^2}{bP+cP+dP},$$

and

$$Q=\alpha P+\beta Q+\gamma Q+\tfrac{aQ^2}{bQ+cQ+dQ}.$$

Therefore,

$$(1-\beta-\gamma)\left(b+c+d\right)P=\alpha\left(b+c+d\right)Q+aP, \tag{14}$$

$$(1-\beta-\gamma)\left(b+c+d\right)Q=\alpha\left(b+c+d\right)P+aQ, \tag{15}$$

Subtracting (15) from (14) gives

$$(1-\beta-\gamma)\left(b+c+d\right)\left(P-Q\right)=\alpha\left(b+c+d\right)\left(Q-P\right)+a(P-Q)$$

$$\left(P-Q\right)\left[(1+\alpha-\beta-\gamma)\left(b+c+d\right)-a\right]=0$$

Since $(1+\alpha-\beta-\gamma)\left(b+c+d\right)-a\neq 0$, then $p=q$. This is a contradiction. Thus, the proof is completed.

Acknowledgements

This article was funded by the Deanship of Scientific Research (DSR), King Abdulaziz University, Jeddah. The authors, therefore, acknowledge with thanks DSR technical and financial support.

REFERENCES

1. V. L. Kocic, and G. Ladas, Global Behavior of Nonlinear Difference Equations of Higher Order with Applications, Kluwer Academic Publishers, Dordrecht, 1993.
2. M. R. S. Kulenovic and G. Ladas, Dynamics of Second Order Rational Difference Equations with Open Problems and Conjectures, Chapman & Hall / CRC Press, 2001.
3. E. A. Grove and G. Ladas, Periodicities in nonlinear difference equations, Vol. 4, Chapman & Hall / CRC Press, 2005.
4. I. Yalcinkaya, On The Global Attractivity of Positive Solutions of A Rational Difference Equation, Selçuk J. Appl. Math., 9(2), (2008), 3-8.
5. E. M. E. Zayed, M. A. El-Moneam, On the Rational Recursive Sequence $x_{n+1}=\frac{\alpha x_n+\beta x_{n-1}+\gamma x_{n-2}+\delta x_{n-3}}{Ax_n+Bx_{n-1}+Cx_{n-2}+Dx_{n-3}}$, J. Appl. Math. Comput., 22, (2006), 247-262.
6. M. m. El-Dessoky, Qualitative behavior of rational difference equation of big Order, Discrete Dyn. Nat. Soc., 2013, (2013), Article ID 495838, 6 pages.
7. Guo-Mei Tang, Lin-Xia Hu, and Xiu-Mei Jia, Dynamics of a Higher-Order Nonlinear Difference Equation, Discrete Dyn. Nat. Soc., 2010, (2010), Article ID 534947, 15 pages.

8. G. Papaschinopoulos, C. J. Schinas, Stefanidou, G., On the nonautonomous difference equation $x_{n+1} = A_n + \frac{x_{n-1}^p}{x_n^q}$, Appl. Math. Comput., 217(12), (2011), 5573-5580.
9. M. M. El-Dessoky, Dynamics and Behavior of the Higher Order Rational Difference equation, J. Comput. Anal. Appl., 21(4), (2016), 743-760.
10. S. Nirmaladevi, and N. Karthikeyan, Dynamics and Behavior of Higher Order Nonlinear Rational Difference Equation, International Journal Of Advance Research And Innovative Ideas In Education, 3 (4) (2017), 2395-4396.
11. Y. Yazlik, D. T. Tollu, N. Taskara, On the Behaviour of Solutions for Some Systems of Difference Equations, J. Comp. Anal. Appl., 18(1), (2015), 166-178.
12. M. A. El-Moneam, S. O. Alamoudy, On Study of the Asymptotic Behavior of Some Rational Difference Equations, Dyn. Contin. Discrete Impuls. Syst. Ser. A Math. Anal., 21, (2014), 89-109.
13. R. Abu-Saris, C. Cinar, I. Yalçınkaya, On the asymptotic stability of $x_{n+1} = \frac{a+x_n x_{n-k}}{x_n+x_{n-k}}$, Comput. Math. Appl., 56, (2008), 1172–1175.
14. C. J. Schinas, G. Papaschinopoulos, G. Stefanidou, On the Recursive Sequence $x_{n+1} = A + \frac{x_{n-1}^p}{x_n^q}$, Adv. Differ. Equ., 2009, (2009), Article ID 327649, 11 page.
15. Mehmet Gűműş, The Periodicity of Positive Solutions of the Nonlinear Difference Equation $x_{n+1} = \alpha + \frac{x_{n-k}^p}{x_n^p}$, Disc. Dyn. Nat. Soc., 2013, (2013), Article ID 742912, 3 pages.
16. M. T. Aboutaleb, M. A. El-Sayed, A. E. Hamza, Stability of the Recursive Sequence $x_{n+1} = \frac{\alpha-\beta x_n}{\gamma+x_{n-1}}$, J. Math. Anal. Appl., 261(1), (2001), 126–133.
17. Mehmet Gűműş and Őzkan Őcalan, Some Notes on the Difference Equation $x_{n+1} = \alpha + \frac{x_{n-1}}{x_n^k}$, Disc. Dyn. Nat. Soc., 2012, (2012), Article ID 258502, 12 pages.
18. A. Brett, E. J. Janowski, M. R. S. Kulenović, Global Asymptotic Stability for Linear Fractional Difference Equation, Journal of Difference Equations, 2014, (2014), Article ID 275312, 11 pages.
19. İlhan Öztürk, Saime Zengin, On the difference equation $y_{n+1} = \frac{\alpha y_n^p}{\beta y_{n-1}^p} - \frac{\gamma y_{n-1}^p}{\beta y_n^p}$, Mathematica Slovaca, 61(6), (2011), 921-932.
20. E. M. Elsayed, M. M. El-Dessoky, Dynamics and global behavior for a fourth-order rational difference equation, Hacettepe J. Math. and Stat., 42(5), (2013), 479–494.
21. R. Abo-Zeid, Global behavior of a higher order difference equation, Mathematica Slovaca, 64(4), (2014), 931-940.
22. E. M. Elsayed, M. M. El-Dessoky, Asim Asiri, Dynamics and Behavior of a Second Order Rational Difference equation, J. Comput. Anal. Appl., 16(4), (2014), 794-807.
23. E. M. Elsayed, M. M. El-Dessoky, E. O. Alzahrani, The Form of The Solution and Dynamic of a Rational Recursive Sequence, J. Comput. Anal. Appl., 17(1), (2014), 172-186.
24. I. Yalcinkaya, A. E. Hamza, C. Cinar, Global Behavior of a Recursive Sequence, Selçuk J. Appl. Math., 14(1), (2013), 3-10.
25. M. A. El-Moneam, On the Dynamics of the Higher Order Nonlinear Rational Difference Equation, Math. Sci. Lett. 3(2), (2014), 121-129.
26. M. M. El-Dessoky and M. A. El-Moneam, On the Higher Order Difference equation $x_{n+1} = Ax_n + Bx_{n-l} + Cx_{n-k} + \frac{\gamma x_{n-k}}{Dx_{n-s}+Ex_{n-t}}$, J. Comput. Anal. Appl., 25(2), (2018), 342-354.
27. M. M. El-Dessoky, On the dynamics of a higher Order rational difference equations, J. Egypt. Math. Soc. 25(1), (2017), 28-36.
28. M. M. El-Dessoky and Aatef Hobiny, Dynamics of a Higher Order Difference Equations $x_{n+1} = \alpha x_n + \beta x_{n-l} + \gamma x_{n-k} + \frac{ax_{n-l}+bx_{n-k}}{cx_{n-l}+dx_{n-k}}$, J. Comput. Anal. Appl., 24(7), (2018), 1353-1365.
29. M. M. El-Dessoky and Aatef Hobiny, On the Difference equation $x_{n+1} = ax_n + bx_{n-1} + \frac{\alpha+cx_{n-2}}{\beta+dx_{n-2}}$, J. Comput. Anal. Appl., 24(4), (2018), 644-655.
30. M. M. El-Dessoky, On the Difference equation $x_{n+1} = ax_{n-l} + bx_{n-k} + \frac{cx_{n-s}}{dx_{n-s}-e}$, Math. Meth. Appl. Sci., 40(3), (2017), 535–545.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 27, NO.6, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC

BEST PROXIMITY POINT OF CONTRACTION TYPE MAPPING IN METRIC SPACE

Kyung Soo Kim

Graduate School of Education, Mathematics Education
Kyungnam University, Changwon, Gyeongnam, 51767, Republic of Korea
e-mail: kksmj@kyungnam.ac.kr

Abstract. The purpose of this article, we consider the existence of a unique best proximity point $x^* \in A$ such that $d(x^*, Tx^*) = dist(A, B)$ for generalized φ-weak contraction mapping $T : A \to B$, where $A, B(\neq \emptyset)$ are subsets of a metric space (X, d).

1. Introduction

Let (X, d) be a metric space. A mapping $T : X \to X$ is a *contraction* if there exists a constant $\alpha \in (0, 1)$ such that

$$d(Tx, Ty) \leq \alpha \cdot d(x, y), \quad \forall\, x, y \in X.$$

A mapping $T : X \to X$ is a *φ-weak contraction* if there exists a continuous and nondecreasing function $\varphi : [0, \infty) \to [0, \infty)$ with $\varphi^{-1}(0) = \{0\}$ and $\lim_{t\to\infty} \varphi(t) = \infty$ such that

$$d(Tx, Ty) \leq d(x, y) - \varphi(d(x, y)), \quad \forall\, x, y \in X. \tag{1.1}$$

If X is bounded, then the infinity condition can be omitted.

The concept of the φ-weak contraction was introduced by Alber and Guerre-Delabriere [1] in 1997, who proved the existence of fixed points in Hilbert spaces. Later Rhoades [14] in 2001, who extended the results of [1] to metric spaces.

Theorem 1.1. ([14]) *Let (X, d) be a complete metric space, $T : X \to X$ be a φ-weak contractive self-map on X. The T has a unique fixed point p in X.*

Remark 1.2. Theorem 1.1 is one of generalizations of the Banach contraction principle because it takes $\varphi(t) = (1 - \alpha)t$ for $\alpha \in (0, 1)$, then φ-weak contraction contains contraction as special cases.

Next, we present a brief discussion about best proximity point which is a interesting topic in best proximity theory.

[0]2010 Mathematics Subject Classification: 54H25, 47H09, 47H10, 41A65.
[0]Keywords: Optimal solution, best proximity point, P-property, generalized φ-weak contraction mapping, fixed point, metric space.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 27, NO.6, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC

Let (X, d) be a metric space and $A(\neq \emptyset)$ be a subset of (X, d). Consider a mapping $T : A \to X$. The solutions to the fixed point equation $Tx = x$ are called *fixed points* of the mapping T. It is clear that $T(A) \cap A \neq \emptyset$ is a necessary (but not sufficient) condition for the existence of a fixed point for the mapping $T : A \to X$. If the necessary condition fails, then

$$d(x, Tx) > 0,$$

for all $x \in A$. This means that the mapping $T : A \to X$ does not have any fixed point, *i.e.*, $Tx = x$ has no solution. This point of view, it give us to think of a point $x \in A$ which is closest to Tx in some sense. Best approximation theory and best proximity point theory are relevant in this perspective. One of the most interesting best approximation theorem is due to Fan [3].

Theorem 1.3. ([3]) *Let $C(\neq \emptyset)$ be a compact convex subset of a normed linear space V and $F : C \to V$ be a continuous function. Then there exists a point $p \in C$ such that $\|p - Fp\| = d(Fp, C) = \inf\{\|Fp - c\| : c \in C\}$.*

Such an element $p \in C$ in Theorem 1.3 is called a *best approximant point* of T in C.

Although a best approximation point acts as an approximate solution of the equation $Fp = p$, the value $\|p - Fp\|$ need not be the optimum, *i.e.*, a best approximant point is not an optimal solution in the sense that

$$\min_{p \in A} \|p - Fp\|.$$

Naturally, let us consider nonempty subsets A, B of a metric space (X, d) and a mapping $T : A \to B$. Then one can think of finding an element $x^* \in A$ such that

$$d(x^*, Tx^*) = \min\{d(x, Tx) : x \in A\}.$$

Since

$$d(x, Tx) \geq dist(A, B) = \inf\{d(a, b) : a \in A, b \in B\}$$

for all $x \in A$, the optimal solution of $\min_{x \in A} d(x, Tx)$ is one for which the value $dist(A, B)$ is attained. A point $x^* \in A$ is said to be a *best proximity point* of $T : A \to B$ if

$$d(x^*, Tx^*) = dist(A, B).$$

So a best proximity point of the mapping T is an approximate solution of the equation $Tx = x$ which is optimal solution in the sense that

$$\min_{x \in A} d(x, Tx).$$

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 27, NO.6, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC

Remark 1.4. It is trivial that all best proximity point theorems work as a natural generalization of fixed point theorems if the mapping T is a self-mapping.

Recently Sultana and Vetrivel [15] obtained the following best proximity point theorem for mapping satisfies (1.1).

Theorem 1.5. ([15], Theorem 3.4) *Let $A, B(\neq \emptyset)$ be two closed subsets of a complete metric space (X, d) such that the pair (A, B) has the P-property and $A_0 \neq \emptyset$ and $T : A \to B$ be a mapping such that $T(A_0) \subseteq B_0$ and it satisfies* (1.1). *Then there exists a unique $p \in A$ such that $d(p, Tp) = dist(A, B)$.*

In 2016, Xue [16] introduced a new contraction type mapping as follows.

Definition 1.6. ([16]) A mapping $T : X \to X$ is a *generalized φ-weak contraction* if there exists a continuous and nondecreasing function $\varphi : [0, \infty) \to [0, \infty)$ with $\varphi(0) = 0$ such that

$$d(Tx, Ty) \leq d(x, y) - \varphi(d(Tx, Ty)), \quad \forall\, x, y \in X \tag{1.2}$$

holds.

We notice immediately that if $T : X \to X$ is φ-weak contraction, then T satisfies the following inequality

$$d(Tx, Ty) \leq d(x, y) - \varphi(d(Tx, Ty)), \quad \forall\, x, y \in X.$$

However, the converse is not true in general.

Example 1.7. Let $X = (-\infty, +\infty)$ be endowed with the Euclidean metric $d(x, y) = |x-y|$ and let $Tx = \frac{2}{5}x$ for each $x \in X$. Define $\varphi : [0, +\infty) \to [0, +\infty)$ by $\varphi(t) = \frac{4}{3}t$. Then T satisfies (1.2), but T does not satisfy inequality (1.1). Indeed,

$$\begin{aligned} d(Tx, Ty) &= \left|\frac{2}{5}x - \frac{2}{5}y\right| \\ &\leq |x - y| - \frac{4}{3} \cdot \frac{2}{5}|x - y| \\ &= d(x, y) - \varphi(d(Tx, Ty)) \end{aligned}$$

and

$$\begin{aligned} d(Tx, Ty) &= \left|\frac{2}{5}x - \frac{2}{5}y\right| \\ &\geq |x - y| - \frac{4}{3}|x - y| \\ &= d(x, y) - \varphi(d(x, y)) \end{aligned}$$

for all $x, y \in X$.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 27, NO.6, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC

Example 1.8. ([16]) Let $X = (-1, +\infty)$ be endowed by $d(x, y) = |x - y|$ and let $Tx = \frac{x}{1+x}$ for each $x \in X$. Define $\varphi : [0, +\infty) \to [0, +\infty)$ by $\varphi(t) = \frac{t^2}{1+t}$. Then

$$\begin{aligned} d(Tx, Ty) &= \left| \frac{x}{1+x} - \frac{y}{1+y} \right| = \frac{|x-y|}{(1+x)(1+y)} \\ &\leq \frac{|x-y|}{1+|x-y|} = |x-y| - \frac{|x-y|^2}{1+|x-y|} \\ &= d(x, y) - \varphi(d(x, y)) \end{aligned}$$

holds for all $x, y \in X$. So T is a φ-weak contraction. However T is not a contraction.

Remark 1.9. The above examples show that the class of generalized φ-weak contractions properly includes the class of φ-weak contractions and the class of φ-weak contractions properly includes the class of contractions.

In fact, let $T : X \to X$ be a contraction, there exists $\alpha \in (0, 1)$ such that

$$d(Tx, Ty) \leq \alpha \cdot d(x, y), \quad \forall\, x, y \in X.$$

Then

$$\begin{aligned} d(Tx, Ty) &\leq \alpha \cdot d(x, y) = d(x, y) - (1 - \alpha) d(x, y) \\ &= d(x, y) - \varphi(d(x, y)), \end{aligned}$$

where, $\varphi(d(x, y)) = (1 - \alpha)d(x, y)$. So, T is a φ-weak contraction. Moreover, let T be a φ-weak contraction, from property of φ, we have $d(Tx, Ty) \leq d(x, y)$ and

$$\varphi(d(Tx, Ty)) \leq \varphi(d(x, y)).$$

From (1.1),

$$\begin{aligned} d(Tx, Ty) &\leq d(x, y) - \varphi(d(x, y)) \\ &\leq d(x, y) - \varphi(d(Tx, Ty)), \quad \forall\, x, y \in X. \end{aligned}$$

Therefore, T is a generalized φ-weak contraction.

In the meantime, if T is a φ-weak contractive self mapping for one mapping φ so we do not expect that the φ-weak contractivity should be satisfied with the same function φ. Let us suppose that T is a φ-weak contractive self mapping and consider

$$\tilde{\varphi}(x) = \min \{\varphi(x/2); x/2\}.$$

Then, if $d(Tx, Ty) > \frac{1}{2} d(x, y)$, we have

$$d(Tx, Ty) \leq d(x, y) - \varphi(d(Tx, Ty)) \leq d(x, y) - \varphi\left(\frac{1}{2} d(x, y)\right)$$

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 27, NO.6, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC

on account of monotonocity of φ and finally

$$d(Tx,Ty) \le d(x,y) - \tilde{\varphi}(d(x,y)).$$

On the other hand, if $d(Tx,Ty) < \frac{1}{2}d(x,y)$, we get

$$d(Tx,Ty) < d(x,y) - \frac{1}{2}d(x,y) \le d(x,y) - \tilde{\varphi}(d(x,y)).$$

So T is just the $\tilde{\varphi}$-weak contractive mapping. The continuity and monotonocity of $\tilde{\varphi}$ follows directly from properties of min function, φ and the metric.

For related results, please see [9], [10], [11] and the references therein ([5], [6], [7], [8]).

The purpose of this article, we consider the existence of a unique best proximity point $x^* \in A$ such that $d(x^*, Tx^*) = dist(A,B)$ for generalized φ-weak contraction mapping $T : A \to B$, where $A, B(\neq \emptyset)$ are subsets of a metric space (X,d).

2. Preliminaries

Let A, B be two nonempty subsets of a metric space (X,d). Let us define the following notation which will be need throughout this article:

$$A_0 = \{x \in A : d(x,y) = dist(A,B) \quad \text{for some} \quad y \in B\},$$
$$B_0 = \{y \in B : d(x,y) = dist(A,B) \quad \text{for some} \quad x \in A\}.$$

In [12], the authors discussed sufficient conditions which guarantee the nonemptiness of A_0 and B_0. Also, in [2], the authors proved that A_0 is contained in the boundary of A.

Let us define the notion of nonself generalized φ-weak contraction mapping as follows.

Definition 2.1. Let A, B be two nonempty subsets of a metric space (X,d). A mapping $T : A \to B$ is said to be a *generalized φ-weak contraction* if

$$d(Tx,Ty) \le d(x,y) - \varphi(d(Tx,Ty)), \quad \forall\, x, y \in A, \tag{2.1}$$

where $\varphi : [0,\infty) \to [0,\infty)$ is a continuous and nondecreasing function such that φ is positive on $(0,\infty)$, $\varphi^{-1}(0) = \{0\}$ and $\lim_{t\to\infty} \varphi(t) = \infty$. If A is bounded, then the infinity condition can be omitted.

The notion called the P-property was introduced in [13].

Definition 2.2. ([13]) Let (A,B) be a pair of nonempty subsets of a metric space (X,d) with $A_0 \neq \emptyset$. Then the pair (A,B) is said to be *has the P-property*

if and only if for any $x_1, x_2 \in A_0$ and $y_1, y_2 \in B_0$,

$$d(x_1, y_1) = dist(A, B) = d(x_2, y_2) \quad \Rightarrow \quad d(x_1, x_2) = d(y_1, y_2).$$

Now we recall the following results from [4] and [15].

Lemma 2.3. ([15]) *Let $\varphi : [0, \infty) \to [0, \infty)$ be a function such that $\varphi^{-1}(0) = \{0\}$ and φ is either nondecreasing or continuous. Then*

$$\varphi(\mu_n) \to 0 \quad \Rightarrow \quad \mu_n \to 0$$

for any bounded sequence $\{\mu_n\}$ of positive reals.

Lemma 2.4. ([4]) *For a given subset D of $\{(x, y) \in \mathbb{R}^2 : x, y \geq 0\}$, the following statements are equivalent:*

(i) *for any $\varepsilon > 0$, there exist $\delta > 0$ and $\gamma \in (0, \varepsilon)$ such that*

$$u < \varepsilon + \delta \quad \Rightarrow \quad v \leq \gamma$$

for all $(u, v) \in D$,

(ii) *there exist a continuous and nondecreasing function $\phi : [0, \infty) \to [0, \infty)$ such that*

$$\phi(u) < u, \ \forall\, u > 0 \quad \text{and} \quad v \leq \phi(u), \ \forall\, (u, v) \in D.$$

3. Main results

Lemma 3.1. *Let A and B be two nonempty subsets of a metric space (X, d) and $\varphi : [0, \infty) \to [0, \infty)$ be a function such that $\varphi^{-1}(0) = \{0\}$ and*

$$\varphi(t_n) \to 0 \quad \Rightarrow \quad t_n \to 0 \tag{3.1}$$

for any bounded sequence $\{t_n\}$ of positive reals. Let $T : A \to B$ be a genearlized φ-weak contraction mapping satisfying (2.1). Then, for any $\varepsilon > 0$, there exist $\delta > 0$ and $\gamma \in (0, \varepsilon)$ such that

$$d(x, y) < \varepsilon + \delta \quad \Rightarrow \quad d(Tx, Ty) \leq \gamma$$

for all $x, y \in A$.

Proof. Suppose that there exists an $\varepsilon_0 > 0$ such that for any $\delta > 0$, $\gamma \in (0, \varepsilon_0)$ and there exist $x, y \in A$ such that

$$d(x, y) < \varepsilon_0 + \delta \quad \Rightarrow \quad d(Tx, Ty) > \gamma.$$

Let

$$\delta_n = \frac{1}{n^2} \quad \text{and} \quad \gamma_n = \frac{n^2}{1 + n^2}\varepsilon_0, \ \forall\, n \in \mathbb{N}.$$

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 27, NO.6, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC

Then there exist $\{x_n\}$ and $\{y_n\}$ in A such that

$$d(x_n, y_n) < \varepsilon_0 + \frac{1}{n^2} \quad \Rightarrow \quad d(Tx_n, Ty_n) > \frac{n^2}{1+n^2}\varepsilon_0. \tag{3.2}$$

From (2.1), we have

$$\begin{aligned} \frac{n^2}{1+n^2}\varepsilon_0 &< d(Tx_n, Ty_n) \\ &\leq d(x_n, y_n) - \varphi(d(Tx_n, Ty_n)) \\ &< \varepsilon_0 + \frac{1}{n^2} - \varphi(d(Tx_n, Ty_n)). \end{aligned}$$

That is

$$\varphi(d(Tx_n, Ty_n)) < \varepsilon_0 + \frac{1}{n^2} - \frac{n^2}{1+n^2}\varepsilon_0 = \frac{1}{n^2} + \frac{\varepsilon_0}{1+n^2}.$$

Hence

$$\varphi(d(Tx_n, Ty_n)) \to 0 \quad \text{as} \quad n \to \infty.$$

Since $d(Tx_n, Ty_n) \leq d(x_n, y_n)$ and $\{d(x_n, y_n)\}$ is bounded, we get $\{d(Tx_n, Ty_n)\}$ is bounded. By the given hypothesis (3.1),

$$d(Tx_n, Ty_n) \to 0 \quad \text{as} \quad n \to \infty.$$

On the other hand, from (3.2),

$$\lim_{n\to\infty} d(Tx_n, Ty_n) \geq \varepsilon_0 > 0.$$

This is a contradiction. Thus Lemma 3.1 holds. □

The following theorem is main result which gives sufficient conditions for the existence of a unique best proximity point for generalized φ-weak contraction mapping.

Theorem 3.2. *Let (A, B) be a pair of two nonempty closed subsets of a complete metric space (X, d) such that $A_0 \neq \emptyset$. Let $T : A \to B$ be a generalized φ-weak contraction mapping such that $T(A_0) \subseteq B_0$. Assume that the pair (A, B) has the P-property. Then there exists a unique x^* in A such that $d(x^*, Tx^*) = dist(A, B)$.*

Proof. Let $x_0 \in A_0$. Since $Tx_0 \in T(A_0) \subseteq B_0$, there exists $x_1 \in A_0$ such that

$$d(x_1, Tx_0) = dist(A, B).$$

Again, since $Tx_1 \in T(A_0) \subseteq B_0$, there exists $x_2 \in A_0$ such that

$$d(x_2, Tx_1) = dist(A, B).$$

Continuing this process, we can find a sequence $\{x_n\}$ in A_0 such that

$$d(x_{n+1}, Tx_n) = dist(A, B), \quad \forall\, n \in \mathbb{N}. \tag{3.3}$$

Since (A, B) has the P-property, from (3.3), we obtain

$$d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n), \quad \forall\, n \in \mathbb{N}. \tag{3.4}$$

By the definition of T and (3.4), we have

$$\begin{aligned} d(x_n, x_{n+1}) &= d(Tx_{n-1}, Tx_n) \\ &\leq d(x_{n-1}, x_n) - \varphi(d(Tx_{n-1}, Tx_n)) \\ &\leq d(x_{n-1}, x_n), \quad \forall\, n \in \mathbb{N}. \end{aligned}$$

Therefore, the sequence $\{d(x_n, x_{n+1})\}$ is monotone nonincresing and bounded. Hence it converges. If we set $\lambda_n = d(x_n, x_{n+1})$ and L be the limit of λ_n, *i.e.*,

$$\lim_{n\to\infty} \lambda_n = \lim_{n\to\infty} d(x_n, x_{n+1}) = L \geq 0.$$

Now, we claim that $L = 0$. Suppose to the contrary that $L > 0$. Since $\{\lambda_n\}$ is nonincreasing sequence, *i.e.*,

$$\lambda_n \geq \lambda_{n+1} \geq \cdots \geq L > 0, \quad \forall\, n \in \mathbb{N}$$

and φ is nondecreasing, we obtain

$$\varphi(\lambda_n) \geq \varphi(L) > 0, \quad \forall\, n \in \mathbb{N}. \tag{3.5}$$

From the inequality

$$\begin{aligned} \lambda_n = d(x_n, x_{n+1}) &\leq d(x_{n-1}, x) - \varphi(d(Tx_{n-1}, Tx_n)) \\ &= \lambda_{n-1} - \varphi(d(Tx_{n-1}, Tx_n)), \end{aligned}$$

(3.4) and (3.5), we have

$$\begin{aligned} \lambda_n &\leq \lambda_{n-1} - \varphi(d(x_n, x_{n+1})) = \lambda_{n-1} - \varphi(\lambda_n) \\ &\leq \lambda_{n-1} - \varphi(L), \quad \forall\, n \in \mathbb{N}. \end{aligned}$$

Since φ is continuous, we get $L \leq L - \varphi(L)$. That is

$$\varphi(L) \leq 0$$

which contradicts condition of φ. Hence

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = L = 0.$$

Now we apply Lemma 2.3 and Lemma 2.4 to the set $D = \{(d(x, y), d(Tx, Ty)) : x, y \in A\}$ on Lemma 3.1, one knows that there exists a function $\phi : [0, \infty) \to [0, \infty)$ such that ϕ is continuous and nondecreasing with

$$\phi(t) < t, \;\; \forall\, t > 0 \quad \text{and} \quad d(Tx, Ty) \leq \phi(d(x, y)), \;\; \forall\, x, y \in A. \tag{3.6}$$

Thus for a given $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$d(x_n, x_{n+1}) \leq \varepsilon - \phi(\varepsilon), \quad \forall\, n \geq N. \tag{3.7}$$

Next we show that $\{x_n\}$ is a Cauchy sequence.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 27, NO.6, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC

Denotes the closed ball with center x and radius ε by $B[x,\varepsilon]$, we will claim the following relations.

Claim I. $T(B[x_N,\varepsilon]\cap A)\subseteq B[Tx_{N-1},\varepsilon]$.

Let $x\in B[x_N,\varepsilon]\cap A$, *i.e.*, $d(x_N,x)\leq\varepsilon$, from (3.4), (3.6) and (3.7), then

$$\begin{aligned} d(Tx,Tx_{N-1}) &\leq d(Tx,Tx_N)+d(Tx_N,Tx_{N-1}) \\ &\leq \phi(d(x,x_N))+d(x_{N+1},x_N) \\ &\leq \phi(\varepsilon)+\{\varepsilon-\phi(\varepsilon)\}=\varepsilon, \end{aligned}$$

which implies that $Tx\in B[Tx_{N-1},\varepsilon]$.

Claim II. $y\in B[Tx_{N-1},\varepsilon]$ with $d(x,y)=dist(A,B)$ for some $x\in A_0$ implies $x\in B[x_N,\varepsilon]\cap A$.

Let $y\in B[Tx_{N-1},\varepsilon]$ with $d(x,y)=dist(A,B)$ for some $x\in A_0$. From (3.3), we have $d(x_N,Tx_{N-1})=dist(A,B)$. Therefore, by using the P-property of (A,B), we obtain that

$$d(x_N,x)=d(Tx_{N-1},y)\leq\varepsilon.$$

Hence Claim II holds.

From (3.7), it is clear that

$$x_{N+1}\in B[x_N,\varepsilon]\cap A.$$

And by Claim I, we have $Tx_{N+1}\in B[Tx_{N-1},\varepsilon]$. From (3.3), $d(x_{N+2},Tx_{N+1})=dist(A,B)$ with $x_{N+2}\in A_0$. From Claim II,

$$x_{N+2}\in B[x_N,\varepsilon]\cap A.$$

Again by Claim I, $Tx_{N+2}\in B[Tx_{N-1},\varepsilon]$ and by (3.3), $d(x_{N+3},Tx_{N+2})=dist(A,B)$ with $x_{N+3}\in A_0$. Again by Claim II,

$$x_{N+3}\in B[x_N,\varepsilon]\cap A.$$

Continuing this process, we can conclude that

$$x_{N+m}\in B[x_N,\varepsilon]\cap A,\quad \forall\, m\in\mathbb{N},$$

i.e., $d(x_N,x_{N+m})\leq\varepsilon$. Hence the sequence $\{x_n\}$ is a Cauchy sequence. Since A is closed subset of the complete metric space (X,d), there exists an element $x^*\in A$ such that $\lim_{n\to\infty}x_n=x^*$. By the definition of T, we have $d(Tx,Ty)\leq d(x,y)$ for all $x,y\in A$ which implies that T is continuous in A. Therefore we obtain

$$\lim_{n\to\infty}Tx_n=Tx^*.$$

Also, from the continuity of the distance function d, we have

$$\lim_{n\to\infty}d(x_n,Tx_n)=d(x^*,Tx^*).$$

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 27, NO.6, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC

Equation (3.3), it means that the sequence $\{d(x_{n+1}, Tx_n)\}$ is a constant sequence with the value $dist(A, B)$. Hence

$$d(x^*, Tx^*) = dist(A, B),$$

$i.e.$, x^* is a best proximity point of T.

Finally, we show that x^* is unique best proximity point of T. Suppose that x_1 and x_2 are two best proximity points of T in A with $x_1 \neq x_2$. Since x_1 and x_2 are two best proximity points of T, we have

$$d(x_1, Tx_1) = dist(A, B) = d(x_2, Tx_2).$$

By the P-property of (A, B), we obtain

$$d(x_1, x_2) = d(Tx_1, Tx_2).$$

Since x_1 and x_2 are distinct elements in A, one can have

$$\varphi(d(x_1, x_2)) > 0. \tag{3.8}$$

From the definition of T and (3.8),

$$\begin{aligned} d(x_1, x_2) = d(Tx_1, Tx_2) &\leq d(x_1, x_2) - \varphi(d(Tx_1, Tx_2)) \\ &= d(x_1, x_2) - \varphi(d(x_1, x_2)) \\ &< d(x_1, x_2). \end{aligned}$$

This is a contradiction. Therefore the uniqueness of the best proximity point follows. $\square$

The following example illustrates that Theorem 3.2 holds.

Example 3.3. Let $X = (-\infty, +\infty)$ be endowed with the Euclidean metric $d(x, y) = |x - y|$. Then (X, d) is a complete metric space. Let $A = [-1, 1]$ and $B = [0, 2]$ be two subsets of (X, d). Define $T : A \to B$ by

$$Tx = \frac{2}{5}x$$

for each $x \in A$. Define $\varphi(t) : [0, +\infty) \to [0, +\infty)$ by

$$\varphi(t) = \frac{4}{3}t.$$

Then, by Example 1.7, T satisfies (1.2). It is easy to check that A and B are closed subsets of complete metric space (X, d), $\emptyset \neq A_0 = [0, 1] = B_0$ and $T(A_0) = [0, \frac{2}{5}] \subseteq [0, 1] = B_0$. Moreover (A, B) has the P-property. Indeed, let $d(x_1, Tx_1) = dist(A, B) = d(x_2, Tx_2)$. By

$$0 = dist(A, B) = \inf\{d(a, b) : a \in A, b \in B\},$$

we have $x_1 = Tx_1$ and $x_2 = Tx_2$. Thus $d(x_1, x_2) = d(Tx_1, Tx_2)$. Hence (A, B) has the P-property. Therefore all the assumption of Theorem 3.2 hold and note that $x^* = 0$ is the unique best proximity point.

Acknowledgments

This work was supported by Kyungnam University Research Fund, 2017.

References

[1] Y.I. Alber and S. Guerre-Delabriere, *Principle of weakly contractive maps in Hilbert spaces*, in: I. Gohberg, Yu. Lyubich(Eds.), New Results in Operator Theory, in: Advances and Appl., vol. 98, Birkhäuser, Basel, 1997, 7–22.

[2] S.S. Basha and P. Veeramani, *Best proximity pair theorems for multifunctions with open fibres*, J. Approx. Theory, **103**(1) (2000), 119–129.

[3] K. Fan, *Extension of two fixed point theorems of F.E. Browder*, Math. Z., **122** (1969), 234–240.

[4] M. Hegedüs and T. Szilágyi, *Equivalent conditions and a new fixed point theorem in the theory of contractive type mappings*, Math. Japon., **25** (1980), 147–157.

[5] J.K. Kim, K.H. Kim and K.S. Kim, *Three-step iterative sequences with errors for asymptotically quasi-nonexpansive mappings in convex metric spaces*, Proc. of RIMS Kokyuroku, Kyoto Univ., **1365** (2004), 156–165.

[6] J.K. Kim, K.S. Kim and S.M. Kim, *Convergence theorems of implicit iteration process for a finite family of asymptotically quasi-nonexpansive mappings in convex metric spaces*, Proc. of RIMS Kokyuroku, Kyoto Univ., **1484** (2006), 40–51.

[7] J.K. Kim, K.S. Kim and Y.M. Nam, *Convergence and stability of iterative processes for a pair of simultaneously asymptotically quasi-nonexpansive type mappings in convex metric spaces*, J. of Compu. Anal. Appl., **9**(2) (2007), 159–172.

[8] K.S. Kim, *Some convergence theorems for contractive type mappings in $CAT(0)$ spaces*, Abstract and Applied Analysis, 2013, Article ID 381715, 9 pages, http://dx.doi.org/10.1155/2013/381715

[9] K.S. Kim, *Convergence and stability of generlaized φ-weak contraction mapping in $CAT(0)$ spaces*, Open Mathematics, **15** (2017), 1063–1074.

[10] K.S. Kim, *Equivalence between some iterative schemes for generalized φ-weak contraction mappings in $CAT(0)$ spaces*, East Asian Math. J., **33**(1) (2017), 11–22.

[11] K.S. Kim, H. Lee, S.J. Park, S.Y. Yu, J.H. Ahn and D.Y. Kwon, *Convergence of generlaized φ-weak contraction mapping in convex metric spaces*, Far East J. Math. Sci., **101**(7) (2017), 1437–1447.

[12] W.A. Kirk, S. Reich and P. Veeramani, *Proximal retracts and best proximity pair theorems*, Numer. Funct. Anal. Optim., **24**(7-8) (2003), 851–862.

[13] V.S. Raj, *A best proximity point theorem for weakly contractive non-self-mappings*, Nonlinear Anal., **74** (2011), 4804–4808.

[14] B.E. Rhoades, *Some theorems on weakly contractive maps*, Nonlinear Anal., **47** (2001), 2683–2693.

[15] A. Sultana and V. Vetrivel, *On the existence of best proximity points for generalized contractions*, Appl. Gen. Topol., **15**(1) (2014), 55–63.

[16] Z. Xue, *The convergence of fixed point for a kind of weak contraction*, Nonlinear Func. Anal. Appl., **21**(3) (2016), 497–500.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 27, NO.6, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC

Explicit viscosity rule of nonexpansive mappings in CAT(0) spaces

Shin Min Kang[1,2], Absar Ul Haq[3], Waqas Nazeer[4,*], Iftikhar Ahmad[5] and Maqbool Ahmad[6]

[1]Department of Mathematics and RINS, Gyeongsang National University, Jinju 52828, Korea
e-mail: smkang@gnu.ac.kr

[2]Center for General Education, China Medical University, Taichung 40402, Taiwan

[3]Department of Mathematics, University of Management and Technology, Sialkot Campus, Lahore 51410, Pakistan
e-mail: absarulhaq@hotmail.com

[4]Division of Science and Technology, University of Education, Lahore 54000, Pakistan
e-mail: nazeer.waqas@ue.edu.pk

[5]Department of Mathematics and Statistics, University of Lahore, Lahore 54000, Pakistan
e-mail: iftikharcheema1122@gmail.com

[6]Department of Mathematics and Statistics, University of Lahore, Lahore 54000, Pakistan
e-mail: maqboolchaudhri@gmail.com

Abstract

In this paper, we present a explicit viscosity technique of nonexpansive mappings in the framework of CAT(0) spaces. The strong convergence theorem of the proposed technique is proved under certain assumptions imposed on the sequence of parameters. The results presented in this paper extend and improve some recent announced in the current literature.

2010 Mathematics Subject Classification: 47J25, 47N20, 34G20, 65J15

Key words and phrases: viscosity rule, CAT(0) space, nonexpansive mapping, variational inequality.

1 Introduction

The study of spaces of nonpositive curvature originated with the discovery of hyperbolic spaces, and flourished by pioneering works of Hadamard and Cartan, etc. in the first decades of the twentieth century. The idea of nonpositive curvature geodesic metric spaces could be traced back to the work of Busemann and Alexandrov, etc. in the 50's. Later on Gromov [9] restated some features of global Riemannian geometry solely based on the so-called CAT(0) inequality. For through discussion of CAT(0) spaces and of fundamental role they play in geometry, we refer the reader to Bridson and Haefliger [5].

* Corresponding author

As we know, iterative methods for finding fixed points of nonexpansive mappings have received vast investigations due to its extensive applications in a variety of applied areas of inverse problem, partial differential equations, image recovery, and signal processing; see [1–3,7,14–17] and the references therein. One of the difficulties in carrying out results from Banach space to complete CAT(0) space setting lies in the heavy use of the linear structure of the Banach spaces. Berg and Nikolaev [4] introduce the notion of an inner product-like notion (quasi-linearization) in complete CAT(0) spaces to resolve these difficulties.

Fixed-point theory in CAT(0) spaces was first studied by Kirk [10,11]. He showed that every nonexpansive (single-valued) mapping defined on a bounded closed convex subset of a complete CAT(0) space always has a fixed point. Since then, the fixed-point theory for single-valued and multivalued mappings in CAT(0) spaces has been rapidly developed.

In 2000, Moudaf's [13] introduce viscosity approximation methods as following

Theorem 1.1. *Let C be a nonempty closed convex subset of the real Hilbert space X. Let T be a nonexpansive mapping of C into itself such that $Fix(T)$ is nonempty. Let f be a contraction of C into itself with coefficient $\theta \in [0,1)$. Pick any $x_0 \in [0,1)$, let $\{x_n\}$ be a sequence generated by*

$$x_{n+1} = \frac{\gamma_n}{1+\gamma_n} f(x_n) + \frac{1}{1+\gamma_n} T(x_n), \quad n \geq 0,$$

where $\{\gamma_n\}$ is a sequence in $(0,1)$ satisfying the following conditions:

(1) $\lim_{n\to\infty} \gamma_n = 0$,

(2) $\sum_{n=0}^{\infty} \gamma_n = \infty$,

(3) $\sum_{n=0}^{\infty} |\frac{1}{\gamma_{n+1}} - \frac{1}{\gamma_n}| = 0$.

Then $\{x_n\}$ converges strongly to a fixed point x^* of the mapping T, which is also the unique solution of the variational inequality

$$\langle x - f(x), x - y \rangle \geq 0, \quad \forall y \in Fix(T).$$

In other words, x^* is the unique fixed point of the contraction $P_{Fix(T)} f$, that is, $P_{Fix(T)} f(x^*) = x^*$.

Shi and Chen [15] studied the convergence theorems of the following Moudaf's viscosity iterations for a nonexpansive mapping in CAT(0) spaces.

$$x_{n+1} = tf(x_n) \oplus (1-t)T(x_n), \tag{1.1}$$

$$x_{n+1} = \alpha_n f(x_n) \oplus (1-\alpha_n)T(x_n). \tag{1.2}$$

They proved that $\{x_n\}$ defned by (1.1) and $\{x_n\}$ defined by (1.2) converged strongly to a fxed point of T in the framework of CAT(0) space.

Zhao et al. [18] applied viscosity approximation methods for the implicit midpoint rule for nonexpansive mappings

$$x_{n+1} = \alpha_n f(x_n) \oplus (1-\alpha_n) T\left(\frac{x_n \oplus x_{n+1}}{2}\right), \quad \forall n \geq 0.$$

Motivated and inspired by the idea of Kwun et al. [12], in this paper, we extend and study the explicit viscosity rules of nonexpansive mappings in CAT(0) spaces

$$\begin{cases} x_{n+1} = (1-\alpha_n) f(x_n) \oplus \alpha_n T(y_n), \\ y_n = (1-\beta_n) x_n \oplus \beta_n T(x_n). \end{cases} \tag{1.3}$$

2 Preliminaries

Let(X, d) be a metric space. A geodesic path joining $x \in X$ to $y \in X$ (or, more briefly, a geodesic from x to y) is a map c from a closed interval $[0, l] \subset R$ to X such that $c(0)= x$, $c(l) = y$, and $d(c(t), c(t')) = |t - t'|$ for all $t, t' \in [0, l]$. In particular, c is an isometry and $d(x, y) = l$. The image α of c is called a geodesic (or metric) segment joining x and y. When it is unique, this geodesic segment is denoted by $[x, y]$. The space (X, d) is said to be a geodesic space if every two points of X are joined by a geodesic, and X is said to be uniquely geodesic if there is exactly one geodesic joining x and y for each $x, y \in X$. A subset $Y \subset X$ is said to be convex if Y includes every geodesic segment joining any two of its points. A geodesic triangle $\triangle(x_1, x_2, x_3)$ in a geodesic metric space (X, d) consists of three points x_1, x_2,and x_3 in X (the vertices of $\triangle$) and a geodesic segment between each pair of vertices (the edges of $\triangle$). A comparison triangle for the geodesic triangle $\triangle(x_1, x_2, x_3$ in (X, d) is a triangle $\overline{\triangle}(x_1, x_2, x_3) := \triangle(\overline{x_1}, \overline{x_2}, \overline{x_3})$ in the Euclidean plane $\mathbb{E}^2$ such that $d_{\mathbb{E}^2} d(x_i, x_j) = d(x_i, x_j)$for $i, j = 1, 2, 3$.
A geodesic space is said to be a CAT(0) space if all geodesic triangles satisfy the following comparison axiom.

Let $\triangle$ be a geodesic triangle in X, and let $\overline{\triangle}$ be a comparison triangle for $\triangle$. Then, $\triangle$ is said to satisfy the $CAT(0)$ *inequality* if for all $x, y \in \triangle$ and all comparison points $\overline{x}, \overline{y} \in \overline{\triangle}$,

$$d(x, y) = d_{\mathbb{E}^2}(\overline{x}, \overline{y}) \tag{2.1}$$

Let $x, y \in X$ and by the Lemma 2.1(iv) of [8] for each $t \in [0, 1]$, there exists a unique point $z \in [x, y]$ such that

$$d(x, z) = td(x, y), \quad d(y, z) = (1 - t)d(x, y). \tag{2.2}$$

From now on, we will use the notation $(1 - t)x \oplus ty$ for the unique fixed point z satisfying the above equation.

We now collect some elementary facts about CAT(0) spaces which will be used in the proofs of our main results.

Lemma 2.1. ([8]) *Let X be a $CAT(0)$ spaces.*
(a) For any $x, y, z \in X$ and $t \in [0, 1]$,

$$d((1 - t)x \oplus ty, z) \leq (1 - t)d(x, z) + td(y, z). \tag{2.3}$$

(b) For any $x, y, z \in X$ and $t \in [0, 1]$,

$$d^2((1 - t)x \oplus ty, z) \leq (1 - t)^2 d(x, z) + td^2(y, z) - t(1 - t)d^2(x, y). \tag{2.4}$$

Complete CAT(0) spaces are often called Hadamard spaces (see [5]). If x, y_1, y_2 are points of a CAT(0) spaces and y_0 is the midpoint of the segment $[y_1, y_2]$, which we will denoted by $\frac{(y_1 \oplus y_2)}{2}$, then the CAT$(0)$ inequality implies

$$d^2\left(x, \frac{y_1 \oplus y_2}{2}\right) \leq \frac{1}{2} d^2(x, y_1) + \frac{1}{2} d^2(x, y_2) - \frac{1}{4} d^2(y_1, y_2). \tag{2.5}$$

This inequality is the (CN) inequality of Bruhat and Tits [6]. In fact, a geodesic metric space is a CAT(0) space if and only if it satisfies the (CN) inequality (cf. [5], page 163).

Definition 2.2. Let X be a CAT(0) space and $T : X \to X$ be a mapping. Then T is called *nonexpensive* if

$$d(T(x), T(y)) \leq d(x, y), \quad x, y \in C$$

Definition 2.3. Let X be a CAT(0) space and $T : X \to X$ be a mapping. Then T is called *contraction* if

$$d(T(x), T(y)) \leq \theta d(x, y), \quad x, y \in C, \, \theta \in [0, 1)$$

Berg and Nikolaev [4] introduce the concept of quasi-linearization as follow. Let us denote the pair $(a, b) \in X \times X$ by the $\overrightarrow{ab}$ and call it a vector. Then, quasi-linearization is defined as a mapping

$$\langle \cdot, \cdot \rangle : (X \times X) \times (X \times X) \longrightarrow \mathbb{R}$$

defined as

$$\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = \frac{1}{2}(d^2(a, d) + d^2(b, c) - d^2(a, c) - d^2(b, d)) \tag{2.6}$$

it is easy to see that $\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = \langle \overrightarrow{cd}, \overrightarrow{ab} \rangle$, $\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = -\langle \overrightarrow{ba}, \overrightarrow{cd} \rangle$ and $\langle \overrightarrow{ax}, \overrightarrow{cd} \rangle + \langle \overrightarrow{xb}, \overrightarrow{cd} \rangle = \langle \overrightarrow{ab}, \overrightarrow{cd} \rangle$ for all $a, b, c, d \in X$. We say that X satisfies the Cauchy-Schwarz inequality if

$$\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle \leq d(a, b)d(a, c)$$

for all $a, b, c, d \in X$. It is well-known [4] that a geodesically connected metric space is a CAT(0) space of and only if it satisfy the Cauchy-Schwarz inequality.
Let C be a non-empty closed convex subset of a complete CAT(0) space X. The metric projection $P_c : X \to C$ is defined by

$$u = P_c(x) \Longleftrightarrow \inf\{d(y, x) : y \in C\}, \quad \forall x \in X$$

Definition 2.4. Let $P_c : X \to C$ is called the *metric projection* if for every $x \in X$ there exist a unique nearest point in C, denoted by $P_c x$, such that

$$d(x, P_c x) \leq d(x, y), \quad y \in C$$

The following theorem gives you the conditions for a projection mapping to be nonexpensive.

Theorem 2.5. *Let C be a non-empty closed convex subset of a real $CAT(0)$ space X and $P_c : X \to X$ a metric projection. Then*
(1) $d(P_c x, P_c y) \leq \langle \overrightarrow{xy}, \overrightarrow{P_c x P_c y} \rangle$ *for all* $x, y \in X$,
(2) P_c *is nonexpensive mapping, that is,* $d(x, p_c x) \leq d(x, y)$ *for all* $y \in C$,
(3) $\langle \overrightarrow{x P_c x}, \overrightarrow{y P_c y} \rangle \leq 0$ *for all* $x \in X$ *and* $y \in C$.

Further if, in addition, C is bounded, then $Fix(T)$ is nonempty.
The following lemmas are very useful for proving our main results:

Lemma 2.6. (*The demiclosedness principle*) *Let C be a nonempty closed convex subset of the real $CAT(0)$ space X and $T : C \to C$ such that*

$$x_n \rightharpoonup x^* \in C \quad \textit{and} \quad (I - T)x_n \to 0.$$

Then $x^* = Tx^*$. Here $\to$ and $\rightharpoonup$ denote strong and weak convergence, respectively.

Moreover, the following result gives the conditions for the convergence of a nonnegative real sequences.

Lemma 2.7. *Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that $a_{n+1} \leq (1-\gamma_n)a_n + \delta_n, \forall n \geq 0$, where $\{\beta_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence with*

(1) $\sum_{n=0}^{\infty} \gamma_n = \infty$,

(2) $\lim_{n\to\infty} \sup \frac{\delta_n}{\gamma_n} \leq 0$ *or* $\sum_{n=0}^{\infty} |\delta_n| < \infty$.

Then $\lim_{n\to\infty} a_n \to 0$.

3 The main result

Theorem 3.1. *Let C be a nonempty closed convex subset of a complete $CAT(0)$ space X. Let $T : C \to C$ be a nonexpansive mapping with $F(T) \neq \phi$ and $f : C \to C$ be a contraction with coefficient $\theta \in [0,1)$. Pick any $x_0 \in C$, let $\{x_n\}$ be a sequence generated by (1.3), where$\{\alpha_n\}$ and $\{\beta_n\}$ are the sequence in $(0,1)$ satisfying the following conditions:*

(1) $\lim_{n\to\infty} \alpha_n = 1$ *and* $\lim_{n\to\infty} \beta_n = 1$,

(2) $\sum_{n=0}^{\infty} \alpha_n = \infty$ *and* $\sum_{n=0}^{\infty} \beta_n = \infty$,

(3) $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ *and* $\sum_{n=0}^{\infty} |\beta_{n+1} - \beta_n| < \infty \forall n \geq 0$,

(4) $\lim_{n\to\infty} d(x_n, T(x_n)) = 0$.

Then $\{x_n\}$ converges strongly to a fixed point x^* of the nonexpansive mapping T which is also the unique solution of the variational inequality

$$\langle \overrightarrow{xf(x)}, \overrightarrow{yx} \rangle \geq 0, \quad \forall y \in F(T).$$

In other words, x^* is the unique fixed point of the contraction $P_{Fix(T)}f$, that is, $P_{Fix(T)}f(x^*)$ $= x^*$.

Proof. We divide the proof into the following five steps.

Step 1. First, we show that x_n is bounded. Indeed, take $p \in F(T)$ arbitrarily, we have

$$\begin{aligned} d(x_{n+1}, p) &= d((1-\alpha_n)f(x_n) \oplus \alpha_n T(y_n), p) \\ &\leq (1-\alpha_n)d(f(x_n), p) + \alpha_n d(T(y_n), p) \\ &\leq (1-\alpha_n)d(f(x_n), f(p)) + (1-\alpha_n)d(f(p), p) + \alpha_n d(y_n, p) \\ &\leq (1-\alpha_n)\theta d(x_n, p) + (1-\alpha_n)d(f(p), p) + \alpha_n d(y_n, p). \end{aligned} \tag{3.1}$$

Now consider

$$\begin{aligned} d(y_n, p) &= ((1-\beta_n)x_n \oplus \beta_n T(x_n), p) \\ &\leq (1-\beta_n)d(x_n, p) + \beta_n d(T(x_n), p) \\ &\leq (1-\beta_n)d(x_n, p) + \beta_n d(x_n, p) \\ &\leq d(x_n, p). \end{aligned}$$

Using this in (3.1) we have

$$\begin{aligned} d(x_{n+1},p) &\le (1-\alpha_n)\theta d(x_n,p) + (1-\alpha_n)d(f(p),p) + \alpha_n d(x_n,p) \\ &= [(1-\alpha_n)\theta + \alpha_n]d(x_n,p) + (1-\alpha_n)d(f(p),p) \\ &= [1-1+\alpha+(1-\alpha_n)\theta]d(x_n,p) + (1-\alpha_n)d(f(p),p) \\ &= [1-(1-\alpha)+(1-\alpha_n)\theta]d(x_n,p) + (1-\alpha_n)d(f(p),p) \\ &= [1-(1-\alpha_n)(1-\theta)]d(x_n,p) + (1-\alpha_n)(1-\theta)\left(\frac{1}{1-\theta}d(f(p),p)\right), \end{aligned}$$

thus we have

$$d(x_{n+1},p) \le \max\left\{d(x_n,p), \left(\frac{1}{1-\theta}d(f(p),p)\right)\right\},$$

similarly

$$d(x_n,p) \le \max\left\{d(x_{n-1},p), \left(\frac{1}{1-\theta}d(f(p),p)\right)\right\}.$$

From this

$$\begin{aligned} &d(x_{n+1},p) \\ &\le \max\left\{d(x_n,p), \left(\frac{1}{1-\theta}d(f(p),p)\right)\right\} \\ &\le \max\left\{d(x_{n-1},p), \left(\frac{1}{1-\theta}d(f(p),p)\right)\right\} \\ &\ \ \vdots \\ &\le \max\left\{d(x_0,p), \left(\frac{1}{1-\theta}d(f(p),p)\right)\right\}, \end{aligned}$$

which shows that $\{x_n\}$ is bounded. From this we deduce immediately that $\{f(x_n)\}$, $\{T(x_n)\}$ are bounded.

STEP 2. Next, we want to prove that $\lim_{n\to\infty} d(x_{n+1},x_n) = 0$.

For this consider

$$\begin{aligned} &d(x_{n+1},x_n) \\ &= d((1-\alpha_n)f(x_n)\oplus\alpha_n T(y_n), (1-\alpha_{n-1})f(x_{n-1})\oplus\alpha_{n-1}T(y_{n-1})) \\ &\le (1-\alpha_n)\theta d(x_n,x_{n-1}) + |\alpha_n-\alpha_{n-1}|d(T(y_{n-1}),f(x_{n-1})) + \alpha_n d(y_n,y_{n-1}). \end{aligned} \tag{3.2}$$

Now consider

$$\begin{aligned} &d(y_n,y_{n-1}) \\ &= d((1-\beta_n)x_n\oplus\beta_n T(x_n), (1-\beta_{n-1})x_{n-1}\oplus\beta_{n-1}T(x_{n-1})) \\ &\le (1-\beta_n)d(x_n,x_{n-1}) + |\beta_n-\beta_{n-1}|d(T(x_{n-1}),x_{n-1}) + \beta_n d(x_n,x_{n-1}) \\ &\le d(x_n,x_{n-1}) + |\beta_n-\beta_{n-1}|d(T(x_{n-1}),x_{n-1}). \end{aligned}$$

Using this in (3.2) we get

$$\begin{aligned} &d(x_{n+1},x_n) \\ &\le (1-\alpha_n)\theta d(x_n,x_{n-1}) + |\alpha_n-\alpha_{n-1}|d(T(y_{n-1}),f(x_{n-1})) \\ &\quad + \alpha_n d(x_n,x_{n-1}) + \alpha_n|\beta_n-\beta_{n-1}|d(T(x_{n-1}),x_{n-1}) \\ &= [(1-\alpha_n)\theta+\alpha_n]d(x_n,x_{n-1}) + |\alpha_n-\alpha_{n-1}|d(T(y_{n-1}),f(x_{n-1})) \\ &\quad + \alpha_n|\beta_n-\beta_{n-1}|d(T(x_{n-1}),x_{n-1}). \end{aligned}$$

Let $\lambda_n = (1-\alpha_n)$ so $\lambda_n \in (0,1)$, since $\alpha_n \in (0,1)$ $\sum_{n=0}^{\infty} \lambda = \infty$, $\sum_{n=0}^{\infty} |\alpha_n - \alpha_{n-1}| < \infty$ and $\sum_{n=0}^{\infty} |\beta_n - \beta_{n-1}|$. By using Lemma 2.7, we get $\lim_{n\to\infty} d(x_{n+1}, x_n) = 0$.

STEP 3. Now, we want to prove that$\lim_{n\to\infty} d(x_n, T(y_n)) \to 0$

$$\begin{aligned} d(x_n, T(y_n)) &\leq d(x_n, T(x_n)) + d(T(x_n), T(y_n)) \\ &\leq d(x_n, T(x_n)) + d(x_n, y_n) \\ &= d(x_n, T(x_n)) + d(x_n, (1-\beta_n)x_n \oplus \beta_n T(x_n)) \\ &\leq d(x_n, T(x_n)) + \beta_n d(x_n, T(x_n)) \\ &\leq (1+\beta_n) d(x_n, T(x_n)) \\ &\to 0 \quad (n \to \infty). \end{aligned}$$

STEP 4. In this step, we claim that $\limsup_{n\to\infty} \langle \overrightarrow{x^* f(x^*)}, \overrightarrow{x^* x_n} \rangle \leq 0$, where $x^* = P_{F(T)} f(x^*)$.

Indeed, we take a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ which converges weakly to a fixed point p of T. Without loss of generality, we may assume that $\{x_{n_i}\} \rightharpoonup p$. From $\lim_{n\to\infty} d(x_n, Tx_n) = 0$ and Lemma 2.6 we have $p = T(p)$. This together with the property of the metric projection implies that

$$\lim_{n\to\infty} \sup \langle \overrightarrow{x^* f(x^*)}, \overrightarrow{x^* x_n} \rangle = \lim_{n\to\infty} \sup \langle \overrightarrow{x^* f(x^*)}, \overrightarrow{x^* x_{n_i}} \rangle = \langle \overrightarrow{x^* f(x^*)}, \overrightarrow{x^* p} \rangle \leq 0.$$

STEP 5. Finally, we show that $x_n \to x^*$ as $n \to \infty$. Here again $x^* \in Fix(T)$ is the unique fixed point of the contraction $P_{Fix(T)} f$. Consider

$$\begin{aligned} &d^2(x_{n+1}, x^*) \\ &= d^2((1-\alpha_n) f(x_n) \oplus \alpha_n T(y_n), x^*) \\ &= (1-\alpha_n)^2_n d^2(f(x_n), x^*) + (1-\alpha_n) d^2(T(x_n), x^*) \\ &\quad + 2\alpha_n(1-\alpha_n) \langle \overrightarrow{f(x_n) x^*}, \overrightarrow{T(y_n) x^*} \rangle \\ &\leq \alpha_n^2 d^2(y_n, x^*) + (1-\alpha_n)^2 d^2(f(x_n), x^*) + 2\alpha_n(1-\alpha_n) \langle \overrightarrow{f(x_n) f(x^*)}, \overrightarrow{T(y_n) x^*} \rangle \\ &\quad + 2\alpha_n(1-\alpha_n) \langle \overrightarrow{f(x^*) x^*}, \overrightarrow{T(y_n) x^*} \rangle \\ &\leq \alpha_n^2 d^2(y_n, x^*) + (1-\alpha_n)^2 d^2(f(x_n), x^*) + 2\alpha_n(1-\alpha_n) d(f(x_n), f(x^*)) d(T(y_n), x^*) \\ &\quad + 2\alpha_n(1-\alpha_n) \langle \overrightarrow{f(x^*) x^*}, \overrightarrow{T(y_n) x^*} \rangle \\ &\leq \alpha_n^2 d^2(y_n, x^*) + 2\alpha_n(1-\alpha_n)\theta d(x_n, x^*) d(y_n, x^*) + (1-\alpha_n)^2 d^2(f(x_n), x^*) \\ &\quad + 2\alpha_n(1-\alpha_n) \langle \overrightarrow{f(x^*) x^*}, \overrightarrow{T(y_n) x^*} \rangle, \end{aligned} \tag{3.3}$$

now consider

$$\begin{aligned} d(y_n, x^*) &= d((1-\beta_n)x_n \oplus \beta_n T(x_n), x^*) \\ &\leq (1-\beta_n) d(x_n, x^*) + \beta_n d(T(x_n), x^*) \\ &\leq (1-\beta_n) d(x_n, x^*) + \beta_n d(x_n, x^*) \\ &\leq d(x_n, x^*), \end{aligned} \tag{3.4}$$

using (3.2) in (3.3) we get

$$\begin{aligned}
&d^2(x_{n+1}, x^*) \\
&\le \alpha_n^2 d^2(x_n, x^*) + 2\alpha_n(1-\alpha_n)\theta d(x_n, x^*) d(x_n, x^*) + (1-\alpha_n)^2 d^2(f(x_n), x^*) \\
&\quad + 2\alpha_n(1-\alpha_n)\langle \overrightarrow{f(x^*)x^*}, \overrightarrow{T(y_n)x^*} \rangle \\
&\le \alpha_n^2 d^2(x_n, x^*) + 2\alpha_n(1-\alpha_n)\theta d(x_n, x^*) d(x_n, x^*) + (1-\alpha_n)^2 d^2(f(x_n), x^*) \\
&\quad + 2\alpha_n(1-\alpha_n)\langle \overrightarrow{f(x^*)x^*}, \overrightarrow{T(y_n)x^*} \rangle \\
&\le [\alpha_n^2 + 2\alpha_n(1-\alpha_n)\theta] d^2(x_n, x^*) + (1-\alpha_n)^2 d^2(f(x_n), x^*) \\
&\quad + 2\alpha_n(1-\alpha_n)\langle f(x^*) - x^*, T(y_n) - x^* \rangle.
\end{aligned} \tag{3.5}$$

Note that $\alpha_n\theta < \alpha_n$ since $\alpha_n \in (0,1)$ and $\theta \in [0,1)$

$$2\alpha_n\theta < 2\alpha_n,$$

which implies that

$$\alpha_n^2 + 2\alpha_n\theta(1-\alpha_n) < \alpha_n^2 + 2\alpha_n(1-\alpha_n),$$

therefore, we have

$$\begin{aligned}
&d^2(x_{n+1}, x^*) \\
&\le [\alpha_n^2 + 2\alpha_n(1-\alpha_n)] d^2(x_n, x^*) + (1-\alpha_n)^2 d^2(f(x_n), x^*) \\
&\quad + 2\alpha_n(1-\alpha_n)\langle \overrightarrow{f(x^*)x^*}, \overrightarrow{T(y_n)x^*} \rangle \\
&\le [2\alpha_n - \alpha_n^2)] d^2(x_n, x^*) + (1-\alpha_n)^2 d^2(f(x_n), x^*) \\
&\quad + 2\alpha_n(1-\alpha_n)\langle \overrightarrow{f(x^*)x^*}, \overrightarrow{T(y_n)x^*} \rangle \\
&\le 2\alpha_n d^2(x_n, x^*) + (1-\alpha_n)^2 d^2(f(x_n), x^*) \\
&\quad + 2\alpha_n(1-\alpha_n)\langle \overrightarrow{f(x^*)x^*}, \overrightarrow{T(y_n)x^*} \rangle \\
&\le 2[1-(1-\alpha_n)] d^2(x_n, x^*) + (1-\alpha_n)^2 d^2(f(x_n), x^*) \\
&\quad + 2\alpha_n(1-\alpha_n)\langle \overrightarrow{f(x^*)x^*}, \overrightarrow{T(y_n)x^*} \rangle,
\end{aligned} \tag{3.6}$$

as by $\lim_{n\to\infty} \alpha_n = 1$ we have

$$\begin{aligned}
&\lim_{n\to\infty} \sup \frac{(1-\alpha_n)^2 d^2(f(x_n), x^*) + 2\alpha_n(1-\alpha_n)\langle \overrightarrow{f(x^*)x^*}, \overrightarrow{T(y_n)x^*} \rangle}{(1-\alpha_n)} \\
&= \lim_{n\to\infty} \sup[(1-\alpha_n) d^2(f(x_n), x^*) + 2\alpha_n \langle \overrightarrow{f(x^*)x^*}, \overrightarrow{T(y_n)x^*} \rangle] \\
&\le 0.
\end{aligned} \tag{3.7}$$

From (3.6), (3.7), and Lemma 2.7 we have

$$\lim_{n\to\infty} d^2(x_{n+1}, x_n) = 0,$$

which implies that $x_n \to x^*$ as $n \to \infty$. This completes the proof. ☐

References

[1] I. Ahmad and M. Ahmad, An implicit viscosity technique of nonexpansive mapping in Cat(0) spaces, *Open J. Math. Anal.*, **1** (2017), 1–12.

[2] I. Ahmad and M. Ahmad, On the viscosity rule for common fixed points of two nonexpansive mappings in CAT(0) spaces, *Open J. Math. Anal.*, **2** (2018) (in press).

[3] M. A. Alghamdi, M. A. Alghamdi, N. Shahzad and H. K. Xu, The implicit midpoint rule for nonexpansive mappings, *Fixed Point Theory Appl.* **2014** (2014), Paper No. 96, 9 pages

[4] I. D. Berg and I. G. Nikolaev, Quasilinearization and curvature of Aleksandrov spaces, *Geom. Dedicata*, **133** (2008), 195–218.

[5] M. R. Bridson and A. Haefliger, *Metric Spaces of Non-Positive Curvature*, Grundlehren der mathematischen Wissenschaften, vol. 319, Springer-Verlag, Berlin, 1999.

[6] F. Bruhat and J. Tits, Groupes réductifs sur un corps local, *Inst. Hautes Études Sci. Publ. Math.*, **41** (1972), 5–251.

[7] H. Dehghan, J. Rooin, A characterization of metric projection in CAT(0) spaces, In: International Conference on Functional Equation, Geometric Functions and Applications (ICFGA 2012), Payame Noor University, Tabriz, 2012, pp. 41-43.

[8] S. Dhompongsa and B. Panyanak, On δ-convergence theorems in CAT(0) spaces, *Comput. Math. Appl.*, **56** (2008), 2572–2579.

[9] M. Gromov, CAT(κ)-spaces: construction and concentration, *J. Math. Sci.*, **119** (2004), 178–200.

[10] W. A. Kirk, Geodesic geometry and fixed point theory, In Seminar of Mathematical Analysis (Malaga/Seville, 2002/2003), **64** (2003), 195–225.

[11] W. A. Kirk, Geodesic geometry and fixed point theory, II, International Conference on Fixed Point Theory and Applications, Yokohama Publ., Yokohama, 2004, pp. 113–142.

[12] Y. C. Kwun, W. Nazeer, M. Munir and S. M. Kang, Explicit viscosity rules and applications of nonexpansive mappings, *J. Comput. Anal. Appl.*, **24** (2018), 1541–1552.

[13] A. Moudafi, Viscosity approximation methods for fixed-points problems, *J. Math. Anal. Appl.*, **241** (2000), 46–55.

[14] S. F. A. Naqvi and M. S. Khan, On the viscosity rule for common fixed points of two nonexpansive mappings in Hilbert spaces, *Open J. Math. Sci.*, **1** (2017), 111–125.

[15] L. Y. Shi and R. D. Chen, Strong convergence of viscosity approximation methods for nonexpansive mappings in CAT(0) spaces, *J. Appl. Math.*, **2012** (2012), Article ID 421050, 11 pages.

[16] K. Shimoji and W. Takahashi, Strong convergence to common fixed points of infinite nonexpansive mappings and applications, *Taiwanese J. Math.*, **5** (2001) 387–404.

[17] H. K. Xu, Viscosity approximation methods for nonexpansive mappings, *J. Math. Anal. Appl.*, **298** (2004) 279–291.

[18] L. Zhao, S. S. Chang, L. Wang and G. Wang, Viscosity approximation methods for the implicit midpoint rule of nonexpansive mappings in CAT(0) Spaces, *J. Nonlinear Sci. Appl.*, **10** (2017), 386–394.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 27, NO.6, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC

The generalized viscosity implicit rules of asymptotically nonexpansive mappings in CAT(0) spaces

Shin Min Kang[1,2], Absar Ul Haq[3], Waqas Nazeer[4,*] and Iftikhar Ahmad[5]

[1]Department of Mathematics and RINS, Gyeongsang National University, Jinju 52828, Korea
e-mail: smkang@gnu.ac.kr

[2]Center for General Education, China Medical University, Taichung 40402, Taiwan

[3]Department of Mathematics, University of Management and Technology, Sialkot Campus, Lahore 51410, Pakistan
e-mail: absarulhaq@hotmail.com

[4]Division of Science and Technology, University of Education, Lahore 54000, Pakistan
e-mail: nazeer.waqas@ue.edu.pk

[5]Department of Mathematics and Statistics, University of Lahore, Lahore 54000, Pakistan
e-mail: iftikharcheema1122@gmail.com

Abstract

In this paper, we establish the generalized viscosity implicit rules of asymptotically nonexpansive mappings in CAT(0) spaces. The strong convergence theorems of the implicit rules proposed are proved under certain assumptions imposed on the control parameters. The results presented in this paper improve and extend some recent corresponding results announced.

2010 Mathematics Subject Classification: 47J25, 47N20, 34G20, 65J15

Key words and phrases: viscosity rule, CAT(0) space, nonexpansive mapping, asymptotically nonexpansive mapping, variational inequality

1 Introduction

The study of spaces of nonpositive curvature originated with the discovery of hyperbolic spaces, and flourished by pioneering works of Hadamard and Cartan, etc. in the first decades of the twentieth century. The idea of nonpositive curvature geodesic metric spaces could be traced back to the work of Busemann and Alexandrov, etc. in the 50's. Later on Gromov [11] restated some features of global Riemannian geometry solely based on the so-called CAT(0) inequality. For through discussion of CAT(0) spaces and of fundamental role they play in geometry, we refer the reader to Bridson and Haefliger [6].

* Corresponding author

As we know, iterative methods for finding fixed points of nonexpansive mappings have received vast investigations due to its extensive applications in a variety of applied areas of inverse problem, partial differential equations, image recovery, and signal processing; see [1–4, 8, 9, 16, 18–21] and the references therein. One of the difficulties in carrying out results from Banach space to complete CAT(0) space setting lies in the heavy use of the linear structure of the Banach spaces. Berg and Nikolaev [5] introduce the notion of an inner product-like notion(quasilinearization) in complete CAT(0) spaces to resolve these difficulties.

Fixed-point theory in CAT(0) spaces was frsc studied by Kirk [13,14]. He showed that every nonexpansive (single-valued) mapping defined on a bounded closed convex subset of a complete CAT(0) space always has a fxed point. Since then, the fxed-point theory for single-valued and multivalued mappings in CAT(0) spaces has been rapidly developed.

In 2000, Moudaf's [15] introduce viscosity approximation methods as following

Theorem 1.1. *Let C be a nonempty closed convex subset of the real Hilbert space X. Let T be a nonexpansive mapping of C into itself such that $Fix(T) = \{x : T(x) = x\}$ is nonempty. Let f be a contraction of C into itself with coefficient $\theta \in [0, 1)$. Pick any $x_0 \in [0, 1)$, let $\{x_n\}$ be a sequence generated by*

$$x_{n+1} = \frac{\gamma_n}{1+\gamma_n} f(x_n) + \frac{1}{1+\gamma_n} T(x_n), \quad n \geq 0,$$

where $\{\gamma_n\}$ is a sequence in $(0, 1)$ satisfying the following conditions:

(1) $\lim_{n\to\infty} \gamma_n = 0$,
(2) $\sum_{n=0}^{\infty} \gamma_n = \infty$,
(3) $\sum_{n=0}^{\infty} \left| \frac{1}{\gamma_{n+1}} - \frac{1}{\gamma_n} \right| = 0$.

Then $\{x_n\}$ converges strongly to a fixed point x^* of the mapping T, which is also the unique solution of the variational inequality

$$\langle x - f(x), x - y\rangle \geq 0, \quad \forall y \in Fix(T).$$

In other words, x^* is the unique fixed point of the contraction $P_{Fix(T)}f$, that is, $P_{Fix(T)}f(x^*) = x^*$.

Shi and Chen [17] studied the convergence theorems of the following Moudaf's viscosity iterations for a nonexpansive mapping in CAT(0) spaces.

$$x_{n+1} = tf(x_n) \oplus (1-t)T(x_n), \tag{1.1}$$

$$x_{n+1} = \alpha_n f(x_n) \oplus (1-\alpha_n)T(x_n). \tag{1.2}$$

They proved that $\{x_n\}$ defned by (1.1) and $\{x_n\}$ defned by (1.2) converged strongly to a fxed point of T in the framework of CAT(0) space.

Zhao et al. [22] applied viscosity approximation methods for the implicit midpoint rule for nonexpansive mappings

$$x_{n+1} = \alpha_n f(x_n) \oplus (1-\alpha_n)T\left(\frac{x_n \oplus x_{n+1}}{2}\right), \forall n \geq 0.$$

Motivated by He et al. [12], in this paper, we study the generalized viscosity implicit rules of asymptotically nonexpansive mappings in the framework of CAT(0) spaces.

More precisely, we consider the following implicit iterative algorithm

$$x_{n+1} = \alpha_n f(x_n) \oplus (1-\alpha_n)T^n(\beta_n x_n \oplus (1-\beta_n)x_{n+1}) \tag{1.3}$$

under suitable conditions, we proved that the sequence $\{x_n\}$ converge strongly to a fixed point of the asymptotically nonexpansive mapping T.

2 Preliminaries

Let(X, d) be a metric space. A geodesic path joining $x \in X$ to $y \in X$ (or, more briefly, a geodesic from x to y) is a map c from a closed interval $[0, l] \subset R$ to X such that $c(0)= x$, $c(l) = y$, and $d(c(t), c(t')) = |t - t'|$ for all $t, t' \in [0, l]$. In particular, c is an isometry and $d(x, y) = l$. The image α of c is called a geodesic (or metric) segment joining x and y. When it is unique, this geodesic segment is denoted by $[x, y]$. The space (X, d) is said to be a geodesic space if every two points of X are joined by a geodesic, and X is said to be uniquely geodesic if there is exactly one geodesic joining x and y for each $x, y \in X$. A subset $Y \subset X$ is said to be convex if Y includes every geodesic segment joining any two of its points. A geodesic triangle $\triangle(x_1, x_2, x_3)$ in a geodesic metric space (X, d) consists of three points x_1, x_2,and x_3 in X (the vertices of $\triangle$) and a geodesic segment between each pair of vertices (the edges of $\triangle$). A comparison triangle for the geodesic triangle $\triangle(x_1, x_2, x_3$ in (X, d) is a triangle $\overline{\triangle}(x_1, x_2, x_3) := \triangle(\overline{x_1}, \overline{x_2}, \overline{x_3})$ in the Euclidean plane $\mathbb{E}^2$ such that $d_{\mathbb{E}^2} d(x_i, x_j) = d(x_i, x_j)$for $i, j = 1, 2, 3$.

A geodesic space is said to be a CAT(0) space if all geodesic triangles satisfy the following comparison axiom.

Let $\triangle$ be a geodesic triangle in X, and let $\overline{\triangle}$ be a comparison triangle for $\triangle$. Then, $\triangle$ is said to satisfy the $CAT(0)$ *inequality* if for all $x, y \in \triangle$ and all comparison points $\overline{x}, \overline{y} \in \overline{\triangle}$,

$$d(x, y) = d_{\mathbb{E}^2}(\overline{x}, \overline{y}). \tag{2.1}$$

Let $x, y \in X$ and by the Lemma 2.1(iv) of [10] for each $t \in [0, 1]$, there exist a unique point $z \in [x, y]$ such that

$$d(x, z) = td(x, y), \quad d(y, z) = (1-t)d(x, y). \tag{2.2}$$

From now on, we will use the notation $(1-t)x \oplus ty$ for the unique fixed point z satisfying the above equation.

We now collect some elementary facts about CAT(0) spaces which will be used in the proofs of our main results.

Lemma 2.1. ([10]) *Let X be a $CAT(0)$ spaces.*

(a) *For any $x, y, z \in X$ and $t \in [0, 1]$,*

$$d((1-t)x \oplus ty, z) \le (1-t)d(x, z) + td(y, z). \tag{2.3}$$

(b) *For any $x, y, z \in X$ and $t \in [0, 1]$,*

$$d^2((1-t)x \oplus ty, z) \le (1-t)^2 d(x, z) + td^2(y, z) - t(1-t)d^2(x, y). \tag{2.4}$$

Complete CAT(0) spaces are often called Hadamard spaces (see [6]). If x, y_1, y_2 are points of a CAT(0) spaces and y_0 is the midpoint of the segment $[y_1, y_2]$, which we will denoted by $\frac{y_1 \oplus y_2}{2}$, then the CAT(0) inequality implies

$$d^2\left(x, \frac{y_1 \oplus y_2}{2}\right) \leq \frac{1}{2}d^2(x, y_1) + \frac{1}{2}d^2(x, y_2) - \frac{1}{4}d^2(y_1, y_2). \tag{2.5}$$

This inequality is the (CN) inequality of Bruhat and Tits [7]. In fact, a geodesic metric space is a CAT(0) space if and only if it satisfies the (CN) inequality (cf. [6], page 163).

Definition 2.2. Let X be a CAT(0) space and $T : X \to X$ be a mapping. Then T is called *nonexpensive* if

$$d(T(x), T(y)) \leq d(x, y), \quad x, y \in C.$$

Definition 2.3. Let X be a CAT(0) space and $T : X \to X$ be a mapping. Then T is called *contraction* if

$$d(T(x), T(y)) \leq \theta d(x, y), \quad x, y \in C\, \theta \in [0, 1).$$

Berg and Nikolaev [5] introduce the concept of quasi-linearization as follow. Let us denote the pair $(a, b) \in X \times X$ by the $\overrightarrow{ab}$ and call it a vector. Then, quasi-linearization is defined as a mapping

$$\langle \cdot, \cdot \rangle : (x \times X) \times (X \times X) \longrightarrow \mathbb{R}$$

defined as

$$\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = \frac{1}{2}(d^2(a, d) + d^2(b, c) - d^2(a, c) - d^2(b, d)), \tag{2.6}$$

it is easy to see that $\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = \langle \overrightarrow{cd}, \overrightarrow{ab} \rangle$, $\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = -\langle \overrightarrow{ba}, \overrightarrow{cd} \rangle$ and $\langle \overrightarrow{ax}, \overrightarrow{cd} \rangle + \langle \overrightarrow{xb}, \overrightarrow{cd} \rangle = \langle \overrightarrow{ab}, \overrightarrow{cd} \rangle$ for all $a, b, c, d \in X$. We say that X satisfies the Cauchy-Schwarz inequality if

$$\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle \leq d(a, b)d(a, c)$$

for all $a, b, c, d \in X$. It is well-known [5] that a geodesically connected metric space is a CAT(0) space of and only if it satisfy the Cauchy-Schwarz inequality.
Let C be a non-empty closed convex subset of a complete CAT(0) space X. The metric projection $P_c : X \to C$ is defined by

$$u = P_c(x) \Longleftrightarrow \inf\{d(y, x) : y \in C\}, \quad \forall x \in X$$

Definition 2.4. Let $P_c : X \to C$ is called the *metric projection* if for every $x \in X$ there exist a unique nearest point in C, denoted by $P_c x$, such that

$$d(x, P_c x) \leq d(x, y), \quad y \in C.$$

The following theorem gives you the conditions for a projection mapping to be nonexpensive.

Theorem 2.5. *Let C be a non-empty closed convex subset of a real $CAT(0)$ space X and $P_c : X \to X$ a metric projection. Then*
(1) $d(P_c x, P_c y) \leq \langle \overrightarrow{xy}, \overrightarrow{P_c x P_c y} \rangle$ *for all* $x, y \in X$,
(2) P_c *is nonexpensive mapping, that is,* $d(x, p_c x) \leq d(x, y)$ *for all* $y \in C$,
(3) $\langle \overrightarrow{xP_c x}, \overrightarrow{yP_c y} \rangle \leq 0$ *for all* $x \in X$ *and* $y \in C$.

Definition 2.6. A mapping $T : C \to C$ is called *asymptotically nonexpensive* if there exist a sequence a sequence $\{k_n\} \subset [0, \infty)$ with $\lim_{n\to\infty} k_n = 1$ such that

$$d(T^n x - T^n y) \leq k_n d(x, y), \quad \forall x, y \in C,\ n \geq 1. \tag{2.7}$$

It is well known that if T is an asymptotically nonexpansive, then $Fix(T)$ is always closed and convex. Further if, in addition, C is bounded, then $Fix(T)$ is nonempty.

The following lemmas are very useful for proving our main results:

Lemma 2.7. (*The demiclosedness principle*) *Let C be a nonempty closed convex subset of the real $CAT(0)$ space X and $T : C \to C$ such that*

$$x_n \rightharpoonup x^* \in C \quad \textit{and} \quad (I - T)x_n \to 0.$$

Then $x^* = Tx^*$. Here $\to$ and $\rightharpoonup$) denote strong and weak) convergence, respectively.

Moreover, the following result gives the conditions for the convergence of a nonnegative real sequence.

Lemma 2.8. *Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that $a_{n+1} \leq (1 - \beta_n)a_n + \delta_n, \forall n \geq 0$, where $\{\beta_n\}$ is a sequence in $(0, 1)$ and $\{\delta_n\}$ is a sequence with*

(1) $\sum_{n=0}^{\infty} \beta_n = \infty$,

(2) $\limsup_{n\to\infty} \sup \frac{\delta_n}{\beta_n} \leq 0$ *or* $\sum_{n=0}^{\infty} |\beta_n| < \infty$.

Then $\lim_{n\to\infty} a_n \to 0$.

3 The main results

Theorem 3.1. *Let C be a non-empty closed convex subset of a complete $CAT(0)$ space X and $T : C \to C$ be an asymptotically nonexpensive mapping with sequence $\{k_n\} \subset [0, +\infty)$ with $\lim_{n\to\infty} k_n = 1$ and $Fix(T) \neq \emptyset$. Let $f : C \longrightarrow C$ be a contraction with coefficient $\theta \in [0, 1)$. For arbitrary initial point $x_0 \in C$, let $\{x_n\}$ be a sequence generated by (1.3), where $\{\alpha_n\}$ and $\{\beta_n\}$ are the sequence in $(0, 1)$ satisfying the following conditions:*

(1) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(2) $\lim_{n\to\infty} \frac{k_n^2 - 1}{\alpha_n} = 0$,

(3) $0 < \tau < \beta_n < \beta_{n+1} < 1$, *for all* $n \geq 0$,

(4) $\lim_{n\to\infty} d(T^n(x_n), (x_n)) = 0$.

Then $\{x_n\}$ converges strongly to the point $x^* = P_{Fix(T)} f(x^*)$ of the mapping T, which is also the unique solution of the variational inequality

$$\langle \overrightarrow{xf(x)}, \overrightarrow{xy} \rangle \geq 0, \quad \forall y \in Fix(T).$$

In other words, x^* is the unique fixed point of the contraction $P_{Fix(T)} f$, that is, $P_{Fix(T)} f(x^*) = x^*$.

Proof. We have divided the proof into four steps.

STEP 1: First, we show that the generalized viscosity implicit rule (**??**) is well-defined

$$S_n(x) = \alpha_n f(x_n) \oplus (1 - \alpha_n) T^n(\beta_n x_n \oplus (1 - \beta_n) x_{n+1}).$$

Consider

$$\begin{aligned}
&d(S_n(x), S_n(y))\\
&= d(\alpha_n f(x_n) \oplus (1-\alpha_n)T^n(\beta_n x_n \oplus (1-\beta_n)x),\\
&\quad\ \alpha_n f(y_n) \oplus (1-\alpha_n)T^n(\beta_n y_n \oplus (1-\beta_n)y))\\
&= (1-\alpha_n)d(T^n(\beta_n x_n \oplus (1-\beta_n)x), T^n(\beta_n y_n \oplus (1-\beta_n)y))\\
&\le (1-\alpha_n)k_n(1-\beta_n)d(x,y).
\end{aligned}$$

Since $\lim\limits_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\lim_{n\to\infty} \frac{k_n^2-1}{\alpha_n} = 0$, or $n \ge 0$. We may assume that $(1-\alpha_n)k_n(1-\beta_n) \le 1-\tau$ for all $n \ge 0$. This implies that S_n is a contraction for each n. Therefore there exists a unique fixed point for S_n by contraction principle, which also implies that (1.3) is well-defined.

Step 2: Now, we show that the sequence $\{x_n\}$ is bounded. Indeed take $p \in Fix(T)$ arbitrary, we have

$$\begin{aligned}
&d(x_{n+1}, p)\\
&= d(\alpha_n f(x_n) \oplus (1-\alpha_n)T^n(\beta_n x_n \oplus (1-\beta_n)x_{n+1}), p)\\
&\le \alpha_n d((f(x_n), p) + (1-\alpha_n)d((\beta_n x_n \oplus (1-\beta_n)x_{n+1}), p)\\
&\le \alpha_n d((f(x_n), f(p)) + \alpha_n d((f(p), p) + (1-\alpha_n)k_n d((\beta_n x_n \oplus (1-\beta_n)x_{n+1}), p)\\
&\le \alpha_n \theta d(x_n, p) + \alpha_n d((f(p), p) + (1-\alpha_n)k_n\beta_n d(x_n, p)\\
&\quad + (1-\beta_n)k_n(1-\beta_n)d(x_{n+1}, p)\\
&\le (\alpha_n\theta + (1-\alpha_n)k_n\beta_n)d(x_n, p) + \alpha_n d((f(p), p)\\
&\quad + (1-\beta_n)k_n(1-\beta_n)d(x_{n+1}, p),
\end{aligned}$$

it follows that

$$\begin{aligned}
&[1-(1-\alpha_n)k_n(1-\beta_n)]d(x_{n+1}, p)\\
&= (\alpha_n\theta + (1-\alpha_n)k_n\beta_n)d(x_n, p) + \alpha_n d((f(p), p).
\end{aligned} \tag{3.1}$$

Since $\{\alpha_n\}$ and $\{\beta_n\}$ are the sequence in $(0,1)$

$$\liminf_{n\to\infty}(1-\alpha_n)k_n(1-\beta_n) \le 1$$

for any given positive number ϵ, $0 < \epsilon < 1-\theta$, there exists a sufficient large positive integer n_0, such that for any $n > n_0$, we have

$$k_n^2 - 1 \le \beta_n \epsilon \alpha_n$$

and

$$k_n - 1 \le \frac{k_n+1}{\beta_n}(k_n-1) \le \frac{k_n^2-1}{\beta_n} \le \epsilon\alpha_n.$$

Moreover, by (3.1)

$$\begin{aligned}
d(x_{n+1},p) &= \frac{\alpha_n\theta+(1-\alpha_n)k_n\beta_n}{1-(1-\alpha_n)k_n(1-\beta_n)}d(x_n,p)+\frac{\alpha_n}{1-(1-\alpha_n)k_n(1-\beta_n)}d(f(p),p)\\
&=\left[1-\frac{\alpha_n(k_n-\theta)-(k_n-1)}{1-(1-\alpha_n)k_n(1-\beta_n)}\right]d(x_n,p)\\
&\quad+\frac{\alpha_n}{1-(1-\alpha_n)k_n(1-\beta_n)}d(f(p),p)\\
&\le\left[1-\frac{\alpha_n(k_n-\theta-\epsilon)}{1-(1-\alpha_n)k_n(1-\beta_n)}\right]d(x_n,p)\\
&\quad+\frac{\alpha_n(k_n-\theta-\epsilon)}{1-(1-\alpha_n)k_n(1-\beta_n)}\left(\frac{1}{(k_n-\theta-\epsilon)}d(f(p),p)\right)\\
&\le\max\left\{d(x_n,p),\frac{1}{k_n-\theta-\epsilon}d(f(p),p)\right\}\\
&\le\max\left\{d(x_n,p),\frac{1}{1-\theta-\epsilon}d(f(p),p)\right\}.
\end{aligned}$$

By applying induction, we obtain

$$d(x_{n+1},p)\le\max\left\{d(x_0,p),\frac{1}{1-\theta-\epsilon}d(f(p),p)\right\}.$$

Hence, we conclude that $\{x_n\}$ is bounded. Consequently, we deduce immediately from it that $\{f(x_n)\}$ and $\{T^n(\beta_n x_n\oplus(1-\beta_n)x_{n+1}\}$ are bounded.

Step 3: Now, we prove that $\lim_{n\to\infty} d(x_{n+1},x_n)=0$

$$\begin{aligned}
d(x_{n+1},x_n) &\le d(x_{n+1},T^n x_n)+d(T^n x_n,x_n)\\
&= d(\alpha_n f(x_n)\oplus(1-\alpha_n)T^n(\beta_n x_n\oplus(1-\beta_n)x_{n+1}),T^n x_n)+d(T^n x_n,x_n)\\
&\le \alpha_n d((f(x_n),T^n x_n)+(1-\alpha_n)d(T^n(x_n(\beta_n x_n\oplus(1-\beta_n)x_{n+1})),T^n x_n)\\
&\quad+d(T^n x_n,x_n)\\
&\le \alpha_n d((f(x_n),T^n x_n)+(1-\alpha_n)k_n d((\beta_n x_n\oplus(1-\beta_n)x_{n+1}),x_n)\\
&\quad+d(T^n x_n,x_n)\\
&\le \alpha_n d((f(x_n),T^n x_n)+(1-\alpha_n)k_n(1-\beta_n)d(x_{n+1},x_n)+d(T^n x_n,x_n)\\
&\le \alpha_n M_1+(1-\alpha_n)k_n(1-\beta_n)d(x_{n+1},x_n)+d(T^n x_n,x_n),
\end{aligned}$$

where $M_1=\sup\{d((f(x_n),T^n x_n),n\ge 1\}$ is constant such that

$$1-(1-\alpha_n)k_n(1-\beta_n)d(x_{n+1},x_n)\le\alpha_n M_1+d(T^n x_n,x_n)$$

It gives

$$\begin{aligned}
d(x_{n+1},x_n)\le&\frac{\alpha_n M_1}{1-(1-\alpha_n)k_n(1-\beta_n)}\\
&+\frac{1}{1-(1-\alpha_n)k_n(1-\beta_n)}d(T^n x_n,x_n)
\end{aligned}$$

Since $1-(1-\alpha_n)k_n(1-\beta_n)\ge\tau$ by virtue of the conditions (1) and (4), we have

$$\lim_{n\to\infty} d(x_{n+1},x_n)=0. \tag{3.2}$$

STEP 4: Now we show that $\lim_{n\to\infty} d(x_n, Tx_n) = 0$.

$$\begin{aligned}
&d(x_n, T^{n-1}x_n)\\
&= d(\alpha_{n-1}f(x_{n-1}) \oplus (1-\alpha_{n-1})T^{n-1}(\beta_{n-1}x_{n-1} \oplus (1-\beta_{n-1})x_n), T^{n-1}x_n)\\
&\le \alpha_{n-1}d((f(x_{n-1}), T^{n-1}x_n) + (1-\alpha_{n-1})k_n d((\beta_{n-1}x_{n-1} \oplus (1-\beta_{n-1})x_n), x_n)\\
&\le \alpha_{n-1}d((f(x_{n-1}), T^{n-1}x_n) + (1-\alpha_{n-1})k_n\beta_{n-1}d(x_n, x_{n-1})\\
&\le \alpha_{n-1}M_1 + (1-\alpha_{n-1})k_n\beta_{n-1}d(x_n, x_{n-1})
\end{aligned}$$

by condition (1) and (3.2) we have

$$\lim_{n\to\infty} d(x_n, T^{n-1}x_n) = 0.$$

Hence we get

$$\begin{aligned}
d(x_n, Tx_n) &\le d(x_n, T^n x_n) + d(T^n x_n, Tx_n)\\
&\le d(x_n, T^n x_n) + k_1 d(T^{n-1}x_n, x_n)\\
&\to 0 \quad (n\to\infty)
\end{aligned} \tag{3.3}$$

Then, it follows from (3.2) and (3.3) that

$$\begin{aligned}
d(T^n(\beta_n x_n \oplus (1-\beta_n)x_{n+1}, x_n) &\le d(T^n(\beta_n x_n \oplus (1-\beta_n)x_{n+1}, Tx_n) + d(Tx_n, x_n)\\
&\le k_n d((\beta_n x_n \oplus (1-\beta_n)x_{n+1}, x_n) + d(Tx_n, x_n)\\
&\le k_n(1-\beta_n)d(x_{n+1}, x_n) + d(Tx_n, x_n)\\
&\le k_n d(x_{n+1}, x_n) + d(Tx_n, x_n)\\
&\to 0 \quad (n\to\infty).
\end{aligned}$$

STEP 5: In this step, we claim that

$$\limsup_{x\to\infty}\langle \overrightarrow{x^*f(x^*)}, \overrightarrow{x^*x_n}\rangle \le 0,$$

where $x^* = P_{Fix(T)}f(x^*)$. Indeed, we take a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ which converges weakly to a fixed point p of T. Without loss of generality, we may assume that $\{x_{n_i}\} \rightharpoonup p$. From $\lim_{n\to\infty} d(x_n, T(x_n) = 0$ and the Lemma 2.7 we have $p = T(p)$. This together, with the property of metric projection implies that

$$\begin{aligned}
\limsup_{x\to\infty}\langle \overrightarrow{x^*f(x^*)}, \overrightarrow{x^*x_n}\rangle &= \limsup_{x\to\infty}\langle \overrightarrow{x^*f(x^*)}, \overrightarrow{x^*x_{n_i}}\rangle\\
&= \limsup_{x\to\infty}\langle \overrightarrow{x^*f(x^*)}, \overrightarrow{x^*p}\rangle\\
&\le 0.
\end{aligned}$$

STEP 6: Finally, we show that $x_n \to x^*$ as $n \to \infty$.

Now, we prove that $\lim_{n\to\infty} d(x_{n+1}, x_n) = 0$. Now, we again take $x^* \in Fix(T)$ is the unique fixed point of the contraction $P_{Fix(T)}f$. Consider

$$\begin{aligned}
d^2(x_n, x_n) &= d^2(\alpha_n f(x_n) \oplus (1-\alpha_n)T^n(\beta_n x_n \oplus (1-\beta_n)x_{n+1}), x^*) \\
&= \alpha_n^2 d^2(f(x_n), x^*) + (1-\alpha_n^2)d^2(T^n(\beta_n x_n \oplus (1-\beta_n)x_{n+1}), x^*) \\
&\quad + 2\alpha_n(1-\alpha_n)\langle \overrightarrow{f(x_n)x^*}, \overrightarrow{T^n(\beta_n x_n \oplus (1-\beta_n)x_{n+1})x^*}\rangle \\
&\le \alpha_n^2 d^2(f(x_n), x^*) + (1-\alpha_n^2)k_n^2 d^2(\beta_n x_n \oplus (1-\beta_n)x_{n+1}), x^*) \\
&\quad + 2\alpha_n(1-\alpha_n)\langle \overrightarrow{f(x_n)f(x^*)}, \overrightarrow{T^n(\beta_n x_n \oplus (1-\beta_n)x_{n+1})x^*}\rangle \\
&\quad + 2\alpha_n(1-\alpha_n)\langle \overrightarrow{f(x^*)x^*}, \overrightarrow{T^n(\beta_n x_n \oplus (1-\beta_n)x_{n+1})x^*}\rangle \\
&\le (1-\alpha_n^2)k_n^2 d^2(\beta_n x_n \oplus (1-\beta_n)x_{n+1}), x^*) \\
&\quad + 2\alpha_n(1-\alpha_n)d(f(x_n)f(x^*))d(T^n(\beta_n x_n \oplus (1-\beta_n)x_{n+1})x^*) + K_n \\
&\le (1-\alpha_n^2)k_n^2 d^2(\beta_n x_n \oplus (1-\beta_n)x_{n+1}), x^*) \\
&\quad + 2\theta\alpha_n(1-\alpha_n)k_n d(x_n, x^*)d(T^n(\beta_n x_n \oplus (1-\beta_n)x_{n+1})x^*) + K_n,
\end{aligned}$$

where

$$K_n = \alpha_n^2 d^2(f(x_n), x^*) + 2\alpha_n(1-\alpha_n)\langle \overrightarrow{f(x^*)x^*}, \overrightarrow{T^n(\beta_n x_n \oplus (1-\beta_n)x_{n+1})x^*}\rangle,$$

it becomes

$$\begin{aligned}
&(1-\alpha_n^2)k_n^2 d^2(\beta_n x_n \oplus (1-\beta_n)x_{n+1}), x^*) \\
&\quad + 2\theta\alpha_n(1-\alpha_n)k_n d(x_n, x^*)d(T^n(\beta_n x_n \oplus (1-\beta_n)x_{n+1})x^*) + K_n + d^2(x_n, x_n) \\
&\ge 0.
\end{aligned}$$

Solving this quadratic inequality for $d((\beta_n x_n \oplus (1-\beta_n)x_{n+1})x^*)$ yields

$$\begin{aligned}
&d((\beta_n x_n \oplus (1-\beta_n)x_{n+1})x^*) \\
&\ge \frac{1}{2(1-\alpha_n)^2k_n^2}\Big\{ -2\theta\alpha_n(1-\alpha_n)k_n d(x_n, x^*) \\
&\qquad + \sqrt{4\theta^2\alpha_n^2(1-\alpha_n)^2k_n^2 d^2(x_n, x^*) - 4(1-\alpha_n)^2k_n^2(K_n - d^2(x_n, x^*))}\Big\} \\
&= \frac{-\theta\alpha_n d(x_n, x^*) + \sqrt{\theta^2\alpha_n^2 d^2(x_n, x^*) - K_n + d^2(x_{n+1}, x^*)}}{(1-\alpha_n)k_n}.
\end{aligned}$$

This implies that

$$\begin{aligned}
&\beta_n d(x_n, x^*) + (1-\beta_n)d(x_{n+1}, x^*) \\
&\ge \frac{-\theta\alpha_n d(x_n, x^*) + \sqrt{\theta^2\alpha_n^2 d^2(x_n, x^*) - K_n + d^2(x_{n+1}, x^*)}}{(1-\alpha_n)k_n},
\end{aligned}$$

namely,

$$\begin{aligned}
&[(1-\alpha_n)k_n\beta_n]d(x_n, x^*) + (1-\alpha_n)k_n(1-\beta_n)d(x_{n+1}, x^*) \\
&\ge \sqrt{\theta^2\alpha_n^2 d^2(x_n, x^*) - K_n + d^2(x_{n+1}, x^*)}.
\end{aligned}$$

Then

$$\begin{aligned}
&\theta^2\alpha_n^2 d^2(x_n, x^*) - K_n + d^2(x_{n+1}, x^*) \\
&\le [(1-\alpha_n)k_n\beta_n + \theta\alpha_n]^2 d^2(x_n, x^*) \\
&\quad + (1-\alpha_n)^2 k_n^2 (1-\beta_n)^2 d^2(x_{n+1}, x^*) \\
&\quad + 2[(1-\alpha_n)k_n\beta_n + \theta\alpha_n](1-\alpha_n)k_n(1-\beta_n) d(x_n, x^*) d(x_{n+1}, x^*) \\
&\le [(1-\alpha_n)k_n\beta_n + \theta\alpha_n]^2 d^2(x_n, x^*) \\
&\quad + (1-\alpha_n)^2 k_n^2 (1-\beta_n)^2 d^2(x_{n+1}, x^*) \\
&\quad + ((1-\alpha_n)k_n\beta_n + \theta\alpha_n)(1-\alpha_n)k_n(1-\beta_n)(d^2(x_n, x^*) + d^2(x_{n+1}, x^*)),
\end{aligned}$$

which is reduced to the inequality

$$\begin{aligned}
&[1 - (1-\alpha_n)^2 k_n^2 (1-\beta_n)^2 - ((1-\alpha_n)k_n\beta_n + \theta\alpha_n)(1-\alpha_n)k_n(1-\beta_n)] d^2(x_{n+1}, x^*) \\
&\le [((1-\alpha_n)k_n(1-\beta_n))^2 + (1-\alpha_n)k_n(1-\beta_n)(1-\alpha_n)k_n(1-\beta_n) \\
&\quad - \theta^2\alpha_n^2] d^2(x_n, x^*) + K_n,
\end{aligned}$$

that is,

$$\begin{aligned}
&[1 - (k_n + \alpha_n(\theta - k_n))(1-\alpha_n)(1-\beta_n)k_n] d^2(x_{n+1}, x^*) \\
&\le [((1-\alpha_n)k_n\beta_n + \theta\alpha_n)(k_n + \alpha_n(\theta - k_n)) - \theta^2\alpha_n^2] d^2(x_n, x^*) + K_n
\end{aligned} \tag{3.4}$$

it follows from (3.4) that

$$\begin{aligned}
&d^2(x_{n+1}, x^*) \\
&\le \frac{[((1-\alpha_n)k_n\beta_n + \theta\alpha_n)(k_n + \alpha_n(\theta - k_n)) - \theta^2\alpha_n^2] d^2(x_n, x^*)}{[1 - (k_n + \alpha_n(\theta - k_n))(1-\alpha_n)(1-\beta_n)k_n]} \\
&\quad + \frac{K_n}{[1 - (k_n + \alpha_n(\theta - k_n))(1-\alpha_n)(1-\beta_n)k_n]}.
\end{aligned} \tag{3.5}$$

Let

$$\begin{aligned}
w_n &= \frac{1}{\alpha_n}\left\{1 - \frac{((1-\alpha_n)k_n\beta_n + \theta\alpha_n)(k_n + \alpha_n(\theta - k_n)) - \theta^2\alpha_n^2}{1 - (k_n + \alpha_n(\theta - k_n))(1-\alpha_n)(1-\beta_n)k_n}\right\} \\
&= \frac{1}{\alpha_n}\frac{1 - k_n^2 - 2\alpha_n k_n(\theta - k_n) - \alpha_n^2(\theta - k_n) - \theta^2\alpha_n^2}{1 - (k_n + \alpha_n(\theta - k_n))(1-\alpha_n)(1-\beta_n)k_n} \\
&\le \frac{-\beta_n\epsilon - 2k_n(\theta - k_n) - \alpha_n(\theta - k_n)^2 + \theta^2\alpha_n}{1 - (k_n + \alpha_n(\theta - k_n))(1-\alpha_n)(1-\beta_n)k_n}
\end{aligned}$$

since $0 < \epsilon < 1 - \theta$ and the sequence $\{\beta_n\}$ satisfies $0 < \tau \le \beta_n \le \beta_{n+1} < 1$ for all $n \ge 0$ and $\lim_{n\to\infty} \beta_n$ exists, assume that

$$\lim_{n\to\infty} \beta_n = \beta^* > 0.$$

Then

$$\lim_{n\to\infty} w_n \le \frac{(2-\beta^*)(1-\theta)}{\beta^*} > 0.$$

Let $0 < \lambda_1 < \frac{(2-\beta^*)(1-\theta)}{\beta^*}$. Then there exists an sufficiently large integer N_1 such that $w_n > \lambda_1$ for all $n > N_1$. Hence, we have

$$\frac{((1-\alpha_n)k_n\beta_n + \theta\alpha_n)(k_n + \alpha_n(\theta - k_n)) - \theta^2\alpha_n^2}{1-(k_n+\alpha_n(\theta-k_n))(1-\alpha_n)(1-\beta_n)k_n} \leq 1 - \lambda_1\alpha_n, \quad \forall n \geq N_1. \tag{3.6}$$

It turns out from (3.5) that

$$d^2(x_{n+1}, x^*) \leq (1-\lambda_1\alpha_n)d^2(x_n, x^*) + \frac{K_n}{[1-(k_n+\alpha_n(\theta-k_n))(1-\alpha_n)(1-\beta_n)k_n]}. \tag{3.7}$$

From (3.5), $\lim_{n\to\infty}\alpha_n = 0$ and Step 4 we have

$$\begin{aligned}
&\limsup_{n\to\infty}\frac{K_n}{\alpha_n\lambda_1[1-(k_n+\alpha_n(\theta-k_n))(1-\alpha_n)(1-\beta_n)k_n]}\\
&= \limsup_{n\to\infty}\frac{\alpha_n^2 d^2(f(x_n), x^*) + 2\alpha_n(1-\alpha_n)\langle\overrightarrow{f(x^*)x^*}, \overrightarrow{T^n(\beta_n x_n \oplus (1-\beta_n)x_{n+1})x^*}\rangle}{\alpha_n\lambda_1[1-(k_n+\alpha_n(\theta-k_n))(1-\alpha_n)(1-\beta_n)k_n]}\\
&= \limsup_{n\to\infty}\frac{\alpha_n d^2(f(x_n), x^*) + 2(1-\alpha_n)\langle\overrightarrow{f(x^*)x^*}, \overrightarrow{T^n(\beta_n x_n \oplus (1-\beta_n)x_{n+1})x^*}\rangle}{\lambda_1[1-(k_n+\alpha_n(\theta-k_n))(1-\alpha_n)(1-\beta_n)k_n]}\\
&\leq 0.
\end{aligned} \tag{3.8}$$

From (3.7) and (3.8) and the Lemma 2.8 we have

$$\lim_{n\to\infty} d(x_{n+1}, x^*) = 0.$$

This implies that $x_n \to x^*$ as $n \to \infty$. This complete the proof. $\square$

The following result is an immediate consequence of the Theorem 3.1.

Theorem 3.2. *Let C be a non-empty closed convex subset of a complete $CAT(0)$ space X and $T : C \to C$ be an nonexpensive mapping with $Fix(T) \neq \emptyset$. Let $f : C \to C$ be a contraction with coefficient $\theta \in [0, 1)$ and for arbitrary initial point $x_0 \in C$. Let $\{x_n\}$ be a sequence generated by*

$$x_{n+1} = \alpha_n f(x_n) \oplus (1-\alpha_n)T(\beta_n x_n \oplus (1-\beta_n)x_{n+1}), \tag{3.9}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are the sequence in $(0, 1)$ satisfying the condition of Theorem 3.1.

Then $\{x_n\}$ converges strongly to the point $x^* = P_{Fix(T)}f(x^*)$ of the mapping T, which is also the unique solution of the variational inequality

$$\langle\overrightarrow{xf(x)}, \overrightarrow{xy}\rangle \geq 0, \quad \forall y \in Fix(T).$$

In other words, x^* is the unique fixed point of the contraction $P_{Fix(T)}f$, that is, $P_{Fix(T)}f(x^*) = x^*$.

References

[1] I. Ahmad and M. Ahmad, An implicit viscosity technique of nonexpansive mapping in Cat(0) spaces, *Open J. Math. Anal.*, **1** (2017), 1–12.

[2] I. Ahmad and M. Ahmad, On the viscosity rule for common fixed points of two nonexpansive mappings in CAT(0) spaces, *Open J. Math. Anal.*, **2** (2018) (in press).

[3] M. A. Alghamdi, M. A. Alghamdi, N. Shahzad and H. K. Xu, The implicit midpoint rule for nonexpansive mappings, *Fixed Point Theory Appl.* **2014** (2014), Paper No. 96, 9 pages

[4] H. Attouch, Viscosity solutions of minimization problems, *SIAM J. Optim.*, **6** (1996), 769–806.

[5] I. D. Berg and I. G. Nikolaev, Quasilinearization and curvature of Aleksandrov spaces, *Geom. Dedicata*, **133** (2008), 195–218.

[6] M. R. Bridson and A. Haefliger, *Metric Spaces of Non-Positive Curvature*, Grundlehren der mathematischen Wissenschaften, vol. 319, Springer-Verlag, Berlin, 1999.

[7] F. Bruhat and J. Tits, Groupes réductifs sur un corps local, *Inst. Hautes Études Sci. Publ. Math.*, **41** (1972), 5–251.

[8] H. Dehghan and J. Rooin, A characterization of metric projection in CAT(0) spaces, In: International Conference on Functional Equation, Geometric Functions and Applications (ICFGA 2012), Payame Noor University, Tabriz, 2012, pp. 41-43.

[9] S. Dhompongsa and W. A. Kirk, B. Panyanak,Nonexpansive set-valued mappings in metric and Banach spaces, *J. Nonlinear Convex Anal.*, **8** (2007), 35–45.

[10] S. Dhompongsa and B. Panyanak, On δ-convergence theorems in CAT(0) spaces, *Comput. Math. Appl.*, **56** (2008), 2572–2579.

[11] M. Gromov, CAT(κ)-spaces: construction and concentration, *J. Math. Sci.*, **119** (2004), 178–200.

[12] S. He, Y. Mao, Z. Zhou, and J. Q. Zhang, The generalized viscosity implicit rules of asymptotically nonexpansive mappings in Hilbert spaces, *Appl. Math. Sci.*, **11** (2017), 549–560.

[13] W. A. Kirk, Geodesic geometry and fixed point theory II, International Conference on Fixed Point Theory and Applications, Yokohama Publ., Yokohama, 2004, pp. 113–142.

[14] W. A. Kirk and B. Panyanak, A concept of convergence in geodesic spaces, *Nonlinear Anal.*, **68** (2008), 3689–3696.

[15] A. Moudafi, Viscosity approximation methods for fixed-points problems, *J. Math. Anal. Appl.*, **241** (2000), 46–55.

[16] S. F. A. Naqvi and M. S. Khan, On the viscosity rule for common fixed points of two nonexpansive mappings in Hilbert spaces, *Open J. Math. Sci.*, **1** (2017), 111–125.

[17] L. Y. Shi and R. D. Chen, Strong convergence of viscosity approximation methods for nonexpansive mappings in CAT(0) spaces, *J. Appl. Math.*, **2012** (2012), Article ID 421050, 11 pages.

[18] K. Shimoji and W. Takahashi, Strong convergence to common fixed points of infinite nonexpansive mappings and applications, *Taiwanese J. Math.*, **5** (2001) 387–404.

[19] D. Wu, S. S. Chang and G. X. Yuan, Approximation of common fixed points for a family of finite nonexpansive mappings in Banach space, *Nonlinear Anal.*, **63** (2005), 987–999.

[20] H. K. Xu, Viscosity approximation methods for nonexpansive mappings, *J. Math. Anal. Appl.*, **298** (2004) 279–291.

[21] Y. Yao and N. Shahzad, New methods with perturbations for non-expansive mappings in Hilbert spaces, *Fixed Point Theory Appl.*, **2011** (2011), Paper No. 79, 9 pages.

[22] L. Zhao, S. S. Chang, L. Wang and G. Wang, Viscosity approximation methods for the implicit midpoint rule of nonexpansive mappings in CAT(0) Spaces, *J. Nonlinear Sci. Appl.*, **10** (2017), 386–394.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 27, NO.6, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC

On some sixth-order rational recursive sequences

M. Folly-Gbetoula * and D. Nyirenda †

Abstract

We study the sixth-order recursive sequences of the form

$$x_{n+1} = \frac{x_{n-5}x_n}{x_{n-4}(a_n + b_n x_{n-5}x_n)},$$

where a_n and b_n are sequences of real numbers, via the technique of Lie group analysis. Symmetry generators associated with the group of transformations that map solutions onto themselves are obtained and exact solutions derived. The 'final constraint' when finding the symmetries, is used to split the solution into different categories. The result of this work generalizes a recent work by Elsayed et al.

Keywords Difference equation; Symmetry; Group invariant solutions
PACS 39A10; 39A13; 39A90

1 Introduction

Among the numerous well-known techniques for solving differential equations, is the powerful Lie symmetry approach. In the nineteenth century, the Norwegian mathematician Sophus Lie [12] developed a systematic algorithm based on the invariance of the ordinary differential equations under a group of transformations (symmetry). In the twentieth century, Maeda [13, 14] demonstrated that this approach can be extended to ordinary difference equations and recently, Hydon [6] used a similar approach to come up with some interesting results. It is now known that Lie's method can be implemented to find symmetries, first integrals (conservation laws) and closed form solutions of difference equations, even in the context of variational equations.

*School of Mathematics, University of the Witwatersrand, Johannesburg, X3, Wits 2050, South Africa
Tel.: +27(0)117176289
Email: Mensah.Folly-Gbetoula@wits.ac.za
†School of Mathematics, University of the Witwatersrand, Johannesburg, X3, Wits 2050, South Africa
Tel.: +27(0)117176224
Email: Darlison.Nyirenda@wits.ac.za

In this paper, we obtain symmetry generators admitted by the difference equations of the form

$$x_{n+1} = \frac{x_{n-5}x_n}{x_{n-4}(a_n + b_n x_{n-5}x_n)}, \tag{1}$$

where a_n and b_n are random sequences, and then proceed to find the solutions in closed form via the invariance of the group of transformations admitted by (2). We first present the solutions in a unified manner and then split them into different categories based on some properties of the 'final constraint'.
This work generalizes the work by Elsayed et. al. [3], where the authors obtained the formulas of the solutions of the difference equations

$$x_{n+1} = \frac{x_{n-5}x_n}{x_{n-4}(\pm + \pm x_{n-5}x_n)}, \quad n = 0, 1, \ldots, \tag{2}$$

in which the initial conditions $x_{-5}, x_{-4}, x_{-3}, x_{-2}, x_{-1}, x_0$ are arbitrary non-zero real numbers.
For similar work on the symmetry approach, see [4, 5, 7, 15, 16] and on different methods, see [1, 2, 8, 10, 19].

1.1 Preliminaries

In this section, we shortly present key elements of Lie group analysis of difference equations. For more understanding of the concepts and notation, we refer the reader to [6, 17] where our definitions and most of our notation are taken from.
Let

$$x^\star = X(x; \varepsilon) \tag{3}$$

be a one parameter Lie group of transformations.

Definition 1.1 *An infinitely differentiable function F is an invariant function of the Lie group of point transformation* (3) *if and only if, for any group transformations,*

$$F(x) = F(x^\star). \tag{4}$$

Definition 1.2 *The infinitesimal generator of the one-parameter Lie group of point transformation* (3) *is the operator*

$$X = X(x) = \xi(x) \times \Delta = \sum_{i=1}^{n} \xi_i(x)\frac{\partial}{\partial x_i}, \tag{5}$$

where Δ is the gradient operator.

Theorem 1.1 *$F(x)$ is invariant under the Lie group of transformations* (3) *if and only if*

$$XF(x) = 0. \tag{6}$$

Now, consider a general kth-order difference equation

$$u_{n+k} = \omega(n, u_n, u_{n+1}, \dots, u_{n+k-1}) \tag{7}$$

for some smooth function ω. We are seeking a one-parameter Lie group of point transformations

$$n^* = n, \tag{8a}$$
$$u_n^* = u_n + \varepsilon\xi(n, u_n) +)(\varepsilon^2), \tag{8b}$$
$$\vdots$$
$$u_{n+k}^* = u_{n+k} + \varepsilon S^k\xi(n, u_n) +)(\varepsilon^2), \tag{8c}$$

where ξ denotes the characteristic, ε (ε is small enough) is the group parameter and $S : n \mapsto n+1$ stands for the shift forward operator . The symmetry criterion is given by

$$u_{n+k}^* = \omega(n, u_n^*, u_{n+1}^*, \dots, u_{n+k-1}^*), \tag{9}$$

whenever (7) holds, and further the substitution of (8) in (9) yields the linearized symmetry condition:

$$S^k\xi(n, u_n) - X\omega = 0 \tag{10}$$

where X, the corresponding prolonged symmetry operator of the group of transformations (8), is given by

$$X = \xi(n, u_n)\frac{\partial}{\partial u_n} + S\xi(n, u_n)\frac{\partial}{\partial u_{n+1}} + \cdots + S^{k-1}\xi(n, u_n)\frac{\partial}{\partial u_{n+k-1}}. \tag{11}$$

The characteristics are obtained by solving the functional equation (10). As simple as (10) may look, its solution is found after a series of steps that require a set of cumbersome calculations.
In this work, we employ the well-known choice of canonical coordinate [9]

$$S_n = \int \frac{du_n}{\xi(n, u_n)} \tag{12}$$

to reduce the order of the difference equation under investigation.

3

2 Main results

Consider the sixth-order difference equations of the form (2). Let

$$u_{n+6} = \omega = \frac{u_n u_{n+5}}{u_{n+1}(A_n + B_n u_n u_{n+5})}, \tag{13}$$

where A_n and B_n are random sequences, be the forward difference equation equivalent to (2).
The linearized symmetry condition (10) imposed on (13) leads to

$$\xi(n+6,\omega) - \frac{A_n u_n \xi(n+5,u_{n+5})}{u_{n+1}(A_n + B_n u_n u_{n+5})^2} + \frac{u_n u_{n+5}\xi(n+1,u_{n+1})}{u_{n+1}^2(A_n + B_n u_n u_{n+5})}$$
$$- \frac{A_n u_{n+5}\xi(n,u_n)}{u_{n+1}(A_n + B_n u_n u_{n+5})^2} = 0. \tag{14}$$

By the means of the first-order partial differential operator

$$L = \frac{\partial}{\partial u_n} - \frac{\omega_{u_n}}{\omega_{u_{n+5}}}\frac{\partial}{\partial u_{n+5}},$$

we can get rid of the first term in (14). This yields the following:

$$\frac{A_n u_{n+5}\xi'(n+5,u_{n+5})}{u_{n+1}(A_n + B_n u_n u_{n+5})^2} - \frac{A_n u_{n+5}\xi'(n,u_n)}{u_{n+1}(A_n + B_n u_n u_{n+5})^2} - \frac{A_n \xi(n+5,u_{n+5})}{u_{n+1}(A_n + B_n u_n u_{n+5})^2}$$
$$+ \frac{A_n u_{n+5}\xi(n,u_n)}{u_n u_{n+1}(A_n + B_n u_n u_{n+5})^2} = 0. \tag{15}$$

Here, it is important to simplify the equation in order to minimize the number of derivations. Thus, we clear fractions in (15) and divide the resulting equation by $u_n u_{n+5}$ to get

$$\xi'(n+5,u_{n+5}) - \frac{1}{u_{n+5}}\xi(n+5,u_{n+5}) - \xi'(n,u_n) + \frac{1}{u_n}\xi(n,u_n) = 0. \tag{16}$$

Differentiating (16) with respect to u_n, keeping u_{n+5} fixed, leads to

$$\frac{d}{du_n}\left[-\xi'(n,u_n) + \frac{1}{u_n}\xi(n,u_n)\right] = 0. \tag{17}$$

Clearly, the solution of (17) is

$$\xi(n,u_n) = f(n)u_n + g(n)u_n \ln u_n \tag{18}$$

4

for some arbitrary functions f and g of n. To ease the computation we shall assume that g is zero. Using the expression of the characteristic given in (18), equation (14) becomes

$$B_n f(n+1)u_n u_{n+5} + B_n f(n+6)u_n u_{n+5} - A_n f(n) + A_n f(n+1) - A_n f(n+5) + A_n f(n+6) = 0. \tag{19}$$

which splits into

$$1: \qquad f(n) + f(n+5) = 0 \tag{20}$$

$$u_n u_{n+5}: \qquad f(n+1) + f(n+6) = 0. \tag{21}$$

The system above reduces to the final constraint:

$$f(n) + f(n+5) = 0. \tag{22}$$

Solving (22) for f, we obtain five independent solutions given by $(-1)^n$, $\exp(\pm n\pi/5)$ and $\exp(\pm 3n\pi/5)$. Therefore, the characteristics are

$$\xi_1 = (-1)^n u_n,\ \xi_2 = \beta^n u_n,\ \xi_3 = \bar{\beta}^n u_n,\ \xi_4 = \theta^n u_n,\ \xi_5 = \bar{\theta}^n u_n, \tag{23}$$

and so the prolonged infinitesimal generators admitted by (13) are

$$\begin{aligned} X_1 =& (-1)^n u_n \partial_{u_n} + (-1)^{n+1} u_{n+1} \partial_{u_{n+1}} + (-1)^{n+2} u_{n+2} \partial_{u_{n+2}} + \\ & (-1)^{n+3} u_{n+3} \partial_{u_{n+3}} + (-1)^{n+4} u_{n+4} \partial_{u_{n+4}} + (-1)^{n+5} u_{n+5} \partial_{u_{n+5}}, \end{aligned} \tag{24a}$$

$$\begin{aligned} X_2 =& \beta^n u_n \partial_{u_n} + \beta^{n+1} u_{n+1} \partial_{u_{n+1}} + \beta^{n+2} u_{n+2} \partial_{u_{n+2}} + \\ & \beta^{n+3} u_{n+3} \partial_{u_{n+3}} + \beta^{n+4} u_{n+4} \partial_{u_{n+4}} + \beta^{n+5} u_{n+5} \partial_{u_{n+5}}, \end{aligned} \tag{24b}$$

$$\begin{aligned} X_3 =& \bar{\beta}^n u_n \partial_{u_n} + \bar{\beta}^{n+1} u_{n+1} \partial_{u_{n+1}} + \bar{\beta}^{n+2} u_{n+2} \partial_{u_{n+2}} + \\ & \bar{\beta}^{n+3} u_{n+3} \partial_{u_{n+3}} + \bar{\beta}^{n+4} u_{n+4} \partial_{u_{n+4}} + \bar{\beta}^{n+5} u_{n+5} \partial_{u_{n+5}}, \end{aligned} \tag{24c}$$

$$\begin{aligned} X_4 =& \theta^n u_n \partial_{u_n} + \theta^{n+1} u_{n+1} \partial_{u_{n+1}} + \theta^{n+2} u_{n+2} \partial_{u_{n+2}} + \\ & \theta^{n+3} u_{n+3} \partial_{u_{n+3}} + \theta^{n+4} u_{n+4} \partial_{u_{n+4}} + \theta^{n+5} u_{n+5} \partial_{u_{n+5}}, \end{aligned} \tag{24d}$$

$$\begin{aligned} X_5 =& \bar{\theta}^n u_n \partial_{u_n} + \bar{\theta}^{n+1} u_{n+1} \partial_{u_{n+1}} + \bar{\theta}^{n+2} u_{n+2} \partial_{u_{n+2}} + \\ & \bar{\theta}^{n+3} u_{n+3} \partial_{u_{n+3}} + \bar{\theta}^{n+4} u_{n+4} \partial_{u_{n+4}} + \bar{\theta}^{n+5} u_{n+5} \partial_{u_{n+5}}. \end{aligned} \tag{24e}$$

Note that $\beta = \exp(\pi/5)$ and $\theta = \exp(3\pi/5)$ Using the generator X_2, we have the canonical coordinate

$$S_n = \int \frac{du_n}{\beta^n u_n} = \frac{1}{\beta^n} \ln |u_n|. \tag{25}$$

Taking advantage of the form of the relation (22), we construct the invariant function $\tilde{V}_n$

$$\tilde{V}_n = S_n \beta^n + S_{n+5} \beta^{n+5} \tag{26}$$

in view of the fact that

$$\begin{aligned} X_1 \tilde{V}_n =& (-1)^n + (-1)^{n+5} = 0, && \text{(27a)} \\ X_2 \tilde{V}_n =& \beta^n + \beta^{n+5} = 0, && \text{(27b)} \\ X_3 \tilde{V}_n =& \bar{\beta}^n + \bar{\beta}^{n+5} = 0, && \text{(27c)} \\ X_4 \tilde{V}_n =& \theta^n + \theta^{n+5} = 0, && \text{(27d)} \\ & && \text{(27e)} \end{aligned}$$

and

$$X_5 \tilde{V}_n = \bar{\theta}^n + \bar{\theta}^{n+5} = 0. \tag{27f}$$

For rational difference equations, it is convenience to use

$$|V_n| = \exp\{-\tilde{V}_n\}, \tag{28}$$

i.e., $V_n = \pm 1/(u_n u_{n+5})$ but we will be using the plus sign. Substituting (28) into equation (13), we reduce it to

$$V_{n+1} = A_n V_n + B_n. \tag{29}$$

We iterate (29) to get its solution in closed form as

$$V_j = V_0 \left(\prod_{k_1=0}^{j-1} A_{k_1} \right) + \sum_{l=0}^{j-1} \left(B_l \prod_{k_2=l+1}^{j-1} A_{k_2} \right). \tag{30}$$

From (25), (26) and (28), we have

$$\begin{aligned}
|u_n| =& \exp\left(\beta_n S_n\right)\\
=& \exp\Bigg[(-1)^n c_1 + \beta^n c_2 + \bar{\beta}^n c_3 + \theta^n c_4 + \bar{\theta}^n c_5 - \frac{1}{5}\sum_{k_1=0}^{n-1}(-1)^{n-k_1}|\tilde{V}_{k_1}|\\
&- \frac{1}{5}\sum_{k_2=0}^{n-1}\beta^n\bar{\beta}^{k_2}|\tilde{V}_{k_2}| - \frac{1}{5}\sum_{k_3=0}^{n-1}\bar{\beta}^n\beta^{k_3}|\tilde{V}_{k_3}| - \frac{1}{5}\sum_{k_4=0}^{n-1}\theta^n\bar{\theta}^{k_4}|\tilde{V}_{k_4}|\\
&- \frac{1}{5}\sum_{k_5=0}^{n-1}\bar{\theta}^n\theta^{k_5}|\tilde{V}_{k_5}|\Bigg]\\
=& \exp\Bigg[(-1)^n c_1 + \beta^n c_2 + \bar{\beta}^n c_3 + \theta^n c_4 + \bar{\theta}^n c_5 + \frac{1}{5}\sum_{k_1=0}^{n-1}(-1)^{n-k_1}\ln|V_{k_1}|\\
&+ \frac{1}{5}\sum_{k_2=0}^{n-1}\beta^n\bar{\beta}^{k_2}\ln|V_{k_2}| + \frac{1}{5}\sum_{k_3=0}^{n-1}\bar{\beta}^n\beta^{k_3}\ln|V_{k_3}| + \frac{1}{5}\sum_{k_4=0}^{n-1}\theta^n\bar{\theta}^{k_4}\ln|V_{k_4}|\\
&+ \frac{1}{5}\sum_{k_5=0}^{n-1}\bar{\theta}^n\theta^{k_5}\ln|\mathcal{V}_{k_5}|\Bigg]
\end{aligned}$$

$$= \exp\left[H_n + \frac{1}{5}\sum_{k=0}^{n-1}\left[(-1)^{n-k} + 2\mathrm{Re}(\gamma_1(n,k)+\gamma_2(n,k))\right]\ln|V_k|\right], \quad (31)$$

where $H_n = (-1)^n c_1 + \beta^n c_2 + \bar{\beta}^n c_3 + \theta^n c_4 + \bar{\theta}^n c_5$, $\gamma_1(n,k) = \beta^n\bar{\beta}^k$ and $\gamma_2(n,k) = \theta^n\bar{\theta}^k$.
The following properties hold:

$$\begin{aligned}
&\gamma_1(0,1)=\bar{\beta}, \gamma_1(0,3)=\bar{\theta}, \gamma_1(0,5)=-1, \gamma_1(0,7)=\theta, \gamma_1(1,0)=\beta,\\
&\gamma_1(3,0)=\theta, \gamma_1(5,0)=-1, \gamma_1(7,0)=\bar{\theta}, \gamma_1(n+9,k)=\gamma_1(n,k+1),\\
&\gamma_1(n,k+9)=\gamma_1(n+1,k), \gamma_1(10n,k)=\gamma_1(0,k), \gamma_1(n,10k)=\gamma_1(n,0);\\
&\gamma_2(0,1)=\bar{\theta}, \gamma_2(0,3)=\beta, \gamma_2(0,5)=-1, \gamma_2(0,7)=\bar{\beta}, \gamma_2(1,0)=\theta,\\
&\gamma_2(3,0)=\bar{\beta}, \gamma_2(5,0)=-1, \gamma_2(7,0)=\beta, \gamma_2(n+9,k)=\gamma_2(n,k+1),\\
&\gamma_2(n,k+9)=\gamma_2(n+1,k), \gamma_2(10n,k)=\gamma_2(0,k), \gamma_2(n,10k)=\gamma_2(n,0). \quad (32)
\end{aligned}$$

From the expression of u_n given in (31) and from the above properties (32),

it is clear that

$$|u_{10n+j}| = \exp\left(H_j + \frac{1}{5}\sum_{k_1=0}^{10n+j-1}\left[(-1)^k + 2\mathrm{Re}(\gamma_1(0,k)+\gamma_2(0,k))\right]\ln|V_{k_1}|\right). \tag{33}$$

For $j = 0$, we have that

$$\begin{aligned}|u_{10n}| &= \exp(H_0 + \ln|V_0| - \ln|V_5| + \ldots + \ln|V_{10n-10}| - \ln|V_{10n-5}|)\\ &= \exp(H_0)\prod_{s=0}^{n-1}\left|\frac{V_{10s}}{V_{10s+5}}\right|.\end{aligned} \tag{34}$$

By setting $n = 0$ in (31), we get $\exp(H_0) = u_0$ and so

$$|u_{10n}| = |u_0|\prod_{s=0}^{n-1}\left|\frac{V_{10s}}{V_{10s+5}}\right|. \tag{35}$$

It can be shown, using (28), that we need not the absolute value function in (36). Similarly, for any $j = 0, 1, \ldots, 9$, we obtain the following:

$$u_{10n+j} = u_j\prod_{s=0}^{n-1}\frac{V_{10s+j}}{V_{10s+j+5}}. \tag{36}$$

Thus, using (30),

$$\begin{aligned}u_{10n+j} &= u_j\prod_{s=0}^{n-1}\frac{V_0\left(\prod\limits_{k_1=0}^{10s+j-1}A_{k_1}\right) + \sum\limits_{l=0}^{10s+j-1}\left(B_l\prod\limits_{k_2=l+1}^{10s+j-1}A_{k_2}\right)}{V_0\left(\prod\limits_{k_1=0}^{10s+j+4}A_{k_1}\right) + \sum\limits_{l=0}^{10s+j+4}\left(B_l\prod\limits_{k_2=l+1}^{10s+j+4}A_{k_2}\right)}\\ &= u_j\prod_{s=0}^{n-1}\frac{\left(\prod\limits_{k_1=0}^{10s+j-1}A_{k_1}\right) + u_0u_5\sum\limits_{l=0}^{10s+j-1}\left(B_l\prod\limits_{k_2=l+1}^{10s+j-1}A_{k_2}\right)}{\left(\prod\limits_{k_1=0}^{10s+j+4}A_{k_1}\right) + u_0u_5\sum\limits_{l=0}^{10s+j+4}\left(B_l\prod\limits_{k_2=l+1}^{10s+j+4}A_{k_2}\right)}.\end{aligned}$$

Hence, the solution to our equation (2) is

$$x_{10n+j-5} = x_{j-5}\prod_{s=0}^{n-1}\frac{\left(\prod\limits_{k_1=0}^{10s+j-1}a_{k_1}\right) + x_{-5}x_0\sum\limits_{l=0}^{10s+j-1}\left(b_l\prod\limits_{k_2=l+1}^{10s+j-1}a_{k_2}\right)}{\left(\prod\limits_{k_1=0}^{10s+j+4}a_{k_1}\right) + x_{-5}x_0\sum\limits_{l=0}^{10s+j+4}\left(b_l\prod\limits_{k_2=l+1}^{10s+j+4}a_{k_2}\right)} \tag{37}$$

where $j = 0, 1, 2, \ldots, 9$, whenever the denominators do not vanish. In the following section, we turn to the special case where a_n and b_n are constant sequences.

3 The case when a_n and b_n are constant sequences

In this case, let $a_n = a$ and $b_n = b$ where $a, b \in \mathbb{R}$.

3.1 The case $a \neq 1$

Using (37), the solution is given by

$$x_{10n+j-5} = \bar{x}_j \prod_{s=0}^{n-1} \frac{a^{10s+j} + bx_{-5}x_0 \frac{1-a^{10s+j}}{1-a}}{a^{10s+j+5} + bx_{-5}x_0 \frac{1-a^{10s+j+5}}{1-a}}, \tag{38}$$

where $j = 0, 1, 2, 3, \ldots, 9$, $\bar{x}_j$ is defined as

$$\bar{x}_j = \begin{cases} x_{j-5}, & 0 \leq j \leq 5; \\ \frac{x_{-5}x_0}{x_{j-10}\left(a^{j-5} + x_{-5}x_0 b \frac{1-a^{j-5}}{1-a}\right)}, & 6 \leq j \leq 9, \end{cases}$$

and for all $(j, s) \in \{0, 1, 2, \ldots, 9\} \times \{0, 1, 2, \ldots, n-1\}$,

$$(1-a)a^{10s+j} + bx_{-5}x_0(1 - a^{10s+j}) \neq 0.$$

3.1.1 The case $a = -1$

In this case, we have

$$x_{10n+j-5} = \bar{x}_j \prod_{s=0}^{n-1} \frac{(-1)^j + bx_{-5}x_0 \frac{1-(-1)^j}{2}}{(-1)^{j+1} + bx_{-5}x_0 \frac{1-(-1)^{j+1}}{2}},$$

where

$$\bar{x}_j = \begin{cases} x_{j-5}, & 0 \leq j \leq 5; \\ \frac{x_{-5}x_0}{x_{j-10}\left((-1)^{j+1} + x_{-5}x_0 b \frac{1-(-1)^{j+1}}{2}\right)}, & 6 \leq j \leq 9. \end{cases}$$

Evaluating the above, we obtain the following solution which, for $b = \pm 1$, appears in [3] (see Theorems 3.1 and 5.1).

$$x_{10n-5} = x_{-5}(-1 + bx_{-5}x_0)^{-n}, \qquad x_{10n-4} = x_{-4}(-1 + bx_{-5}x_0)^{n},$$

$$x_{10n-3} = x_{-3}(-1 + bx_{-5}x_0)^{-n}, \qquad x_{10n-2} = x_{-2}(-1 + bx_{-5}x_0)^{n},$$

$$x_{10n-1} = x_{-1}(-1 + bx_{-5}x_0)^{-n}, \qquad x_{10n} = x_0(-1 + bx_{-5}x_0)^{n},$$

$$x_{10n+1} = \frac{x_{-5}x_0}{x_{-4}(-1 + bx_{-5}x_0)^{n+1}}, \qquad x_{10n+2} = \frac{x_{-5}x_0}{x_{-3}}(-1 + bx_{-5}x_0)^{n},$$

$$x_{10n+3} = \frac{x_{-5}x_0}{x_{-2}(-1 + x_{-5}x_0 b)^{n+1}}, \qquad x_{10n+4} = \frac{x_{-5}x_0}{x_{-1}}(-1 + bx_{-5}x_0)^{n},$$

where $bx_{-5}x_0 \neq 1$.

However, the solution can be written in a more compact form, i.e.,

$$x_{10n-j+5} = \begin{cases} x_{j-5}(-1 + bx_{-5}x_0)^{(-1)^{j+1}n}, & 0 \leq j \leq 5; \\ \frac{x_{-5}x_0}{x_{j-10}}(-1 + bx_{-5}x_0)^{\frac{1-(-1)^j}{2} + (-1)^{j+1}n}, & 6 \leq j \leq 9; \end{cases}$$

as long as $bx_{-5}x_0 \neq 1$.

3.2 The case $a = 1$

Using (37), the solution, which for $b = \pm 1$ appears in [3] (see Theorems 2.1 and 4.1), is given by

$$x_{10n-5} = x_{-5}\prod_{s=0}^{n-1}\frac{1 + 10sbx_{-5}x_0}{1 + (10s+5)bx_{-5}x_0}, \; x_{10n-4} = x_{-4}\prod_{s=0}^{n-1}\frac{1 + (10s+1)bx_{-5}x_0}{1 + (10s+6)bx_{-5}x_0},$$

$$x_{10n-3} = x_{-3}\prod_{s=0}^{n-1}\frac{1 + (10s+2)bx_{-5}x_0}{1 + (10s+7)bx_{-5}x_0}, \; x_{10n-2} = x_{-2}\prod_{s=0}^{n-1}\frac{1 + (10s+3)bx_{-5}x_0}{1 + (10s+8)bx_{-5}x_0},$$

10

$$x_{10n-1} = x_{-1} \prod_{s=0}^{n-1} \frac{1+(10s+4)bx_{-5}x_0}{1+(10s+9)bx_{-5}x_0}, \quad x_{10n} = x_0 \prod_{s=0}^{n-1} \frac{1+(10s+5)bx_{-5}x_0}{1+(10s+10)bx_{-5}x_0},$$

$$x_{10n+1} = \frac{x_{-5}x_0}{x_{-4}(1+bx_{-5}x_0)} \prod_{s=0}^{n-1} \frac{1+(10s+6)bx_{-5}x_0}{1+(10s+11)bx_{-5}x_0},$$

$$x_{10n+2} = \frac{x_{-5}x_0}{x_{-3}(1+2bx_{-5}x_0)} \prod_{s=0}^{n-1} \frac{1+(10s+7)bx_{-5}x_0}{1+(10s+12)bx_{-5}x_0},$$

$$x_{10n+3} = \frac{x_{-5}x_0}{x_{-2}(1+3bx_{-5}x_0)} \prod_{s=0}^{n-1} \frac{1+(10s+8)bx_{-5}x_0}{1+(10s+13)bx_{-5}x_0},$$

$$x_{10n+4} = \frac{x_{-5}x_0}{x_{-1}(1+4bx_{-5}x_0)} \prod_{s=0}^{n-1} \frac{1+(10s+9)bx_{-5}x_0}{1+(10s+14)bx_{-5}x_0},$$

where $jbx_{-5}x_0 \neq -1$ for all $j = 5, 6, 7, \ldots, 10n+4$.

More compactly, the solution can be written as

$$x_{10n+j-5} = \begin{cases} x_{j-5} \prod_{s=0}^{n-1} \frac{1+(10s+j)bx_{-5}x_0}{1+(10s+j+5)bx_{-5}x_0}, & 0 \leq j \leq 5; \\ \frac{x_{-5}x_0}{x_{j-10}(1+b(j-5)x_{-5}x_0)} \prod_{s=0}^{n-1} \frac{1+(10s+j)bx_{-5}x_0}{1+(10s+j+5)bx_{-5}x_0}, & 6 \leq j \leq 9. \end{cases}$$

4 Conclusion

In this paper, we derived symmetry generators for the difference equations (2) and explicit formulas for the solutions of the equations were obtained. As a recent result, Theorems 2.1, 3.1, 4.1 and 5.1 of Elsayed et al. [3] were generalized.

References

[1] C. Cinar, On the positive solutions of the difference equation $x_{n+1} = ax_{n-1}/(1+bx_nx_{n-1})$, Applied Mathematics and Computational **156**, 587-590 (2004).

[2] E. M. Elsayed and T.F. Ibrahim, Periodicity and solutions for some systems of nonlinear rational difference equations, Hacet. J. Math. Stat. **44:6**, 1361–1390 (2015).

[3] E.M. Elsayed, F. Alzahrani and H.S. Alayachi, Formulas and properties of some class of nonlinear difference equations, J. Computational Analysis and Applications **24:8**, (2018).

[4] M. Folly-Gbetoula and A.H. Kara, Symmetries, conservation laws, and 'integrability' of difference equations, Advances in Difference Equations **2014**, (2014).

[5] M. Folly-Gbetoula , Symmetry, reductions and exact solutions of the difference equation $u_{n+2} = (au_n)/(1 + bu_nu_{n+1})$, Journal of Difference Equations and Applications **23:6** (2017).

[6] P. E. Hydon, Difference Equations by Differential Equation Methods, Cambridge University Press, Cambridge, 2014.

[7] P. E. Hydon, Symmetries and first integrals of ordinary difference equations, Proc. Roy. Soc. Lond. A **456**, 2835-2855 (2000).

[8] T. F. Ibrahim and M. A. El-Moneam, Global stability of a higher-order difference equation, Iran J. Sci. Technol. Trans. Sci. **41:1**, 51–58 (2017).

[9] N. Joshi and P. Vassiliou, The existence of Lie Symmetries for First-Order Analytic Discrete Dynamical Systems, Journal of Mathematical Analysis and Applications **195**, 872-887 (1995).

[10] A. Khaliq and E.M. Elsayed, The dynamics and solution of some difference equations, J. Nonlinear Sci. Appl. **9**, 1052–1063 (2016).

[11] D. Levi, L. Vinet and P. Winternitz, Lie group formalism for difference equations, J. Phys. A: Math. Gen. **30**, 633-649 (1997).

[12] S. Lie, Classification und Integration von gewohnlichen Differentialgleichungen zwischen xy, die eine Gruppe von Transformationen gestatten I , Math. Ann. **22** , 213–253 (1888).

[13] S. Maeda, Canonical structure and symmetries for discrete systems, Math. Japonica **25**, 405–420 (1980).

[14] S. Maeda, The similarity method for difference equations, IMA J. Appl. Math.**38**, 129-134 (1987).

[15] N. Mnguni, D. Nyirenda and M. Folly-Gbetoula, On solutions of some fifth-order difference equations, Far East Journal of Mathematical Sciences **102:12**, 3053-3065 (2017).

[16] D. Nyirenda and M. Folly-Gbetoula, Invariance analysis and exact solutions of some sixth-order difference equations, J. Nonlinear Sci. Appl. **10**, 6262-6273 (2017).

[17] P. J. Olver, Applications of Lie Groups to Differential Equations, Second Edition, Springer, New York, 1993.

[18] G. R. W. Quispel and R. Sahadevan, Lie symmetries and the integration of difference equations, Physics Letters A, **184**, 64-70 (1993).

[19] I. Yalcinkaya, On the global attractivity of positive solutions of a rational difference equation, Selcuk J. Appl. Math., **9:2** (2008) 3-8.

TABLE OF CONTENTS, JOURNAL OF COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 27, NO. 6, 2019

Volume 27, Number 7 December 2019
ISSN:1521-1398 PRINT,1572-9206 ONLINE

Journal of Computational Analysis and Applications

EUDOXUS PRESS,LLC

Journal of Computational Analysis and Applications
ISSNno.'s:1521-1398 PRINT,1572-9206 ONLINE

SCOPE OF THE JOURNAL

An international publication of Eudoxus Press, LLC
(fifteen times annually)

Editor in Chief: George Anastassiou
Department of Mathematical Sciences,
University of Memphis, Memphis, TN 38152-3240, U.S.A
ganastss@memphis.edu
http://www.msci.memphis.edu/~ganastss/jocaaa

The main purpose of "J.Computational Analysis and Applications" is to publish high quality research articles from all subareas of Computational Mathematical Analysis and its many potential applications and connections to other areas of Mathematical Sciences. Any paper whose approach and proofs are computational,using methods from Mathematical Analysis in the broadest sense is suitable and welcome for consideration in our journal, except from Applied Numerical Analysis articles. Also plain word articles without formulas and proofs are excluded. The list of possibly connected mathematical areas with this publication includes, but is not restricted to: Applied Analysis, Applied Functional Analysis, Approximation Theory, Asymptotic Analysis, Difference Equations, Differential Equations, Partial Differential Equations, Fourier Analysis, Fractals, Fuzzy Sets, Harmonic Analysis, Inequalities, Integral Equations, Measure Theory, Moment Theory, Neural Networks, Numerical Functional Analysis, Potential Theory, Probability Theory, Real and Complex Analysis, Signal Analysis, Special Functions, Splines, Stochastic Analysis, Stochastic Processes, Summability, Tomography, Wavelets, any combination of the above, e.t.c.

"J.Computational Analysis and Applications" is a peer-reviewed Journal. See the instructions for preparation and submission of articles to JoCAAA. Assistant to the Editor: Dr.Razvan Mezei,mezei_razvan@yahoo.com, Madison,WI,USA.

Journal of Computational Analysis and Applications(JoCAAA) is published by **EUDOXUS PRESS,LLC**,1424 Beaver Trail Drive,Cordova,TN38016,USA,anastassioug@yahoo.com http://www.eudoxuspress.com. **Annual Subscription Prices**:For USA and Canada,Institutional:Print $800, Electronic OPEN ACCESS. Individual:Print $400. For any other part of the world add $160 more(handling and postages) to the above prices for Print. No credit card payments.

JoCAAA is reviewed and abstracted by AMS Mathematical Reviews,MATHSCI,and Zentralblaat MATH.

Dartmouth College
8000 Cummings Hall,
Hanover, NH 03755-8000
603-646-3843 (X 3546 Secr.)
e-mail:george.cybenko@dartmouth.edu
Approximation Theory and Neural Networks

Sever S. Dragomir
School of Computer Science and Mathematics, Victoria University,
PO Box 14428,
Melbourne City,
MC 8001, AUSTRALIA
Tel. +61 3 9688 4437
Fax +61 3 9688 4050
sever.dragomir@vu.edu.au
Inequalities, Functional Analysis, Numerical Analysis, Approximations, Information Theory, Stochastics.

Oktay Duman
TOBB University of Economics and Technology,
Department of Mathematics, TR-06530,
Ankara, Turkey,
oduman@etu.edu.tr
Classical Approximation Theory, Summability Theory, Statistical Convergence and its Applications

Saber N. Elaydi
Department Of Mathematics
Trinity University
715 Stadium Dr.
San Antonio, TX 78212-7200
210-736-8246
e-mail: selaydi@trinity.edu
Ordinary Differential Equations, Difference Equations

J .A. Goldstein
Department of Mathematical Sciences
The University of Memphis
Memphis, TN 38152
901-678-3130
jgoldste@memphis.edu
Partial Differential Equations, Semigroups of Operators

H. H. Gonska
Department of Mathematics
University of Duisburg
Duisburg, D-47048
Germany
011-49-203-379-3542
e-mail: heiner.gonska@uni-due.de
Approximation Theory, Computer Aided Geometric Design

John R. Graef
Department of Mathematics
University of Tennessee at Chattanooga
Chattanooga, TN 37304 USA
John-Graef@utc.edu
Ordinary and functional differential equations, difference equations, impulsive systems, differential inclusions, dynamic equations on time scales, control theory and their applications

Weimin Han
Department of Mathematics
University of Iowa
Iowa City, IA 52242-1419
319-335-0770
e-mail: whan@math.uiowa.edu
Numerical analysis, Finite element method, Numerical PDE, Variational inequalities, Computational mechanics

Tian-Xiao He
Department of Mathematics and Computer Science
P.O. Box 2900, Illinois Wesleyan University
Bloomington, IL 61702-2900, USA
Tel (309)556-3089
Fax (309)556-3864
the@iwu.edu
Approximations, Wavelet, Integration Theory, Numerical Analysis, Analytic Combinatorics

Margareta Heilmann
Faculty of Mathematics and Natural Sciences, University of Wuppertal
Gaußstraße 20
D-42119 Wuppertal, Germany,
heilmann@math.uni-wuppertal.de
Approximation Theory (Positive Linear Operators)

Xing-Biao Hu
Institute of Computational Mathematics
AMSS, Chinese Academy of Sciences
Beijing, 100190, CHINA
hxb@lsec.cc.ac.cn

Instructions to Contributors
Journal of Computational Analysis and Applications

An international publication of Eudoxus Press, LLC, of TN.

Editor in Chief: George Anastassiou

Department of Mathematical Sciences
University of Memphis
Memphis, TN 38152-3240, U.S.A.

1. Manuscripts files in Latex and PDF and in English, should be submitted via email to the Editor-in-Chief:

Prof.George A. Anastassiou
Department of Mathematical Sciences
The University of Memphis
Memphis,TN 38152, USA.
Tel. 901.678.3144
e-mail: ganastss@memphis.edu

Authors may want to recommend an associate editor the most related to the submission to possibly handle it.

Also authors may want to submit a list of six possible referees, to be used in case we cannot find related referees by ourselves.

2. Manuscripts should be typed using any of TEX,LaTEX,AMS-TEX,or AMS-LaTEX and according to EUDOXUS PRESS, LLC. LATEX STYLE FILE. (Click HERE to save a copy of the style file.)They should be carefully prepared in all respects. Submitted articles should be brightly typed (not dot-matrix), double spaced, in ten point type size and in 8(1/2)x11 inch area per page. Manuscripts should have generous margins on all sides and should not exceed 24 pages.

3. Submission is a representation that the manuscript has not been published previously in this or any other similar form and is not currently under consideration for publication elsewhere. A statement transferring from the authors(or their employers,if they hold the copyright) to Eudoxus Press, LLC, will be required before the manuscript can be accepted for publication.The Editor-in-Chief will supply the necessary forms for this transfer.Such a written transfer of copyright,which previously was assumed to be implicit in the act of submitting a manuscript,is necessary under the U.S.Copyright Law in order for the publisher to carry through the dissemination of research results and reviews as widely and effective as possible.

4. The paper starts with the title of the article, author's name(s) (no titles or degrees), author's affiliation(s) and e-mail addresses. The affiliation should comprise the department, institution (usually university or company), city, state (and/or nation) and mail code.

The following items, 5 and 6, should be on page no. 1 of the paper.

5. An abstract is to be provided, preferably no longer than 150 words.

6. A list of 5 key words is to be provided directly below the abstract. Key words should express the precise content of the manuscript, as they are used for indexing purposes.

The main body of the paper should begin on page no. 1, if possible.

7. All sections should be numbered with Arabic numerals (such as: 1. INTRODUCTION) .
Subsections should be identified with section and subsection numbers (such as 6.1. Second-Value Subheading).
If applicable, an independent single-number system (one for each category) should be used to label all theorems, lemmas, propositions, corollaries, definitions, remarks, examples, etc. The label (such as Lemma 7) should be typed with paragraph indentation, followed by a period and the lemma itself.

8. Mathematical notation must be typeset. Equations should be numbered consecutively with Arabic numerals in parentheses placed flush right, and should be thusly referred to in the text [such as Eqs.(2) and (5)]. The running title must be placed at the top of even numbered pages and the first author's name, et al., must be placed at the top of the odd numbed pages.

9. Illustrations (photographs, drawings, diagrams, and charts) are to be numbered in one consecutive series of Arabic numerals. The captions for illustrations should be typed double space. All illustrations, charts, tables, etc., must be embedded in the body of the manuscript in proper, final, print position. In particular, manuscript, source, and PDF file version must be at camera ready stage for publication or they cannot be considered.

Tables are to be numbered (with Roman numerals) and referred to by number in the text. Center the title above the table, and type explanatory footnotes (indicated by superscript lowercase letters) below the table.

10. List references alphabetically at the end of the paper and number them consecutively. Each must be cited in the text by the appropriate Arabic numeral in square brackets on the baseline.
References should include (in the following order):
initials of first and middle name, last name of author(s)
title of article,

name of publication, volume number, inclusive pages, and year of publication.

Authors should follow these examples:

Journal Article

1. H.H.Gonska,Degree of simultaneous approximation of bivariate functions by Gordon operators, (journal name in italics) *J. Approx. Theory*, 62,170-191(1990).

Book

2. G.G.Lorentz, (title of book in italics) *Bernstein Polynomials* (2nd ed.), Chelsea,New York,1986.

Contribution to a Book

3. M.K.Khan, Approximation properties of beta operators,in(title of book in italics) *Progress in Approximation Theory* (P.Nevai and A.Pinkus,eds.), Academic Press, New York,1991,pp.483-495.

11. All acknowledgements (including those for a grant and financial support) should occur in one paragraph that directly precedes the References section.

12. Footnotes should be avoided. When their use is absolutely necessary, footnotes should be numbered consecutively using Arabic numerals and should be typed at the bottom of the page to which they refer. Place a line above the footnote, so that it is set off from the text. Use the appropriate superscript numeral for citation in the text.

13. After each revision is made please again submit via email Latex and PDF files of the revised manuscript, including the final one.

14. Effective 1 Nov. 2009 for current journal page charges, contact the Editor in Chief. Upon acceptance of the paper an invoice will be sent to the contact author. The fee payment will be due one month from the invoice date. The article will proceed to publication only after the fee is paid. The charges are to be sent, by money order or certified check, in US dollars, payable to Eudoxus Press, LLC, to the address shown on the Eudoxus homepage.

No galleys will be sent and the contact author will receive one (1) electronic copy of the journal issue in which the article appears.

15. This journal will consider for publication only papers that contain proofs for their listed results.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 27, NO.7, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC

Common fixed point theorems in G_b-metric space

Youqing Shen, Chuanxi Zhu*, Zhaoqi Wu

Department of Mathematics, Nanchang University, Nanchang, 330031, P. R. China

jillshen123@163.com (Y. Q. Shen), chuanxizhu@126.com (C. X. Zhu)

Abstract In this paper, we introduce a new type of common fixed point for three mappings in G_b-complete G_b-metric space. On the other hand, we prove that the theory is also established in G-metric space and several corollaries and examples are listed.

Keywords: G_b-metric space; common fixed point; G-metric space

1 Preliminaries

Mustafa and Sims [1] generalized the concept of metric space and Mustafa [2,3,7] obtained some fixed point theorems in his papers. After that, many authors established fixed point and common fixed point theorems for different contractive-type condition in G-metric space. In 1998, Czerwik [10] introduced the notion of b-metric space, and then Aghajani [12] based on the notion gave the concept of G_b-metric space and some authors obtained the existence and uniqueness fixed point in G_b-metric space [7,11].

Fixed point theory has a large number of applications in many branches of nonlinear analysis and has been extended in many different directions. Let A, B and C are self mappings of a nonempty set X, if there exists a $p \in X$, such that $Ap = Bp = Cp = p$, then we call p is a common fixed point of A, B and C. For a mapping T on nonempty set X to itself, we have $Tx = x$, and x is unique then we call x is a Picard operator.

In this paper, we mainly obtain a unique common fixed point for three mappings in G_b-metric space. First, we recall some basic properties of G_b-metric space.

Let $\mathbb{R} = (-\infty, \infty)$, $\mathbb{R}^+ = [0, \infty)$ and $\mathbb{N}$ be the set of all natural numbers. Denote $\mathbb{N}^+$ the set of all positive integers.

Definition 1.1 ([12]) Let X be a nonempty set and $s \geq 1$ be a given real number, and let the function $G : X \times X \times X \to [0, \infty)$ satisfy the following properties:

(G_b1) $G(x, y, z) = 0$ if $x = y = z$ whenever $x, y, z \in X$;

(G_b2) $0 \leq G(x, x, y)$ for all $x, y \in X$ with $x \neq y$;

(G_b3) $G(x, x, y) \leq G(x, y, z)$ for all $x, y, z \in X$ with $y \neq z$;

(G_b4) $G(x, y, z) = G(p\{x, y, z\})$, where p is a permutation of x, y, z;

(G_b5) $G(x, y, z) \leq s(G(x, a, a) + G(a, y, z))$ for all $x, y, z \in X$.

†*Correspondence author. Chuanxi Zhu. Email address: chuanxizhu@126.com. Tel:+8613970815298.

Then G is called a G_b-metric on X, and (X, G) is called a G_b-metric space.

Definition 1.2 ([12]) A G_b-metric space G is said to be symmetric if $G(x, x, y) = G(y, x, x)$ for all $x, y \in X$.

Proposition 1.3 ([12]) Let X be a G_b-metric space, then for each $x, y, z, a \in X$ it follows that:

(1) if $G(x, y, z) = 0$ then $x = y = z$;

(2) $G(x, y, z) \leq sG(G(x, y, y) + G(x, x, z))$;

(3) $G(x, y, y) \leq 2s(G(y, x, x))$;

(4) $G(x, y, z) \leq s(G(x, a, a) + G(a, y, z))$.

Definition 1.4 ([12]) Let X be a G_b-metric space. A sequence $\{x_n\}$ in X is said to be:

(1) G_b-Cauchy if for each $\varepsilon > 0$, there exists a positive integer n_0 such that for all $m, n, l \geq n_0$, $G(x_n, x_m, x_l) < \varepsilon$;

(2) G_b-convergent to a point $x \in X$ if for each $\varepsilon > 0$, there exists a positive integer n_0 such that for all $m, n, \geq n_0$, $G(x_n, x_m, x) < \varepsilon$;

Definition 1.5 ([12]) A G_b-metric space X is called complete if every G_b-Cauchy sequence is G_b-convergent in X.

lemma 1.6 ([11]) Let $(X, , G)$ be a G_b-metric space with $s > 1$.

(1) Suppose that $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ are G_b-convergent to x, y and z, respectively. Then we have

$$\frac{1}{s^3}G(x, y, z) \leq \liminf_{n\to\infty} G(x_n, y_n, z_n) \leq \limsup_{n\to\infty} G(x_n, y_n, z_n) \leq s^3 G(x, y, z).$$

(2) If $\{z_n\} = c$ is constant, then

$$\frac{1}{s^2}G(x, y, c) \leq \liminf_{n\to\infty} G(x_n, y_n, c) \leq \limsup_{n\to\infty} G(x_n, y_n, c) \leq s^2 G(x, y, c).$$

(3) If $\{y_n\} = b$ and $\{z_n\} = c$ are constant, then

$$\frac{1}{s}G(x, b, c) \leq \liminf_{n\to\infty} G(x_n, b, c) \leq \limsup_{n\to\infty} G(x_n, b, c) \leq sG(x, b, c).$$

2 Common fixed point theorems in G_b-metric space

Theorem 2.1 Let (X, G) be a G_b-complete G_b-metric space and A, B and C are mappings from X to itself. Suppose that A, B and C satisfy the following condition:

$$G(Ax, By, Cz) \leq \frac{G(x, Ax, Ax) + G(x, By, By) + G(z, Cz, Cz)}{G(x, Ax, By) + G(y, By, Cz) + G(z, Cz, Ax) + 1})G(x, y, z) \tag{2.1}$$

for all $x, y, z \in X$. Then either one of A, B and C has a fixed point, or, A, B and C have a unique common fixed point.

Proof. Define the sequence $\{x_n\}$ as $x_{3n+1} = Ax_{3n}, x_{3n+2} = Bx_{3n+1}, x_{3n+3} = Bx_{3n+2}$ for all $n = 0, 1, 2, \cdots$.

If $x_{3n} = x_{3n+1}$, then x_{3n} is a fixed point of A.

If $x_{3n+1} = x_{3n+2}$, then x_{3n+1} is a fixed point of B.

If $x_{3n+2} = x_{3n+3}$, then x_{3n+2} is a fixed point of C.

If the above conclusions are not true, then we assume that $x_n \neq x_{n+1}$ for all n. Let $d_n = G(x_n, x_{n+1}, x_{n+2})$, then for (2.1) we have

$$\begin{aligned}
&G(Ax_{3n}, Bx_{3n+1}, Cx_{3n+2})\\
\leq&\frac{G(x_{3n}, Ax_{3n}, Ax_{3n}) + G(x_{3n+1}, Bx_{3n+1}, Bx_{3n+1}) + G(x_{3n+2}, Cx_{3n+2}, Cx_{3n+2})}{G(x_{3n}, Ax_{3n}, Bx_{3n+1}) + G(x_{3n+1}, Bx_{3n+1}, Cx_{3n+2}) + G(x_{3n+2}, Cx_{3n+2}, Ax_{3n}) + 1})G(x_{3n}, x_{3n+1}, x_{3n+2})\\
=&\frac{G(x_{3n}, x_{3n+1}, x_{3n+1}) + G(x_{3n+1}, x_{3n+2}, x_{3n+2}) + G(x_{3n+2}, x_{3n+3}, , x_{3n+3})}{G(x_{3n}, x_{3n+1}, x_{3n+2}) + G(x_{3n+1}, x_{3n+2}, x_{3n+3}) + G(x_{3n+2}, x_{3n+3}, x_{3n+1}) + 1})G(x_{3n}, x_{3n+1}, x_{3n+2})\\
\leq&\frac{G(x_{3n}, x_{3n+1}, x_{3n+2}) + G(x_{3n+1}, x_{3n+2}, x_{3n+3}) + G(x_{3n+1}, x_{3n+2}, , x_{3n+3})}{G(x_{3n}, x_{3n+1}, x_{3n+2}) + G(x_{3n+1}, x_{3n+2}, x_{3n+3}) + G(x_{3n+2}, x_{3n+3}, x_{3n+1}) + 1})G(x_{3n}, x_{3n+1}, x_{3n+2})
\end{aligned}$$

so we have

$$d_{3n+1} \leq \frac{d_{3n} + 2d_{3n+1}}{d_{3n} + 2d_{3n+1} + 1} d_{3n}$$

Let

$$\alpha_{3n} = \frac{d_{3n} + 2d_{3n+1}}{d_{3n} + 2d_{3n+1} + 1}$$

so we have

$$d_{3n+1} \leq \alpha_{3n} d_{3n}$$

by introduction, we have

$$d_{3n+1} \leq \alpha_{3n}\alpha_{3n-1} \cdots \alpha_1 d_1$$

It is obvious that for any natural number $n \in \mathbb{N}$, we have $0 < \alpha_n < 1$,and so

$$d_n \leq d_{n-1}$$

then we have

$$\begin{aligned}
d_n \leq d_{n-1} &\Rightarrow d_n + d_{n+1} \leq d_{n-1} + d_n\\
&\Rightarrow 1 + \frac{1}{d_{n-1} + 2d_n} \leq 1 + \frac{1}{d_n + 2d_{n+1}}\\
&\Rightarrow \frac{1}{\alpha_{n-1}} \leq \frac{1}{\alpha_n}
\end{aligned}$$

Hence, we can get

$$\alpha_{n-1} \geq \alpha_n$$

so we can obtain

$$\alpha_{3n}\alpha_{3n-1} \cdots \alpha_1 \leq {\alpha_1}^{3n}$$

taking the limit as $n \to \infty$, so we have

$$\lim_{n\to\infty} d_{3n+1} \leq \lim_{n\to\infty} \alpha_{3n}\alpha_{3n-1}\cdots\alpha_1 d_1 \leq \lim_{n\to\infty} {\alpha_1}^{3n} d_1 = 0$$

so

$$\lim_{n\to\infty} G(x_n, x_{n+1}, x_{n+2}) = 0.$$

Next,we will show that $\{x_n\}$ is a G_b-Cauchy sequence. on the other hand, according to (G_b3) we have

$$\lim_{n\to\infty} G(x_n, x_{n+1}, x_{n+1}) \leq \lim_{n\to\infty} G(x_n, x_{n+1}, x_{n+2}) = 0 \tag{2.2}$$

for any $n, m \in \mathbb{N}$, $m > n$, using (G_b5), so we have

$$\begin{aligned}
G(x_n, x_m, x_m) &\leq sG(x_n, x_{n+1}, x_{n+1}) + sG(x_{n+1}, x_m, x_m) \\
&\leq sG(x_n, x_{n+1}, x_{n+1}) + s^2G(x_{n+1}, x_{n+2}, x_{n+2}) + s^2G(x_{n+2}, x_m, x_m) \\
&\leq sG(x_n, x_{n+1}, x_{n+1}) + s^2G(x_{n+1}, x_{n+2}, x_{n+2}) + \cdots + s^{m-n}G(x_{m-1}, x_{m-1}, x_m) \\
&\leq sG(x_n, x_{n+1}, x_{n+2}) + s^2G(x_{n+1}, x_{n+2}, x_{n+3}) + \cdots + s^{m-n}G(x_{m-1}, x_m, x_{m+1}) \\
&= d_1(s{\alpha_1}^n + s^2{\alpha_1}^{n+1} + \cdots + s^{m-n}{\alpha_1}^{m-1}) \\
&= d_1\frac{s{\alpha_1}^n(1-(s\alpha)^{m-n-1})}{1-s{\alpha_1}^n}
\end{aligned}$$

taking the limit as $n \to \infty$, then we have

$$\lim_{n\to\infty} G(x_n, x_m, x_m) \leq \lim_{n\to\infty} d_1\frac{s{\alpha_1}^n(1-(s\alpha)^{m-n-1})}{1-s{\alpha_1}^n} = 0$$

so $\{x_n\}$ is a G_b-Cauchy sequence.

Since X is complete, so there exists a $p \in X$, such that $\{x_n\}$ is a G_b-Cauchy sequence and G_b-converges to p such that

$$\begin{aligned}
\lim_{n\to\infty} x_{3n+1} &= \lim_{n\to\infty} Ax_{3n} = \lim_{n\to\infty} x_{3n+2} = \lim_{n\to\infty} Bx_{3n+1} \\
&= \lim_{n\to\infty} x_{3n+3} = \lim_{n\to\infty} Cx_{3n+2} = p.
\end{aligned}$$

Now we prove that p is a common fixed point of A, B and C.

Using Lemma 1.6 and (2.1), taking the upper limit as $n \to \infty$, we get

$$\begin{aligned}
&G(Ap, p, p) \\
&\leq s^2 \lim_{n\to\infty}\sup G(Ap, Bx_{3n+1}, Cx_{3n+2}) \\
&\leq s^2 \lim_{n\to\infty}\sup \frac{G(p, Ap, Ap) + G(x_{3n+1}, Bx_{3n+1}, Bx_{3n+1}) + G(x_{3n+2}, Cx_{3n+2}, Cx_{3n+2})}{G(p, Ap, Bx_{3n+1}) + G(x_{3n+1}, Bx_{3n+1}, Cx_{3n+2}) + G(x_{3n+2}, Cx_{3n+2}, Ap) + 1} G(p, x_{3n+1}, x_{3n+1}) \\
&\leq s^4 \lim_{n\to\infty}\sup \frac{G(p, Ap, Ap) + G(x_{3n+1}, Bx_{3n+1}, Bx_{3n+1}) + G(x_{3n+2}, Cx_{3n+2}, Cx_{3n+2})}{G(p, Ap, Bx_{3n+1}) + G(x_{3n+1}, Bx_{3n+1}, Cx_{3n+2}) + G(x_{3n+2}, Cx_{3n+2}, Ap) + 1} G(p, p, p) \\
&= 0
\end{aligned}$$

then we get $G(Ap, p, p) = 0$. Hence by (1) of Proposition 1.1, we can get $Ap = p$. Similarly, letting $x = x_{3n}$, $y = p$, $z = x_{3n+2}$ and $x = x_{3n}$, $y = x_{3n+1}$, $z = p$ we can get $Bp = p$ and $Cp = p$ respectively, so we have $Ap = Bp = Cp = p$.

Now, we show that the common fixed point of A, B and C is unique. Assume on contrary that q is another fixed point, i.e. $Aq = Bq = Cq = q$ such that $p \neq q$. Then, by our assumption, we apply (2.1) to obtain

$$\begin{aligned} G(p,p,q) &= G(Ap,Bp,Cq) \\ &\leq \frac{G(p,Ap,Ap)+G(p,Bp,Bp)+G(q,Cq,Cq)}{G(p,Ap,Bp)+G(p,Bp,Cq)+G(q,Cq,Ap)+1}G(p,p,q) \\ &= \frac{G(p,p,p)+G(p,p,p)+G(q,q,q)}{G(p,p,p)+G(p,p,q)+G(q,q,p)+1}G(p,p,q) \\ &= 0 \end{aligned}$$

so by the Proposition 1.1, we have $G(p,p,q) = 0$, then $p = q$.

Corollary 2.2 Let (X, G) be a G_b-complete G_b-metric space and T be a mapping from X to itself. Suppose that T satisfy the following condition:

$$G(Tx,Ty,Tz) \leq \frac{G(x,Tx,Tx)+G(x,Tx,Tx)+G(x,Tx,Tx)}{G(x,Tx,Ty)+G(y,Ty,Tz)+G(z,Tz,Tx)+1})G(x,y,z)$$

for all $x, y, z \in X$. Then T has a unique fixed point.

Proof. Taking $A = B = C = T$, the result follow from Theorem 2.1.

Theorem 2.3 Let (X, G) be a G_b-complete G_b-metric space and A, B and C are mappings from X to itself. Suppose that A, B and C satisfy the following condition:

$$G(Ax,By,Cz) \leq \alpha\frac{\min\{G(y,By,By),G(z,Cz,Cz)\}}{G(z,Cz,Ax)+1}G(x,Ax,Ax)+\beta G(x,y,z) \tag{2.3}$$

for all $x, y, z \in X$, where $\alpha + \beta \leq 1$.

Then either one of A, B and C has a fixed point, or, A, B and C have a unique common fixed point.

Proof. Let $d_n = G(x_n, x_{n+1}, x_{n+2})$, then for (2.3) we have

$$\begin{aligned} G(Ax_{3n},Bx_{3n+1},Cx_{3n+2}) &\leq \alpha\frac{\min\{G(x_{3n+1},Bx_{3n+1},Bx_{3n+1}),G(x_{3n+2},Cx_{3n+2},Cx_{3n+2})\}}{G(x_{3n+2},Cx_{3n+2},Ax_{3n})+1}) \\ &\quad G(x_{3n},Ax_{3n},Ax_{3n})+\beta G(x_{3n},x_{3n+1},x_{3n+2}) \\ &= \alpha\frac{\min\{G(x_{3n+1},x_{3n+2},x_{3n+2}),G(x_{3n+2},x_{3n+3},x_{3n+3})\}}{G(x_{3n+2},x_{3n+3},x_{3n+1})+1}) \\ &\quad G(x_{3n},x_{3n+1},x_{3n+1})+\beta G(x_{3n},x_{3n+1},x_{3n+2}) \\ &\leq \alpha\frac{d_{3n+1}}{d_{3n+1}+1}d_{3n}+\beta d_{3n} \\ &= (\alpha\frac{d_{3n+1}}{d_{3n+1}+1}+\beta)d_{3n} \end{aligned}$$

Since $\alpha\frac{d_{3n+1}}{d_{3n+1}+1} + \beta \leq 1$, the following proof is similar to Theorem 2.1.

Corollary 2.4 Let (X, G) be a G_b-complete G_b-metric space and T be mapping from X to itself. Suppose that T satisfy the following condition:

$$G(Tx,Ty,Tz) \leq \alpha(\frac{\min\{G(y,Ty,Ty),G(z,Tz,Tz)\}}{G(z,Tz,Tx)+1})G(x,Tx,Tx)+\beta G(x,y,z)$$

for all $x, y, z \in X$, where $\alpha + \beta \leq 1$.

Then T has a unique fixed point.

Proof. Taking $A = B = C = T$, the result follow from Theorem 2.4.

Theorem 2.5 Let (X, G) be a G-complete G-metric space and A, B and C are mappings from X to itself. Suppose that A, B and C satisfy the following condition:

$$G(Ax, By, Cz) \leq \frac{G(x, Ax, Ax) + G(y, By, By) + G(z, Cz, Cz)}{G(x, Ax, By) + G(y, By, Cz) + G(z, Cz, Ax) + 1})G(x, y, z)$$

for all $x, y, z \in X$. Then either one of A, B and C has a fixed point, or, A, B and C have a unique common fixed point.

Proof. The proof is similar to Theorem 2.1. There is a little difference between them.

First, when we prove that $\{x_n\}$ is a G_b-Cauchy sequence, we have

$$\begin{aligned} G(x_n, x_m, x_m) &\leq G(x_n, x_{n+1}, x_{n+1}) + G(x_{n+1}, x_m, x_m) \\ &\leq G(x_n, x_{n+1}, x_{n+1}) + G(x_{n+1}, x_{n+2}, x_{n+2}) + G(x_{n+2}, x_m, x_m) \\ &\leq G(x_n, x_{n+1}, x_{n+1}) + G(x_{n+1}, x_{n+2}, x_{n+2}) + \cdots + G(x_{m-1}, x_{m-1}, x_m) \\ &\leq \sum_{i=0}^{m-n} G(x_{n+i}, x_{n+i+1}, x_{n+i+1}) \end{aligned}$$

so we can get

$$\begin{aligned} G(x_n, x_m, x_m) &\leq \sum_{i=0}^{m-n} G(x_{n+i}, x_{n+i+1}, x_{n+i+1}) \\ &\leq \sum_{i=0}^{m-n} G(x_{n+i}, x_{n+i+1}, x_{n+i+2}) \\ &= \sum_{i=0}^{m-n} d_{n+i} \end{aligned}$$

taking the limit as $n \to \infty$, then we have

$$\lim_{n\to\infty} G(x_n, x_m, x_m) \leq \lim_{n\to\infty} \sum_{i=0}^{m-n} {\alpha_1}^{n+i} d_1 = 0$$

so $\{x_n\}$ is a G-Cauchy sequence.

Secondly, since that G-metric space is continuous so when we prove that p is a common fixed point in G-metric space we have

$$G(Ap, p, p) \leq \frac{G(p, Ap, Ap) + G(p, Bp, Bp) + G(p, Cp, Cp)}{G(p, Ap, Bp) + G(p, Bp, Cp) + G(p, Cp, Ap) + 1} G(p, p, p) = 0$$

Corollary 2.6 Let (X, G) be a G-complete G-metric space and T be a mapping from X to itself. Suppose that T satisfy the following condition:

$$G(Tx, Ty, Tz) \leq \frac{G(x, Tx, Tx) + G(x, Tx, Tx) + G(x, Tx, Tx)}{G(x, Tx, By) + G(y, Ty, Tz) + G(z, Tz, Tx) + 1})G(x, y, z)$$

for all $x, y, z \in X$. Then T has a unique fixed point.

Proof. Taking $A = B = C = T$, the result follow from Theorem 2.6.

3 An example

Example 3.1 Let $G(x,y,z) = (\max\{|x-y|,|y-x|,|z-x|\})^2$ for all $x,y,z \in X$ then G is a G_b-metric on X where $s=2$. Define self-mappings A,B and C on x by

$$A(x) = 1, B(x) = 1, C(x) = \frac{7+x}{8}$$

Then we have

$$\begin{aligned} G(Ax,By,Cz) &= (\max\{|1-1|,|1-\frac{7+z}{8}|,|1-\frac{7+z}{8}|\})^2 \\ &= (\frac{1}{8}-\frac{z}{8})^2 \end{aligned}$$

and we also have

$$G(x,Ax,Ax) = (1-x)^2, G(y,By,By) = (1-y)^2$$
$$G(z,Cz,Cz) = (\frac{7-7z}{8})^2, G(x,Ax,By) = (1-x)^2$$
$$G(y,By,Cz) = \max\{(1-y)^2,(\frac{1-z}{8})^2,(\frac{7+z}{8}-y)^2\}$$
$$G(z,Cz,Ax) = (1-z)^2$$
$$G(x,y,z) = \max\{|x-y|^2,|y-z|^2,|z-x|^2\}$$

Case1: when $z \leq x, z \leq y$ $\alpha \geq \frac{1}{64}, \beta = 0$ and $y \leq \frac{1+7z}{8}$, we have the (2.3) established. Then $x=y=z=1$ is a common fixed point.

Case2: when $y \geq \frac{1+7z}{8}$ $\alpha = 1, \beta = 0$ and $z \geq -6+7x$, we have the (2.3) established. Then $x=y=z=1$ is a common fixed point.

Acknowledgements

This work is supported by the Natural Science Foundation of China (11771198, 11361042, 11071108, 11461045, 11701259), the Natural Science Foundation of Jiangxi Province of China (20132BAB201001,20142BAB211016) and the Scientific Program of the Provincial Education Department of Jiangxi (GJJ150008) and the Innovation Program of the Graduate student of Nanchang University(colonel-level project).

References

[1] Z. Mustafa, B. Sims, A new approach to generalized metric space. J. Nonlinear Convex Anal. 7, 289-297, (2006).

[2] Z. Mustafa, B. Sims, Fixed point theorems for contractive mappings in complete G-metric space, Fixed Point Theory Appl. 2009, 977175, (2013).

[3] H. Obiedat, Z. Mustafa, Fixed point result on a nonsymmetric G-metric space, Jordan J. Math. Stat. 3, 65-79, (2010).

[4] A. Azam, N.Mehmood, Fixed point theorems for multivalued mappings in G-cone metric space, J. Inequal. Appl. 2013, 354, (2013).

[5] Y. U. Caba, Fixed point theorems in G-metric space, J. Math. Anal. Appl. 455, 528-537, (2017).

[6] P. N. Dutta, B. S. Choudhury, K. Das, Some fixed point results in Menger spaces, Surv. Math. Appl. 4, 41-52, (2009).

[7] Z. Mustafa, J. R. Roshan, Coupled coincidence point result for (ψ, φ)-weakly contractive mappings in partially ordered G_b-metric spaces. Fixed Point Theory Appl. 2013 , 206, (2013).

[8] J. R. Roshan, V. Sedghi, Common fixed point of almost generalized $(\psi, \varphi)_s$-contractive mappings in ordered b-metric space. Fixed Point Theory Appl. 2013, 159, (2013).

[9] V.Parvaneh, J. R. Radenović, Existence of tripled coincidence points in ordered b-metric spaces and an application to a system of integral equations, Fixed Point Theory Appl. 2013, 130, (2013).

[10] S. Czerwik, Nonlinear set-valued contraction mappings in b-metric space, Atti Sem Mat Fis Univ Modena. 46, 236-276, (1998).

[11] R. R. Jamal, S. Nabioliah, Common fixed point theorems for three maps in discontinuous G_b-metric spaces, Act Math Sci. 34, 1643-1654, (2014).

[12] A. Aghajani, M. Abbas, J. R. Roshan, C common fixed point of generalized weak contractive mappings in partially ordered G_b-metric space. Filomat. 28, 1087-1101, (2014).

[13] I. A. Baskhtin, The contraction mapping principle in quasimetric spaces, Func Anal Unianowsk Gos Ped Inst. 30, 26-37, (1989).

[14] C. X. Zhu, Several nonlinear operator problems in Menger PN space, Nonlinear Anal. 65 1281-1284 (2006).

[15] C. X. Zhu, Research on some problems for nonlinear operator, Nonlinear Anal. 71, 4568-4571, (2009).

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 27, NO.7, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC

A modified collocation method for weakly singular Fredholm integral equations of second kind*

Guang Zeng[a, b]†, Chaomin Chen[a,]‡, Li Lei[a, b]§, Xi Xu[b,]¶

[a] *Fundamental Science on Radioactive Geology and Exploration Technology Laboratory, East China University of Technology, Nanchang, Jiangxi, 330013, P.R. China*

[b] *School of Scinence, East China University of Technology, Nanchang, Jiangxi, 330013, P.R. China*

Abstract

In this paper, a collocation method with high precision by using the polynomial basis functions is proposed to solve the Fredholm integral equation of second kind with weakly singular kernel. We introduce the polynomial basis functions and use it to reduce the given equation to a system of linear algebraic equation. Thus, we can simplify the solving of the equation. The error analysis are given. Numerical examples are given to illustrate the efficiency of our method.

Keyword: Weakly Singular · Fredholm Integral Equation · Polynomial basis function Method

AMS subject classification: 65D10 · 65D32

1 Introduction

This paper is concerned with collocation method for weakly singular Fredholm integral equations of the second kind as follows

*The work is supported by National Natural Science Foundation of China(11661005,11301070).

†Corresponding author: zengguang5340@sina.com (G. Zeng)

‡cmchen93@163.com (C. Chen)

§betterleili@163.com (L. Lei)

¶xuxi93@163.com (X. Xu)

$$\phi(x) + \lambda \int_a^b \kappa(x,t)\phi(t)dt = f(x), 0 \leq x \leq 1, \tag{1.1}$$

where $\kappa(x,t) = \frac{H(x,t)}{|x-t|^\alpha}, 0 < \alpha < 1, H(x,t), f(x)$ are continue and bounded functions and $\phi(x)$ is the function to be determined.

Numerical methods for weakly singular Fredholm integral equations of the second kind have been developed by many scholars in recent years because of their important applications in science and engineering. These methods can be classified into two types. One type is through making approximations to the analytical solutions directly. For instance, Tricomi used successive approximations method to solve the integral equations in his book [1]. Variational iteration method and Adomian decomposition method were introduced in [2] and [3] respectively. Also, The homotopy analysis method was proposed by Liao [4] and has applied it in [5] et. Another type is through shifting the equations into a form which easier to solve than the original equations. For example, Taylor expansion collocation methods are presented to solve integral equations in [6-8]. In [9], the orthogonal triangular basis functions were used by Babolian et al. to solve some integral equations systems. And Legendre wavelets method was proposed by Jafari et al. in [10] to find the numerical solutions of linear integral equations systems. Moreover, in [12] architecture artificial neural networks was suggested to approximate the solutions of linear integral equations systems. Furthermore, Jafarian et al. [13] using the Bernstein polynomials to obtain the numerical solutions of linear Fredholm and Volterra integral equations systems of the second kind. And application of Bernstein polynomial have been made by scholar for solving both differential equations and integral equations, see [11]. And piecewise polynomial collocation method were applied to solve the Volterra integro-differential equations with weakly singular kernel in [14] respectively. And the stability of piecewise polynomial collocation methods for solving weakly singular integral equations of the second kind has been discussed by Kangro et al. in [15]. Besides, Baratella et al. [16] had proposed an approach with product integration to solve the weakly singular Volterra integral equations. Kolk et al. And Pallaw et al. [17] used the quadratic spline collocation to solve the smoothed weakly singular Fredholm integral equations. However, these methods introduced above do not provide a good accuracy in the solution near the singular points.

In this paper, we are going to use polynomial basis functions collocation method to approximate the solution of singular Fredholm integral equations of the second kind. The proposed approach converted the given equation with unique solution into a system of linear algebraic equations in general case. To do this, first the polynomial basis functions of certain degree n of unknown functions are substituted in the given integral equations. So that the solution of the unknown function of given equations have converted into the solutions of the coefficients of the unknown polynomial basis functions, such that we can solve the integral equations in a convenient way.

The layout of this paper is as follows: In section 2 we presented the procedure of the polynomial basis functions collocation method to obtain the approximate solution of the weakly singular Fredholm integral equation. In section 3, we had demonstrated that the proposed method is convergent to all the weakly singular Fredholm integral equations of second kind. In section 4, we give numerical example to test the effectiveness and efficiency of the method. Finally, Numerical examples are given to illustrate the efficiency of our method.

2 The Polynomial Basis Function Method

We are going to use the polynomial basis functions to solve the eq.(1.1). The form of the functions are as follows:

$$U = \sum_{k=0}^{m-1} x^k,$$

where the polynomial basis functions $1, x, x^2, \cdots, x^{m-1}$ are linear independent.

Since eq.(1.1) is a weakly singular integral equation, the singularity of the e-quation must be removed such that the procedure of solving the problem can be move on. But since the proposed method of this paper is belong to the colloca-tion method, which can smooth the singular points of the discretion, so that we can use the method directly. Then we provided the procedure of using polynomial basis functions to solve the kind of the integral equations proposed in this paper concretely as follows:

Step 1. Choosing the basis functions $u = [1, x, x^2, ..., x^k], (k = 0, 1, 2, ..., m-1)$ the unknown function $\phi(x)$ is substituted by the following polynomials

$$\phi(x) \approx \phi_m(x) = \sum_{k=0}^{m-1} a_k x^k, \tag{2.1}$$

Step 2. Substituting (2.1) into (1.1) we have

$$\sum_{k=0}^{m-1} a_k x^k + \lambda \sum_{k=0}^{m-1} a_k x^k \int_a^b \kappa(x,t) t^k dt = f(x), \tag{2.2}$$

Step 3. Discrete the interval [a,b] into n sections uniformly, we obtained the systems of the coefficient a_k as follows

$$\sum_{k=0}^{m-1} a_k x_j^k + \lambda \sum_{k=0}^{m-1} a_k x_j^k \int_a^b \kappa(x,t) t^k dt = f(x_j), \tag{2.3}$$

where $j = 1, 2, ..., n, x_j = a + j(b-a)/n$. We transformed the equations into the form of linear matrix as follows

$$(U + KU)A = f, \tag{2.4}$$

where

$$U = \begin{pmatrix} 1 & x_1 & \cdots & x_1^{m-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & \cdots & x_n^{m-1} \end{pmatrix}, A = \begin{pmatrix} a_0 \\ \vdots \\ a_{m-1} \end{pmatrix}, f = \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix}, \tag{2.5}$$

and $K = \int_a^b \kappa(x,t)dt$ which is the integral operator.

Step 4. Solve the system we obtained the solutions of the coefficients of a_k as follows

$$a_0, a_1, ..., a_{m-1}.$$

Substituting them into eq.(2.1) we obtained the approximate solution $\phi_m(x)$.

3 Convergence and Error Analysis

In this section, we are going to prove that the approximate method we proposed in this paper is convergent to the analytic solution of eq.(1.1).

Firstly, we rewrite the form of the weakly singular kernel as follows

$$K(x,t) = \frac{H(x,t)}{|x-t|^\alpha}.$$

Let $0 < \alpha \leq \frac{1}{2}$, and $H(x,t)$ is continuously bounded. Then the eigenvalue integral equation with weakly singular kernel is as follows

$$\lambda\phi(x) = \int_a^b K(x,t)\phi(t)dt, 0 \leq x \leq 1, \tag{3.1}$$

where $K(x,t)$ is the weakly singular kernel, λ is the eigenvalue of the $K(x,t)$, $\phi(x)$ is the eigenfunction of λ.

Lemma 3.1 [14]. If $x_1, x_2 \in C^{m,v}(0,T], m \in N, v < 1$, then $x_1x_2 \in C^{m,v}(0,T]$, and

$$\|x_1x_2\|_{m,v} \leq c\|x_1\|_{m,v}\|x_2\|_{m,v}$$

with a constant c which is independent of x_1 and x_2.

Proof. See [14].

Lemma 3.2 [18] Suppose that the function $\phi_m(x)$ obtained by the polynomial basis function is the approximation of eq.(1) and eq.(1) is with bounded first

derivative, then eq.(1) can be expanded as an infinite sum of the polynomial basis functions, that is, $\phi(x) = \sum_{k=0}^{m-1} c_k x^k$, and the coefficients c_k are bounded as

$$c_k < \frac{K}{(m+1)2^{\frac{3k}{2}}}$$

where K is a constant.

Proof. See [18].

Let the linear operator $\mathbf{K} : L^1_{[0,1]} \to L^1_{[0,1]}$

$$(\mathbf{K}\phi)(x) = \int_0^1 K(x,t)\phi(t)dt, 0 \leq x \leq 1$$

then (3.1) can be written as

$$\mathbf{K}\phi = \lambda\phi, \tag{3.2}$$

using $\mathbf{K}$ operating two sides of (3.2) we yield

$$\mathbf{K}^2\phi = \lambda^2\phi$$

where

$$\mathbf{K}^2\phi(x) = \int_0^1 K_2(x,t)\phi(t)dt$$

and $K_2(x,t)$ is the iterative kernel of $K(x,t)$

$$K^2(x,t) = \int_0^1 K_1(x,r)K_1(r,t)dr$$

$$K_1(x,r) = K(x,r).$$

Theorem 3.3. Let $\phi_m(x)$ be the polynomial basis function of degree $m-1$ and whose coefficients has been obtained by solving linear system (2.4), the given polynomial basis function is converge to the analytical solution of the weak singular Fredholm integral equations of the second kind (1.1), when $m \to \infty$.

Proof. Since

$$\phi(x) = \lim_{m\to\infty} \phi_m(x),$$

substitute $\phi_m(x)$ into eq.(1.1), we have

$$\phi_m(x) + \lambda \int_a^b K(x,t)\phi_m(t)dt = f(x), 0 \leq x \leq 1. \tag{3.3}$$

We defined the error function $\|e_m\|$ by subtracting (2.1) and (2.5) as follows

$$\|e_m\| = \|\phi_m(x) - \phi(x)\| + |\lambda| \| \int_0^1 \|K(x,t)\| \cdot \|\phi_m(t) - \phi(t)\| dt\|,$$

According to lemma 3.1 and 3.2, since the subinterval of integral equation is compact and the coefficients obtained by the polynomial basis functions are bounded and the kernel $K(x,t)$ can be continuous and bounded through iteration, therefore, whether

$$\|e_m\| \to 0$$

depends on

$$\|\phi_m(x) - \phi(x)\| \to 0,$$

since

$$\phi(x) = \lim_{m\to\infty} \phi_m(x),$$

that is,

$$\|e_m\| \to 0$$

when $m \to \infty$.

Thus, the proof is completed. $\Box$

Remark 3.4. When we use this method we can find that it is similar to the piecewise linear spline function interpolation method which is convergent and numerical stable. The speed of the convergency is accelerated with the increasing of the degree m of the polynomial basis function.

4 Numerical Experiments

Example 1: Consider the following Fredholm integral equations of the second kind with weakly singular kernel

$$\phi(x) - \frac{1}{10}\int_0^1 K(x,t)\phi(t)dt = f(x), 0 \le x \le 1, \tag{4.1}$$

where $K(x,t) = |x-t|^{-\frac{1}{3}}$,

$$f(x) = x^2(1-x)^2 - \frac{27}{30800}[x^{\frac{8}{3}}(54x^2 - 126x + 77) + (1-x)^{\frac{8}{3}}(54x^2 + 18x + 5)].$$

the exact solution of eq.(4.1) is $\phi(x) = x^2(1-x)^2$.

Using the method we proposed in section 2 and the successive approximation method and using MATLAB writing the program codes we obtained the figures and tables so that we can make a comparison for the accuracy of the two methods.

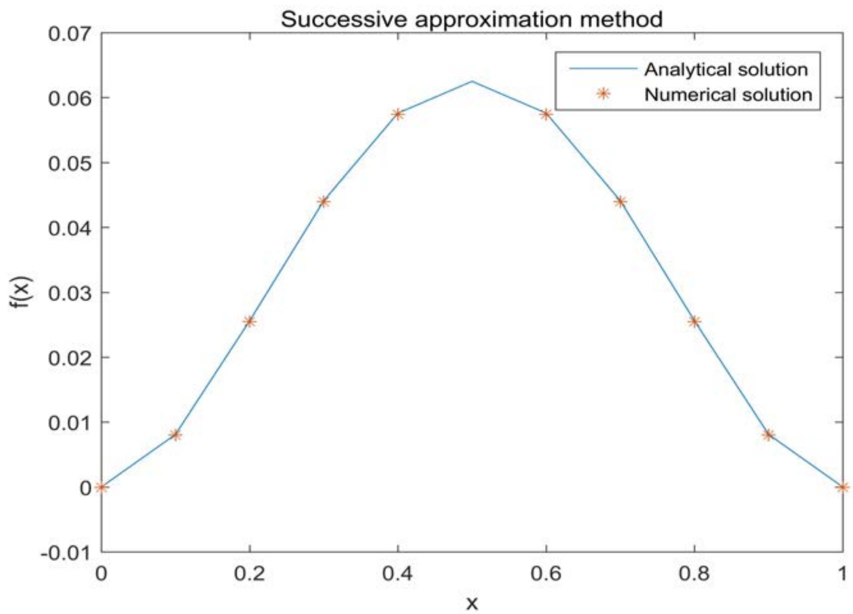


Figure 1: The nodes are 11, iterations are 6, The result of Successive approximation method and the analytical solutions.

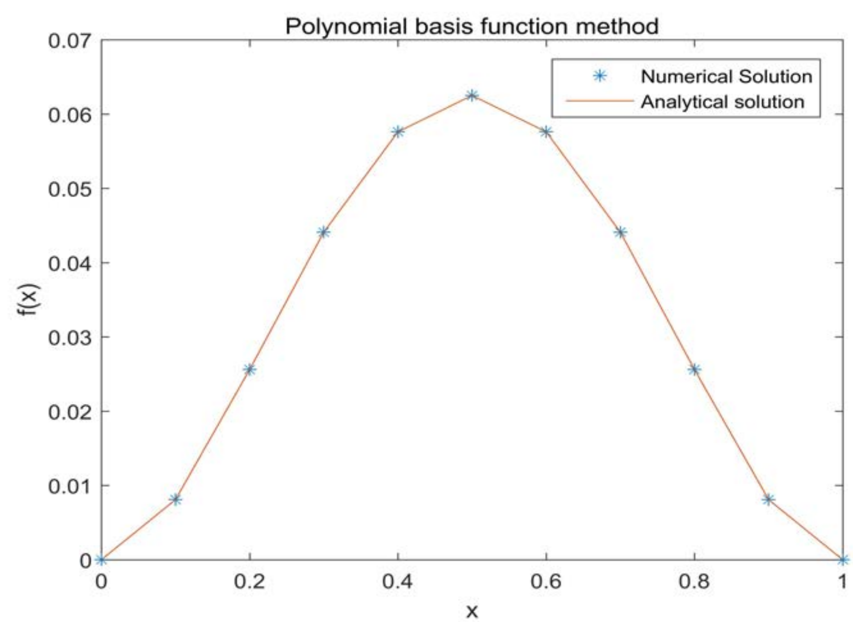


Figure 2: The nodes are 11, k=4, The result of Polynomial basis function method and the analytical solutions.

Firstly, we obtained the figures of the results of the Polynomial basis function method and the successive approximation method

From the figures we can find that both of the curves of successive approximation method and the polynomial basis functions method are simulated very well, but there is a defection of the numerical solution of successive approximation method that the singular point in the function can not be removed. But the polynomial basis functions method almost accordant with the analytical solutions. Namely, the accuracy of polynomial basis functions method is better than the successive approximation methods.

Table 1: The comparison of the solutions of the two kinds of methods.

Node	Exact Solution	Successive Approximation	Polynomial Basis Function Method
0	0	-2.8235e-04	-1.136e-016
0.1	8.1000e-003	7.7634e-03	8.1000e-003
0.2	2.5600e-002	2.5228e-02	2.5600e-002
0.3	4.4100e-002	4.3685e-02	4.4100e-002
0.4	5.7600e-002	5.7144e-02	5.7600e-002
0.5	6.2500e-002	NaN	6.2500e-002
0.6	5.7600e-002	5.7144e-02	5.7600e-002
0.7	4.4100e-002	4.3685e-02	4.4100e-002
0.8	2.5600e-002	2.5228e-02	2.5600e-002
0.9	8.1000e-003	7.7634e-03	8.1000e-003
1	0	-2.8235e-04	0

The Table 1 shows the results of the solutions of the example 1 using successive approximation method with the iterations k=8 and the polynomial basis function method with the orders m=5 of the polynomial basis function, respectively. From the table we can find that the results of the polynomial basis function method is more approximate to the exact solutions than the successive approximation method.

From the Table 2 we can easily find that with the increasing of the iterations of k, there is little increasing of the error accuracy of the successive approximation method. And it is obvious that there is a singular point of the discrete interval.

The Table 3 shows the errors accuracy results of the polynomial basis function method when the orders of the polynomials are n=3,4,5,6, respectively. We can find that the results is much superior than the successive approximation method. The best error effectiveness of successive approximation is $O(10^{-4})$, but we obtained the high accuracy of the polynomial basis function method when the orders of the polynomial basis functions is $n = 5$ and the effective errors have reached $O(10^{-16})$, which is much better than the successive approximation methods.

Table 2: The error comparison of Successive Approximation methods.

Node	Exact solution	k=2	k=4	k=6	k=8
0	0	9.2298e-03	6.6592e-05	2.7807e-04	2.8225e-04
0.1	8.1000e-03	1.0586e-02	7.7066e-05	3.3105e-04	3.3646e-04
0.2	2.5600e-02	1.1068e-02	1.1413e-04	3.6606e-04	3.7179e-04
0.3	4.4100e-02	1.1342e-02	1.5033e-04	4.0921e-04	4.1510e-04
0.4	5.7600e-02	1.1480e-02	1.8759e-04	4.4996e-04	4.5593e-04
0.5	6.2500e-02	NaN	NaN	NaN	NaN
0.6	5.7600e-02	1.1480e-02	1.8759e-04	4.4996e-04	4.5593e-04
0.7	4.4100e-02	1.1342e-02	1.5033e-04	4.0921e-04	4.1510e-04
0.8	2.5600e-02	1.1068e-02	1.1413e-04	3.6606e-04	3.7179e-04
0.9	8.1000e-03	1.0586e-02	7.7066e-05	3.3105e-04	3.3646e-04
1	0	9.2298e-03	6.6592e-05	2.7807e-04	2.8225e-04

Table 3: The error comparison of Polynomial basis function methods.

Node	Exact solution	n=3	n=4	n=5	n=6
0	0	6.7365e-03	6.7365e-03	2.0322e-16	3.6580e-16
0.1	8.1000e-03	7.5301e-03	7.5301e-03	8.3267e-17	2.9490e-16
0.2	2.5600e-02	7.4263e-03	7.4263e-03	4.1633e-17	2.1164e-16
0.3	4.4100e-02	1.3522e-03	1.3522e-03	6.2450e-17	1.8041e-16
0.4	5.7600e-02	4.6922e-03	4.6922e-03	4.8572e-17	1.3184e-16
0.5	6.2500e-02	7.1071e-03	7.1071e-03	1.3878e-17	9.7145e-17
0.6	5.7600e-02	4.6922e-03	4.6922e-03	2.7756e-17	8.3267e-17
0.7	4.4100e-02	1.3522e-03	1.3522e-03	3.4694e-17	1.3878e-17
0.8	2.5600e-02	7.4263e-03	7.4263e-03	1.4572e-16	1.5613e-16
0.9	8.1000e-03	7.5301e-03	7.5301e-03	2.2204e-16	2.7756e-16
1	0	6.7365e-03	6.7365e-03	0	1.5260e-16

Example 2: Consider the following Fredholm integral equations of the second kind with weakly singular kernel

$$\phi(x) - \frac{1}{10}\int_0^1 K(x,t)\phi(t)dt = f(x), 0 \leq x \leq 1, \tag{4.2}$$

where $K(x,t) = |x-t|^{-\frac{1}{2}}$,

$$f(x) = x^2(1-x)^2 - \frac{27}{30800}[x^{\frac{8}{3}}(54x^2 - 126x + 77) + (1-x)^{\frac{8}{3}}(54x^2 + 18x + 5)].$$

We have not the exact solutions of the example 2, but we compared the accuracy of the two methods through the error accuracy when the iterations increased of

Table 4: The error comparison of succcessive approximation methods.

Node	n=2	n=4	n=6	errors of (c3-c2)	errors of (c4-c3)
0.0139	5.5529e-04	7.8461e-04	7.9539e-04	2.2932e-04	1.0780e-05
0.0556	3.3207e-03	3.5784e-03	3.5903e-03	2.5772e-04	1.1917e-05
0.1250	1.2872e-02	1.3386e-02	1.3399e-02	5.1414e-04	1.2785e-05
0.2222	3.2102e-02	3.2388e-02	3.2402e-02	2.8611e-04	1.3450e-05
0.3472	5.4893e-02	5.5188e-02	5.5202e-02	2.9552e-04	1.3437e-05
0.5000	NaN	NaN	NaN	NaN	NaN
0.6528	5.4893e-02	5.5188e-02	5.5202e-02	2.9552e-04	1.3437e-05
0.7778	3.2102e-02	3.2388e-02	3.2402e-02	2.8611e-04	1.3450e-05
0.8750	1.2872e-02	1.3386e-02	1.3399e-02	5.1414e-04	1.2785e-05
0.9444	3.3207e-03	3.5784e-03	3.5903e-03	2.5772e-04	1.1917e-05
0.9861	5.5529e-04	7.8461e-04	7.9539e-04	2.2932e-04	1.0780e-05

the successive approximation method and when the orders of the polynomial basis function increased, respectively. The column 2 to column 4 of Table 4 shows the solutions of the method when the iterations k=2,4,6, respectively, and it shows the error accuracy of the solutions of column 3 minus column 2 and column 4 minus column 3 and we get column 5 and column 6, respectively. From the Table 4 we can easily find that, with the increasing of the iterations of the successive approximation method, the error accuracy increased accordingly.

Table 5: The error comparison of polynomial basis function methods.

Node	n=4	n=5	n=6	errors of (c3-c2)	errors of (c4-c3)
0.0139	-1.2677e-04	6.3355e-03	6.3355e-03	6.4623e-03	1.8388e-16
0.0556	1.1224e-02	9.5719e-03	9.5719e-03	-1.6520e-03	3.1225e-17
0.1250	2.7883e-02	2.0391e-02	2.0391e-02	-7.4917e-03	-1.3878e-17
0.2222	4.6462e-02	4.0972e-02	4.0972e-02	-5.4890e-03	9.7145e-17
0.3472	6.2217e-02	6.5462e-02	6.5462e-02	3.2455e-03	1.8041e-16
0.5000	6.9050e-02	7.8091e-02	7.8091e-02	9.0411e-03	6.9389e-17
0.6528	6.2217e-02	6.5462e-02	6.5462e-02	3.2455e-03	-1.8041e-16
0.7778	4.6462e-02	4.0972e-02	4.0972e-02	-5.4890e-03	-1.0408e-16
0.8750	2.7883e-02	2.0391e-02	2.0391e-02	-7.4917e-03	2.0470e-16
0.9444	1.1224e-02	9.5719e-03	9.5719e-03	-1.6520e-03	4.5103e-17
0.9861	-1.2677e-04	6.3355e-03	6.3355e-03	6.4623e-03	-8.5001e-17

Table 5 shows the results of the solutions of the method we proposed in this paper. It shows the results of the solutions of the proposed method from the column 2 to column 4, and the error accuracy results obtained by column 3 minus column

2 and column 4 minus column 3 and we get column 5 and column 6, respectively. From the data of the table 5 we can easily find that the polynomial basis function method is much superior than the successive approximation method. The best error effectiveness of successive approximation we finally obtained is $O(10^{-5})$, but we obtained the high accuracy of the polynomial basis function method when we let $n = 5$ and the effective errors have reached $O(10^{-16})$, which is much nearly to the exact solutions.

5 Acknowledgements

This research was supported in part by the National Natural Science Foundation of China(11661005,11301070,11661004), the Natural Science Foundation of Jiangxi Province of China, the Science Foundation of Education Committee of Jiangxi for Young Scholar.

References

[1] Tricomi F.G., Integral Equations. Dover, New York, 1982.

[2] Lan X., Variational iteration method for solving integral equations. Comput. Math. Appl. 54:1071-1078, 2007.

[3] Babolian E., Sadeghi Goghary S., Abbasbandy S., Numerical solution of linear Fredholm fuzzy integral equations of the second kind by Adomian method. Appl. Math. Comput. 161:733-744, 2005.

[4] Liao S.J., Beyond Perturbation: Introduction to the Homotopy Analysis Method. Chapman & Hall/CRC Press, Boca Raton, 2003.

[5] Abbanbandy S., Numerical solution of integral equations: homotopy perturbation method and Adomin's decomposition methond. Appl. Math. Comput. 161:733-744, 2006.

[6] Kanwal R.P., Liu K.C.: A Taylor expansion approach for soiving integral equations. Int. J. Math. Educ. Sci. Technol. 2:411-414, 1989.

[7] Maleknejad K., Aghazadh N., Numerical solution of Volterra integral equations of the second kind with convolution kernel by using Taylor-series expansion method. Appl. Math. Comput. 161:915-922, 2005.

[8] Nas S,, Yalcynbas S., Sezer M., A Taylor polynomial approach for soving higher-order linear Fredholm integro-differential equations. Int. J. Math. Educt. Sci. Technol. 31:213-225, 2000.

[9] Babolian E., Masouri Z., Hatamzadeh-Varmazyar S., A direct method for numerically solving integral equations system using orthogonal triangular functions. lint. J. lind. Math. 2: 1365-145, 2009.

[10] Jafari H., Hosseinzadeh H., Mohamadzadeh S., Numerical solution system of linear integral equations by using Legendre wavelets. lnt. J. Open probl. Comput. Sci. Math. 5:63-71, 2010.

[11] Farouki R.T., Goodman T.N.T., On the optimal stability of the Bernstein basis. Math. Comput. 65(216):1553-1566, 1996.

[12] Navot I., A further extension of the Euler-Maclaurin summation formula. J. Math. Phy., 41:155-163, 1962.

[13] Hochstadt H., Integral Equations. Wiley, New York, 1973.

[14] Brunner H., Pedas A., Vaainikko G., Piecewise polynomical collocation methods for linear Volterra integro-differential equations with weakly singular kernels. SIAM J Numer Anal, 39(3):957-982, 2001.

[15] Kangro I., Kangro R., On the stability of piecewise polynomial collocation methods for solving weakly singular integral equations of the second kind. Math Model Anal, 13(1): 29-36, 2008.

[16] Baratella P., Orsi A. P., A new approach to the numerical solution of weakly singular Volterra integral equations. J Comput Appl Math, 163:401-418, 2004.

[17] Pallaw R., pedas A., Quadratic spline collocation for the smoothed weakly singular Fredholm integral equations. Numer Funct Anal Optim, 30(9-10):1048-1064, 2009.

[18] M.A. Yan, L. Wang, H. Wang and X. Zhang, The reseach of eigenvalue numerical solution methods for weakly singular integral equation in L^1 space, Mathematics in Practice and Theory, 43(2):199-207, 2013.

SHARP COEFFICIENT ESTIMATES FOR NON-BAZILEVIČ FUNCTIONS

JI HYANG PARK, VIRENDRA KUMAR, AND NAK EUN CHO

Abstract. The class $\bar{\mathcal{B}}(\alpha)$ of non-Bazilevič functions was introduced by Obradović. Later, estimates on the second coefficient and Fekete–Szegö functional for normalized analytic functions in the class $\bar{\mathcal{B}}(\alpha)$ were investigated by Tuneski and Darus. In the present work, sharp estimate on third to eighth coefficients for normalized analytic functions $f(z) = z + a_2z^2 + a_3z^3 + \cdots \in \bar{\mathcal{B}}(\alpha)$ are investigated. Further sharp estimate on the functional $|a_2a_3 - a_4|$ is also obtained.

1. Introduction

The class of analytic functions defined in the unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ and having the Taylor series expansion of the form

$$f(z) = z + a_2z^2 + a_3z^3 + \cdots \tag{1.1}$$

is denoted by $\mathcal{A}$. The subclass of $\mathcal{A}$ consisting of univalent functions is denoted by $\mathcal{S}$. De Branges, in 1984, proved that if $f \in \mathcal{S}$, then $|a_n| \leq n$. This result was put before by Bieberbach in 1916 and is popularly known as the Bieberbach conjecture. Among the many subclasses of $\mathcal{S}$, the class of starlike and convex functions are the most investigated. The class of starlike and convex functions are defined, respectively, by $\mathcal{S}^* := \{f \in \mathcal{S} : \mathrm{Re}(zf'(z)/f(z)) > 0\}$ and $\mathcal{K} := \{f \in \mathcal{S} : \mathrm{Re}(1 + zf''(z)/f'(z)) > 0\}$. Thomas [15], in 1967, introduced a general form of the class of starlike functions. Thomas [15], for a starlike functions g, defined the class $\mathcal{B}_\alpha := \{f \in \mathcal{S} : \mathrm{Re}(zf'(z)f(z)^{\alpha-1}/g(z)^\alpha) > 0\}$. This class is popularly known as the class of Bazilevič functions of type α. In 1973, Singh [12] investigated a special case of $\mathcal{B}_\alpha$. For $\alpha \geq 0$ and setting $g(z) = z$, he considered a subclass of $\mathcal{B}_\alpha$ defined by

$$\mathcal{B}_1(\alpha) := \left\{ f \in \mathcal{A} : \mathrm{Re}\left(f'(z) \left(\frac{f(z)}{z} \right)^{\alpha-1} \right) > 0 \right\}.$$

In his paper, he obtained the sharp radius estimates for certain integral operator to be a member of the class $\mathcal{B}_1(\alpha)$ and he also obtained the sharp upper bound on the first four initial coefficients. He also investigated the sharp bound on the Fekete-Szegö functional for functions in this class. It should be noted that the class $\mathcal{B}_1(1)$ is a subclass of close-to-convex

2010 *Mathematics Subject Classification.* 30C45, 30C50.

Key words and phrases. Univalent function, Coefficient bound, Hankel determinant.

functions and hence univalent in $\mathbb{D}$. Moreover, $\mathcal{B}_1(0) = \mathcal{S}^*$. In 2015, Thomas [13] proved the sharp bound $|a_2a_4 - a_3^2| \leq 4/(2+\alpha)^2$ for functions in the class $\mathcal{B}_1(\alpha)$ for $\alpha \in [0,1]$. In 2017, Marjono *et al.* [6] investigated the sharp upper bound on fifth and sixth coefficients. They also conjectured that if $f \in \mathcal{B}_1(\alpha)$, then

$$|a_n| \leq \frac{2}{n-1+\alpha} \quad (n = 2,3,4,\cdots)$$

holds for all $\alpha \geq 1$. This conjecture for the fifth coefficient, for certain range of α, was recently settled by Cho and Kumar [1]. For many results related to the Bazilevič functions we refer the reader to the papers [11, 14, 15, 17] and the references cited therein. A class $\mathcal{B}(\alpha,\beta)$ with stronger conditions was considered by Ponnusamy [8]. For $\alpha > 0$ and $0 < \beta < 1$, he defined

$$\mathcal{B}(\alpha,\beta) := \left\{ f \in \mathcal{A} : \left| f'(z) \left(\frac{f(z)}{z} \right)^{\alpha-1} - 1 \right| < \beta \right\}.$$

For the negative value of $\alpha \in (-1,0)$, the class $\mathcal{B}(\alpha,\beta)$ can be rewritten as

$$\bar{\mathcal{B}}(\alpha,\beta) := \left\{ f \in \mathcal{A} : \left| f'(z) \left(\frac{z}{f(z)} \right)^{\alpha+1} - 1 \right| < \beta \right\}.$$

This class was introduced and investigated by Obradović, in 1998. He obtained the conditions on the parameter β that embeds this class into the class of starlike functions. Later in 2002, Tuneski and Darus [16], for $0 < \alpha < 1$, considered the class

$$\bar{\mathcal{B}}(\alpha) := \left\{ f \in \mathcal{A} : \operatorname{Re} \left(f'(z) \left(\frac{z}{f(z)} \right)^{\alpha+1} \right) > 0 \right\}.$$

This class, as mentioned by Obradović in the conference "Computational Methods and Function Theory 2001" is called to be class of functions of non-Bazilevič type, see [16]. Tuneski and Darus investigated the sharp bounds on $|a_2|$ and the Fekete-Szegö functional $|a_3 - \mu a_2^2|$. Some typographical errors in the result [16, Theorem 1, p. 64] were reported by Kumar and Kumar [5]. For a more general result and the correct version of their result one can refer to [5]. Starlikeness of multivalent non-Bazilevič functions were investigated by Guo *et al.* [2]. Estimate on the second Hankel determinant for the class of functions $f \in \mathcal{A}$ satisfying $\operatorname{Re}\left(f'(z)\,(z/f(z))^{\alpha}\right) > 0$ for $\alpha \in (0, 1/3]$ was obtained by Krishna and Reddy [4].

Motivated by the above works, in this paper, sharp bound on the third to eighth coefficients of functions in the class $\bar{\mathcal{B}}(\alpha)$ are investigated. Moreover, sharp bound on the functional $|a_2a_3 - a_4|$ for functions in the class $\bar{\mathcal{B}}(\alpha)$ is also obtained.

Let $\mathcal{P}$ be the class of analytic functions having the Taylor series of the form $p(z) = 1 + p_1z + p_2z^2 + p_3z^3 + \cdots$ and mapping the unit disk $\mathbb{D}$ onto the right-half of the complex plane i.e. satisfying the condition $\operatorname{Re} p(z) > 0$ $(z \in \mathbb{D})$. Let $\mathbf{B}$ be the class of Schwarz functions consisting of analytic functions of the form $w(z) = c_1z + c_2z^2 + c_3z^3 + \cdots$ $(z \in \mathbb{D})$

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 27, NO.7, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC

and satisfying the condition $|w(z)| < 1$ for $z \in \mathbb{D}$. The following correspondence between the classes $\mathbf{B}$ and $\mathcal{P}$ holds:

$$p \in \mathcal{P} \text{ if and only if } w(z) = \frac{p(z)-1}{p(z)+1} \in \mathbf{B}. \tag{1.2}$$

Comparing coefficients in (1.2), we have

$$c_1 = \frac{p_1}{2},\ c_2 = \frac{2p_2 - p_1^2}{4},\ c_3 = \frac{4p_3 - 4p_1p_2 + p_1^3}{8},\ c_4 = \frac{8p_4 - 8p_1p_3 - 4p_2^2 + 6p_1^2p_2 - p_1^4}{16}. \tag{1.3}$$

Lemma 1.1. [3](see also [10]) *If $p \in \mathcal{P}$, then, for any complex number ν,*

$$|p_2 - \nu p_1^2| \le 2\max\{1; |2\nu - 1|\}$$

and the equality holds for the functions given by

$$p(z) = \frac{1+z^2}{1-z^2} \ \text{ and } \ p(z) = \frac{1+z}{1-z}.$$

Consider the functional $\Psi(\mu,\nu) = |c_3 + \mu c_1 c_2 + \nu c_1^3|$ for $w \in \mathbf{B}$ and μ, $\nu \in \mathbb{R}$. Let us assume that the symbols Ω_k's are defined as follows:

$$\Omega_1 := \left\{(\mu,\nu) \in \mathbb{R}^2 : |\mu| \le 1/2, \ \ |\nu| \le 1\right\},$$

$$\Omega_2 := \left\{(\mu,\nu) \in \mathbb{R}^2 : \frac{1}{2} \le |\mu| \le 2, \ \ \frac{4}{27}(|\mu|+1)^3 - (|\mu|+1) \le \nu \le 1\right\},$$

$$\Omega_3 := \left\{(\mu,\nu) \in \mathbb{R}^2 : |\mu| \le \frac{1}{2}, \ \ \nu \le -1\right\}, \Omega_4 := \left\{(\mu,\nu) \in \mathbb{R}^2 : |\mu| \ge 1/2, \ \ \nu \le -\frac{2}{3}(|\mu|+1)\right\},$$

$$\Omega_5 := \left\{(\mu,\nu) \in \mathbb{R}^2 : |\mu| \le 2, \ \ \nu \ge 1\right\}, \Omega_6 := \left\{(\mu,\nu) \in \mathbb{R}^2 : 2 \le |\mu| \le 4, \ \ \nu \ge \frac{1}{12}(\mu^2+8)\right\},$$

$$\Omega_7 := \left\{(\mu,\nu) \in \mathbb{R}^2 : |\mu| \ge 4, \ \nu \ge \frac{2}{3}(|\mu|-1)\right\},$$

$$\Omega_8 := \left\{(\mu,\nu) \in \mathbb{R}^2 : \frac{1}{2} \le |\mu| \le 2, \ -\frac{2}{3}(|\mu|+1) \le \nu \le \frac{4}{27}(|\mu|+1)^3 - (|\mu|+1)\right\},$$

$$\Omega_9 := \left\{(\mu,\nu) \in \mathbb{R}^2 : |\mu| \ge 2, \ -\frac{2}{3}(|\mu|+1) \le \nu \le \frac{2|\mu|(|\mu|+1)}{\mu^2+2|\mu|+4}\right\},$$

$$\Omega_{10} := \left\{(\mu,\nu) \in \mathbb{R}^2 : 2 \le |\mu| \le 4, \ \frac{2|\mu|(|\mu|+1)}{\mu^2+2|\mu|+4} \le \nu \le \frac{1}{12}(\mu^2+8)\right\},$$

$$\Omega_{11} := \left\{(\mu,\nu) \in \mathbb{R}^2 : |\mu| \ge 4, \ \frac{2|\mu|(|\mu|+1)}{\mu^2+2|\mu|+4} \le \nu \le \frac{2|\mu|(|\mu|-1)}{\mu^2-2|\mu|+4}\right\},$$

$$\Omega_{12} := \left\{(\mu,\nu) \in \mathbb{R}^2 : |\mu| \ge 4, \ \frac{2|\mu|(|\mu|-1)}{\mu^2-2|\mu|+4} \le \nu \le \frac{2}{3}(|\mu|-1)\right\}.$$

The following result is due to Prokhorov and Szynal [9] which we need in our investigation.

Lemma 1.2. [9, Lemma 2, p. 128] *If $w \in \mathbf{B}$, then for any real numbers μ and ν, we have*

$$|\Psi(\mu,\nu)| \le \begin{cases} 1, & (\mu,\nu) \in \Omega_1 \cup \Omega_2 \cup \{(2,1)\}; \\ |\nu|, & (\mu,\nu) \in \bigcup_{k=3}^{7} \Omega_k; \\ \frac{2}{3}(|\mu|+1)\left(\frac{|\mu|+1}{3(|\mu|+\nu+1)}\right)^{1/2}, & (\mu,\nu) \in \Omega_8 \cup \Omega_9; \\ \frac{1}{3}\nu\left(\frac{\mu^2-4}{\mu^2-4\nu}\right)\left(\frac{\mu^2-4}{3(\nu-1)}\right)^{1/2}, & (\mu,\nu) \in \Omega_{10} \cup \Omega_{11} \setminus \{(2,1)\}; \\ \frac{2}{3}(|\mu|-1)\left(\frac{|\mu|-1}{3(|\mu|-\nu-1)}\right)^{1/2}, & (\mu,\nu) \in \Omega_{12}. \end{cases}$$

The extremal functions, up to rotations, are of the form

$$w_1(z) = z^3, \;\; w_2(z) = z, \;\; w_3(z) = \frac{z(t_1 - z)}{1 - t_1 z}, \;\; w_4(z) = \frac{z(t_2 + z)}{1 + t_2 z}$$

and $w_5(z) = c_1 z + c_2 z^2 + c_3 z^3 + \cdots$, where the parameters t_1, t_2 and the coefficients c_i are given by

$$t_1 = \left(\frac{|\mu|+1}{3(|\mu|+\nu+1)}\right)^{1/2}, \;\; t_2 = \left(\frac{|\mu|-1}{3(|\mu|-\nu-1)}\right)^{1/2}, \;\; c_1 = \left(\frac{2\nu(\mu^2+2)-3\mu^2}{3(\nu-1)(\mu^2-4\nu)}\right)^{1/2},$$

$$c_2 = (1-c_1^2)e^{i\theta_0}, \;\; c_3 = -c_1 c_2 e^{i\theta_0}, \;\; \theta_0 = \pm \arccos\left[\frac{\mu}{2}\left(\frac{\nu(\mu^2+8)-2(\mu^2+2)}{2\nu(\mu^2+2)-3\mu^2}\right)^{1/2}\right].$$

2. Coefficient Estimates

The following theorem gives the sharp estimates on $|a_3|$, $|a_4|$ and on the functional $|a_2a_3 - a_4|$ for functions in the class $\bar{\mathcal{B}}(\alpha)$.

Theorem 2.1. *Let $\alpha_0 \approx 2.36, \alpha_1 \approx 2.68$ and $\alpha_2 \approx 2.71$ are the smallest positive roots of the equations $3\alpha^4 - 11\alpha^3 + \alpha^2 + 11\alpha + 20 = 0$, $\alpha^6 - 11\alpha^5 + 56\alpha^4 - 138\alpha^3 + 151\alpha^2 - 7\alpha - 148 = 0$ and $\alpha^3 - 5\alpha^2 + 11\alpha - 13 = 0$, respectively. Let $f \in \bar{\mathcal{B}}(\alpha)$ has the from (1.1). Then, the following sharp inequalities hold:*

$$|a_3| \le \begin{cases} \frac{2}{\alpha-2}, & \text{if } \alpha \in (0,3] \setminus \{1,2\}; \\ \frac{2(\alpha-3)}{(\alpha-2)(\alpha-1)^2}, & \text{if } \alpha > 3, \end{cases} \tag{2.1}$$

$$|a_4| \le \begin{cases} \frac{2\left(\alpha^4-5\alpha^3+11\alpha^2-19\alpha+36\right)}{3(\alpha-1)(\alpha-2)(\alpha-3)}, & \text{if } \alpha \in (0,\alpha_0] \setminus \{1,2\} \text{ or } \alpha_2 \le \alpha < 3; \\ \frac{4(|a|-1)^{3/2}}{3(\alpha-3)(|a|-b-1)^{1/2}}, & \text{if } \alpha_0 \le \alpha \le \alpha_1; \\ \frac{2(\alpha-1)^2(a^2-4)^{3/2}}{(\alpha-3)(a^2-4b)(3(b-1))^{1/2}}, & \text{if } \alpha_1 \le \alpha \le \alpha_2; \\ \frac{2}{\alpha-3}, & \text{if } 3 < \alpha, \end{cases} \tag{2.2}$$

where a and b are given by

$$a := -\frac{2(\alpha-5)}{(\alpha-1)(\alpha-2)} \quad \text{and} \quad b := \frac{\alpha^4-5\alpha^3+11\alpha^2-19\alpha+36}{3(\alpha-1)^3(\alpha-2)}.$$

Proof. Since $f \in \bar{\mathcal{B}}(\alpha)$, it follows that there exists $p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots \in \mathcal{P}$ such that

$$f'(z)\left(\frac{z}{f(z)}\right)^{\alpha+1} = p(z). \tag{2.3}$$

Comparing coefficients of like-power terms in (2.3), we get

$$a_2 = -\frac{p_1}{\alpha-1} \quad \text{and} \quad a_3 = \frac{(\alpha-2)(\alpha+1)p_1^2 - 2(\alpha-1)^2 p_2}{2(\alpha-2)(\alpha-1)^2}. \tag{2.4}$$

Now consider

$$\begin{aligned} a_3 &= \frac{(\alpha-2)(\alpha+1)p_1^2 - 2(\alpha-1)^2 p_2}{2(\alpha-2)(\alpha-1)^2} \\ &= -\frac{1}{\alpha-2}\left[p_2 - \frac{(\alpha-2)(\alpha+1)}{2(\alpha-1)^2}p_1^2\right]. \end{aligned} \tag{2.5}$$

An application of Lemma 1.1 on (2.5), gives

$$|a_3| \le \frac{2}{\alpha-2}\max\left\{1; \frac{|\alpha-3|}{(\alpha-1)^2}\right\}$$

which equivalently can be written as

$$|a_3| \le \begin{cases} \frac{2}{\alpha-2}, & \alpha \in (0,3] \setminus \{1,2\}; \\ \frac{2(\alpha-3)}{(\alpha-2)(\alpha-1)^2}, & \alpha > 3. \end{cases}$$

This is the required bound on third coefficient as stated in the theorem. In the first case of (2.1), equality occurs for the function $f_0 \in \bar{\mathcal{B}}(\alpha)$ defined by

$$f_0'(z)\left(\frac{z}{f_0(z)}\right)^{\alpha+1} = \frac{1+z^2}{1-z^2}, \tag{2.6}$$

whereas in the second case of (2.2), equality holds for the function $\tilde{f}_0 \in \bar{\mathcal{B}}(\alpha)$ defined by

$$\tilde{f}_0{}'(z)\left(\frac{z}{\tilde{f}_0(z)}\right)^{\alpha+1} = \frac{1+z}{1-z}. \tag{2.7}$$

Next we shall find the estimate on $|a_4|$. From (2.3), we have

$$a_4 = \frac{-(\alpha-3)(\alpha-2)(\alpha+1)(2\alpha+1)p_1^3 + 6(\alpha-1)^2(\alpha-3)(\alpha+1)p_1p_2 - 6(\alpha-2)(\alpha-1)^3 p_3}{6(\alpha-1)^3(\alpha-2)(\alpha-3)}. \tag{2.8}$$

In view of the interconnections in (1.2) and (1.3), Eqn. (2.8) can be rewritten as:

$$a_4 = -\frac{2\left[(\alpha^4-5\alpha^3+11\alpha^2-19\alpha+36)\,c_1^3-6(\alpha-5)(\alpha-1)^2c_1c_2+3(\alpha-2)(\alpha-1)^3c_3\right]}{3(\alpha-1)^3(\alpha-2)(\alpha-3)} \tag{2.9}$$

or equivalently

$$a_4 = -\frac{2}{\alpha-3}\left[c_3+ac_1c_2+bc_1^3\right],$$

where the parameters a and b are given by

$$a := -\frac{2(\alpha-5)}{(\alpha-1)(\alpha-2)} \ \text{ and } \ b := \frac{\alpha^4-5\alpha^3+11\alpha^2-19\alpha+36}{3(\alpha-1)^3(\alpha-2)}. \tag{2.10}$$

Assume that Ω_i's are defined as in Lemma 1.2 with the settings $\mu = a$ and $\nu = b$. We now proceed further in the proof with the following steps:

(1) Assume that $\alpha \geq (\sqrt{73}-1)/2 \approx 3.772$. In this case, we see that $-1/2 \leq a \leq 1/2$ holds. Moreover, $b \leq 1$ holds if and only if $\alpha^4-5\alpha^3+8\alpha^2-\alpha-15 \geq 0$, which holds for all $\alpha \geq 3$. Thus for all $\alpha \geq (\sqrt{73}-1)/2$, we conclude that $(a,b) \in \Omega_1$.

(2) Next assume that $3 < \alpha \leq (\sqrt{73}-1)/2$. Then, we see that the condition $-1/2 \leq a \leq 2$ holds for all such α and $(4/27)(a+1)^3-(a+1) \leq b \leq 1$ all $\alpha > 3$. Therefore, for $3 < \alpha \leq (\sqrt{73}-1)/2$, we must have $(a,b) \in \Omega_2$.

(3) Let

$$\alpha_2 := \frac{1}{3}\left(\sqrt[3]{53+9\sqrt{41}}-\frac{8}{\sqrt[3]{53+9\sqrt{41}}}+5\right) \approx 2.71$$

and

$$\alpha_0 := \frac{11}{12}+\frac{1}{12}\sqrt{4\ (2^{2/3})\sqrt[3]{8989+9\sqrt{14717}}+4\sqrt[3]{35956-36\sqrt{14717}}+113}-\frac{1}{2}\sqrt{\hat{C}+\hat{D}} \approx 2.36$$

with

$$\hat{C} := \frac{113}{18}-\frac{1}{9}(2^{2/3})\sqrt[3]{8989+9\sqrt{14717}}-\frac{1}{9}\sqrt[3]{35956-36\sqrt{14717}},$$

and

$$\hat{D} := \frac{407}{18\sqrt{4\ (2^{2/3})\sqrt[3]{8989+9\sqrt{14717}}+4\sqrt[3]{35956-36\sqrt{14717}}+113}}$$

are the smallest positive roots of the equations $\alpha^3-5\alpha^2+11\alpha-13=0$ and $3\alpha^4-11\alpha^3+\alpha^2+11\alpha+20=0$, respectively. Now assume that $0 < \alpha < 1$ or $2 < \alpha \leq \alpha_0$. Then $a \geq 4$ and $b \geq 2(a-1)/3$ hold and hence $(a,b) \in \Omega_7$. Moreover, $a \leq -1/2$ and $b \leq -2(-a+1)/3$ holds whenever $1 < \alpha < 2$. Therefore, $(a,b) \in \Omega_4$ whence $1 < \alpha < 2$. Also it can be easily seen that $2 \leq a \leq 4$ and $b \geq (a^2+8)/12$ hold for $\alpha_2 \leq \alpha < 3$.

(4) Let $2.69 \approx (5+\sqrt{33})/4 \leq \alpha \leq \alpha_0$. Then a and b satisfy $2 \leq a \leq 4$ and
$$\frac{2a(a+1)}{a^2+2a+4} \leq b \leq \frac{a^2+8}{12}.$$
Therefore, for this range of α, we see that $(a,b) \in \Omega_{10}$. Let $\alpha_1 \approx 2.68$ is the smallest positive root of $\alpha^6 - 11\alpha^5 + 56\alpha^4 - 138\alpha^3 + 151\alpha^2 - 7\alpha - 148 = 0$. Further, when $\alpha_1 \leq \alpha \leq (\sqrt{33}+5)/4$, the parameters a and b satisfy $a \geq 4$ and
$$\frac{2a(a+1)}{a^2+2a+4} \leq b \leq \frac{2a(a-1)}{a^2-2a+4}.$$
Hence, in view of Lemma 1.2, we have $(a,b) \in \Omega_{11}$.

(5) Assume that $\alpha_0 \leq \alpha \leq \alpha_1$. In this case, it is a simple matter to check that $a \geq 4$ and
$$\frac{2a(a-1)}{a^2-2a+4} \leq b \leq \frac{2(a-1)}{3}.$$
Therefore, Lemma 1.2 gives $(a,b) \in \Omega_{12}$.

In the light of the above discussions, an application of Lemma 1.2 gives the desired estimates on $|a_4|$. In the first case of (2.2), the equality holds for the function f_0 defined in (2.6), whereas in the forth case of (2.2), the equality holds for the function function $\tilde{f}_0$ defined in (2.7). In the case third of (2.2), the extremal function f_1 is given by
$$f_1'(z)\left(\frac{z}{f_1(z)}\right)^{\alpha+1} = \frac{1+w(z)}{1-w(z)} \tag{2.11}$$
with choice of the Schwarz function (up to rotation) $w(z) = c_1 z + c_2 z^2 + c_3 z^3 + \cdots \in \mathbf{B}$, where the coefficients c_i are given by
$$c_1 = \left(\frac{2b(a^2+2)-3a^2}{3(b-1)(a^2-4b)}\right)^{1/2}, \; c_2 = (1-c_1^2)e^{i\theta_0}, \; c_3 = -c_1 c_2 e^{i\theta_0},$$
with
$$\theta_0 = \pm \arccos\left[\frac{a}{2}\left(\frac{b(a^2+8)-2(a^2+2)}{2b(a^2+2)-3a^2}\right)^{1/2}\right],$$
where a and b are given by (2.10). Finally, in the second case of (2.2), the equality holds for the function $\tilde{f}_1$ defined by
$$\tilde{f}_1'(z)\left(\frac{z}{\tilde{f}_1(z)}\right)^{\alpha+1} = \frac{1+w(z)}{1-w(z)} \tag{2.12}$$
with the Schwarz function given by $w(z) = z(\kappa+z)/(1+\kappa z)$, where
$$\kappa := \left(\frac{|a|-1}{3\,(|a|-b-1)}\right)^{1/2}.$$
This completes the proof. ∎

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 27, NO.7, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC

The following theorem provides sharp bound on the fifth, sixth, seventh and eighth coefficients for functions in the class $\bar{\mathcal{B}}(\alpha)$.

Theorem 2.2. *Let us denote*

$$\Psi := 2\alpha^6 - 28\alpha^5 + 137\alpha^4 - 331\alpha^3 + 437\alpha^2 - 433\alpha + 360,$$

$$\hat{\Psi} := -6\alpha^9+96\alpha^8-674\alpha^7+2836\alpha^6-8942\alpha^5+22504\alpha^4-40886\alpha^3+45124\alpha^2-30132\alpha+21600,$$

$$\begin{aligned}\chi := 23\alpha^{12} - 756\alpha^{11} + 10218\alpha^{10} - 77686\alpha^9 + 376014\alpha^8 - 1243398\alpha^7 + 2969824\alpha^6 \\ - 5401638\alpha^5 + 7729083\alpha^4 - 8432486\alpha^3 + 6389238\alpha^2 - 3333636\alpha + 1360800,\end{aligned}$$

and

$$\begin{aligned}\hat{\chi} := -(45\alpha^{15} - 1530\alpha^{14} + 23641\alpha^{13} - 221500\alpha^{12} + 1438032\alpha^{11} - 7061480\alpha^{10} + 27696314\alpha^9 \\ - 88000680\alpha^8 + 222370901\alpha^7 - 435300650\alpha^6 + 653299149\alpha^5 - 763502860\alpha^4 \\ + 703545502\alpha^3 - 473136900\alpha^2 + 206026416\alpha - 76204800).\end{aligned}$$

If $f \in \bar{\mathcal{B}}(\alpha)$ has the from (1.1), *then for $0 < \alpha < 1$, the following sharp inequalities hold:*

$$|a_5| \le \frac{2\Psi}{3(\alpha-4)(\alpha-3)(\alpha-2)^2(\alpha-1)^4},$$

$$|a_6| \le \frac{\hat{\Psi}}{15(\alpha-5)(\alpha-4)(\alpha-3)(\alpha-2)^2(\alpha-1)^5},$$

$$|a_7| \le \frac{2\chi}{45(\alpha-6)(\alpha-5)(\alpha-4)(\alpha-3)^2(\alpha-2)^3(\alpha-1)^6}$$

and

$$|a_8| \le \frac{2\hat{\chi}}{315(\alpha-7)(\alpha-6)(\alpha-5)(\alpha-4)(\alpha-3)^2(\alpha-2)^3(\alpha-1)^7}.$$

Proof. From (2.3), on comparing the coefficients, we have

$$a_5 = \frac{\tau_1 p_4 + \tau_2 p_1^2 p_2 + \tau_3 p_2^2 + \tau_4 p_1 p_3 + \tau_5 p_1^4}{24(\alpha-4)(\alpha-3)(\alpha-2)^2(\alpha-1)^4}, \tag{2.13}$$

where τ_i's are given by

$$\tau_1 := -24(\alpha-3)(\alpha-2)^2(\alpha-1)^4, \tau_2 := -12(\alpha-4)(\alpha-3)(\alpha-2)(\alpha-1)^2(\alpha+1)(2\alpha+1),$$
$$\tau_3 := 12(\alpha-3)(\alpha-4)(\alpha-1)^4(\alpha+1), \tau_4 := 24(\alpha-4)(\alpha-2)^2(\alpha-1)^3(\alpha+1),$$
$$\tau_5 := (\alpha-4)(\alpha-3)(\alpha-2)^2(\alpha+1)(2\alpha+1)(3\alpha+1).$$

Similarly, the sixth coefficient is given by

$$a_6 = -\frac{\hat{\tau}_1 p_5 + \hat{\tau}_2 p_2^2 p_1 + \hat{\tau}_3 p_2 p_3 + \hat{\tau}_4 p_1^3 p_2 + \hat{\tau}_5 p_1^2 p_3 + \hat{\tau}_6 p_1 p_4 + \hat{\tau}_7 p_1^5}{120(\alpha-5)(\alpha-4)(\alpha-3)(\alpha-2)^2(\alpha-1)^5}, \tag{2.14}$$

where $\hat{\tau}_i$'s are defined by

$$\hat{\tau}_1 := 120(\alpha-4)(\alpha-3)(\alpha-2)^2(\alpha-1)^5, \hat{\tau}_2 := 60(\alpha-5)(\alpha-4)(\alpha-3)(\alpha-1)^4(\alpha+1)(2\alpha+1),$$

$$\hat{\tau}_3 := -120(\alpha-5)(\alpha-4)(\alpha-2)(\alpha-1)^5(\alpha+1),$$

$$\hat{\tau}_4 := -20(\alpha-5)(\alpha-4)(\alpha-3)(\alpha-2)(\alpha-1)^2(\alpha+1)(2\alpha+1)(3\alpha+1),$$

$$\hat{\tau}_5 := 60(\alpha-5)(\alpha-4)(\alpha-2)^2(\alpha-1)^3(\alpha+1)(2\alpha+1),$$

$$\hat{\tau}_6 := -120(\alpha-5)(\alpha-3)(\alpha-2)^2(\alpha-1)^4(\alpha+1),$$

$$\hat{\tau}_7 := (\alpha-5)(\alpha-4)(\alpha-3)(\alpha-2)^2(\alpha+1)(2\alpha+1)(3\alpha+1)(4\alpha+1).$$

To find the estimate on $|a_5|$, we observe from (2.13) that the coefficients τ_i $(i = 1, 2, 3, 4, 5, 6, 7)$ of $p_4, p_1^2p_2, p_2^2, p_1p_3$ and p_1^4 are positive. Hence applying triangle inequality in (2.13) and using the fact that $|p_j| \leq 2$, we get the required estimate on $|a_5|$. A similar argument can be used to obtained the estimates on $|a_6|$, $|a_7|$ and $|a_8|$. In all the cases, equality hold for the function $\tilde{f}_0$ given by (2.7). This completes the proof. ∎

The following theorem gives the sharp bound on the functional $|a_2a_3-a_4|$ for the functions in the class $\bar{\mathcal{B}}(\alpha)$.

Theorem 2.3. *Let $f \in \bar{\mathcal{B}}(\alpha)$ has the form (1.1). Then, the following sharp result holds:*

$$|a_2a_3-a_4| \leq \begin{cases} \frac{2(\alpha^3-4\alpha^2+\alpha+18)}{3(\alpha-1)^2(\alpha-2)(\alpha-3)}, & \text{if } \alpha \in (0,2)\setminus\{1\}; \\ \frac{2(\alpha^3-4\alpha^2+\alpha+18)}{3(\alpha-1)^2(\alpha-2)(3-\alpha)}, & \text{if } 2<\alpha<3; \\ \frac{2}{\alpha-3}, & \text{if } \alpha>3. \end{cases} \tag{2.15}$$

Proof. Proceeding as in the proof of previous theorem and using (2.4) and (2.9), we can write

$$a_2a_3-a_4 = \frac{2\left[(\alpha^3-4\alpha^2+\alpha+18)\,c_1^3+12(\alpha-1)c_1c_2+3(\alpha-2)(\alpha-1)^2c_3\right]}{3(\alpha-3)(\alpha-2)(\alpha-1)^2}. \tag{2.16}$$

By setting

$$s := \frac{4}{(\alpha-1)(\alpha-2)} \quad \text{and} \quad t := \frac{\alpha^3-4\alpha^2+\alpha+18}{3(\alpha-2)(\alpha-1)^2}$$

the expression in (2.16) can be written as

$$a_2a_3-a_4 = \frac{2}{\alpha-3}\left[c_3+sc_1c_2+tc_1^3\right].$$

Assume that the symbols Ω_i's are as defined in Lemma 1.2 with the settings $\mu = s$ and $\nu = t$. Now the proof is accomplished in the following steps:

(1) Let $(3+\sqrt{33})/2 \leq \alpha$. Then it can be easily verified that
$$-\frac{1}{2} \leq s \leq \frac{1}{2} \text{ and } -1 \leq t \leq 1.$$
Therefore, for the range $(3+\sqrt{33})/2 \leq \alpha$, we have $(s,t) \in \Omega_1$. Further, when $3 < \alpha \leq (3+\sqrt{33})/2$, wee see that $(s,t) \in \Omega_2$.
(2) Let $0 < \alpha < 1$ or $1 < \alpha < 2$. Then in a similar way we have $(s,t) \in \Omega_4$. Further if $(3+\sqrt{5})/2 \leq \alpha < 3$, then $(s,t) \in \Omega_6$ and when $2 < \alpha \leq (3+\sqrt{5})/2$, then $(s,t) \in \Omega_7$.

In the light of the above discussions, an application of Lemma 1.2, establish the required estimate on $|a_2a_3 - a_4|$. In the first two cases of (2.15), the equality hold for the function $\tilde{f}_0 \in \bar{\mathcal{B}}(\alpha)$ defined by (2.7). In the third case of (2.15), the equality holds for the function f_2 defined by
$$f_2'(z)\left(\frac{z}{f_2(z)}\right)^{\alpha+1} = \frac{1+z^3}{1-z^3}. \tag{2.17}$$
This completes the proof. ∎

Remark 2.4. It would be interesting to find out the sharp bound on $|a_i|$ $(i = 5,6,7,8)$ for the functions $\tilde{f} \in \bar{\mathcal{B}}(\alpha)$ in the case when $\alpha > 1$.

Acknowledgement

The second author was supported by the Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education, Science and Technology (No. 2016R1D1A1A09916450).

References

[1] N. E. Cho and V. Kumar, On a coefficient conjecture for Bazilevič functions, preprint.
[2] L. Guo, Y. Ling and G. Bao, On the starlikeness for the class of multivalent non-Bazilevic functions, South Asian Journal of Mathematics **3** (2013), no. 1, 67–70.
[3] F. R. Keogh and E. P. Merkes, A coefficient inequality for certain classes of analytic functions, Proc. Amer. Math. Soc. **20** (1969), 8–12.
[4] D. V. Krishna and T. R. Reddy, An upper bound to the second Hankel functional for non-Bazilevic functions, Far East J. Math. Sci. **67** (2012), no. 2, 187–199.
[5] S. S. Kumar and V. Kumar, Fekete-Szegö problem for a class of analytic functions defined by convolution, Tamkang J. Math. **44** (2013), no. 2, 187–195.
[6] Marjono, J. Sokól and D. K. Thomas, The fifth and sixth coefficients for Bazilevič functions $\mathcal{B}_1(\alpha)$, Mediterr. J. Math. **14** (2017), no. 4, Art. ID. 158, 11 pp.
[7] M. Obradović, A class of univalent functions, Hokkaido Math. J. **27** (1998), no. 2, 329–335.
[8] S. Ponnusamy, Convolution properties of some classes of meromorphic univalent functions, Proc. Indian Acad. Sci. Math. Sci. **103** (1993), no. 1, 73–89.
[9] D. V. Prokhorov and J. Szynal, Inverse coefficients for (α,β)-convex functions, Ann. Univ. Mariae Curie-Skłodowska Sect. A **35** (1981), 125–143.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 27, NO.7, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC

[10] V. Ravichandran, Y. Polatoglu, M. Bolcal and A. Sen, Certain subclasses of starlike and convex functions of complex order, Hacet. J. Math. Stat. **34** (2005), 9–15.
[11] T. Sheil-Small, On Bazilevič functions, Quart. J. Math. Oxford Ser. **23** (1972), no. 2, 135–142.
[12] R. Singh, On Bazilevič functions, Proc. Amer. Math. Soc. **38** (1973), 261–271.
[13] D. K. Thomas, On the coefficients of Bazilevič functions with logarithmic growth, Indian J. Math. **57** (2015), no. 3, 403–418.
[14] D. K. Thomas, On a subclass of Bazilevič functions, Internat. J. Math. Math. Sci. **8** (1985), no. 4, 779–783.
[15] D. K. Thomas, On starlike and close-to-convex univalent functions, J. London Math. Soc. **42** (1967), 427–435.
[16] N. Tuneski and M. Darus, Fekete-Szegö functional for non-Bazilevič functions, Acta Math. Acad. Paedagog. Nyházi. (N.S.) **18** (2002), no. 2, 63–65.
[17] J. Zamorski, On Bazilevič schlicht functions, Ann. Polon. Math. **12** (1962), 83–90.

(J. H. Park) Department of Applied Mathematics, Pukyong National University, Busan 48513, South Korea

E-mail address: `jihyang1022@naver.com`

(V. Kumar) Department of Applied Mathematics, Pukyong National University, Busan 48513, South Korea

E-mail address: `vktmaths@yahoo.in`

(N. E. Cho) Corresponding Author, Department of Applied Mathematics, Pukyong National University, Busan 48513, South Korea

E-mail address: `necho@pknu.ac.kr`

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 27, NO.7, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC

A new extragradient method for the split feasibility and fixed point problems *

Ming Zhao[1]†and Yunfei Du[2]

[1]School of Science, China University of Geosciences(Beijing), Beijing 100083, China

[2]LMIB-School of Mathematics and Systems Science, Beihang University, Beijing 100191, China

Abstract: In this paper, we propose a new extragradient method with regularization for finding a common element of the solution set Γ of the split feasibility problem and the set $\mathrm{Fix}(S)$ of fixed points of a nonexpansive mapping S in infinite-dimensional Hilbert spaces, combining the regularization method and the technique of averaged operator, we prove the sequences generated by the proposed algorithm converge weakly to an element of $\mathrm{Fix}(S)\bigcap\Gamma$ under mild conditions.

Keywords: split feasibility problem , extragradient, regularization.

1. Introduction

Throughout this paper, let H be a Hilbert space, $\langle\cdot,\cdot\rangle$ denotes the inner product, and $\|\cdot\|$ denotes for the corresponding norm. The split feasibility problem (SFP) which was first introduced by Censor and Elfving [1] in 1994 for modeling inverse problems arising from phase retrievals and in medical image reconstruction. Let C and Q be closed convex sets in the infinite-dimensional real Hilbert spaces H_1 and H_2, respectively. The SFP is to find a vector x^* satisfying

$$x^* \in C \quad \text{such that} \quad Ax^* \in Q, \tag{1.1}$$

where $A \in B(H_1, H_2)$ which denotes the family of all bounded linear operators from H_1 to H_2. Some related work in the infinite-dimensional setting can be found in [2, 3, 4, 5, 9, 10, 12] and the references therein.

Many methods have been developed to solve the SFP, The basic algorithm have CQ algorithm proposed by Byrne [2], the relaxed CQ algorithm proposed by Yang [9], the half-space relaxation projection method proposed by Qu and Xiu [11], the variable Krasnosel skii-Mann algorithm proposed by Xu [12]. The projections of a point onto C and Q are difficult to compute when C and Q fail to have closed-form expressions, though theoretically we can prove the (weak) convergence of the algorithm.

Very recently, Xu [6] gave a continuation of the study on the CQ algorithm and its convergence. He applied Mann's algorithm to the SFP and proposed an averaged CQ algorithm which was proved to be weakly convergent to a solution of the SFP. On the other hand, Korpelevich

*This work was supported by the Fundamental Research Funds for the Central Universities.

†Corresponding Author. Email address: mingzhao@cugb.edu.cn(M.Zhao)

[7] introduced the so-called extragradient method for finding a solution of a saddle point problem. He proved that the sequences generated by the proposed iterative algorithm converge to a solution of a saddle.

Motivated by the idea of an extragradient method, Nadezhina and Takahashi [8] introduced an iterative algorithm for finding a common element of the set of fixed points of a nonexpansive mapping and the solution set of a variational inequality problem [13] for a monotone, Lipschitz continuous mapping in a real Hilbert space. They obtained a weak convergence theorem for two sequence generated by the proposed algorithm.

In our paper, we introduce and analyze a new extragradient iterative algorithm to find a common element of the solution set Γ of the split feasibility problem and the set $\mathrm{Fix}(S)$ of fixed points of a nonexpansive mapping S in infinite-dimensional Hilbert spaces, furthermore, we prove its convergence. The results of this paper represent the improvement of the corresponding results in [6] and [14].

2. Preliminaries

Throughout this paper, we use $x_n \to x$ and $x_n \rightharpoonup x$ to denote strong and weak convergence to x of the sequence x_n, respectively. Let K be a nonempty closed convex subset of H. Recall that the projection (nearest point or metric) from H onto K, denoted by P_K, is defined in such a way that, for each $x \in H$, $P_K x$ is the unique point in K with the property

$$\| x - P_K x \| = \inf_{y \in K} \| x - y \| =: d(x, K),$$

i.e.

$$P_K(x) = \operatorname{argmin}\{\| x - y \| \mid y \in K\}.$$

Some important properties of projections are gathered in the following Lemma.

Lemma 2.1 *For given $x \in H$ and $z \in K$, the following properties hold:*
(1) $x \in K \Leftrightarrow P_K(x) = x$;
(2) $\langle x - P_K(x), z - P_K(x)\rangle \le 0, \ \forall x \in H$ *and* $\forall z \in K$;
(3) $\langle x - y, P_K(x) - P_K(y)\rangle \ge \|P_K(x) - P_K(y)\|^2, \ \forall x, \ y \in H$;
(4) $\|P_K(x) - z\|^2 \le \|x - z\|^2 - \|P_K(x) - x\|^2, \ \forall x \in H$ *and* $\forall z \in K$;
(5) $\|P_K(x) - P_K(y)\| \le \|x - y\|, \ \forall x, \ y \in H$.

Proof. See Facchinei and Pang [15].

Definition 2.1 Let T be a mapping from $K \subseteq H$ into H, then
(a) T is called monotone on K if

$$\langle T(x) - T(y), x - y\rangle \ge 0, \ \forall \ x, \ y \in K.$$

(b) T is called strongly monotone on K if there is a $\mu > 0$, such that

$$\langle T(x) - T(y), x - y\rangle \ge \mu\|x - y\|^2, \ \forall \ x, \ y \in K.$$

(c) F is called co-coercive (or ν-inverse strongly monotone) on K if there is a $\nu > 0$, such that

$$\langle T(x) - T(y), x - y\rangle \ge \nu\|T(x) - T(y)\|^2, \ \forall \ x, \ y \in K.$$

(d) F is called pseudo-monotone on K if

$$\langle T(y), x - y\rangle \ge 0 \Rightarrow \langle T(x), x - y\rangle \ge 0, \ \forall \ x, \ y \in K.$$

(e) T is called Lipschitz continuous on K if there exists a constant $L > 0$ such that

$$\|T(x) - T(y)\| \le L\|x - y\|, \ \forall \ x, \ y \in K.$$

Definition 2.2 A mapping $T : H \to H$ is said to be:
(a) nonexpansive if

$$\| Tx - Ty \| \leq \| x - y \|, \ \forall x, \ y \in H;$$

(b) firmly nonexpansive if $2T - I$ is nonexpansive, or equivalently,

$$\langle x - y, Tx - Ty \rangle \geq \| Tx - Ty \|, \ \forall x, \ y \in H,$$

or alternatively, T is firmly nonexpansive if and only if T can be expressed as

$$T = \frac{1}{2}(I + S)$$

where $S : H \to H$ is nonexpansive.
Remark 2.1 From Lemma 2.1 and Definition 2.1-2.2, we can infer that if S is nonexpansive, then I-S is monotone; A monotone mapping is pseudo-monotone mapping; An inverse strongly monotone mapping is monotone and Lipschitz continuous; A Lipschitz continuous and strongly monotone mapping is an inverse strongly monotone mapping; The projection operator is 1-ism and nonexpansive.
Lemma 2.2 *A mapping T is 1-ism if and only if the mapping I-T is 1-ism, where I is the identity operator.*
Proof. See [16, Lemma 2.3].
Remark 2.2 If T is an inverse strongly monotone mapping, then T is a nonexpansive mapping.

Definition 2.3 A mapping $T : H \to H$ is said to be an averaged mapping if it can be written as the average of the identity I and a nonexpansive mapping S, that is,

$$T = (1 - \alpha)I + \alpha S \tag{2.1}$$

where $\alpha \in (0, 1)$ and $S : H \to H$ is nonexpansive. More precisely, when (2.1) holds, we say that T is α-averaged. Thus firmly nonexpansive mappings (for example, projections) are $\frac{1}{2}$-averaged mappings.
Proposition 2.1 *([16]). Let $T : H \to H$ be a given mapping:*
(1) T is nonexpansive if and only if the complement I-T is $\frac{1}{2}$-ism.
(2) If T is μ-ism, then for $\gamma > 0$, γT is $\frac{\nu}{\gamma}$-ism.
(3) T is averaged if and only if the complement I-T is ν-ism for some $\nu > \frac{1}{2}$. Indeed, for $\alpha \in (0, 1)$, T is α-averaged if and only if I-T is $\frac{1}{2\alpha}$-ism.
Proposition 2.2 *([16, 17]). Let $S, \ T, \ V : H \to H$ be given operators.*
(1) If $T = (1 - \alpha)S + \alpha V$ for some $\alpha \in (0, 1)$ and if S is averaged and V is nonexpansive, then T is averaged.
(2) T is firmly nonexpansive if and only if the complement I-T is firmly nonexpansive.
(3) If $T = (1 - \alpha)S + \alpha V$ for some $\alpha \in (0, 1)$ and if S is firmly nonexpansive and V is nonexpansive, then T is averaged.
(4) The composite of finitely many averaged mappings is averaged. That is, if each of the mappings $\{T_i\}_{i=1}^N$ is averaged, then so is the composite $T_1 \circ \cdots \circ T_N$. In particular, if T_1 is α_1-averaged and T_2 is α_2-averaged, where $\alpha_1, \ \alpha_2 \in (0, 1)$, then the composite $T_1 \circ T_2$ is α-averaged, where $\alpha = \alpha_1 + \alpha_2 - \alpha_1\alpha_2$.
(5) If the mapping $\{T_i\}_i^N$ are averaged and have a common fixed point, then

$$\bigcap_{i=1}^{N} \mathrm{Fix}(T_i) = \mathrm{Fix}(T_1 \circ \cdots \circ T_N).$$

The notation Fix(T) *denotes the set of all fixed points of the mapping* T*, that is* $\text{Fix}(T) = \{x \in H : Tx = x\}$.

The so-called demiclosedness principle plays an important role in our argument.

Definition 2.4 Let $T : H \to H$ be an operator. We say that I-T is demiclosed (at zero), if for any sequence x_n in H, there holds the following implication:

$$x_n \rightharpoonup x \text{ and } (I\text{-}T)x_n \to 0 \Rightarrow (I\text{-}T)x = 0.$$

Lemma 2.3 *([18]). Let H be a Hilbert space. Then for all $x,\ y \in H$ and $\lambda \in [0,1]$,*

$$\| \lambda x + (1-\lambda)y \|^2 = \lambda \| x \|^2 + (1-\lambda) \| y \|^2 - \lambda(1-\lambda) \| x - y \|^2 .$$

Lemma 2.4 *([19]). Let $\{a_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$ and $\{\delta_n\}_{n=1}^{\infty}$ be sequences of nonnegative real numbers satisfying the inequality*

$$a_{n+1} \le (1+\delta_n)a_n + b_n,\ \forall n \ge 1.$$

If $\sum_{n=1}^{\infty} \delta_n < \infty$ and $\sum_{n=1}^{\infty} b_n < \infty$, then $\lim_{n\to\infty} a_n$ exists.

Corollary 2.1 *([20]). Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be two sequences of nonnegative real numbers satisfying the inequality*

$$a_{n+1} \le a_n + b_n,\ \forall n \ge 1.$$

If $\sum_{n=1}^{\infty} b_n < \infty$, then $\lim_{n\to\infty} a_n$ exists.

Recall that a Banach space X is said to satisfy the Opial condition [22] if for any sequence $\{x_n\}$ in X the condition that $\{x_n\}$ converges weakly to $x \in X$ implies that the inequality

$$\liminf_{n\to\infty} \| x_n - x \| < \liminf_{n\to\infty} \| x_n - y \|$$

holds for every $y \in X$ with $y \ne x$.

It is well-known that every Hilbert space satisfies the Opial condition.

3. Main results

Throughout this paper, we assume that the SFP is consistent, that is, the solution set Γ of the SFP is nonempty.

It is easy to see that SFP is equivalent to the following minimization problem

$$\min_{x\in C} f(x) := \frac{1}{2}\|Ax - P_Q Ax\|^2, \tag{3.1}$$

where $f : H_1 \to R$ is a continuous differentiable function, however it is ill-posed. Therefore, Xu [6] considered the following Tikhonov regularized problem:

$$\min_{x\in C} f_\alpha(x) := \frac{1}{2}\|Ax - P_Q Ax\|^2 + \frac{1}{2}\alpha\|x\|^2, \tag{3.2}$$

where $\alpha > 0$ is the regularization parameter.

We observe that the gradient

$$\nabla f_\alpha(x) = \nabla f(x) + \alpha I = A^*(I - P_Q)A + \alpha I \tag{3.3}$$

is $(\alpha + \|A\|^2)$-Lipschitz continuous and α-strongly monotone.

Proposition 3.1 *([21]) Given $x^* \in H_1$, the following statements are equivalent:*
(1) *x^* solves the SFP;*
(2) *x^* solves the fixed point equation*

$$P_C(I - \lambda\nabla f) = P_C[I - \lambda A^*(I - P_Q)A]x^* = x^* \qquad (3.4).$$

(3) *x^* solves the variational inequality problem (VIP) of finding $x^* \in C$ such that*

$$\langle \nabla f(x^*), x - x^* \rangle \geq 0, \ \forall x \in C, \tag{3.5}$$

where $\nabla f = A^*(I - P_Q)A$ and A^* is the adjoint of A.
Remark 3.1. It is clear from Proposition 3.1 that

$$\Gamma = \mathrm{Fix}(P_C(I - \lambda \nabla f)) = VI(C, \nabla f)$$

for any $\lambda > 0$, where $\mathrm{Fix}(P_C(I - \lambda \nabla f))$ and $VI(C, \nabla f)$ denote the set of fixed points of $P_C(I - \lambda \nabla f)$ and the solution set of VIP(3.5).

Next, we will present our method for solving the SFP and prove its convergence.
Theorem 3.1 *Let $S : C \to C$ be a nonexpansive mapping such that $\mathrm{Fix}(S) \cap \Gamma \neq \emptyset$ in Hilbert space. Let $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ be the sequences in C generated by the following extragradient algorithm:*

$$\begin{cases} x_0 \in C \ \ \textit{chosen arbitrarily}, \\ z_n = (1 - \gamma_n)x_n + \gamma_n P_C(I - \lambda_n \nabla f_{\alpha_n})x_n, \\ y_n = (1 - \beta_n)z_n + \beta_n S P_C(I - \lambda_n f_{\alpha_n})z_n, \\ x_{n+1} = (1 - \mu_n)y_n + \mu_n S P_C(I - \lambda_n \nabla f_{\alpha_n})y_n, \ \forall n > 0, \end{cases} \tag{3.6}$$

where the sequences of parameters $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ and $\{\mu_n\}$ satisfy the following conditions:
(a) $\sum_{n=1}^{\infty} \alpha_n < \infty$;
(b) $\{\lambda_n\} \subset \left(0, \frac{1}{\|A\|^2}\right)$ *and* $0 < \liminf_{n\to\infty} \lambda_n \leq \limsup_{n\to\infty} \lambda_n < \frac{1}{\|A\|^2}$;
(c) $\{\gamma_n\} \subset (0, 1)$, *and* $0 < \liminf_{n\to\infty} \gamma_n \leq \limsup_{n\to\infty} \gamma_n < 1$;
(d) $\{\beta_n\} \subset (0, 1)$, *and* $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;
(e) $\{\mu_n\} \subset (0, 1)$, *and* $0 < \liminf_{n\to\infty} \mu_n \leq \limsup_{n\to\infty} \mu_n < 1$.
Then, the sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ are all converge weakly to an element $\bar{x} \in \mathrm{Fix}(S) \bigcap \Gamma$.
Proof. It [21] has been proved $P_C(I - \lambda \nabla f_\alpha)$ is ζ-averaged for each $\lambda \in \left(0, \frac{2}{\alpha + \|A\|^2}\right)$, where $\zeta = \frac{2 + \lambda\left(\alpha + \|A\|^2\right)}{4}$, so $P_C(I - \lambda \nabla f_\alpha)$ is nonexpansive. Furthermore, for $\{\lambda_n\} \subset \left(0, \frac{1}{\|A\|^2}\right)$, we have

$$0 < \liminf_{n\to\infty} \lambda_n \leq \limsup_{n\to\infty} \lambda_n < \frac{1}{\|A\|^2} = \lim_{0\to\infty} \frac{1}{\alpha_n + \|A\|^2}.$$

Without loss of generality, we may assume that

$$0 < \liminf_{n\to\infty} \lambda_n \leq \limsup_{n\to\infty} \lambda_n < \frac{1}{\alpha_n + \|A\|^2}, \quad \forall n \geq 0.$$

Consequently, $P_C(I - \lambda_n \nabla f_{\alpha_n})$ is ζ_n-averaged for each integer $n \geq 0$, where

$$\zeta_n = \frac{2 + \lambda_n(\alpha_n + \|A\|^2)}{4} \in (0, 1).$$

This implies that $P_C(I - \lambda_n \nabla f_{\alpha_n})$ is nonexpansive for all $n \geq 0$.

Next, we show the sequences $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ generated in Theorem 3.1 are bounded. Indeed, take a fixed $p \in \mathrm{Fix}(S) \bigcap \Gamma$ arbitrarily. Then, we get $Sp = p$ and $P_C(I - \lambda \nabla f)p = p$ for

$\lambda \in \left(0, \frac{1}{\|A\|^2}\right)$. From (3.6), it follows that

$$\begin{aligned} \|z_n - p\| &= \|(1-\gamma_n)(x_n - p) + \gamma_n[P_C(I - \lambda_n \nabla f_{\alpha_n})x_n - p]\| \\ &\leq (1-\gamma_n)\|x_n - p\| + \gamma_n\|P_C(I - \lambda_n \nabla f_{\alpha_n})x_n - p\| \\ &= (1-\gamma_n)\|x_n - p\| + \gamma_n\|P_C(I - \lambda_n \nabla f_{\alpha_n})x_n - P_C(I - \lambda_n \nabla f)p\| \\ &= (1-\gamma_n)\|x_n - p\| + \gamma_n\|P_C(I - \lambda_n \nabla f_{\alpha_n})x_n - P_C(I - \lambda_n \nabla f_{\alpha_n})p \\ &\quad + P_C(I - \lambda_n \nabla f_{\alpha_n})p - P_C(I - \lambda_n \nabla f)p\| \\ &\leq (1-\gamma_n)\|x_n - p\| + \gamma_n(\|P_C(I - \lambda_n \nabla f_{\alpha_n})x_n - P_C(I - \lambda_n \nabla f_{\alpha_n})p\| \\ &\quad + \|P_C(I - \lambda_n \nabla f_{\alpha_n})p - P_C(I - \lambda_n \nabla f)p\|) \\ &\leq (1-\gamma_n)\|x_n - p\| + \gamma_n(\|x_n - p\| + \|(I - \lambda_n \nabla f_{\alpha_n})p - (I - \lambda_n \nabla f)p\|) \\ &= \|x_n - p\| + \lambda_n \alpha_n \gamma_n \|p\|, \end{aligned} \tag{3.7}$$

$$\begin{aligned} \|y_n - p\| &= \|(1-\beta_n)(z_n - p) + \beta_n[SP_C(I - \lambda_n \nabla f_{\alpha_n})z_n - p]\| \\ &\leq (1-\beta_n)\|z_n - p\| + \beta_n\|P_C(I - \lambda_n \nabla f_{\alpha_n})z_n - p\| \\ &= (1-\beta_n)\|z_n - p\| + \beta_n\|P_C(I - \lambda_n \nabla f_{\alpha_n})z_n - P_C(I - \lambda_n \nabla f)p\| \\ &= (1-\beta_n)\|z_n - p\| + \beta_n\|P_C(I - \lambda_n \nabla f_{\alpha_n})z_n - P_C(I - \lambda_n \nabla f_{\alpha_n})p \\ &\quad + P_C(I - \lambda_n \nabla f_{\alpha_n})p - P_C(I - \lambda_n \nabla f)p\| \\ &\leq (1-\beta_n)\|z_n - p\| + \beta_n(\|P_C(I - \lambda_n \nabla f_{\alpha_n})z_n - P_C(I - \lambda_n \nabla f_{\alpha_n})p\| \\ &\quad + \|P_C(I - \lambda_n \nabla f_{\alpha_n})p - P_C(I - \lambda_n \nabla f)p\|) \\ &\leq (1-\beta_n)\|z_n - p\| + \beta_n(\|z_n - p\| + \|(I - \lambda_n \nabla f_{\alpha_n})p - (I - \lambda_n \nabla f)p\|) \\ &= \|z_n - p\| + \lambda_n \alpha_n \beta_n \|p\|, \end{aligned} \tag{3.8}$$

and

$$\begin{aligned} \|x_{n+1} - p\| &= \|(1-\mu_n)(y_n - p) + \mu_n[SP_C(I - \lambda_n \nabla f_{\alpha_n})y_n - p]\| \\ &\leq (1-\mu_n)\|y_n - p\| + \mu_n\|P_C(I - \lambda_n \nabla f_{\alpha_n})y_n - p\| \\ &= (1-\mu_n)\|y_n - p\| + \mu_n\|P_C(I - \lambda_n \nabla f_{\alpha_n})y_n - P_C(I - \lambda_n \nabla f)p\| \\ &= (1-\mu_n)\|y_n - p\| + \mu_n\|P_C(I - \lambda_n \nabla f_{\alpha_n})x_n - P_C(I - \lambda_n \nabla f_{\alpha_n})p \\ &\quad + P_C(I - \lambda_n \nabla f_{\alpha_n})p - P_C(I - \lambda_n \nabla f)p\| \\ &\leq (1-\mu_n)\|y_n - p\| + \mu_n(\|P_C(I - \lambda_n \nabla f_{\alpha_n})z_n - P_C(I - \lambda_n \nabla f_{\alpha_n})p\| \\ &\quad + \|P_C(I - \lambda_n \nabla f_{\alpha_n})p - P_C(I - \lambda_n \nabla f)p\|) \\ &\leq (1-\mu_n)\|y_n - p\| + \mu_n(\|y_n - p\| + \|(I - \lambda_n \nabla f_{\alpha_n})p - (I - \lambda_n \nabla f)p\|) \\ &= \|y_n - p\| + \lambda_n \alpha_n \mu_n \|p\| \\ &\leq \|x_n - p\| + \lambda_n \alpha_n (\gamma_n + \beta_n + \mu_n)\|p\|, \end{aligned} \tag{3.9}$$

where the last inequality follows from (3.7) and (3.8).

Since $\Sigma_{n=1}^{\infty} \alpha_n < \infty$, and $\{\lambda_n\}$, $\{\gamma_n\}$, $\{\beta_n\}$, $\{\mu_n\}$ are bounded, then from Corollary 2.1, we conclude that

$$\lim_{n\to\infty} \|x_n - p\| \ \text{ exists for each } p \in \mathrm{Fix}(S) \bigcap \Gamma. \tag{3.10}$$

Hence $\{x_n\}$ is bounded and so are $\{y_n\}$ and $\{z_n\}$.

In the following, we will show

$$\lim_{n\to\infty} \|x_n - y_n\| = \lim_{n\to\infty} \|y_n - z_n\| = \lim_{n\to\infty} \|x_n - u_n\| = \lim_{n\to\infty} \|y_n - Sw_n\| = \lim_{n\to\infty} \|z_n - Sv_n\| = 0,$$

where $u_n = P_C(I - \lambda_n \nabla f_{\alpha_n})x_n$, $v_n = P_C(I - \lambda_n \nabla f_{\alpha_n})z_n$, $w_n = P_C(I - \lambda_n \nabla f_{\alpha_n})y_n$.

Note that

$$\begin{aligned} \|u_n - p\| &= \|P_C(I - \lambda_n \nabla f_{\alpha_n})x_n - p\| \\ &= \|P_C(I - \lambda_n \nabla f_{\alpha_n})x_n - P_C(I - \lambda_n \nabla f_{\alpha_n})p \\ &\quad + P_C(I - \lambda_n \nabla f_{\alpha_n})p + P_C(I - \lambda_n \nabla f)p\| \\ &\leq \|P_C(I - \lambda_n \nabla f_{\alpha_n})x_n - P_C(I - \lambda_n \nabla f_{\alpha_n})p\| \\ &\quad + \|P_C(I - \lambda_n \nabla f_{\alpha_n})p + P_C(I - \lambda_n \nabla f)p\| \\ &\leq \|x_n - p\| + \lambda_n \alpha_n \|p\|. \end{aligned} \tag{3.11}$$

Similarly, we can obtain that

$$\|v_n - p\| \le \|z_n - p\| + \lambda_n \alpha_n \|p\| \tag{3.12}$$

and

$$\|w_n - p\| \le \|y_n - p\| + \lambda_n \alpha_n \|p\|. \tag{3.13}$$

Indeed, observe that

$$\begin{aligned}
\|z_n - p\|^2 &= \|(1-\gamma_n)(x_n - p) + \gamma_n(u_n - p)\|^2 \\
&= (1-\gamma_n)\|x_n - p\|^2 + \gamma_n\|u_n - p)\|^2 - \gamma_n(1-\gamma_n)\|x_n - u_n\|^2 \\
&\le (1-\gamma_n)\|x_n - p\|^2 + \gamma_n(\|x_n - p\| + \lambda_n\alpha_n\|p\|)^2 - \gamma_n(1-\gamma_n)\|x_n - u_n\|^2 \\
&= (1-\gamma_n)\|x_n - p\|^2 + \gamma_n(\|x_n - p\|^2 + 2\lambda_n\alpha_n\|p\|\|x_n - p\| + \lambda_n^2\alpha_n^2\|p\|^2) \\
&\quad -\gamma_n(1-\gamma_n)\|x_n - u_n\|^2 \\
&= \|x_n - p\|^2 + \alpha_n\gamma_n(2\lambda_n\|p\|\|x_n - p\| + \alpha_n\lambda_n^2\|p\|^2) - \gamma_n(1-\gamma_n)\|x_n - u_n\|^2 \\
&\le \|x_n - p\|^2 + \alpha_n M_1 - \gamma_n(1-\gamma_n)\|x_n - u_n\|^2,
\end{aligned} \tag{3.14}$$

where $M_1 = \sup_{n\ge 0}\{\gamma_n(2\lambda_n\|p\|\|x_n - p\| + \alpha_n\lambda_n^2\|p\|^2)\} < \infty$ and the first inequality follows from (3.11).

Also, observe that

$$\begin{aligned}
\|y_n - p\|^2 &= \|(1-\beta_n)(z_n - p) + \beta_n(Sv_n - p)\|^2 \\
&= (1-\beta_n)\|z_n - p\|^2 + \beta_n\|Sv_n - p)\|^2 - \beta_n(1-\beta_n)\|z_n - Sv_n\|^2 \\
&\le (1-\beta_n)\|z_n - p\|^2 + \beta_n\|v_n - p)\|^2 - \beta_n(1-\beta_n)\|z_n - Sv_n\|^2 \\
&\le (1-\beta_n)\|z_n - p\|^2 + \beta_n(\|z_n - p\| + \lambda_n\alpha_n\|p\|)^2 - \beta_n(1-\beta_n)\|z_n - Sv_n\|^2 \\
&= (1-\beta_n)\|z_n - p\|^2 + \beta_n(\|z_n - p\|^2 + 2\lambda_n\alpha_n\|p\|\|z_n - p\| + \lambda_n^2\alpha_n^2\|p\|^2) \\
&\quad -\beta_n(1-\beta_n)\|z_n - Sv_n\|^2 \\
&= \|z_n - p\|^2 + \alpha_n\beta_n(2\lambda_n\|p\|\|z_n - p\| + \alpha_n\lambda_n^2\|p\|^2) - \beta_n(1-\beta_n)\|z_n - Sv_n\|^2 \\
&\le \|z_n - p\|^2 + \alpha_n M_2 - \beta_n(1-\beta_n)\|z_n - Sv_n\|^2,
\end{aligned} \tag{3.15}$$

where $M_2 = \sup_{n\ge 0}\{\beta_n(2\lambda_n\|p\|\|z_n - p\| + \alpha_n\lambda_n^2\|p\|^2)\} < \infty$ and the second inequality follows from (3.12).

And

$$\begin{aligned}
\|x_{n+1} - p\|^2 &= \|(1-\mu_n)(y_n - p) + \mu_n(Sw_n - p)\|^2 \\
&= (1-\mu_n)\|y_n - p\|^2 + \mu_n\|Sw_n - p)\|^2 - \mu_n(1-\mu_n)\|y_n - Sw_n\|^2 \\
&\le (1-\mu_n)\|y_n - p\|^2 + \mu_n\|w_n - p)\|^2 - \mu_n(1-\mu_n)\|y_n - Sw_n\|^2 \\
&\le (1-\mu_n)\|y_n - p\|^2 + \mu_n(\|y_n - p\| + \lambda_n\alpha_n\|p\|)^2 - \mu_n(1-\mu_n)\|y_n - Sw_n\|^2 \\
&= (1-\mu_n)\|y_n - p\|^2 + \mu_n(\|y_n - p\|^2 + 2\lambda_n\alpha_n\|p\|\|y_n - p\| + \lambda_n^2\alpha_n^2\|p\|^2) \\
&\quad -\mu_n(1-\mu_n)\|y_n - Sw_n\|^2 \\
&= \|y_n - p\|^2 + \alpha_n\mu_n(2\lambda_n\|p\|\|y_n - p\| + \alpha_n\lambda_n^2\|p\|^2) - \mu_n(1-\mu_n)\|y_n - Sw_n\|^2 \\
&\le \|y_n - p\|^2 + \alpha_n M_3 - \mu_n(1-\mu_n)\|y_n - Sw_n\|^2,
\end{aligned} \tag{3.16}$$

where $M_3 = \sup_{n\ge 0}\{\mu_n(2\lambda_n\|p\|\|y_n - p\| + \alpha_n\lambda_n^2\|p\|^2)\} < \infty$ and the second inequality follows from (3.13).

Substitute (3.14) and (3.15) into (3.16), we have

$$\begin{aligned}
\|x_{n+1} - p\|^2 &\le \|x_n - p\|^2 + \alpha_n(M_1 + M_2 + M_3) - \gamma_n(1-\gamma_n)\|x_n - u_n\|^2 \\
&\quad -\beta_n(1-\beta_n)\|z_n - Sv_n\|^2 - \mu_n(1-\mu_n)\|y_n - Sw_n\|^2.
\end{aligned} \tag{3.17}$$

Hence, it follows that

$$\begin{aligned}&\gamma_n(1-\gamma_n)\|x_n-u_n\|^2+\beta_n(1-\beta_n)\|z_n-Sv_n\|^2+\mu_n(1-\mu_n)\|y_n-Sw_n\|^2\\&\le\|x_n-p\|^2-\|x_{n+1}-p\|^2+\alpha_n(M_1+M_2+M_3).\end{aligned}\tag{3.18}$$

Since $\Sigma_{n=1}^{\infty}\alpha_n<\infty$, $0<\liminf_{n\to\infty}\gamma_n\le\limsup_{n\to\infty}\gamma_n<1$, $0<\liminf_{n\to\infty}\beta_n\le\limsup_{n\to\infty}\beta_n<1$, and $0<\liminf_{n\to\infty}\mu_n\le\limsup_{n\to\infty}\mu_n<1$, we deuce from the existence of $\lim_{n\to\infty}\|x_n-p\|$ that

$$\lim_{n\to\infty}\|x_n-u_n\|=\lim_{n\to\infty}\|y_n-Sw_n\|=\lim_{n\to\infty}\|z_n-Sv_n\|=0.\tag{3.19}$$

Then, utilizing (3.6) we get

$$\lim_{n\to\infty}\|z_n-x_n\|=\lim_{n\to\infty}\gamma_n\|u_n-x_n\|=0,\tag{3.20}$$

$$\lim_{n\to\infty}\|y_n-z_n\|=\lim_{n\to\infty}\beta_n\|Sv_n-z_n\|=0,\tag{3.21}$$

and

$$\lim_{n\to\infty}\|x_n-y_n\|=\lim_{n\to\infty}\mu_n\|Sw_n-y_n\|=0.\tag{3.22}$$

This implies that

$$\lim_{n\to\infty}\|x_n-y_n\|=\lim_{n\to\infty}\|y_n-z_n\|=\lim_{n\to\infty}\|x_n-u_n\|=\lim_{n\to\infty}\|y_n-Sw_n\|=\lim_{n\to\infty}\|z_n-Sv_n\|=0.$$

Furthermore, note that

$$\begin{aligned}\|Sv_n-v_n\|&\le\|Sv_n-z_n\|+\|z_n-x_n\|+\|x_n-u_n\|+\|u_n-v_n\|\\&=\|Sv_n-z_n\|+\|z_n-x_n\|+\|x_n-u_n\|\\&\quad+\|P_C(I-\lambda_n\nabla f_{\alpha_n})x_n-P_C(I-\lambda_n\nabla f_{\alpha_n})z_n\|\\&\le\|Sv_n-z_n\|+\|z_n-x_n\|+\|x_n-u_n\|+\|x_n-z_n\|.\end{aligned}$$

From (3.20-3.22), we can get that

$$\lim_{n\to\infty}\|u_n-v_n\|=\lim_{n\to\infty}\|Sv_n-v_n\|=0.\tag{3.23}$$

Similarly, we can prove

$$\lim_{n\to\infty}\|u_n-w_n\|=\lim_{n\to\infty}\|Sw_n-w_n\|=0.\tag{3.24}$$

As $\{x_n\}$ is bounded, there is a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ that converges weakly to some $\bar{x}$. Next, we will show $\bar{x}\in \mathrm{Fix}(S)\bigcap\Gamma$. We first show $\bar{x}\in\Gamma$, let $T=P_C(I-\lambda_n\nabla f)$, then

$$\begin{aligned}\|x_n-Tx_n\|&\le\|x_n-u_n\|+\|u_n-Tx_n\|\\&=\|x_n-u_n\|+\|P_C(I-\lambda_n\nabla f_{\alpha_n})x_n-P_C(I-\lambda_n\nabla f)x_n\|\\&\le\|x_n-u_n\|+\|(I-\lambda_n\nabla f_{\alpha_n})x_n-(I-\lambda_n\nabla f)x_n\|\\&=\|x_n-u_n\|+\lambda_n\alpha_n\|x_n\|.\end{aligned}\tag{3.25}$$

From $\lim_{n\to\infty}\|x_n-u_n\|=0$, $\lim_{n\to\infty}\|\alpha_n\|=0$ and $\{\lambda_n\}$, $\{x_n\}$ are bounded, we can get that $\lim_{n\to\infty}\|x_n-Tx_n\|=0$.

Taking into account $x_{n_i}\rightharpoonup\bar{x}$ and Definition 2.4, we obtain $\bar{x}\in\mathrm{Fix}(T)$. Thus, utilizing Remark 3.1, we have $\bar{x}\in\Gamma$. On the other hand, since

$$\lim_{n\to\infty}\|x_n-u_n\|=\lim_{n\to\infty}\|u_n-v_n\|=\lim_{n\to\infty}\|Sv_n-v_n\|=0,$$

there is subsequence v_{n_j} of v_n that converges weakly to $\bar{x}$ and $\lim_{n\to\infty}\|Sv_{n_j}-v_{n_j}\|=0$. Then from Definition 2.4, we have $\bar{x}\in \mathrm{Fix}(S)$. Therefore, we get $\bar{x}\in \mathrm{Fix}(S)\bigcap\Gamma$.

Let $\{x_{n_j}\}$ be another subsequence of $\{x_n\}$ such that $x_{n_j}\rightharpoonup\tilde{x}$. Then, $\tilde{x}\in \mathrm{Fix}(S)\bigcap\Gamma$. Next, we prove $\tilde{x}=\bar{x}$. Assume that $\tilde{x}\neq\bar{x}$. From the Opial condition [22], we have

$$\begin{aligned}\lim_{n\to\infty}\|x_n-\tilde{x}\| &= \liminf_{i\to\infty}\|x_{n_i}-\tilde{x}\| < \liminf_{i\to\infty}\|x_{n_i}-\bar{x}\|\\ &= \lim_{n\to\infty}\|x_n-\bar{x}\| = \liminf_{j\to\infty}\|x_{n_j}-\bar{x}\|\\ &< \liminf_{j\to\infty}\|x_{n_j}-\tilde{x}\| = \lim_{n\to\infty}\|x_n-\tilde{x}\|,\end{aligned}$$

which is a contradiction. Thus, we have $\tilde{x}=\bar{x}$. This implies $x_n\rightharpoonup\bar{x}\in\mathrm{Fix}(S)\bigcap\Gamma$. Furthermore, from $\lim_{n\to\infty}\|x_n-y_n\|=\lim_{n\to\infty}\|z_n-x_n\|=0$, we can get $y_n\rightharpoonup\bar{x}$ and $z_n\rightharpoonup\bar{x}$. This shows that the sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ are all converge weakly to an element $\bar{x}\in \mathrm{Fix}(S)\bigcap\Gamma$.

Theorem 3.2 *Let $S: C\to C$ be a nonexpansive mapping such that $\mathrm{Fix}(S)\cap\Gamma\neq\emptyset$ in Hilbert space. Let $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ be the sequences in C generated by the following extragradient algorithm:*

$$\begin{cases} x_0\in C \quad \textit{chosen arbitrarily},\\ z_n=(1-\gamma_n)x_n+\gamma_nP_C(I-\lambda_n\nabla f)x_n,\\ y_n=(1-\beta_n)z_n+\beta_nSP_C(I-\lambda_n\nabla f)z_n,\\ x_{n+1}=(1-\mu_n)y_n+\mu_nSP_C(I-\lambda_n\nabla f)y_n,\ \forall n>0,\end{cases} \tag{3.26}$$

where the sequences of parameters $\{\beta_n\}$, $\{\gamma_n\}$ and $\{\mu_n\}$ satisfy the following condition:
(a) $\{\lambda_n\}\subset\left(0,\frac{1}{\|A\|^2}\right)$ *and* $0<\liminf_{n\to\infty}\lambda_n\le\limsup_{n\to\infty}\lambda_n<\frac{1}{\|A\|^2}$;
(b) $\{\gamma_n\}\subset(0,1)$, *and* $0<\liminf_{n\to\infty}\gamma_n\le\limsup_{n\to\infty}\gamma_n<1$;
(c) $\{\beta_n\}\subset(0,1)$, *and* $0<\liminf_{n\to\infty}\beta_n\le\limsup_{n\to\infty}\beta_n<1$;
(d) $\{\mu_n\}\subset(0,1)$, *and* $0<\liminf_{n\to\infty}\mu_n\le\limsup_{n\to\infty}\mu_n<1$.
Then, the sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ are all converge weakly to an element $\bar{x}\in \mathrm{Fix}(S)\bigcap\Gamma$.

Proof. Let α_n=0 in Theorem 3.1, then we can obtain the desired result.

Remark 3.2. Our iteration method improves the corresponding results of [6], [8] and [14].

References

[1] Y. Censor, T. Elfving, A multiprojection algorithm using Bregman projections in a product space, Numer. Algorithms 8(1994)221-239.

[2] C. Byrne, Iterative oblique projection onto convex subsets and the split feasibility problem, Inverse Problems 18(2002)441-453.

[3] B. Qu, N. Xiu, A note on the CQ algorithm for the split feasibility problem, Inverse Problems 21(2005)1655-1665.

[4] Y. Censor, T. Elfving, N. Kopf, T. Bortfeld, The multiple-sets split feasibility problem and its applications for inverse problems, Inverse Problem 21(2005)2071-2084.

[5] C. Byrne, A unified treatment of some iterative algorithms in signal processing and image reconstruction, Inverse Problem 20(2004)103-120.

[6] H.K. Xu, Iterative methods for the split feasibility problem in infinite-dimensional Hilbert spaces, Inverse Problems 26(2010)1-17.

[7] G.M. Korpelevich, An extragradient method for finding saddle points and for other problem, Ekonomika Mat. Metody 12(1976)747-756.

[8] N. Nadezhkina, W. Takahasi, Weak convergence theorem by an extragradient method for nonexpansive mapping and monotone mapping,J. Optim. Theory Appl. 128(2006)191-201.

[9] Q. Yang, The relaxed CQ algorithm solving the split feasibility problem, Inverse Problems 20(2004)1261-1266.

[10] J. Zhao, Q. Yang, severral solution merhods for the split feasibility problem, Inverse Problems 21(2005)1791-1799.

[11] B. Qu, N. Xiu, A new half space-relaxation projection method for the split feasibility problem, Linear Algebr. Appl. 428(2008)1218-1229.

[12] H. Xu, A variable Krasnosel-skii-Mann algorithm and the multiple-set split feasibility problem, Inverse Problems 22(2006) 2021-2034

[13] Kinderlehrar, G. Stampacchia, An introduction to variational Inequalities and their applications, Academic Press, New York, 1980.

[14] L.C. Ceng, Q.H. Ansarib, J.C. Yao, Relaxed extragradient method for finding minnimum-norm solution of the split feasibility problem, Nonlinear Analysis: Theory, Methods and Applications. 75(4)(2012), 2116-2125.

[15] F. Facchinei, J.S. Pang, Finite-Dimensional Variational Inequalities and Complementarity Problems, vols I and II (Berlin: Springer), 2003.

[16] C. Byrne, A unified treatment of some iterative algorithms in signal processing and image reconstruction, Inverse Problem. 20(2004)103-120.

[17] P.L. Combettes, Solving monotone inclusions via compositions of nonexpansive averaged operators, Optimization 53(5-6)(2004)475-504.

[18] K. Geobel, W.A. Kirk, Topics in Metric Fixed Point Theory, Cambridge Studies in Advanced Mathematics, vol.28, Cambridge University Press, 1990.

[19] M.O. Osilike, S.C. Aniagbosor, B.G. Akuchu, Fixed points of asymptotically demicontractive mappings in arbitrary Banach space, Panamer. Math. J. 12(2002)77-88.

[20] K.K. Tan, H.K. Xu, Approximating fixed points of nonexpansive mappings by the Ishikawa iteration process, J. Math. Anal. Appl. 178(1993)301-308.

[21] L.C. Ceng, Q.H. Ansaribc, J.C. Yao, An extragradient method for solviing split feasibility and fixed point problem, Computers and Mathematics with Applications. 64(4)(2012)633-642.

[22] Z. Opial, Weak convergence of the sequence of successive approximations for nonexpansive mappings, Bull. Amer. Math. Soc. 73(1967)591-597.

Behavior of Meromorphic Solutions of Composite Functional-Difference Equations *†

Man-Li Liu[a] and Ling-Yun Gao[b†]

[a] School of Mathematics, Shandong University
Jinan, Shandong, 250100,P.R.China
e-mail: lml6641@163.com

[b] Department of Mathematics,Jinan University
Guangzhou,Guangdong, 510632,P.R.China
e-mail:tgaoly@jnu.edu.cn

Abstract In view of Nevanlinna value distribution theory, we will investigate the behavior of meromorphic solutions of four types of composite functional-difference equations, and a type of system of composite functional-difference equations, some results are obtained. Moreover, we also give some examples to show that the conditions of our theorems are accurate.

Key words: meromorphic solutions; composite functional-difference equations; behavior; growth order

MR(2010) Subject Classification: 30D35,39B32

1.Introduction

Recently, with the establishment of the difference analogues of Nevanlinna value distribution theory, researchers obtained many interesting theorems about the existence and growth of solutions of difference equations, functional equations and so on([3-6]). To state the results, a number of basic definition and standard notations should be introduced. We shall assume that the reader is familiar with the standard notations and results of Nevanlinna value distribution theory such as $m(r, f(z)), n(r, f(z)), N(r, f(z))$ and $T(r, f(z))$([15,18,22]) denote the proximity function, the non-integrated counting function, the counting function and the characteristic function of $f(z)$, respectively. For the integrated counting function for distinct poles of $f(z)$ we use the notations $\overline{N}(r, f(z))$, and $N_1(r, f) = N(r, f) - \overline{N}(r, f)$.

In this article, a meromorphic function means meromorphic in the whole complex plane. Given a meromorphic function $f(z)$, recall that a meromorphic function $h(z)$ is said to be a small function of $f(z)$, if $T(r, h(z)) = S(r, f)$,where $S(r, f)$ is used to denote

*This work was partially supported by NSFC of China(No.11271227,11271161), PCSIRT(No.IRT1264), and the Fundamental Research Funds of Shandong University (No.2017JC019).

†Corresponding author:Gao Lingyun

1

any quantity that satisfies $S(r,f) = o(T(r,f))$ as $r \to \infty$, possibly outside of a set of r of finite logarithmic measure.

Let c be a fixed, non-zero complex number, $\Delta_c f(z) = f(z+c) - f(z)$, and $\Delta_c^n f(z) = \Delta_c(\Delta_c^{n-1} f(z)) = \Delta_c^{n-1} f(z+c) - \Delta_c^{n-1} f(z)$ for each integer $n \geq 2$.Equations written with the above difference operators $\Delta_c^n f(z)$ are difference equations. Let E be a subset on the positive real axis. We define the logarithmic measure of E to be

$$\log(E) = \int_{E\cap(1,+\infty)} \frac{dr}{r}.$$

A set $E \in (1,+\infty)$ is said to have finite logarithmic measure if $\log(E) < \infty$.

Difference equations have been studied in many aspects see e.g.,[1],[5-6],[17]. Some expositions consider (system of) difference equations in real domains, or discrete domain. So far, the previous researches are only on complex differential equations (systems) or difference equations (systems)[5,6], but not on composite functional-difference equations (systems). Therefore, it is very important and meaningful to study the cases of composite functional-difference equations (systems). That will be an innovative contribution of this paper.

The remainder of the paper is organised as follows. In section 2, we will study the existence of meromorphic solutions or the form on some type of composite functional-difference equations, and obtain three theorems, some examples are give to show that our results hold. In section 3, we will discuss the growth order of meromorphic solutions on some types of composite functional-difference equations or system of composite functional-difference equations, which extend the result of Theorem B.

2. Existence of meromorphic solutions of difference equations and form of difference equations

In 2003, H.Silvennoinen[21] was devoted to considering many types of composite functional equations, he got some good results, for example, the following theorem A is one of his results.

Theorem A([21]) The composite functional equation

$$f(p(z)) = \frac{a_0(z) + a_1(z)f(z)}{b_0(z) + b_1(z)f(z)}$$

where the coefficients a_i, b_j are of growth $S(r,f)$ such that $a_0(z)b_1(z) - a_1(z)b_0(z) \neq 0$ and $p(z)$ is a polynomial of $\deg p(z) = k \geq 2$, does not have meromorphic solutions.

A question is,whether or not the assertion of Theorem A remains valid, if we replace the equation

$$f(p(z)) = \frac{a_0(z) + a_1(z)f(z)}{b_0(z) + b_1(z)f(z)}$$

with the following form

$$\sum_{(i)} a_{(i)}(z)(f(z))^{i_0}(\Delta_c f(z))^{i_1} \cdots (\Delta_c^n f(z))^{i_n} = \frac{a_0(z) + a_1(z)f(p(z))}{b_0(z) + b_1(z)f(p(z))}.$$

2

In this section, the authors will pay attention to considering the properties of meromorphic solutions on three types of composite functional difference equations in complex domain, and extend the results obtained by H.Silvennoinen [21] to types of composite functional-difference equations (1)-(3) of the following forms, which are different from the complex differential equations or systems of complex difference equations.

At this point we pause briefly to introduce the notation used in this paper. Let I be a finite set of multi-indexes $i = (i_0, ..., i_n)$,J be a finite set of multi-indexes $j = (j_0, ..., j_n)$. Difference polynomials $\Omega_1(z, f), \Omega_2(z, f)$ of a meromorphic function $f(z)$ are defined as

$$\Omega_1(z,f) = \sum_{(i)\in I} a_{(i)}(f(z))^{i_0}(\Delta_c f(z))^{i_1}\cdots(\Delta_c^n f(z))^{i_n},$$

$$\Omega_2(z,f) = \sum_{(j)\in J} b_{(j)}(f(z))^{j_0}(\Delta_c f(z))^{j_1}\cdots(\Delta_c^n f(z))^{j_n},$$

where each $\{a_{(i)}(z)\}, \{b_{(j)}(z)\}$ is a small meromorphic function with respect to f.

We denote that

$$u_1 = \max\{\sum_{l=0}^{n}(l+1)i_l\}, u_2 = \max\{\sum_{l=0}^{n}(l+1)j_l\}.$$

First, we will investigate the existence of meromorphic solutions of a type of composite functional-difference equations of the form

$$\sum_{(i)} a_{(i)}(z)(f(z))^{i_0}(\Delta_c f(z))^{i_1}\cdots(\Delta_c^n f(z))^{i_n} = \frac{a_0(z)+a_1(z)f(p(z))}{b_0(z)+b_1(z)f(p(z))}, \tag{1}$$

where the coefficients $\{a_i(z)\}$,$\{b_j(z)\}(i, j = 0, 1)$ and $\{a_{(i)}(z)\}$ are of growth $S(r, f)$ such that $a_0(z)b_1(z) - a_1(z)b_0(z) \not\equiv 0$, $p(z) = c_k z^k + \cdots + c_0$,$\deg p(z) \geq 2$.

For the composite functional-difference equations (1), the main theorem can be stated as follows.

Theorem 2.1 Let $u_1 < k$. The composite function-difference equation (1) does not have meromorphic solutions.

Remark 1 The example 1 shows that Theorem 2.1 does not hold if at least $a_i(z), b_j(z)$ and $a_{(i)}(z)$ are not of growth $S(r, f)$,there may exist a rational solution.

Example 1 Let $p(z) = z^2, c = 1$.Then function $f(z) = \frac{1}{z-1}$ is a solution of the following equation

$$\frac{1 - z^2 f(p(z))}{(1 + (z-1)f(p(z)))} = \frac{z(z-1)}{2+z} f\Delta_c f.$$

Second, we will study the properties of $p(z)$ of composite functional-difference equations of the following

$$\sum_{i=0}^{l} a_i(z)f(p(z))^i = \frac{\Omega_1(z,f)}{\Omega_2(z,f)}, \tag{2}$$

where $p(z)$ is an entire function,$\{a_i(z)\}, \{a_{(i)}(z)\}, \{b_{(j)}(z)\}$ are small functions.

We obtain the following result

Theorem 2.2 Let f be a non-constant meromorphic solution of the composite functional-difference equations (2).Then $p(z)$ is a polynomial.

Third, we shall consider the growth and characteristic estimate of meromorphic solutions of the following composite functional-difference equation

$$\sum_{(i)\in I} a_{(i)}(z)f^{i_0}(\Delta_c f)^{i_1}\cdots(\Delta_c^n f)^{i_n} = \sum_{i=0}^{m} a_i(z)(f(p(z)))^i, \tag{3}$$

where $\{a_i(z)\}$ are meromorphic functions, $a_{(i)} \not\equiv 0, a_m(z) \neq 0$, $p(z)$ is a polynomial of degree $k \geq 2$.

We get the main result below.

Theorem 2.3 Let $f(z)$ be a finite order transcendental meromorphic solution of (3),$\{a_{(i)}(z)\}$ be polynomials,

$$T(r, a_i) < KT(r^s, f), i = 0, 1, 2, \cdots, m,$$

where K and s are positive constants, r is large enough. If $s < k$, then for given $\varepsilon > 0$,

$$T(r, f) = O((\log r)^{\alpha+\varepsilon}),$$

where

$$\alpha = \frac{\log((m+1)K + \frac{u_1}{ms})}{\log\frac{k}{s}}, if \quad 1 \leq s < k,$$

and

$$\alpha = \frac{\log\frac{u_1+m(m+1)Ks}{m}}{\log k}, if \quad s < 1 < k,$$

where $u_1 = \max\{\sum\limits_{l=0}^{n}(l+1)i_l\}$.

Remark 2 The example 2 shows that the condition $s < k$ in Theorem 2.3 is best possible.

Example 2 Let $p(z) = c_k z^k + \cdots + c_0$,$\deg p(z) \geq 2$,

$$a_i(z) = C_m^i \frac{e^{2z}}{(1+e^{p(z)})^m}, i = 0, 1, 2, ..., m.$$

Then

$$\sum_{i=0}^{m} a_i(z)f(p(z))^i = \frac{e^{2z}}{(1+e^{p(z)})^m}\sum_{i=0}^{m} C_m^i f(p(z))^i,$$

$f = e^z$ is a transcendental meromorphic solution of the composite functional-difference equation of the form

$$\frac{1+z(e^c-1)^2}{(e^c-1)^3}(\Delta_c f)(\Delta_c^2 f)^2 - f(\Delta_c f)^2 - z(\Delta_c f)^2(\Delta_c^2 f) + (e^c-1)^3 f^2(\Delta_c f)$$
$$-f(\Delta_c^2 f)^2 + f^2 = \sum_{i=0}^{m} a_i(z)(f(p(z)))^i.$$

In this case, $f(z)$ satisfirs

$$T(r, f(z)) = \frac{r}{\pi} + O(1).$$

However, by $k \geq 2$,we have

$$T(r, a_i(z)) = (1+o(1))\frac{m|c_k|r^k}{\pi},$$

4

it shows that Theorem 2.3 does not hold if $s = k$.

To prove Theorem 2.1-2.3, we need some lemmas as follows.

Lemma 2.1([13]) Let f be a transcendental meromorphic function and $p(z) = a_k z^k + a_{k-1}z^{k-1} + \ldots + a_1(z) + a_0, a_k \neq 0, k \geq 1$,be a polynomial of degree k.Given $0 < \delta < |a_k|$,let $\lambda = |a_k| + \delta, \mu = |a_k| - \delta$.Then, given $\varepsilon > 0$,for any $a \in \mathbf{C} \cup \{\infty\}$ and for r large enough,we have

$$kn(\mu r^k, \frac{1}{f-a}) \leq n(r, \frac{1}{f(p)-a}) \leq kn(\lambda r^k, \frac{1}{f-a}),$$

$$N(\mu r^k, \frac{1}{f-a}) + O(\log r) \leq N(r, \frac{1}{f(p)-a}) \leq N(\lambda r^k, \frac{1}{f-a}) + O(\log r),$$

$$(1-\varepsilon)T(\mu r^k, f) \leq T(r, f(p)) \leq (1+\varepsilon)T(\lambda r^k, f).$$

Lemma 2.2([12]) Let ψ:$[r_0, +\infty) \to (0, +\infty)$ be positive and bounded in every finite interval. Suppose that

$$\psi(\mu r^m) \leq A\psi(r) + B, (r \geq r_0),$$

where $\mu > 0, m > 1, A > 1$ and B are real constants.Then

$$\psi(r) = O((\log r)^\alpha),$$

where

$$\alpha = \frac{\log A}{\log m}.$$

Lemma 2.3([18]) Let $R(z,f) = \frac{\sum_{i=0}^{p} a_i(z)f^i}{\sum_{j=0}^{q} b_j(z)f^j}$ be an irreducible rational function in $f(z)$ with the meromorphic coefficients $\{a_i(z)\}$ and $\{b_j(z)\}$.If $f(z)$ is a meromorphic function,then

$$T(r, R(z,f)) = \max\{p,q\}T(r,f) + O\{\sum T(r,a_i) + \sum T(r,b_j)\}.$$

Lemma 2.4([3]) Let f be a non-constant meromorphic function and let g be a transcendental entire function.Then there exists an increasing sequence,$r_n \to \infty$,such that

$$T(r, f(g(z))) \geq T((M(\frac{r}{4}, g))^{\frac{1}{30}}, f)$$

holds for $r = r_n$.

Lemma 2.5([18]) Let g:$(0, +\infty) \to \mathbf{R}$, h:$(0, +\infty) \to \mathbf{R}$ be monotone increasing functions such that $g(r) \leq h(r)$ outside of an exceptional set E of finite linear measure.Then,for any $\alpha > 1$,there exists r_0 such that $g(r) \leq h(\alpha r)$ for all $r > r_0$.

Lemma 2.6([17]) Let $T : [0, +\infty) \to [0, +\infty)$ be a non-decreasing continuous function,let $\delta \in (0,1)$,and let $s \in (0, \infty)$.If T is of finite order,i.e.,

$$\lim_{r\to\infty} \frac{\log T(r)}{\log r} < \infty,$$

then

$$T(r+s) = T(r) + o(\frac{T(r)}{r^\delta}),$$

where r runs to infinity outside of a set of finite logarithmic measure.

Lemma 2.7 Let f be a meromorphic function of finite order,

$$\Omega_1(z,f) = \sum_{(i)\in I} a_{(i)}(z) f^{i_0}(\Delta_c f)^{i_1}\cdots(\Delta_c^n f)^{i_n},$$

$$\Omega_2(z,f) = \sum_{(j)\in I} b_{(j)}(z) f^{j_0}(\Delta_c f)^{j_1}\cdots(\Delta_c^n f)^{j_n}.$$

Then

$$T(r,\Omega_1(z,f)) \leq u_1 T(r,f) + S_1(r,f) + \sum_{(i)\in I} T(r,a_{(i)}),$$

and

$$T(r,\frac{\Omega_1(z,f)}{\Omega_2(z,f)}) \leq (u_1+u_2)T(r,f) + S_1(r,f) + \sum_{(i)\in I} T(r,a_{(i)}) + \sum_{(j)\in J} T(r,b_{(j)}),$$

where $u_1 = \max\{\sum\limits_{l=0}^{n}(l+1)i_l\}, u_2 = \max\{\sum\limits_{l=0}^{n}(l+1)j_l\}$, the exceptional set E associated to $S(r,f)$ is of finite logarithmic measure $\int_E \frac{dr}{r} < +\infty$.

Proof It follows from

$$\Delta_c^n f(z) = \Delta_c(\Delta_c^{n-1} f(z)) = \Delta_c^{n-1} f(z+c) - \Delta_c^{n-1} f(z)$$

that

$$\Delta_c^m f(z) = \sum_{i=0}^{m} C_m^i (-1)^{m-i} f(z+ci).$$

Similar to the proof of Lemma 4.2 in [16](pp. 181-182), we have

$$m(r,\Omega(z,f)) = \lambda m(r,f) + S(r,f),$$

where $\lambda = \sum\limits_{l=0}^{n} i_l$.

In order to estimate the poles of $\Omega(z,f)$, we consider the term of

$$\Omega_{(i)}(z,f) = a_{(i)}(z) f^{i_0}(\Delta_c f)^{i_1}\cdots(\Delta_c^n f)^{i_n}.$$

Noting that

$$n(r,f(z+c)) \leq n(r+C,f) + S(r,f) = n(r,f) + S(r,f), C = |lc|,$$

it is easy to get that

$$n(r,\Omega_{(i)}(z,f)) \leq \sum_{l=0}^{n} i_l(l+1) n(r,f(z+lc)) + n(r,a_{(i)}(z)).$$

6

Hence, we get

$$n(r,\Omega(z,f)) \le \max(\sum_{l=0}^{n} i_l(l+1))n(r,f(z)) + S(r,f) + \sum_{(i)} n(r,a_{(i)}(z)).$$

By the above equality, we get

$$T(r,\Omega_1(z,f)) \le uT(r,f) + S(r,f) + \sum_{(i)} T(r,a_{(i)}(z)),$$

where $u_1 = \max\{\sum\limits_{l=0}^{n}(l+1)i_l\}$,$r$ runs to infinity outside of a set of finite logarithmic measure.

Further,we have

$$T(r,\Omega_2(z,f)) \le u_2T(r,f) + S(r,f) + \sum_{(j)\in J} T(r,b_{(j)}),$$

where $u_2 = \max\{\sum\limits_{l=0}^{n}(l+1)j_l\}$.

Hence,we obtain

$$\begin{aligned} T(r,\tfrac{\Omega_1(z,f)}{\Omega_2(z,f)}) \le & \quad T(r,\Omega_1(z,f)) + T(r,\tfrac{1}{\Omega_2(z,f)}) \\ \le & \quad (u_1+u_2)T(r,f) + S(r,f) + \sum_{(i)\in I} T(r,a_{(i)}) + \sum_{(j)\in I} T(r,b_{(j)}). \end{aligned}$$

Lemma 2.8([21]) Let $P(z,f) = \sum\limits_{i=0}^{p} a_i(z)f^i$ be polynomial in $f(z)$ with the meromorphic coefficients $\{a_i(z)\}$.If $f(z)$ is a meromorphic function,then

$$T(r,P(z,f)) \le pT(r,f) + \sum_{i=0}^{p} T(r,a_i) + O(1),$$

$$T(r,P(z,f)) \ge p(T(r,f) - \sum_{i=0}^{p} T(r,a_i)) + O(1).$$

Lemma 2.9([21]) Let f be a meromorphic function.Then $T(r,f)$ is an increasing function of $\log r$ and convex function of $\log r$,$\frac{T(r,f)}{\log r}$ is an incresing function of r.

Proof of Theorem 2.1 First, we suppose that there is a transcendental meromorphic solution $f(z)$ of composite functional-difference equation (1) .

For a sufficiently small $\varepsilon > 0$, by Lemma 2.1, Lemma 2.3 and Lemma 2.7, we get

$$(1-\varepsilon)T(\mu r^k, f) \le T(r, f(p(z))) \le (u_1+\varepsilon)T(r,f),$$

where $u_1 = \max\{\sum\limits_{l=0}^{n}(l+1)i_l\}$, $\mu = |c_k|(1-\varepsilon)$, outside a possible exceptional set of finite logarithmic measure.

Hence, for $\alpha > 1$ and for r large enough

$$(1-\varepsilon)T(\mu r^k, f) \le (u_1+\varepsilon)T(\alpha r, f).$$

7

Set $t = \alpha r$. Then
$$T(\frac{\mu}{\alpha^k}t^k, f) \leq \frac{u_1+\varepsilon}{(1-\varepsilon)}T(t,f).$$
By Lemma 2.2 we obtain
$$T(t,f) = O((\log t)^{\alpha_1}),$$
where
$$\alpha_1 = \frac{\log \frac{u_1+\varepsilon}{(1-\varepsilon)}}{\log k} < 1,$$
there is a contradiction.

Second, we suppose that $f(z)$ is a rational solution of (1). Then the coefficients $a_{(i)}(z), a_0(z),\ a_1(z), b_0(z), b_1(z)$ must be constants.

Set
$$f(z) = \frac{P(z)}{Q(z)} = \frac{\alpha_p z^p + \alpha_{p-1}z^{p-1} + \cdots + \alpha_0}{\beta_q z^q + \beta_{q-1}z^{q-1} + \cdots + \beta_0},$$
where $\alpha_p \neq 0, \beta_q \neq 0, \deg w(z) = \max\{p,q\} = l$.

If $p \neq q$, we immediately have $\deg(\frac{a_0(z)+a_1(z)f(p(z))}{b_0(z)+b_1(z)f(p(z))}) = kl$.

If $p = q$, we have

$$\begin{aligned}\frac{a_0(z)+a_1(z)f(p(z))}{b_0(z)+b_1(z)f(p(z))} = \frac{a_0+a_1f(p(z))}{b_0+b_1f(p(z))} &= \frac{a_0+a_1\frac{\alpha_p(p(z))^p+\alpha_{p-1}(p(z))^{p-1}+\cdots+\alpha_0}{\beta_q(p(z))^q+\beta_{q-1}(p(z))^{q-1}+\cdots+\beta_0}}{b_0+b_1\frac{\alpha_p(p(z))^p+\alpha_{p-1}(p(z))^{p-1}+\cdots+\alpha_0}{\beta_q(p(z))^q+\beta_{q-1}(p(z))^{q-1}+\cdots+\beta_0}} \\ &= \frac{(a_0\beta_q+a_1\alpha_p)(p(z))^p+(a_0\beta_{q-1}+a_1\alpha_{p-1})(p(z))^{p-1}+\cdots+(a_0\beta_0+a_1\alpha_0)}{(b_0\beta_q+b_1\alpha_p)(p(z))^p+(b_0\beta_{q-1}+b_1\alpha_{p-1})(p(z))^{p-1}+\cdots+(b_0\beta_0+b_1\alpha_0)}.\end{aligned}$$

It follows from the equation above that $a_0\beta_q + a_1\alpha_p = 0$ and $b_0\beta_q + b_1\alpha_p = 0$ can not hold at the same time. Otherwise $\frac{a_0(z)+a_1(z)w(p(z))}{b_0(z)+b_1(z)w(p(z))} = c$, c is a constant.

Hence, we get

$$\begin{aligned} kl =& \deg(\tfrac{a_0(z)+a_1(z)f(p(z))}{b_0(z)+b_1(z)f(p(z))}) \\ =& \deg(\textstyle\sum_{(i)} a_{(i)}(z)(f(z))^{i_0}(\Delta_c f(z))^{i_1}\cdots(\Delta_c^n f(z))^{i_n}) \\ \leq& \max\{i_0 + 2i_1 + \cdots + (n+1)i_n\}l = u_1 l.\end{aligned}$$

So, $u_1 \geq k$, there is also a contradiction. Thus, $f(z)$ is not a rational solution of (1).

Combined with the first and second steps above, the assertion follows.

Proof of Theorem 2.2 Suppose that $p(z)$ is transcendental entire function, we have
$$\liminf_{r\to\infty} \frac{\log M(r, p(z))}{\log r} = \infty.$$
Hence, for any given $K > 30$ and for r large enough
$$M(r,p) > r^K.$$
There exists an increasing sequence $r_n \to \infty$, as in Lemma 2.4, for any n such that
$$M(\frac{r_n}{4}, p) > (\frac{r_n}{4})^K.$$

8

Applying Lemma 2.3 and Lemma 2.7 to equation (2), we have

$$lT(r, f(p(z))) \le (u_1 + u_2)T(r, f) + S(r, f),$$

outside a possible exceptional set of finite linear measure. According to Lemma 2.5, for $\forall\alpha > 1, r \ge r_\alpha$, we obtain

$$T(r, f(p(z))) \le \frac{(u_1 + u_2)(1 + o(1))}{l} T(\alpha r, f). \tag{4}$$

It follows from Lemma 2.4 that

$$T(r_n, f(p(z))) \ge T((\frac{r_n}{4})^{\frac{K}{30}}, f). \tag{5}$$

Note that $\frac{T(r,f)}{\log r}$ is an increasing function of r. As

$$(\frac{r_n}{4})^{\frac{K}{30}} > \alpha r_n,$$

for sufficiently large n, we have

$$T((\frac{r_n}{4})^{\frac{K}{30}}, f) > \frac{K/30(\log r_n - \log 4)}{\log r_n + \log\alpha} T(\alpha r_n, f) > \frac{K}{40} T(\alpha r_n, f), \tag{6}$$

as $n \to \infty$. By (4),(5) and (6), we get

$$\frac{(u_1 + u_2)(1 + o(1))}{l} T(\alpha r_n, f) \ge T((\frac{r_n}{4})^{\frac{K}{30}}, f) > \frac{K}{40} T(\alpha r_n, f), \tag{7}$$

as $n \to \infty$.

Because K can be arbitrarily large, this is a contradiction in (7). This shows that $p(z)$ is a polynomial.

Proof of Theorem 2.3 By the equation (3), Lemma 2.7 and Lemma 2.8, we have

$$mT(r, f(p(z))) - m\sum_{i=0}^{m} T(r, a_i(z)) \le (u_1 + \varepsilon)T(r, f),$$

i.e.,

$$mT(r, f(p(z))) \le (u_1 + \varepsilon)T(r, f) + m\sum_{i=0}^{m} T(r, a_i(z)). \tag{8}$$

Combining (8) and

$$T(r, a_i(z)) < KT(r^s, f), i = 0, 1, 2, \cdots, m,$$

we obtain

$$T(r, f(p(z))) \le \frac{u_1 + \varepsilon}{m} T(r, f) + (m + 1)KT(r^s, f), \tag{9}$$

where K is a positive constant.

Case (1): If $s \geq 1$, by Lemma 2.9, we have $\frac{T(r,f)}{\log r}$ is increasing functions of r, we can obtain for any positive constant C and any $t \geq 1$

$$\frac{T(Cr^t,f)}{T(r,f)} \geq \frac{\log C + t\log r}{\log r} > (1-\varepsilon)t.$$

Hence, for r sufficiently large,

$$T(r,f) < \frac{1}{(1-\varepsilon)t}T(Cr^t,f).$$

Let $s = t, C = 1$. Then

$$T(r,f) < \frac{1}{(1-\varepsilon)s}T(r^s,f). \tag{10}$$

It follows from (9)and (10) that

$$\begin{aligned} T(r,f(p)) \leq &\ (m+1)KT(r^s,f) + \tfrac{u_1+\varepsilon}{(1-\varepsilon)ms}T(r^s,f) \\ \leq &\ ((m+1)K + \tfrac{u_1}{ms} + \varepsilon_1)T(r^s,f). \end{aligned}$$

By Lemma 2.1

$$(1-\varepsilon)T(\mu r^k,f) \leq ((m+1)K + \frac{u_1}{ms} + \varepsilon_1)T(r^s,f).$$

From the above inequality we further get

$$(1-\varepsilon)T(\mu r^{\frac{k}{s}},f) \leq ((m+1)K + \frac{u_1}{ms} + \varepsilon_2)T(r,f). \tag{11}$$

Since $k > s$, then by (11) and Lemma 2.2, we obtain

$$T(r,f(z)) = O((\log r)^{\alpha_1+\varepsilon}),$$

where

$$\alpha_1 = \frac{\log((m+1)K + \frac{u_1}{ms})}{\log\frac{k}{s}}.$$

Case (2):If $s < 1$, by Lemma 2.9, since $\frac{T(r,f)}{\log r}$ is increasing function of r, we obtain

$$\frac{T(r,f)}{\log r} \geq \frac{T(r^s,f)}{\log r^s},$$

i.e.

$$\frac{T(r,f)}{T(r^s,f)} \geq \frac{1}{s}. \tag{12}$$

From (9) and (12) we get

$$T(r,f(p(z))) \leq (\frac{u_1 + m(m+1)Ks + \varepsilon_3}{m})T(r,f).$$

According to Lemma 2.1, we obtain

$$T(\mu r^k,f) \leq (\frac{u_1 + m(m+1)Ks + \varepsilon_4}{m})T(r,f).$$

We obtain from Lemma 2.2

$$T(r,f(z)) = O((\log r)^{\alpha_2+\varepsilon}),$$

where

$$\alpha_2 = \frac{\log\frac{u_1+m(m+1)Ks}{m}}{\log k}.$$

Combining case (1) and case (2), we get the proof of Theorem 2.3.

3. Growth of meromorphic solutions

Since the 1970's, R.Goldstein[10-13], W.Bergweiler[2-4], J.Heittokangas[16] et al had investigated the existence and growth of meromorphic solutions on composite functional equations in the whole complex plane and a number of important results were obtained. Particularly, J.Rieppo [20] discussed the growth on meromorphic solutions of many types of functional equations, he also obtained some interesting results, for example, the following theorem B is one of his some results.

For the following functional equations

$$Q(z, f(az+b)) = R(z, f(z)), \tag{$*$}$$

where $Q(z,f), R(z,f)$ are rational functions in f with small meromorphic coefficients relative to f such that $0 < q = \deg_f^Q \leq d = \deg_f^R$ and $a, b \in \mathbf{C}, a \neq 0$ and $|a| \neq 1$.

He obtained

Theorem B([20]) Suppose that f is a transcendental meromorphic solution of the equation $(*)$. Then

$$\mu(f) = \rho(f) = \frac{\log d - \log q}{\log |a|}.$$

It is known that when treating the meromorphic solutions of difference equations, the basic task is to estimate their growth order, while in the case of complex composite functional difference equations, considering the growth order of them is also an interesting task. Hence, this section is devoted to investigating the growth order of meromorphic solutions on two types of composite functional-difference equations (3), (13) and systems of difference equations (14) in complex domain.

As regards the growth order of meromorphic solutions of complex composite functional-difference equations (3), we obtain Theorem 3.1.

Theorem 3.1 Let $\{a_i(z)\}, \{a_{(i)}(z)\}$ be of growth order of $S(r,f)$, $u_1 \geq km$. Then the lower order and the order of meromorphic solution f of the equation (3) satisfy

$$\rho(f) = \mu(f) = 0.$$

In the following, we will also investigate the growth of meromorphic solutions about a type of composite functional-difference equations of the form

$$\frac{\sum\limits_{i=0}^{l} d_i f(a_{1i}z + b_{1i})^i}{\sum\limits_{j=0}^{t} e_j f(a_{2j}z + b_{2j})^j} = \frac{\Omega_1(z,f)}{\Omega_2(z,f)}, \tag{13}$$

where $\{a_{1i}\}, \{a_{2i}\}, \{b_{1j}\}, \{b_{2j}\}, \{d_i\}, \{e_j\}$ are constants ,$\{a_{(i)}(z)\}, \{b_{(j)}(z)\}$ are small functions and $a_{(i)}(z) \not\equiv 0, b_{(j)}(z) \not\equiv 0$.

For complex composite functional-difference equations (13), we obtain the following main result.

Theorem 3.2 Suppose that f is a transcendental meromorphic solution of composite functional-difference equations (13), $a_{1i}, a_{2j}, b_{1i}, b_{2j} \in \mathbf{C}$, $|a_{1i}| > 1, |a_{2j}| > 1$, and the coefficients $a_{(i)}(z)$ are of growth $S(r,f)$.

(i). If $l > t$, then

$$\rho(f) \leq \frac{\log \frac{u_1+u_2}{l}}{\log|a_{1l}|};$$

(ii). If $l < t$, then

$$\rho(f) \leq \frac{\log \frac{u_1+u_2}{t}}{\log|a_{2t}|};$$

(iii). If $l = t$, then

$$\rho(f) \leq \frac{\log \frac{u_1+u_2}{l}}{\log|a|},$$

where $|a| = \max\{|a_{1l}|, |a_{2t}|\}$.

Remark 3 The example 3 shows that the upper bound in Theorem 3.2 can be reached.

Example 3 $f(z) = e^z$ is a meromorphic solution of the following equation

$$\frac{(e^c-1)^2 f(6z+c)}{e^c f(5z+c)} = \frac{f\Delta_c^2 f}{\Delta_c f + f}.$$

We see that $u_1 = 4, u_2 = 2, \rho(f) = 1 = \frac{\log \frac{u_1+u_2}{\max\{l,t\}}}{\log|a_{12}|} = \frac{\log \frac{6}{1}}{\log 6} = \frac{\log 6}{\log 6}$.

By using the Nevanlinna value distribution theory of meromorphic functions, difference equation theory, a large number of papers also have considered the properties of meromorphic solutions of some types of system of functional equations, and obtained some results([7-9]). Now, we consider the problem of the growth order on a class of system of composite functional equations as follows

$$\begin{cases} \sum\limits_{i=0}^{l} d_i f_1(c_{1i}z + d_{1i})^i = \dfrac{\sum\limits_{\mu=0}^{m_1} a_{1\mu}(z) f_2(z)^\mu}{\sum\limits_{\nu=0}^{n_1} a_{2\nu}(z) f_2(z)^\nu}, \\ \sum\limits_{j=0}^{t} e_j f_2(c_{2j}z + d_{2j})^j = \dfrac{\sum\limits_{s=0}^{m_2} b_{1s}(z) f_1(z)^s}{\sum\limits_{k=0}^{n_2} b_{2k}(z) f_1(z)^k}, \end{cases} \tag{14}$$

where $\{c_{1i}\}, \{c_{2j}\}, \{d_{1i}\}, \{d_{2j}\}, d_i, e_j$ are constants,$\{a_{1\mu}(z)\}, \{a_{2\nu}(z)\}, \{b_{1s}(z)\}, \{b_{2k}(z)\}$ are small functions,$|c_{1l}| > 1, |c_{2t}| > 1$.

The growth order of meromorphic solutions (f_1, f_2) of (14) is defined by

$$\rho(f_1, f_2) = \max\{\rho(f_1), \rho(f_2)\},$$

$$\rho(f_k) = \limsup_{r\to\infty} \frac{\log^+ T(r, f_k)}{\log r}, k = 1, 2.$$

The lower order of meromorphic function $f_i, i = 1, 2$ are defined by

$$\mu(f_k) = \liminf_{r\to\infty} \frac{\log^+ T(r, f_k)}{\log r}, k = 1, 2.$$

As regards the complex composite functional-difference equation (14), we obtain Theorem 3.3 and Theorem 3.4 as follows.

Theorem 3.3 Suppose that f is a transcendental meromorphic solution of the system (14), $c_{ij}, d_{ij} \in \mathbf{C}$, $|c_{1l}| > 1, |c_{2t}| > 1$, and the coefficients $a_{ij}(z)$ and $b_{ij}(z)$ are of growth $S(r, f_i)$.Then

$$\rho(f_1, f_2) \leq \frac{\log \frac{\max\{m_1,n_1\}\max\{m_2,n_2\}}{lt}}{\log |c_{1l}||c_{2t}|}.$$

Example 4 Let $b \in \mathbf{C}$ be a constant such that $b \neq \frac{m\pi}{2}$,where $m \in \mathbf{Z}$. We see that $(f_1(z), f_2(z)) = (\tan z, -\tan z)$ is a meromorphic solution of the following system of composite functional equations of the form

$$\begin{cases} f_1(2z+b) = \frac{-2f_2(z)-C(1-f_2^2)}{1-f_2^2-2Cf_2}, \\ f_2(2z+b) = \frac{2f_1(z)-C(1-f_1^2)}{1-f_1^2+2Cf_1}, \end{cases}$$

where $C = -\tan b \neq 0, \infty$.

In this case, $|a_{1l}||a_{2t}| = 4, \max\{m_1, n_1\} \max\{m_2, n_2\}\} = 4, lt = 1$, thus,

$$\rho(f_1, f_2) = 1 = \frac{\log \frac{\max\{m_1,n_1\}\max\{m_2,n_2\}}{lt}}{\log |a_{1l}||a_{2t}|} = \frac{\log 4}{\log 4}.$$

It shows that the upper bound in Theorem 3.3 can be reached.

Theorem 3.4 Let (f_1, f_2) be a transcendental meromorphic solution of the system (14), and $\mu(f_1), \mu(f_2)$ be the lower order of f_1, f_2, respectively. Then

$$\mu(f_1) + \mu(f_2) \geq \frac{\log \frac{\max\{m_1,n_1\}\max\{m_2,n_2\}}{lt}}{\log |c_{1l}||c_{2t}|},$$

where $\{a_{1\mu}(z)\}, \{a_{2\nu}(z)\}, \{b_{1s}(z)\}, \{b_{2k}(z)\}$ are small functions are small functions.

In order to prove Theorems 3.1-3.4, we need the following Lemmas.

Lemma 3.1([14]) Let $\Phi : (1, \infty) \to (0, \infty)$ be a monotone increasing function,and let f be a nonconstant meromorphic function.If for some real constant $\alpha \in (0, 1)$,there exist real constants $K_1 > 0$ and $K_2 \geq 1$ such that

$$T(r, f) \leq K_1\Phi(\alpha r) + K_2T(\alpha r, f) + S(\alpha r, f),$$

then

$$\rho(f) \leq \frac{\log K_2}{-\log \alpha} + \limsup_{r\to\infty} \frac{\log \Phi(r)}{\log r}.$$

Lemma 3.2([3]) Suppose that a meromorphic function f has finite lower order λ.Then for every constant $c > 1$ and a given ε there exists a sequence $r_n = r_n(c, \varepsilon) \to \infty$ such that

$$T(cr_n, f) \leq c^{\lambda+\varepsilon}T(r_n, f).$$

Proof of Theorem 3.1 For a sufficiently small $\varepsilon > 0$, by Lemma 2.1 and Lemma 2.3, we get

$$m(1-\varepsilon)T(\mu r^k, f) \leq mT(r, f(p(z))) \leq (u_1 + \varepsilon)T(r, f),$$

where $\mu = |c_k|(1-\varepsilon), u_1 = \max\{\sum_{l=0}^{n}(l+1)i_l\}$, outside a possible exceptional set of finite logarithmic measure of r.

Hence, for $\alpha > 1$ and for r large enough

$$m(1-\varepsilon)T(\mu r^k, f) \leq (u_1+\varepsilon)T(\alpha r, f).$$

Set $t = \alpha r$. Then

$$T(\frac{\mu}{\alpha^k}t^k, f) \leq \frac{u_1+\varepsilon}{m(1-\varepsilon)}T(t, f).$$

By Lemma 2.2 we obtain

$$T(t, w) = O((\log t)^{\alpha_1}),$$

where

$$\alpha_1 = \frac{\log \frac{u_1}{m}}{\log k} + \varepsilon_1.$$

From the above equation, we can obtain that

$$\rho(f) = \limsup_{r\to\infty} \frac{\log^+ T(r, f)}{\log r} = 0,$$

$$\mu(f) = \liminf_{r\to\infty} \frac{\log^+ T(r, f)}{\log r} = 0.$$

Thus, we have completed the proof of Theorem 3.1.

Proof of Theorem 3.2 Applying Lemma 2.3 and Lemma 2.7 to equation (13),we get

$$\max\{l,t\}T(r, f(a_{sk}z+b_{sk})) = T(r, \frac{\sum_{i=0}^{l} d_i f(a_{1i}z+b_{1i})^i}{\sum_{j=0}^{t} e_j f(a_{2j}z+b_{2j})^j}) \leq (u_1+u_2)T(r, f) + S(r, f),$$

where $s = 1$ or 2, $k = \max\{l, t\}$.

Applying Lemma 2.1 to equation (13), we get

$$(1-\varepsilon)\max\{l,t\}T(\mu r, f) \leq (u_1+u_2)T(r, f) + S(r, f),$$

that is

$$T(\mu r, f) \leq \frac{u_1+u_2}{(1-\varepsilon)\max\{l,t\}}T(r, f) + S(r, f),$$

where $\mu = |a| - \delta > 1, |a| = \max\{|a_{1k}|, |a_{2k}|\}, \delta > 0$. Denoting $\alpha = \frac{1}{\mu}$, we have $0 < \alpha < 1$, and we deduce that

$$T(r, f) \leq \frac{u_1+u_2}{(1-\varepsilon)\max\{l,t\}}T(\alpha r, f) + S(\alpha r, f).$$

By Lemma 3.1, we obtain

$$\rho(f) \leq \frac{\log \frac{u_1+u_2}{(1-\varepsilon)\max\{l,t\}}}{-\log \alpha}.$$

14

Let $\varepsilon \to 0$ and $\delta \to 0$. Then

$$\rho(f) \leq \frac{\log \frac{u_1+u_2}{\max\{l,t\}}}{\log |a|}.$$

Proof of Theorem 3.3 Applying Lemma 2.3 to system (14), we get

$$lT(r, f_1(c_{1l}z + d_{1l})) = \max\{m_1, n_1\}T(r, f_2) + S(r, f_2). \tag{15}$$

$$tT(r, f_2(c_{2t}z + d_{2t})) = \max\{m_2, n_2\}T(r, f_1) + S(r, f_1). \tag{16}$$

Applying Lemma 2.1 to equations (15) and (16), we get

$$(1 - \varepsilon)lT(\mu_1 r, f_1) \leq \max\{m_1, n_1\}T(r, f_2) + S(r, f_2),$$

$$(1 - \varepsilon)tT(\mu_2 r, f_2) \leq \max\{m_2, n_2\}T(r, f_1) + S(r, f_1),$$

that is

$$T(\mu_1 r, f_1) \leq \frac{\max\{m_1, n_1\}}{(1-\varepsilon)l}T(r, f_2) + S(r, f_2),$$

$$T(\mu_2 r, f_2) \leq \frac{\max\{m_2, n_2\}}{(1-\varepsilon)t}T(r, f_1) + S(r, f_1),$$

where $\mu_1 = |c_{1l}| - \delta_1 > 1, \delta_1 > 0, \mu_2 = |c_{2t}| - \delta_2 > 1, \delta_2 > 0$.

Denoting $\alpha_1 = \frac{1}{\mu_1}, \alpha_2 = \frac{1}{\mu_2}$, we have $0 < \alpha_1 < 1, 0 < \alpha_2 < 1$, and we deduce that

$$T(r, f_1) \leq \frac{\max\{m_1, n_1\}}{(1-\varepsilon)l}T(\alpha_1 r, f_2) + S(\alpha_1 r, f_2), \tag{17}$$

$$T(r, f_2) \leq \frac{\max\{m_2, n_2\}}{(1-\varepsilon)t}T(\alpha_2 r, f_1) + S(\alpha_2 r, f_1), \tag{18}$$

outside a possible exceptional set of finite logarithmic measure of r.

Combining (17) and (18), it yields

$$T(r, f_1) \leq \frac{(1 + o(1))\max\{m_1, n_1\}\max\{m_2, n_2\}}{(1-\varepsilon)^2 lt}T(\alpha_1\alpha_2 r, f_1) + S(\alpha_1\alpha_2 r, f_1),$$

outside a possible exceptional set of finite logarithmic measure of r.

By Lemma 3.1, we obtain

$$\rho(f_1) \leq \frac{\log \frac{\max\{m_1,n_1\}\max\{m_2,n_2\}}{(1-\varepsilon)^2 lt}}{-\log \alpha_1\alpha_2}.$$

By a similar reasoning as to above, we also can get

$$\rho(f_2) \leq \frac{\log \frac{\max\{m_1,n_1\}\max\{m_2,n_2\}}{(1-\varepsilon)^2 lt}}{-\log \alpha_1\alpha_2}.$$

Let $\varepsilon \to 0$ and $\delta_i \to 0, i = 1, 2$. Then Theorem 3.3 is proved.

Proof of Theorem 3.4 We assume conversely that f_1, f_2 are transcendental meromorphic functions.

By Lemma 2.3 and $T(r, f(z+c)) \leq (1+o(1))T(r+|c|, f) + M([17])$, where M is a constant, we have

$$\begin{cases} \max\{m_1, n_1\}T(r, f_2) & \leq lT(r, f_1(c_{1l}(z+\frac{d_{1l}}{c_{1l}}))) + S(r, f_2) \\ & \leq (1+o(1))lT(|c_{1l}|r + |\frac{d_{1l}}{c_{1l}}|, f_1) + S(r, f_2), \\ \max\{m_2, n_2\}T(r, f_1) & \leq tT(r, f_2(c_{2t}(z+\frac{d_{2t}}{c_{2t}}))) + S(r, f_1) \\ & \leq (1+o(1))tT(|c_{2t}|r + |\frac{d_{2t}}{c_{2t}}|, f_2) + S(r, f_1). \end{cases} \tag{19}$$

There are two constants $c_1 = |c_{1l}| + \varepsilon_1, c_2 = |c_{2t}| + \varepsilon_2, \varepsilon_i > 0, i = 1, 2$, such that

$$T(|c_{1l}|r + |\frac{d_{1l}}{c_{1l}}|, f_1) \leq T(c_1 r, f_1), T(|c_{2t}|r + |\frac{d_{2t}}{c_{2t}}|, f_2) \leq T(c_2 r, f_2). \tag{20}$$

When r is large enough, we can obtain from (19) and (20)

$$\begin{cases} \max\{m_1, n_1\}T(r, f_2) \leq (1+o(1))lT(c_1 r, f_1) + S(r, f_2), \\ \max\{m_2, n_2\}T(r, f_1) \leq (1+o(1))tT(c_2 r, f_2) + S(r, f_1), \end{cases}$$

outside a possible exceptional set of finite linear measure of r.

According to Lemma 2.5, for given $\sigma_1 > 1, \sigma_2 > 1$,

$$\begin{cases} \max\{m_1, n_1\}T(r, f_2) \leq (1+o(1))lT(\sigma_1 c_1 r, f_1) + S(r, f_2), \\ \max\{m_2, n_2\}T(r, f_1) \leq (1+o(1))tT(\sigma_2 c_2 r, f_2) + S(r, f_1). \end{cases} \tag{21}$$

Let $\mu(f_1), \mu(f_2)$ be the finite lower order in f_1, f_2,respectively. By Lemma 3.2, for any given $\varepsilon_i > 0, i = 1, 2$, there exists a sequence $r_n \to \infty$ such that for $r_n > r_0$

$$T(c_1 r_n, f_1) \leq c_1^{\mu(f_1)+\varepsilon_1} T(r_n, f_1), T(c_2 r_n, f_2) \leq c_2^{\mu(f_2)+\varepsilon_2} T(r_n, f_2).$$

By (21)

$$\begin{cases} \max\{m_1, n_1\}T(r_n, f_2) \leq (1+o(1))l(\sigma_1 c_1)^{\mu(f_1)+\varepsilon_1} T(r_n, f_1) + S(r_n, f_2), \\ \max\{m_2, n_2\}T(r_n, f_1) \leq (1+o(1))t(\sigma_2 c_2)^{\mu(f_2)+\varepsilon_2} T(r_n, f_2) + S(r_n, f_1). \end{cases} \tag{22}$$

From (22), we get

$$\begin{cases} \max\{m_1, n_1\} \leq (1+o(1))l(\sigma_1 c_1)^{\mu(f_1)+\varepsilon_1} \frac{T(r_n, f_1)}{T(r_n, f_2)} + \frac{S(r_n, f_2)}{T(r_n, f_2)}, \\ \max\{m_2, n_2\} \leq (1+o(1))t(\sigma_2 c_2)^{\mu(f_2)+\varepsilon_2} \frac{T(r_n, f_2)}{T(r_n, f_1)} + \frac{S(r_n, f_1)}{T(r_n, f_1)}. \end{cases} \tag{23}$$

Taking lower limit as $n \to \infty$, and $\lim\limits_{n\to\infty} \inf \frac{S(r_n, f_i)}{T(r_n, f_i)} = 0, i = 1, 2$. Then (23) becomes

$$\max\{m_1, n_1\} \max\{m_2, n_2\} \leq lt(\sigma_1 c_1)^{\mu(f_1)+\varepsilon_3} (\sigma_2 c_2)^{\mu(f_2)+\varepsilon_3},$$

where $\varepsilon_3 = \max\{\varepsilon, \varepsilon_1, \varepsilon_2\}, \varepsilon_3 \to 0, \sigma_1 \to 1, \sigma_2 \to 1$. Hence

$$\mu(f_1) + \mu(f_2) \geq \frac{\log \frac{\max\{m_1, n_1\} \max\{m_2, n_2\}}{lt}}{\log |c_{1l}||c_{2t}|}.$$

Thus, we have completed the proof of Theorem 3.4.

Reference

[1] Ablowitz,M.J. Halburd R,Herbst B,On the extension of the Painleve property to difference equations.Nonlinearity, 2000, 13:889-905

[2] Bergweiler,W. Untersuchungen des Wachstums Zusammengesetzter meromorpher Funktionen, Dissertation, Aachen, 1986.

[3] Bergweiler,W. Ishizaki,K., Yanagihara,N. Growth of meromorphic solutions of some functional equations I, Aequations Math., 2002, 63(1-2):140-151

[4] Bergweiler,W. Ishizaki,K., Yanagihara,N. Meromorphic solutions of some functional equations, Methods Appl.Anal., 1998, 5(3):248-258

[5] Chen Zongxuan,Growth and zeros of meromorphic solution of some linear difference equations.Journal of Mathematical Analysis and Applications,2011,373:235–241.

[6] Chen Z.X.,K.H.Shon,On zeros and fixed points of differences of meromorphic functions, J. Math. Anal. Appl. 2008,344:373-383.

[7] Gao Lingyun. On meromorphic solutions of a type of system of composite functional equations, Acta Mathematica Scientia, 2012, 32B(2):800-806

[8] Gao Lingyun. On solutions of a type of system of complex differential-difference equations, Chinese Journal of Contemporary Mathematics, 2017, 381: 23-30

[9] Gao Lingyun. On admissible solutions of two types of systems of differential equations in the complex plane. Acta Mathematica Sinica,2000,43(1):149-156

[10] Goldstein,R. On certain compositions of functions of a complex variable, Aequations Math, 1970, 4:103-126

[11] Goldstein,R. On meromorphic solutions of a functional equations, Aequations Math, 1972, 8:82-94

[12] Goldstein,R. On meromorphic solutions of certain functional equations, Aequations Math, 1978, 18:112-157

[13] Goldstein,R. Some results on factorisation of meromorphic functions, J.London Math.Soc., 1971, 4(2):357-364

[14] Gundersen,R.Heittokangas,J.,Laine,I.,Rieppo,J.,D.Yang,Meromorphic solutions of generalized Schroder equations,Aequationes Math.,2002,63(1-2):110-135

[15] He Yuzan, Xiao Xiuzhi. Algebroid function and ordinary differential equations. Beijing:Science Press, 1988

[16] Heittokangas,J. Laine,I., Rieppo,J., D.Yang. Meromorphic solutions of some linear functional equations, Aequations Math., 2000, 60:148-166

[17] Korhonen,R.A new Clunie type theorem for difference polynomials,J. Difference Equ.Appl.,2011,17(3):387-400

[18] Laine,I. Nevanlinna theory and complex differential equations. Berlin:Walter de Gruyter, 1993

[19] Mokhonko A. Z and Mokhonko V. D. Estimates for the Nevanlinna characteristics of some classes of meromorphic functions and their aplications to differential equations, Siberian Math. J., 1974,15, 921-934.

[20] Rieppo J. On a class of complex functional equations, Ann.Acad.Sci.Fenn., 2007,32:151-170

[21] Silvennoinen H. Meromorphic solutions of some composite functional equations. Ann Acad Sci Fenn, Helsinki: Mathematica Dissertations, 2003, 133

[22] Yi Hongxun, Yang C C. Theory of the uniqueness of meromorphic functions(in Chinese). Beijing:Science Press, 1995

Locally and globally small Riemann sums and Henstock-Stieltjes integral for n-dimensional fuzzy-number-valued functions

Muawya Elsheikh Hamid[a,b*]
[a] School of Mathematical Science, Yangzhou University, Yangzhou 225002, China
[b] Faculty of Engineering, University of Khartoum, Khartoum, Sudan

Abstract: In this paper, we study locally and globally small Riemann sums with respect to α for n-dimensional fuzzy-number-valued functions. And we prove that a fuzzy-number-valued functions in n-dimensional is Henstock-Stieltjes (HS) integrable on $[a,b]$ if and only if it has $(LSRS)$ with respect to α on $[a,b]$. Also we shall prove that a fuzzy-number-valued functions in n-dimensional is Henstock-Stieltjes (HS) integrable on $[a,b]$ if and only if it has $(GSRS)$ with respect to α on $[a,b]$.
Keywords: Fuzzy-number-valued functions in E^n; Henstock-Stieltjes integral (HS); locally small Riemann sums $(LSRS)$; globally small Riemann sums $(GSRS)$.

1 Introduction

Since the concept of fuzzy sets was firstly introduced by Zadeh in 1965 [13], it has been studied extensively from many different aspects of the theory and applications, such as fuzzy topology, fuzzy analysis, fuzzy decision making and fuzzy logic, information science and so on.

The locally and globally small Riemann sums have been introduced by many authors from different points of views including [3, 4, 5, 7, 8, 10, 11]. In 1986, Schurle characterized the Lebesgue integral in $(LSRS)$ (locally small Riemann sums) property [10]. The $(LSRS)$ property has been used to characterized the Perron (P) integral on $[a,b]$ [11]. By considering the equivalency between the (P) integral and the Henstock-Kurzweil (HK) integral, the $(LSRS)$ property has been used to characterized the (HK) integral on $[a,b]$ [8]. In 2015, Indrati [7] introduced a countably Lipschitz condition of a function which is simpler than the ACG*, and proved that the (HK) integrable function or it's primitive could be characterized in countably Lipschitz condition. Also, by considering the characterization of the (HK) integral in the $(GSRS)$ property, it showed that the relationship between $(GSRS)$ property and countably Lipschitz condition of an (HK) integrable function on $[a,b]$. In 2018, Hamid et al. [5] introduced locally and globally small Riemann sums for fuzzy-number-valued functions and established two main theorems: (i) A fuzzy-number-valued functions $\tilde{f}(x)$ is (HS) integrable on $[a,b]$ iff $\tilde{f}(x)$ has $(LSRS)$. (ii) A fuzzy-number-valued functions $\tilde{f}(x)$ is (HS) integrable on $[a,b]$ iff $\tilde{f}(x)$ has $(GSRS)$.

In this paper, the concept of locally small Riemann sums for n-dimensional fuzzy-number-valued functions with respect to α is introduced and discussed. Furthermore, we provide a characterizations of globally small Riemann sums in n-dimensional fuzzy-number-valued functions with respect to α.

The rest of this paper is organized as follows. To make our analysis possible, in Section 2 we shall review the relevant concepts and properties of fuzzy-number-valued functions in E^n and the definition of Henstock-Stieltjes (HS) integral for fuzzy-number-valued functions in E^n. In Section 3, we introduce the support function characterizations of locally small Riemann sums and (HS) integral for fuzzy-number-valued functions in E^n. In section 4, we shall discuss the support function characterizations of globally small Riemann sums and (HS) integral for fuzzy-number-valued functions in E^n. The last section provides the Conclusions.

2 Preliminaries

In this paper the close interval $[a,b]$ denotes a compact interval on R. The set of intervals-point $\{([a_1,b_1],\ \xi_1),$ $([a_2,b_2],\xi_2),\cdots,([a_k,b_k],\xi_k)\}$ is called a division of $[a,b]$ that is $\xi_1,\xi_2,\cdots,\xi_k\in[a,b]$, intervals $[a_1,b_1],[a_2,b_2],\cdots,[a_k,b_k]$ are non-intersect and $\bigcup\limits_{i=1}^{k}[a_i,b_i]=[a,b]$. Marking the division of $[a,b]$ as $P=\{([a_1,b_1],\xi_1),([a_2,b_2],\xi_2),\cdots,([a_k,b_k],\xi_k)\}$, shortening as $P=\{[u,v];\xi\}$ [9].

*Corresponding author. Tel.: +8613218977118. E-mail address: mowia-84@hotmail.com, muawya.ebrahim@gmail.com (M.E. Hamid).

Definition 2.1 [6, 8] Let $\delta : [a,b] \to \mathbb{R}^+$ be a positive real-valued function. $P = \{[x_{i-1}, x_i]; \xi_i\}$ is said to be a δ-fine division, if the following conditions are satisfied:

(1) $a = x_0 < x_1 < x_2 < ... < x_n = b$;

(2) $\xi_i \in [x_{i-1}, x_i] \subset (\xi_i - \delta(\xi_i), \xi_i + \delta(\xi_i))(i = 1, 2, \cdots, n)$.

For brevity, we write $P = \{[u,v]; \xi\}$, where $[u,v]$ denotes a typical interval in P and ξ is the associated point of $[u,v]$.

Definition 2.2 [12] E^n is said to be a fuzzy number space if $E^n = \{u : \ R^n \to [0,1] : u$ satisfies (1)-(4) below$\}$:

(1) u is normal, i.e., there exists a $x_0 \in R^n$ such that $u(x_0) = 1$;

(2) u is a convex fuzzy set, i.e., $u(rx + (1-r)y) \geqslant \min(u(x), u(y)),\ x, y \in R^n,\ r \in [0,1]$;

(3) u is upper semi-continuous;

(4) $[u]^0 = \overline{\{x \in R^n : u(x) > 0\}}$ is compact, for $0 < r \leq 1$, denote $[u]^r = \{x : x \in R^n \ and\ u(x) \geqslant r\}$, $[u]^0 = \overline{\bigcup_{r\in(0,1]}[u]^r}$.

From (1)-(4), it follows that for any $u \in E^n$ and $r \in [0,1]$ the $r-$level set $[u]^r$ is a compact convex set. For any $u, v \in E^n$

$$D(u,v) = \sup_{r\in[0,1]} d([u]^r, [v]^r), \tag{1}$$

where d is Hausdorff metric. It is well known that (E^n, d) is an metric space [12]. The norm of fuzzy number $u \in E^n$ is defined by

$$\|u\| = D(u, \tilde{0}) = \sup_{\alpha\in[u]^0} |\alpha|, \tag{2}$$

where the $\|\cdot\|$ is norm on E^n, $\tilde{0}$ is fuzzy number on E^n and $\tilde{0} = \chi_{\{0\}}$.

Definition 2.3 [12] For $A \in P_k(R^n),\ x \in S^{n-1}$, define the support function of A as $\sigma(x, A) = \sup_{y\in A}\langle y, x\rangle$, where S^{n-1} is the unit sphere of R^n, i.e., $S^{n-1} = \{x \in R^n : \|x\| = 1\}$, $\langle\cdot,\cdot\rangle$ is the inner product in R^n.

Definition 2.4 [2] Let $\alpha : [a,b] \to \mathbb{R}$ be an increasing function. A fuzzy-number-valued function $\tilde{f} : [a,b] \to E^n$ is said to be fuzzy Henstock-Stieltjes (FHS) integrable with respect to α on $[a,b]$, if there exists $\tilde{A} \in E^n$, for every $\varepsilon > 0$, there is a function $\delta(\xi) > 0$, such that for any δ-fine division $P = \{[u,v], \xi\}$ of $[a,b]$, we have

$$D\Big(\sum_{(P)} \tilde{f}(\xi)[\alpha(v) - \alpha(u)], \tilde{A}\ \Big) < \varepsilon. \tag{3}$$

We write $(FHS)\int\limits_a^b \tilde{f}(x)\mathrm{d}\alpha = \widetilde{A}$.

Lemma 2.1 [12] If $u, v \in E^n,\ k \in R,$ for any $r \in [0,1]$, we have

$$[u+v]^r = [u]^r + [v]^r,\ [ku]^r = k[u]^r. \tag{4}$$

Lemma 2.2 [12] Suppose $u \in E^n$, then

(1) $u^*(r, x+y) \leq u^*(r,x) + u^*(r,y)$,

(2) if $u, v \in E^n$, $r \in [0,1]$, then

$$d([u]^r, [v]^r) = \sup_{x\in S^{n-1}} |u^*(r,x) - v^*(r,x)|, \tag{5}$$

(3) $(u+v)^*(r,x) = u^*(r,x) + v^*(r,x)$,

(4) $(ku)^*(r,x) = ku^*(r,x),\ k \geq 0$.

Lemma 2.3 [1, 12] Given $u, v \in E^n$ the distance $D : E^n \times E^n \to [0, +\infty)$ between u and v is defined by the equation $D(u,v) = \sup\limits_{r\in[0,1]} d([u]^r, [v]^r)$, then

(1) (E^n, D) is a complete metric space,

(2) $D(u+w, v+w) = D(u,v)$,

(3) $D(u+v, w+e) \leqslant D(u,w) + D(v,e)$,

(4) $D(ku, kv) = |k|D(u,v), k \in R$,

(5) $D(u+v, \widetilde{0}) \leqslant D(u, \widetilde{0}) + D(v, \widetilde{0})$,

(6) $D(u+v, w) \leqslant D(u,w) + D(v, \widetilde{0})$.

Where $u, v, w, e, \widetilde{0} \in E^n,\ \widetilde{0} = \mathcal{X}_{(\{0\})}$.

Lemma 2.4 [2] Let $\alpha : [a,b] \to \mathbb{R}$ be an increasing function. A fuzzy-number-valued function $\widetilde{F} : [a,b] \to E^n$ is (FHS) integrable with respect to α on $[a,b]$ if and only if $F^*(t)(r,x)$ is (RHS) integrable with respect to α on $[a,b]$ uniformly for any $r \in [0,1]$ and $x \in S^{n-1}$, we have

$$\bigg((FHS)\int_a^b \widetilde{F}(t)\mathrm{d}\alpha\bigg)^*(r,x) = (RHS)\int_a^b F^*(t)(r,x)\mathrm{d}\alpha. \tag{6}$$

Uniformly for any $r \in [0,1]$.

3 Support function characterizations of locally small Riemann sums and (HS) integral for fuzzy-number-valued functions in E^n

In this section, we shall define locally small Riemann sums or in short $(LSRS)$ with respect to α on $[a,b]$ by using support function $f^*(\xi)(r,x)$ and show that it is the necessary and sufficient condition for $\tilde{f}$ to be (HS) integrable with respect to α on $[a,b]$.

Definition 3.1 Let $\alpha : [a,b] \to \mathbb{R}$ be an increasing function. A fuzzy-number-valued function $\tilde{f} : [a,b] \to E^n$ is said to be have locally small Riemann sums or $(LSRS)$ with respect to α on $[a,b]$ if for every $\varepsilon > 0$ there is a $\delta(\xi) > 0$ such that for every $t \in [a,b]$, we have

$$\Big\| \sum \tilde{f}(\xi)[\alpha(v) - \alpha(u)] \Big\|_{E^n} < \varepsilon, \tag{7}$$

whenever $P = \{[u,v];\xi\}$ is a δ-fine division of an interval $C \subset (t-\delta(t), t+\delta(t))$, $t \in C$ and Σ sums over P. (Where $C = [y,z]$).

The following Theorem 3.1 shows that $\tilde{f}$ has $(LSRS)$ with respect to α on $[a,b]$ is equal to the type of it's support functions.

Theorem 3.1 Let $\alpha : [a,b] \to \mathbb{R}$ be an increasing function and let $\tilde{f} : [a,b] \to E^n$ be a fuzzy-number-valued function, the support-function-wise $f^*(\xi)(r,x)$ of $\tilde{f}$ has locally small Riemann sums or $(LSRS)$ with respect to α on $[a,b]$ if and only if for every $\varepsilon > 0$, there is a $\delta(\xi) > 0$ such that for every $t \in [a,b]$, we have

$$\Big| \sum f^*(\xi)(r,x)[\alpha(v) - \alpha(u)] \Big| < \varepsilon, \tag{8}$$

uniformly for any $r \in [0,1]$ and $x \in S^{n-1}$, whenever $P = \{[u,v];\xi\}$ is a δ-fine division of an interval $C \subset (t-\delta(t), t+\delta(t))$, $t \in C$ and Σ sums over P.

Proof Let $\tilde{0} \in E^n$ denote the (FHS) integral of $\tilde{f}$ with respect to α on $[a,b]$. Given $\varepsilon > 0$ there is a $\delta(\xi) > 0$ such that for any δ-fine division $P = \{[u,v];\xi\}$ of $[a,b]$, we have

$$D\Big(\sum \tilde{f}(\xi)[\alpha(v) - \alpha(u)], \tilde{0} \Big) < \varepsilon. \tag{9}$$

That is

$$\sup_{r \in [0,1]} d\Big(\big[\sum \tilde{f}(\xi)[\alpha(v) - \alpha(u)] \big]^r, [\tilde{0}]^r \Big) < \varepsilon. \tag{10}$$

By Lemma 2.2 we have

$$\sup_{r \in [0,1]} \sup_{x \in S^{n-1}} \Big| \big(\sum \tilde{f}(\xi)[\alpha(v) - \alpha(u)] \big)^*(r,x) - \sigma(x,0) \Big| < \varepsilon. \tag{11}$$

Furthermore, by $\sigma(x,A) = \sup_{y \in A} \langle y, x \rangle$, we have

$$\sup_{r \in [0,1]} \sup_{x \in S^{n-1}} \Big| \sum f^*(\xi)(r,x)[\alpha(v) - \alpha(u)] - \sigma(x,0) \Big| < \varepsilon. \tag{12}$$

Hence, for any $r \in [0,1]$, $x \in S^{n-1}$ and for any δ-fine division P we have

$$\Big| \sum f^*(\xi)(r,x)[\alpha(v) - \alpha(u)] \Big| < \varepsilon. \tag{13}$$

Where $\sigma(x,0) = 0$.
This completes the proof. □

Lemma 3.1 (Henstock Lemma). Let $\alpha : [a,b] \to \mathbb{R}$ be an increasing function and let $\tilde{f} : [a,b] \to E^n$ be a fuzzy-number-valued function and (HS) integrable to $\tilde{A}$ with respect to α on $[a,b]$. Then, the support-function-wise $f^*(\xi)(r,x)$ of $\tilde{f}$ on $[a,b]$ is (HS) integrable to $A^*(r,x)$ with respect to α on $[a,b]$ uniformly for any $r \in [0,1]$, $x \in S^{n-1}$ and $\tilde{A} \in E^n$, i.e., for every $\varepsilon > 0$ there is a positive function $\delta(\xi) > 0$, for δ-fine division $P = \{[u,v];\xi\}$ of $[a,b]$ and for any $x \in S^{n-1}$, we have

$$\Big| \sum f^*(\xi)(r,x)[\alpha(v) - \alpha(u)] - A^*(r,x) \Big| < \varepsilon. \tag{14}$$

Furthermore, for any sum of parts $\sum_1$ from $\sum$ we have

$$\Big| \sum_1 f^*(\xi)(r,x)[\alpha(v) - \alpha(u)] - A^*(r,x) \Big| < \varepsilon. \tag{15}$$

Proof Let $\tilde{A} \in E^n$ denote the (FHS) integral of $\tilde{f}$ with respect to α on $[a,b]$. Given $\varepsilon > 0$ there is a $\delta(\xi) > 0$ such that for any δ-fine division $P = \{[u,v];\xi\}$ of $[a,b]$, we have

$$D\bigg(\sum \tilde{f}(\xi)[\alpha(v)-\alpha(u)], \tilde{A}\bigg) < \varepsilon. \tag{16}$$

That is

$$\sup_{r\in[0,1]} d\bigg(\big[\sum \tilde{f}(\xi)[\alpha(v)-\alpha(u)]\big]^r, [\tilde{A}]^r\bigg) < \varepsilon. \tag{17}$$

By Lemma 2.2 we have

$$\sup_{r\in[0,1]}\sup_{x\in S^{n-1}} \bigg|\big(\sum \tilde{f}(\xi)[\alpha(v)-\alpha(u)]\big)^*(r,x) - A^*(r,x)\bigg| < \varepsilon. \tag{18}$$

Furthermore, by $A^*(r,x) = \sup_{y\in[A]^r}\langle y,x\rangle$, we have

$$\sup_{r\in[0,1]}\sup_{x\in S^{n-1}} \bigg|\sum f^*(\xi)(r,x)[\alpha(v)-\alpha(u)] - A^*(r,x)\bigg| < \varepsilon. \tag{19}$$

Hence, for any $r \in [0,1]$, $x \in S^{n-1}$ and for any δ-fine division P we have

$$\bigg|\sum f^*(\xi)(r,x)[\alpha(v)-\alpha(u)] - A^*(r,x)\bigg| < \varepsilon. \tag{20}$$

For proof

$$\bigg|\sum_1 f^*(\xi)(r,x)[\alpha(v)-\alpha(u)] - A^*(r,x)\bigg| < \varepsilon, \tag{21}$$

the proof is similar to the Theorem 3.7 in [8].
This completes the proof. □

Theorem 3.2 Let $\alpha : [a,b] \to \mathbb{R}$ be an increasing function and let $\tilde{f} : [a,b] \to E^n$ be a fuzzy-number-valued function. If $\tilde{f}$ is (HS) integrable to $\tilde{F}([a,b])$ with respect to α on $[a,b]$, then $\tilde{f}$ has $LSRS$ with respect to α on $[a,b]$.

Proof Since $\tilde{f}$ is (HS) integrable to $\tilde{F}([a,b])$ with respect to α on $[a,b]$, by Theorem 3.1 the support-function-wise $f^*(\xi)(r,x)$ of $\tilde{f}$ on $[a,b]$ is (HS) integrable to $F^*([a,b])(r,x)$ with respect to α on $[a,b]$ uniformly for any $r \in [0,1]$, $x \in S^{n-1}$, i.e., for every $\varepsilon > 0$ there is a positive function $\delta(\xi) > 0$, for δ-fine division $P = \{[u,v];\xi\}$ of $[a,b]$ and for any $x \in S^{n-1}$, we have

$$\bigg|\sum f^*(\xi)(r,x)[\alpha(v)-\alpha(u)] - F^*([a,b])(r,x)\bigg| < \frac{\varepsilon}{2}. \tag{22}$$

For each $t \in [a,b]$, there is a closed interval $C = [y,z] \subset (t-\delta(t), t+\delta(t))$ such that

$$\bigg|F^*([y,z])(r,x)\bigg| < \frac{\varepsilon}{2}. \tag{23}$$

According to Henstock Lemma, for each $t \in [a,b]$ and δ-fine division $P = \{[u,v];\xi\}$ of $C \subset (t-\delta(t), t+\delta(t))$, we have

$$\begin{aligned}\bigg|\sum f^*(\xi)(r,x)[\alpha(v)-\alpha(u)]\bigg| &\le \bigg|\sum f^*(\xi)(r,x)[\alpha(v)-\alpha(u)] - F^*([a,b])(r,x)\bigg| + \bigg|F^*([y,z])(r,x)\bigg| \\ &< \varepsilon.\end{aligned}$$

Applies Theorem 3.1 again $\tilde{f}$ has $LSRS$ with respect to α on $[a,b]$.
This completes the proof. □

Lemma 3.2 Let $\alpha : [a,b] \to \mathbb{R}$ be an increasing function and let $\tilde{f} : [a,b] \to E^n$ be a fuzzy-number-valued function. If $\tilde{f}$ is (FHS) integrable with the $\tilde{F}$ as primitive then for each number $\varepsilon > 0$ there is a positive function $\delta(\xi) > 0$, such that for any $[u,v] \subset [a,b]$ with $\big(\alpha(v)-\alpha(u)\big) < \delta(\xi)$, we have

$$\bigg\|\tilde{F}([u,v])\bigg\|_{E^n} = \bigg\|(FHS)\int_{[u,v]} \tilde{f}\,\mathrm{d}\alpha\bigg\|_{E^n} < \varepsilon. \tag{24}$$

Proof The continuity follows from Lemma 3.1 and the following inequality:

$$\begin{aligned}\bigg\|\tilde{F}([u,v])\bigg\|_{E^n} &= D\bigg(\tilde{F}(u), \tilde{F}(v)\bigg) \\ &\le D\bigg(\tilde{F}([u,v]), \tilde{f}(\xi)[\alpha(v)-\alpha(u)]\bigg) + \bigg\|\tilde{f}(\xi)[\alpha(v)-\alpha(u)]\bigg\|_{E^n} \\ &< \varepsilon.\end{aligned}$$

We only need set $\delta(\xi) < \frac{\varepsilon}{2(\|\tilde{f}(\xi)\|_{E^n}+1)}$.
This completes the proof. □

Theorem 3.3 Let $\alpha : [a,b] \to \mathbb{R}$ be an increasing function and let a fuzzy-number-valued function $\tilde{f} : [a,b] \to E^n$ has $LSRS$ with respect to α on $[a,b]$, then $\tilde{f}$ is (FHS) integrable with respect to α on $[a,b]$.

Proof Given any $\varepsilon > 0$ and $P = \{([a,b],\xi)\} = \{([a_1,b_1],\xi_1), ([a_2,b_2],\xi_2), \cdots, ([a_n,b_n],\xi_n)\}$ is a δ-fine partition of $[a,b]$. For each $i(i = 1,2,\cdots,n)$ there is a positive function δ_i with $P_i = \{([u_i,v_i],\xi_i)\}$ is a δ_i-fine partition of $[a_i,b_i]$. Since $\tilde{f}$ has $LSRS$ with respect to α on $[a_i,b_i]$, then we have

$$\left\| \sum_{P_i} \tilde{f}(\xi)[\alpha(v) - \alpha(u)] \right\|_{E^n} < \frac{\varepsilon}{2n}. \tag{25}$$

Taken $\eta = \max\{\delta(\xi),\ \xi \in [a,b]\}$, according to the Lemma 3.2 we have

$$\left\| \tilde{F}([a_i,b_i]) \right\|_{E^n} = \left\| (FHS) \int_{[a_i,b_i]} \tilde{f} \mathrm{d}\alpha \right\|_{E^n} < \frac{\varepsilon}{2n}. \tag{26}$$

Therefore, for any δ_i-fine partition $P_i = \{([u_i,v_i],\xi_i)\}$ of $[a_i,b_i]$, we have

$$\begin{aligned} \left(\sum_{P_i} \tilde{f}(\xi)[\alpha(v) - \alpha(u)], \tilde{F}([a_i,b_i]) \right) &\leq \left\| \sum_{P_i} \tilde{f}(\xi)[\alpha(v) - \alpha(u)] \right\|_{E^n} + \left\| \tilde{F}([a_i,b_i]) \right\|_{E^n} \\ &< \frac{\varepsilon}{2n} + \frac{\varepsilon}{2n} = \frac{\varepsilon}{n}, \end{aligned}$$

for each i.

Subsequently taken $\delta^*(\xi) = \min\{\delta(\xi), \delta_i(\xi)\}$, then $P = \bigcup_{i=1}^{n} P_i$ denote δ^*-fine partition of $[a,b]$.

Therefore we have

$$\begin{aligned} \left(\sum_{P} \tilde{f}(\xi)[\alpha(v) - \alpha(u)], \tilde{F}([a,b]) \right) &= \sum_{i=1}^{n} D\left(\sum_{P_i} \tilde{f}(\xi)[\alpha(v) - \alpha(u)], \tilde{F}([a_i,b_i]) \right) \\ &< n \cdot \frac{\varepsilon}{n} = \varepsilon. \end{aligned}$$

Then $\tilde{f}$ is (FHS) integral with respect to α on $[a,b]$.
This completes the proof. □

4 Support function characterizations of globally small Riemann sums and (HS) integral for fuzzy-number-valued functions in E^n

In this section, we shall define globally small Riemann sums or in short $(GSRS)$ integral with respect to α on $[a,b]$ by using support function $f^*(\xi)(r,x)$ and show that it is the necessary and sufficient condition for $\tilde{f}$ to be (HS) integrable on $[a,b]$.

Definition 4.1 Let $\alpha : [a,b] \to \mathbb{R}$ be an increasing function. A fuzzy-number-valued function $\tilde{f} : [a,b] \to E^n$ is said to be have globally small Riemann sums or $(GSRS)$ with respect to α on $[a,b]$ if for every $\varepsilon > 0$ there exists a positive integer N such that for every $n \geqslant N$ there is a $\delta_n(\xi) > 0$ and for every δ_n-fine division $P = \{[u,v];\xi\}$ of $[a,b]$, we have

$$\left\| \sum_{\|\tilde{f}(\xi)\|_{E^n} > n} \tilde{f}(\xi)[\alpha(v) - \alpha(u)] \right\|_{E^n} < \varepsilon, \tag{27}$$

where the $\sum$ is taken over P and for which $\left\|\tilde{f}(\xi)\right\|_{E^n} >_n$.

The following Theorem 4.1 shows that $\tilde{f}$ has $(GSRS)$ with respect to α on $[a,b]$ is equal to the type of it's support functions.

Theorem 4.1 Let $\alpha : [a,b] \to \mathbb{R}$ be an increasing function and let $\tilde{f} : [a,b] \to E^n$ be a fuzzy-number-valued function, the support-function-wise $f^*(\xi)(r,x)$ of $\tilde{f}$ has globally small Riemann sums or $(GSRS)$ with respect to α on $[a,b]$ if and only if for every $\varepsilon > 0$, there exists a positive integer N such that for every $n \geqslant N$ there is a $\delta_n(\xi) > 0$ and for every δ_n-fine division $P = \{[u,v];\xi\}$ of $[a,b]$, we have

$$\left| \sum_{|f^*(\xi)(r,x)| > n} f^*(\xi)(r,x)[\alpha(v) - \alpha(u)] \right| < \varepsilon, \tag{28}$$

uniformly for any $r \in [0,1]$ and $x \in S^{n-1}$, where the $\sum$ is taken over P and for which $\left|f^*(\xi)(r,x)\right| > n$.

Proof First, we can prove the following statements are equivalent:
(1) $\|\tilde{f}(\xi)\|_{E^n} > n$.
(2) $|f^*(\xi)(r,x)| > n$.
In fact

$$\begin{aligned}\|\tilde{f}(\xi)\|_{E^n} > n &= \sup_{r\in[0,1]} d\big([\tilde{f}(\xi)]^r, [\tilde{0}]^r\big)\\ &= \sup_{r\in[0,1]}\sup_{x\in S^{n-1}} \big|f^*(\xi)(r,x)\big|.\end{aligned}$$

Second, let $\tilde{0}\in E^n$ denote the (FHS) integral of $\tilde{f}$ with respect to α on $[a,b]$. Given $\varepsilon>0$ there exists a positive integer N such that for every $n\geqslant N$ there is a $\delta_n(\xi)>0$ and for every δ_n-fine division $P=\{[u,v];\xi\}$ of $[a,b]$, we have

$$D\bigg(\sum_{\|\tilde{f}(\xi)\|_{E^n}>n} \tilde{f}(\xi)[\alpha(v)-\alpha(u)], \tilde{0}\bigg) < \varepsilon. \tag{29}$$

That is

$$\sup_{r\in[0,1]} d\bigg(\big[\sum_{\|\tilde{f}_r(\xi)\|_{E^n}>n} \tilde{f}(\xi)[\alpha(v)-\alpha(u)]\big]^r, [\tilde{0}]^r\bigg) < \varepsilon. \tag{30}$$

By Lemma 2.2 we have

$$\sup_{r\in[0,1]}\sup_{x\in S^{n-1}} \bigg|\big(\sum_{|f^*(\xi)(r,x)|>n} f(\xi)[\alpha(v)-\alpha(u)]\big)^*(r,x) - \sigma(x,0)\bigg| < \varepsilon. \tag{31}$$

Furthermore, by $\sigma(x,A)=\sup_{y\in A}\langle y,x\rangle$, we have

$$\sup_{r\in[0,1]}\sup_{x\in S^{n-1}} \bigg|\sum_{|f^*(\xi)(r,x)|>n} f^*(\xi)(r,x)[\alpha(v)-\alpha(u)] - \sigma(x,0)\bigg| < \varepsilon. \tag{32}$$

Hence, for any $r\in[0,1]$, $x\in S^{n-1}$ and for any δ-fine division P we have

$$\bigg|\sum_{|f^*(\xi)(r,x)|>n} f^*(\xi)(r,x)[\alpha(v)-\alpha(u)]\bigg| < \varepsilon. \tag{33}$$

Where $\sigma(x,0)=0$.
This completes the proof. □

Theorem 4.2 Let $\alpha:[a,b]\to\mathbb{R}$ be an increasing function and let $\tilde{f}:[a,b]\to E^n$ be a fuzzy-number-valued function. If $\tilde{f}$ has $GSRS$ with respect to α on $[a,b]$ then $\tilde{f}$ is (HS) integrable with respect to α on $[a,b]$.

Proof Because $\tilde{f}$ has $GSRS$ with respect to α on $[a,b]$, then by Theorem 4.1 for every $\varepsilon>0$, there exists a positive integer N such that for every $n\geqslant N$ there is a $\delta_n(\xi)>0$ and for every δ_n-fine division $P=\{[u,v];\xi\}$ of $[a,b]$, we have

$$\bigg|\sum_{|f^*(\xi)(r,x)|>n} f^*(\xi)(r,x)[\alpha(v)-\alpha(u)]\bigg| < \varepsilon. \tag{34}$$

uniformly for any $r\in[0,1]$ and $x\in S^{n-1}$, where the $\sum$ is taken over P and for which $|f^*(\xi)(r,x)|>n$.
For each two δ-fine divisions $P_1=\{[u_1,v_1];\xi_1\}$, $P_2=\{[u_2,v_2];\xi_2\}$ of $[a,b]$, we have

$$\begin{aligned}&\Big|\sum f^*(\xi_1)(r,x)[\alpha(v_1)-\alpha(u_1)] - \sum f^*(\xi_2)(r,x)[\alpha(v_2)-\alpha(u_2)]\Big|\\
\leq\ &\Big|\sum f^*(\xi_1)(r,x)[\alpha(v_1)-\alpha(u_1)]\Big| + \Big|\sum f^*(\xi_2)(r,x)[\alpha(v_2)-\alpha(u_2)]\Big|\\
\leq\ &\bigg|\sum_{|f^*(\xi_1)(r,x)|>n} f^*(\xi_1)(r,x)[\alpha(v_1)-\alpha(u_1)]\bigg| + \bigg|\sum_{|f^*(\xi_1)(r,x)|\leq n} f^*(\xi_1)(r,x)[\alpha(v_1)-\alpha(u_1)]\bigg|\\
+\ &\bigg|\sum_{|f^*(\xi_2)(r,x)|>n} f^*(\xi_2)(r,x)[\alpha(v_2)-\alpha(u_2)]\bigg| + \bigg|\sum_{|f^*(\xi_2)(r,x)|\leq n} f^*(\xi_2)(r,x)[\alpha(v_2)-\alpha(u_2)]\bigg|\\
<\ &4\varepsilon.\end{aligned}$$

According to the properties of Cauchy, $\tilde{f}$ is (HS) integrable on $[a,b]$.
This completes the proof. □

Theorem 4.3 Let $\alpha : [a,b] \to \mathbb{R}$ be an increasing. Given a fuzzy-number-valued function $\tilde{f} : [a,b] \to E^n$, for each $r \in [0,1]$ and $x \in S^{n-1}$ defined the support function $f_n^*(\xi)(r,x)$ of $\tilde{f}_n$ by the formula:

$$f_n^*(\xi)(r,x) = \begin{cases} f^*(\xi)(r,x), \xi \in [a,b] & \text{if } |f^*(\xi)(r,x)| \le n, \\ 0, & \text{others.} \end{cases}$$

A fuzzy-number-valued function $\tilde{f}$ is (HS) integrable with respect to α on $[a,b]$ if and only if $\tilde{f}$ has $GSRS$ with respect to α on $[a,b]$ and $\tilde{F}_n([a,b]) \to \tilde{F}([a,b])$ as $n \to \infty$. (Where $\tilde{F}([a,b])$ and $\tilde{F}_n([a,b])$ the integral of $\tilde{f}$ and $\tilde{f}_n$ with respect to α on $[a,b]$ respectively).

Proof First we shall prove the necessity. Because a fuzzy-number-valued function $\tilde{f}$ is (HS) integrable with respect to α on $[a,b]$ uniformly for any $r \in [0,1]$ and $x \in S^{n-1}$, i.e., for every $\varepsilon > 0$ there is a positive function δ^*, for δ^*-fine division $P = \{[u,v];\xi\}$ of $[a,b]$, we have

$$\left| \sum f^*(\xi)(r,x)[\alpha(v) - \alpha(u)] - F^*([a,b])(r,x) \right| < \frac{\varepsilon}{3}. \tag{35}$$

For each $n \in \mathbb{N}$, there is a positive function δ_n, for δ_n-fine division $P = \{[u,v];\xi\}$ of $[a,b]$, we have

$$\left| \sum f_n^*(\xi)(r,x)[\alpha(v) - \alpha(u)] - F_n^*([a,b])(r,x) \right| < \frac{\varepsilon}{3}, \tag{36}$$

for each $r \in [0,1]$ and $x \in S^{n-1}$.

Because $\{F_n^*([a,b])(r,x)\}$ converge to $F^*([a,b])(r,x)$ of $[a,b]$ then there is a positive number N so if $n \ge N$ we have

$$\left| F_n^*([a,b])(r,x) - F^*([a,b])(r,x) \right| < \frac{\varepsilon}{3}. \tag{37}$$

For $n \ge N$, defined a positive function δ on $[a,b]$ by the formula:

$$\delta(\xi) = \min\{\delta^*(\xi), \delta_n(\xi)\}. \tag{38}$$

Therefor, for each δ-fine division $P = \{[u,v];\xi\}$ of $[a,b]$, we have

$$\begin{aligned} & \left| \sum_{|f^*(\xi)(r,x)|>n} f^*(\xi)(r,x)[\alpha(v) - \alpha(u)] \right| \\ = & \left| \sum f^*(\xi)(r,x)[\alpha(v) - \alpha(u)] - \sum f_n^*(\xi)(r,x)[\alpha(v) - \alpha(u)] \right| \\ \le & \left| \sum f^*(\xi)(r,x)[\alpha(v) - \alpha(u)] - F^*([a,b])(r,x) \right| + \left| F_n^*([a,b])(r,x) - F^*([a,b])(r,x) \right| \\ + & \left| F^*([a,b])(r,x) - \sum f_n^*(\xi)(r,x)[\alpha(v) - \alpha(u)] \right| \\ < & \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

Then $\tilde{f}$ has $GSRS$ with respect to α on $[a,b]$.

Second we shall prove the sufficiency. Because $\tilde{f}$ has $GSRS$ with respect to α on $[a,b]$, then by Theorem 4.1 for every $\varepsilon > 0$, there exists a positive integer N such that for every $n \geqslant N$ there is a $\delta_n(\xi) > 0$ and for every δ_n-fine division $P = \{[u,v];\xi\}$ of $[a,b]$, we have

$$\left| \sum_{|f^*(\xi)(r,x)|>n} f^*(\xi)(r,x)[\alpha(v) - \alpha(u)] \right| < \varepsilon, \tag{39}$$

uniformly for any $r \in [0,1]$ and $x \in S^{n-1}$, where the $\sum$ is taken over P and for which $\left| f^*(\xi)(r,x) \right| > n$.

Note that $\tilde{f}_n$, is Henstock-Stieltjes integrable with respect to α on $[a,b]$ for all n. Choose N so that whenever $n, m \geqslant N$ we have

$$\left| F_n^*([a,b])(r,x) - F_m^*([a,b])(r,x) \right| < \varepsilon. \tag{40}$$

Then for $n, m \geqslant N$ and a suitably chosen δ-fine division $P = \{[u,v];\xi\}$, we have

$$\begin{aligned} & \left| F_n^*([a,b])(r,x) - F_m^*([a,b])(r,x) \right| \\ \le & \left| F_n^*([a,b])(r,x) - \sum_{|f^*(\xi)(r,x)|\le n} f^*(\xi)(r,x)[\alpha(v) - \alpha(u)] \right| + \left| \sum_{|f^*(\xi)(r,x)|>n} f^*(\xi)(r,x)[\alpha(v) - \alpha(u)] \right| \\ + & \left| \sum_{|f^*(\xi)(r,x)|\le m} f^*(\xi)(r,x)[\alpha(v) - \alpha(u)] - F_m^*([a,b])(r,x) \right| + \left| \sum_{|f^*(\xi)(r,x)|>m} f^*(\xi)(r,x)[\alpha(v) - \alpha(u)] \right| \\ < & 4\varepsilon. \end{aligned}$$

That is, $\{F_n^*([a,b])(r,x)\}$ converge to $F^*([a,b])(r,x)$, as $n \to \infty$. Again, for suitably chosen N and $\delta(\xi)$ and for every δ-fine division $P = \{[u,v];\xi\}$, we have

$$\begin{aligned}
& \left|\sum f^*(\xi)(r,x)[\alpha(v)-\alpha(u)] - F^*([a,b])(r,x)\right| \\
\leq & \left|\sum f^*(\xi)(r,x)[\alpha(v)-\alpha(u)] - F_N^*([a,b])(r,x)\right| + \left|F_N^*([a,b])(r,x) - F^*([a,b])(r,x)\right| \\
\leq & \left|\sum_{|f^*(\xi)(r,x)|\leq N} f^*(\xi)(r,x)[\alpha(v)-\alpha(u)] - F_N^*([a,b])(r,x)\right| + \left|\sum_{|f^*(\xi)(r,x)|>N} f^*(\xi)(r,x)[\alpha(v)-\alpha(u)]\right| \\
+ & \left|F_N^*([a,b])(r,x) - F^*([a,b])(r,x)\right| \\
< & \ 3\varepsilon.
\end{aligned}$$

That is, $\tilde{f}$ is (HS) integrable on $[a,b]$.
This completes the proof. □

5 conclusions

In this paper, the notions of locally and globally small Riemann sums modifications with respect to fuzzy-number-valued functions in E^n are introduced and studied. The basic properties and characterizations are presented. In particular, it is proved that a fuzzy-number-valued functions in E^n is (HS) integrable on $[a,b]$ iff it has $(LSRS)$, and also it is proved that a fuzzy-number-valued functions in E^n is (HS) integrable on $[a,b]$ iff it has $(GSRS)$.

References

[1] S.X. Hai and Z.T. Gong, On Henstock integral of fuzzy-number-valued functions in R^n, International Journal of Pure and Applied Mathematics, **7**(1)(2003), 111-121.

[2] M.E. Hamid and Z.T Gong, The Henstock-Stieltjes Integral for n-dimensional Fuzzy-Number-Valued Functions, International Journal of Mathematics And its Applications, **5**(1-B)(2017), 171-185.

[3] M.E. Hamid, L.S. Xu and Z.T. Gong, Locally and globally small Riemann sums and Henstock integral of fuzzy-number-valued functions, Journal of Computational Analysis and Applications, **25**(1)(2018), 11-18.

[4] M.E. Hamid, L.S. Xu, Locally and globally small Riemann sums and Henstock integral of fuzzy- number-valued functions in E^n, Journal of Computational Analysis and Applications, in Press.

[5] M.E. Hamid, L.S. Xu and Z.T. Gong, Locally and globally small Riemann sums and Henstock-Stieltjes integral of fuzzy- number-valued functions, Journal of Computational Analysis and Applications, **25**(6)(2018), 1107-1115.

[6] R. Henstock, Theory of Integration, Butterworth, London, 1963.

[7] C.R. Indrati, Some Characteristics of the Henstock-Kurzweil in Countably Lipschitz Condition, The 7th SEAMS-UGM Conference 2015.

[8] P.Y. Lee, Lanzhou Lectures on Henstock Integration, World Scientific, Singapore, 1989.

[9] P.Y. Lee and R. Vyborny, The Integral: An Easy Approach after Kurzweil and Henstock, Cambridge University Press, 2000.

[10] A.W. Schurle, A new property equivalent to Lebesgue integrability, Proceedings of the American Mathematical Society, **96**(1)(1986), 103-106.

[11] A.W. Schurle, A function is Perron integrable if it has locally small Riemann sums, Journal of the Australian Mathematical Society (Series A), **41**(2)(1986), 224-232.

[12] C.X. Wu, M. Ma and J.X. Fang, Structure Theory of Fuzzy Analysis, Guizhou Scientific Publication (1994), In Chinese.

[13] L.A. Zadeh, Fuzzy sets, Information Control, **8**(1965), 338-353.

Solving Systems of Nonhomogeneous Coupled Linear Matrix Differential Equations in Terms of Mittag-Leffler Matrix Functions

Rungpailin Kongyaksee, Pattrawut Chansangiam*

Department of Mathematics, Faculty of Science,
King Mongkut's Institute of Technology Ladkrabang,
Bangkok 10520, Thailand.

Abstract

In this paper, we investigate systems of nonhomogeneous coupled linear matrix differential equations. Applying Kronecker products, the vector operator, and matrix convolution product, we obtain explicit formula of the general solution to this system in terms of matrix series concerning exponentials and Mittag-Leffler functions.

Keywords: linear matrix differential equation, Kronecker product, vector operator, matrix convolution product, Mittag-Leffler function.
Mathematics Subject Classifications 2010: 15A16, 15A69, 33E12, 34A30, 44A35.

1 Introduction

Theory of linear matrix differential equations can be applied in a broad range of scientific fields, e.g. statistics [2, 6, 8], game theory [4], ecometrics and Leondief model [6, 8, 11], control and system theory [3, 7]. The simplest first-order homogeneous linear matrix differential equation with time-invariant coefficient is given by

$$X'(t) = AX(t). \tag{1.1}$$

Here, A is a given square matrix and $X(t)$ is an unknown matrix-valued function to be solved. The system (1.1) has been widely studied, and the solution relies on the computation of e^{tA}; see more information in [12, 13]. The nonhomogeneous case appears in the form

$$X'(t) = AX(t) + U(t), \tag{1.2}$$

*Corresponding author. Email: pattrawut.ch@kmitl.ac.th

here $U(t)$ is a given matrix-valued function. In fact, the equation (1.2) has a general solution given by a one-parameter matrix-valued function

$$X(t) = e^{(t-t_0)A}X(t_0) + e^{tA} * U(t), \tag{1.3}$$

where $*$ denotes the matrix convolution product. See related works on nonhomogeneous case in [10, 15] and references therein.

Coupled matrix differential equations have numerous applications in pure and applied mathematics. For example, to obtain the solution of an optimal control problem with performance index we need to solve the system [7]

$$\begin{aligned} X'(t) &= AX(t) + BY(t), \\ Y'(t) &= CX(t) - A^T Y(t). \end{aligned}$$

A general system of nonhomogeneous coupled linear matrix differential equations with time-invariant coefficient takes the form

$$\begin{aligned} X'(t) &= AX(t)B + CY(t)D + U(t), \\ Y'(t) &= EX(t)F + GY(t)H + V(t). \end{aligned} \tag{1.4}$$

In [5], a homogeneous case of (1.4) when $E = C$, $F = D$, $G = A$, $H = B$ was investigated under the assumption that $AC = CA$ and $BD = DB$. In this case, the solution is given in terms of Kronecker products, the vector operator, and matrix series concerning exponentials and hyperbolic functions. A nonhomogeneous case of (1.4) was discussed in [1].

In this work, we investigate the system (1.4) under the assumption that $AC = CG$, $GE = EA$, $DB = HD$, $FH = BF$. We apply Kronecker products and the vector operator to reduce our complex system to the simplest form. Thus, an explicit formula of the general solution to this system is obtained in terms of Mittag-Leffler matrix functions. In particular, we obtain general solution of several special cases of the main system. When initial conditions are imposed to these problems, its solution is uniquely determined. Our results also include the previous works [1, 5].

This paper is structured as follows. In Section 2, we supply useful facts for solving linear matrix differential equations, including matrix functions defined by power series, Kronecker product, vector operator, and matrix convolution product. The main part of the paper, Section 3, deals with solving the system (1.4) and its interesting special cases. In Sections 4, we treat an initial value problem related to (1.4) and illustrate it with a numerical example.

2 Preliminaries

In this section, we provide adequate tools for solving system of linear matrix differential equations. We shall denote the set of all m-by-n complex matrices by $M_{m,n}$, and we set $M_n = M_{n,n}$.

2.1 Functions of a matrix defined by power series

Consider $A \in M_n$ and a holomorphic function f defined on a region in the complex plane containing the origin and the spectrum of A. Let $R > 0$ be such that f admits the Taylor series expansion

$$f(z) = \sum_{k=0}^{\infty} a_k z^k \quad \text{for } |z| < R,$$

where $a_0 = f(0)$ and $a_k = f^{(k)}(0)/k!$ for any $k \in \mathbb{N}$. If the spectral radius of A is less than R, then the matrix power series $\sum_{k=0}^{\infty} a_k A^k$ converges, denoted by $f(A)$. Hence if f is an entire function then $f(A)$ is a well-defined matrix for any $A \in M_n$. In particular, the following matrix series converge for any $A \in M_n$:

$$\sinh(A) = \sum_{k=0}^{\infty} \frac{1}{(2k+1)!} A^{2k+1}, \quad \cosh(A) = \sum_{k=0}^{\infty} \frac{1}{(2k)!} A^{2k}.$$

Recall that the two-parameter Mittag-Leffler functions (e.g. [14]) is defined by

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)} \tag{2.1}$$

where Γ is the Gamma function. The power series (2.1) converges for all complex numbers z.

The Mittag-Leffler function of a matrix $A \in M_n$ with parameters $\alpha > 0$ and $\beta > 0$ is defined by

$$E_{\alpha,\beta}(A) = \sum_{k=0}^{\infty} \frac{1}{\Gamma(\alpha k + \beta)} A^k = I_n + \frac{1}{\Gamma(\alpha + \beta)} A + \frac{1}{\Gamma(2\alpha + \beta)} A^2 + \cdots .$$

The class of matrix Mittag-Leffler functions include the following functions:

$$E_{1,1}(A) = \sum_{k=0}^{\infty} \frac{1}{k!} A^k = e^A, \quad E_{2,1}(A^2) = \sum_{k=0}^{\infty} \frac{1}{(2k)!} A^{2k} = \cosh(A).$$

An expansion shows that $\big(E_{2,2}(A^2)\big)A = \sum_{k=0}^{\infty} \dfrac{1}{(2k+1)!} A^{2k+1} = \sinh(A)$.

Lemma 2.1 (see e.g. [9])**.** *If (A, B) is a pair of commuting complex matrices, then $e^{A+B} = e^A e^B$.*

The next lemma is useful for deriving explicit formulas of solutions for system of linear matrix differential equations in Section 3.

Lemma 2.2. *For any $A \in M_n(\mathbb{C})$ and $B \in M_n(\mathbb{C})$, we have*

$$e^{\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}} = \begin{bmatrix} E_{2,1}(AB) & \big(E_{2,2}(AB)\big)A \\ \big(E_{2,2}(BA)\big)B & E_{2,1}(BA) \end{bmatrix}.$$

3

Proof. A computation using matrix analysis reveals that

$$
\begin{aligned}
e^{\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}} &= \sum_{k=0}^{\infty} \frac{1}{k!} \begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}^k \\
&= \sum_{k=0}^{\infty} \frac{1}{(2k)!} \begin{bmatrix} (AB)^k & 0 \\ 0 & (BA)^k \end{bmatrix} + \sum_{k=0}^{\infty} \frac{1}{(2k+1)!} \begin{bmatrix} 0 & (AB)^k A \\ (BA)^k B & 0 \end{bmatrix} \\
&= \begin{bmatrix} \sum_{k=0}^{\infty} \frac{1}{(2k)!} (AB)^k & 0 \\ 0 & \sum_{k=0}^{\infty} \frac{1}{(2k)!} (BA)^k \end{bmatrix} \\
&\quad + \begin{bmatrix} 0 & \sum_{k=0}^{\infty} \frac{1}{(2k+1)!} (AB)^k A \\ \sum_{k=0}^{\infty} \frac{1}{(2k+1)!} (BA)^k B & 0 \end{bmatrix} \\
&= \begin{bmatrix} \sum_{k=0}^{\infty} \frac{1}{\Gamma(2k+1)} (AB)^k & \sum_{k=0}^{\infty} \frac{1}{\Gamma(2k+2)} (AB)^k A \\ \sum_{k=0}^{\infty} \frac{1}{\Gamma(2k+2)} (BA)^k B & \sum_{k=0}^{\infty} \frac{1}{\Gamma(2k+1)} (BA)^k \end{bmatrix} \\
&= \begin{bmatrix} E_{2,1}(AB) & \big(E_{2,2}(AB)\big)A \\ \big(E_{2,2}(BA)\big)B & E_{2,1}(BA) \end{bmatrix}. \qquad \square
\end{aligned}
$$

2.2 Kronecker product and vector operator

Given two matrices $A = [a_{ij}] \in M_{m,n}$ and $B = [b_{ij}] \in M_{p,q}$ the Kronecker product of A and B is defined by

$$A \otimes B = [a_{ij} B]_{ij} \in M_{mp,nq}.$$

The the vector operator Vec : $M_{m,n} \to \mathbb{C}^{mn}$ is defined for each $A = [a_{ij}]$ by

$$\operatorname{Vec} A = [a_{11} \dots a_{m1} \dots a_{12} \dots a_{m2} \dots a_{1m} \dots a_{mn}]^T.$$

It is clear that Vec is a linear isomorphism. Algebraic properties of the Kronecker product and the vector operator used in this paper are as follows:

Lemma 2.3 (see e.g. [9])**.** *The map $(A, B) \mapsto A \otimes B$ is bilinear. The following properties hold for matrices of appropriate sizes:*

1. $I_m \otimes I_n = I_{mn}$,
2. $(A \otimes B)(C \otimes D) = AC \otimes BD$,
3. $\operatorname{Vec}(AXB) = (B^T \otimes A) \operatorname{Vec} X$.

The Kronecker product is compatible with holomorphic functions in the following sense.

Lemma 2.4 (see e.g.[9]). *Let f be a holomorphic function defined on a region including the origin and the spectrum of $A \in M_n$. Then $f(I \otimes A) = I \otimes f(A)$ and $f(A \otimes I) = f(A) \otimes I$. In particular, the following relations hold for any $A \in M_n$:*

$$\begin{array}{lll} E_{\alpha,\beta}(A \otimes I) = E_{\alpha,\beta}(A) \otimes I & \textit{and} & E_{\alpha,\beta}(I \otimes A) = I \otimes E_{\alpha,\beta}(A), \\ \sinh(A \otimes I) = \sinh(A) \otimes I & \textit{and} & \sinh(I \otimes A) = I \otimes \sinh(A), \\ \cosh(A \otimes I) = \cosh(A) \otimes I & \textit{and} & \cosh(I \otimes A) = I \otimes \cosh(A). \end{array}$$

2.3 Matrix convolution product

Let $\Omega = [0, \infty)$ or $\Omega = [0, b]$ for some $b > 0$. The convolution is a binary operation assigned to each pair of integrable function f and g defined by

$$(f * g)(t) = \int_0^t f(\tau)g(t-\tau)d\tau, \quad t \in \Omega.$$

The convolution is bilinear and commutative. Given two integrable matrix-valued functions $A : \Omega \to M_{m,n}(\mathbb{R})$, $A(t) = [a_{ij}(t)]$ and $B : \Omega \to M_{n,p}(\mathbb{R})$, $B(t) = [b_{ij}(t)]$, we define the matrix convolution product of A and B by

$$(A * B)(t) \ = \ \left[\sum_{k=1}^{n} a_{ik}(t) * b_{kj}(t)\right] \in M_{m,p}(\mathbb{R}), \quad t \in \Omega.$$

We may write $A(t) * B(t)$ for $(A * B)(t)$. The matrix convolution product is bilinear, but not commutative in general.

3 General solutions of systems of nonhomogeneous coupled linear matrix differential equations

From now on, let $A, B, C, D, E, F, G, H, J, K \in M_n(\mathbb{C})$ be given constant matrices and let $U, V : \Omega \to M_n(\mathbb{C})$ be given matrix-valued functions. We wish to solve certain systems of linear matrix differential equations in unknown matrix-valued functions $X, Y : \Omega \to M_n(\mathbb{C})$.

Theorem 3.1. *Assume that $DB = HD$, $AC = CG$, $FH = BF$, $GE = EA$. Then the general solution of the system of nonhomogeneous coupled linear matrix differential equations:*

$$\begin{aligned} X'(t) \ &= \ AX(t)B + CY(t)D + U(t), \\ Y'(t) \ &= \ EX(t)F + GY(t)H + V(t) \end{aligned} \tag{3.1}$$

is given by

$$\begin{aligned}
\operatorname{Vec} X(t) &= e^{(t-t_0)(B^T\otimes A)}\big\{\big(E_{2,1}((t-t_0)^2M)\big)\operatorname{Vec} X(t_0)\\
&\quad+\big(t-t_0\big)\big(E_{2,2}((t-t_0)^2M)\big)\big(D^T\otimes C\big)\operatorname{Vec} Y(t_0)\\
&\quad+\big(E_{2,1}((t-t_0)^2M)\big)*\operatorname{Vec} U(t)\\
&\quad+\big(t-t_0\big)\big(E_{2,2}((t-t_0)^2M)\big)\big(D^T\otimes C\big)*\operatorname{Vec} V(t)\big\},\\
\operatorname{Vec} Y(t) &= e^{(t-t_0)(H^T\otimes G)}\big\{(t-t_0)\big(E_{2,2}((t-t_0)^2N\big)\big(F^T\otimes E\big)\operatorname{Vec} X(t_0)\\
&\quad+\big(E_{2,1}((t-t_0)^2N)\big)\operatorname{Vec} Y(t_0)\\
&\quad+\big(t-t_0\big)\big(E_{2,2}((t-t_0)^2N)\big)\big(F^T\otimes E\big)*\operatorname{Vec} U(t)\\
&\quad+\big(E_{2,1}((t-t_0)^2N)\big)*\operatorname{Vec} V(t)\big\},
\end{aligned} \tag{3.2}$$

where $M=(FD)^T\otimes CE$ *and* $N=(DF)^T\otimes EC$.

Proof. Using Lemma 2.3, we can transform the system (3.1) into the vector form:

$$\begin{bmatrix}\operatorname{Vec} X'(t)\\ \operatorname{Vec} Y'(t)\end{bmatrix}=\begin{bmatrix}B^T\otimes A & D^T\otimes C\\ F^T\otimes E & H^T\otimes G\end{bmatrix}\begin{bmatrix}\operatorname{Vec} X(t)\\ \operatorname{Vec} Y(t)\end{bmatrix}+\begin{bmatrix}\operatorname{Vec} U(t)\\ \operatorname{Vec} V(t)\end{bmatrix}.$$

Let us denote $P=\begin{bmatrix}B^T\otimes A & 0\\ 0 & H^T\otimes G\end{bmatrix}$ and $Q=\begin{bmatrix}0 & D^T\otimes C\\ F^T\otimes E & 0\end{bmatrix}$. From (1.3), this system has the following solution:

$$\begin{bmatrix}\operatorname{Vec} X(t)\\ \operatorname{Vec} Y(t)\end{bmatrix}=e^{(t-t_0)S}\begin{bmatrix}\operatorname{Vec} X(t_0)\\ \operatorname{Vec} Y(t_0)\end{bmatrix}+e^{(t-t_0)S}*\begin{bmatrix}\operatorname{Vec} U(t)\\ \operatorname{Vec} V(t)\end{bmatrix},$$

where $S=P+Q$. Now, we will compute e^S. Since $DB=HD$, $AC=CG$, $FH=BF$ and $GE=EA$, by Lemma 2.3 we have $PQ=QP$. From which it follows from Lemma 2.1 that $e^S=e^{P+Q}=e^Pe^Q$. By expanding the power series of matrix exponential, we have

$$e^P=\begin{bmatrix}e^{B^T\otimes A} & 0\\ 0 & e^{H^T\otimes G}\end{bmatrix}.$$

By Lemma 2.2, we have

$$e^Q=\begin{bmatrix}E_{2,1}(M) & \big(E_{2,2}(M)\big)\big(D^T\otimes C\big)\\ \big(E_{2,2}(N)\big)\big(F^T\otimes E\big) & E_{2,1}(N)\end{bmatrix}.$$

Thus

$$\begin{aligned}
e^S &= \begin{bmatrix}e^{B^T\otimes A} & 0\\ 0 & e^{H^T\otimes G}\end{bmatrix}\begin{bmatrix}E_{2,1}(M) & \big(E_{2,2}(M)\big)\big(D^T\otimes C\big)\\ \big(E_{2,2}(N)\big)\big(F^T\otimes E\big) & E_{2,1}(N)\end{bmatrix}\\
&= \begin{bmatrix}e^{B^T\otimes A}E_{2,1}(M) & e^{B^T\otimes A}\big(E_{2,2}(M)\big)\big(D^T\otimes C\big)\\ e^{H^T\otimes G}\big(E_{2,2}(N)\big)\big(F^T\otimes E\big) & e^{H^T\otimes G}E_{2,1}(N)\end{bmatrix}.
\end{aligned}$$

Denoting

$$\begin{aligned} R_1 &= e^{(t-t_0)(B^T\otimes A)}E_{2,1}((t-t_0)^2M), \\ R_2 &= e^{(t-t_0)(B^T\otimes A)}\big(t-t_0\big)\big(E_{2,2}((t-t_0)^2M)\big)\big(D^T\otimes C\big), \\ R_3 &= e^{(t-t_0)(H^T\otimes G)}\big(t-t_0\big)\big(E_{2,2}((t-t_0)^2N)\big)\big(F^T\otimes E\big), \\ R_4 &= e^{(t-t_0)(H^T\otimes G)}E_{2,1}((t-t_0)^2N), \end{aligned}$$

we obtain

$$e^{(t-t_0)S}\begin{bmatrix}\operatorname{Vec}X(t_0)\\ \operatorname{Vec}Y(t_0)\end{bmatrix} = \begin{bmatrix}R_1 & R_2\\ R_3 & R_4\end{bmatrix}\begin{bmatrix}\operatorname{Vec}X(t_0)\\ \operatorname{Vec}Y(t_0)\end{bmatrix} = \begin{bmatrix}R_1\operatorname{Vec}X(t_0)+R_2\operatorname{Vec}Y(t_0)\\ R_3\operatorname{Vec}X(t_0)+R_4\operatorname{Vec}Y(t_0)\end{bmatrix}.$$

We also have

$$e^{(t-t_0)S}*\begin{bmatrix}\operatorname{Vec}U(t)\\ \operatorname{Vec}V(t)\end{bmatrix} = \begin{bmatrix}R_1 & R_2\\ R_3 & R_4\end{bmatrix}*\begin{bmatrix}\operatorname{Vec}U(t)\\ \operatorname{Vec}V(t)\end{bmatrix} = \begin{bmatrix}R_1*\operatorname{Vec}U(t)+R_2*\operatorname{Vec}V(t)\\ R_3*\operatorname{Vec}U(t)+R_4*\operatorname{Vec}V(t)\end{bmatrix}.$$

Therefore, the general solution of (3.1) is given by (3.2). □

Corollary 3.2. *Assume that $DB = HD$, $AC = CG$, $FH = BF$, $GE = EA$. Then the general solution of the system*

$$\begin{aligned} X'(t) &= AX(t)B + CY(t)D, \\ Y'(t) &= EX(t)F + GY(t)H \end{aligned}$$

is given by

$$\begin{aligned} \operatorname{Vec}X(t) =& e^{(t-t_0)(B^T\otimes A)}\big\{\big(E_{2,1}((t-t_0)^2M)\big)\operatorname{Vec}X(t_0) \\ &+ \big(t-t_0\big)\big(E_{2,2}((t-t_0)^2M)\big)\big(D^T\otimes C\big)\operatorname{Vec}Y(t_0), \\ \operatorname{Vec}Y(t) =& e^{(t-t_0)(H^T\otimes G)}\big\{\big(t-t_0\big)\big(E_{2,2}((t-t_0)^2N)\big)\big(F^T\otimes E\big)\operatorname{Vec}X(t_0) \\ &+ \big(E_{2,1}((t-t_0)^2N)\big)\operatorname{Vec}Y(t_0)\big\} \end{aligned} \tag{3.3}$$

where $M = (FD)^T \otimes CE$ and $N = (DF)^T \otimes EC$.

Proof. Put $U(t) = V(t) = 0$ in (3.2) and then use Lemma 2.3. □

The next result was firstly established in [1].

Corollary 3.3. *The general solution of the system*

$$\begin{aligned} X'(t) &= AX(t)B + CY(t)D + U(t), \\ Y'(t) &= CX(t)D + AY(t)B + V(t) \end{aligned} \tag{3.4}$$

under the assumption that $AC = CA$ and $BD = DB$, is given by

$$\begin{aligned} \operatorname{Vec}X(t) =& e^{(t-t_0)(B^T\otimes A)}\big\{\cosh L\operatorname{Vec}X(t_0) + \sinh L\operatorname{Vec}Y(t_0) \\ &+ \cosh L*\operatorname{Vec}U(t) + \sinh L*\operatorname{Vec}V(t)\big\}, \\ \operatorname{Vec}Y(t) =& e^{(t-t_0)(B^T\otimes A)}\big\{\sinh L\operatorname{Vec}X(t_0) + \cosh L\operatorname{Vec}Y(t_0) \\ &+ \sinh L*\operatorname{Vec}U(t) + \cosh L*\operatorname{Vec}V(t)\big\}, \end{aligned} \tag{3.5}$$

7

where $L = (t-t_0)(D^T \otimes C)$.

Proof. Put $E = C$, $F = D$, $G = A$ and $H = B$ in (3.2), and use Lemma 2.3. $\square$

The corresponding homogeneous system of (3.4) is given by

$$\begin{aligned} X'(t) &= AX(t)B + CY(t)D, \\ Y'(t) &= CX(t)D + AY(t)B. \end{aligned} \tag{3.6}$$

If $AC = CA$ and $BD = DB$, then the general solution of (3.6) is reduced to

$$\begin{aligned} \operatorname{Vec} X(t) &= e^{(t-t_0)(B^T\otimes A)}\big\{ \cosh L \operatorname{Vec} X(t_0) + \sinh L \operatorname{Vec} Y(t_0)\big\}, \\ \operatorname{Vec} Y(t) &= e^{(t-t_0)(B^T\otimes A)}\big\{ \sinh L \operatorname{Vec} X(t_0) + \cosh L \operatorname{Vec} Y(t_0)\big\}. \end{aligned}$$

This result was firstly obtained in [5].

Corollary 3.4. *The general solution of the system*

$$\begin{aligned} X'(t) &= AX(t)B + CY(t) \quad + U(t), \\ Y'(t) &= EX(t) \quad + GY(t)B + V(t) \end{aligned}$$

under the condition $AC = CG, GE = EA$, *is given by*

$$\begin{aligned} \operatorname{Vec} X(t) &= e^{(t-t_0)(B^T\otimes A)} \operatorname{Vec}\Big\{ \big(E_{2,1}(K_1)\big)X(t_0) + \big(t-t_0\big)\big(E_{2,2}(K_1)\big)CY(t_0)\Big\} \\ &\quad + e^{(t-t_0)(B^T\otimes A)}\Big\{ \big(I_n \otimes E_{2,1}(K_1)\big) * \operatorname{Vec} U(t) \\ &\quad + \big(I_n \otimes \big(t-t_0\big)\big(E_{2,2}(K_1)\big)C\big) * \operatorname{Vec} V(t)\Big\}, \\ \operatorname{Vec} Y(t) &= e^{(t-t_0)(B^T\otimes G)} \operatorname{Vec}\Big\{ \big(t-t_0\big)\big(E_{2,2}(K_2)\big)EX(t_0) + \big(E_{2,1}(K_2)\big)Y(t_0)\Big\} \\ &\quad + e^{(t-t_0)(B^T\otimes G)}\Big\{ \big(I_n \otimes \big(t-t_0\big)\big(E_{2,2}(K_2)\big)E\big) * \operatorname{Vec} U(t) \\ &\quad + \big(I_n \otimes E_{2,1}(K_2)\big) * \operatorname{Vec} V(t)\Big\}, \end{aligned}$$

where $K_1 = (t-t_0)^2 CE$ *and* $K_2 = (t-t_0)^2 EC$.

Proof. Put $H = B$, $D = F = I_n$ in (3.2) and then use Lemmas 2.3 and 2.4. $\square$

Corollary 3.5. *The general solution of the system*

$$\begin{aligned} X'(t) &= AX(t)B + \quad Y(t) \quad + U(t), \\ Y'(t) &= \quad X(t) \quad + AY(t)B + V(t) \end{aligned}$$

is given by

$$\begin{aligned} \operatorname{Vec} X(t) &= e^{(t-t_0)(B^T\otimes A)}\big\{ \cosh(t-t_0) \operatorname{Vec} X(t_0) + \sinh(t-t_0) \operatorname{Vec} Y(t_0) \\ &\quad + \cosh(t-t_0)\big(I_{n^2} * \operatorname{Vec} U(t)\big) + \sinh(t-t_0)\big(I_{n^2} * \operatorname{Vec} V(t)\big)\big\}, \\ \operatorname{Vec} Y(t) &= e^{(t-t_0)(B^T\otimes A)}\big\{ \sinh(t-t_0) \operatorname{Vec} X(t_0) + \cosh(t-t_0) \operatorname{Vec} Y(t_0) \\ &\quad + \sinh(t-t_0)\big(I_{n^2} * \operatorname{Vec} U(t)\big) + \cosh(t-t_0)\big(I_{n^2} * \operatorname{Vec} V(t)\big)\big\}. \end{aligned}$$

Proof. Put $C = D = I_n$ in (3.5) and then use Lemma 2.3. □

Corollary 3.6. *The general solution of the system*

$$\begin{aligned} X'(t) &= AX(t)B + U(t), \\ Y'(t) &= EX(t)F + GY(t)H + V(t) \end{aligned}$$

under the condition $FH = BF$ and $GE = EA$, is given by

$$\begin{aligned} \operatorname{Vec} X(t) &= e^{(t-t_0)(B^T \otimes A)} \big\{ \operatorname{Vec} X(t_0) + I * \operatorname{Vec} U(t) \big\}, \\ \operatorname{Vec} Y(t) &= e^{(t-t_0)(H^T \otimes G)} \operatorname{Vec} \big\{ (t-t_0) E X(t_0) F + Y(t_0) \big\} \\ &\quad + e^{(t-t_0)(H^T \otimes G)} \big\{ (t-t_0)(F^T \otimes E) * \operatorname{Vec} U(t) + I * \operatorname{Vec} V(t) \big\}. \end{aligned}$$

Proof. Put $C = D = 0$ in (3.2) and then use Lemma 2.3. □

Corollary 3.7. *The general solution of equation $X'(t) = AX(t)B + U(t)$ is given by* $\operatorname{Vec} X(t) = e^{(t-t_0)(B^T \otimes A)} \big\{ \operatorname{Vec} X(t_0) + I * \operatorname{Vec} U(t) \big\}$.

Proof. Put $E = F = 0$ in Corollary 3.6. □

4 Unique solution of initial value problem and a numerical example

Consider the following initial value problem associated with the system (3.1):

$$\begin{aligned} X'(t) &= AX(t)B + CY(t)D + U(t), \\ Y'(t) &= EX(t)F + GY(t)H + V(t) \end{aligned}$$

subject to initial conditions $X(0) = J$ and $Y(0) = K$. Suppose $DB = HD$, $AC = CG$, $FH = BF$, $GE = EA$. In this case, the solution of this problem is unique and given by

$$\begin{aligned} \operatorname{Vec} X(t) &= e^{t(B^T \otimes A)} \big\{ \big(E_{2,1}(t^2 M)\big) \operatorname{Vec} J + t\big(E_{2,2}(t^2 M)\big)\big(D^T \otimes C\big) \operatorname{Vec} K \\ &\quad + \big(E_{2,1}(t^2 M)\big) * \operatorname{Vec} U(t) + t\big(E_{2,2}(t^2 M)\big)\big(D^T \otimes C\big) * \operatorname{Vec} V(t) \big\}, \\ \operatorname{Vec} Y(t) &= e^{t(H^T \otimes G)} \big\{ t\big(E_{2,2}(t^2 N)\big(F^T \otimes E\big) \operatorname{Vec} J + \big(E_{2,1}(t^2 N)\big) \operatorname{Vec} K \\ &\quad + t\big(E_{2,2}(t^2 N)\big)\big(F^T \otimes E\big) * \operatorname{Vec} U(t) + \big(E_{2,1}(t^2 N)\big) * \operatorname{Vec} V(t) \big\}, \end{aligned}$$

where $M = (FD)^T \otimes CE$ and $N = (DF)^T \otimes EC$.

Let us see a numerical example.

Example 4.1. *The initial value problem*

$$\begin{aligned} X'(t) &= AX(t)B + \quad Y(t) \quad + U(t), \\ Y'(t) &= \quad X(t) \quad + AY(t)B + V(t) \\ & X(0) = J \quad \textit{and} \quad Y(0) = K \end{aligned}$$

9

with $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, B = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}, J = \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}, K = \begin{bmatrix} 3 & 1 \\ 1 & -1 \end{bmatrix}$,
$U(t) = \begin{bmatrix} -e^{2t} & 1 \\ 1 & \sin t \end{bmatrix}, V(t) = \begin{bmatrix} 1 & e^{2t} \\ \cos t & \sin 2t \end{bmatrix}$ *has a unique solution given by*

$$\operatorname{Vec} X(t) = e^{tW} \operatorname{Vec} \begin{bmatrix} w_1(t)\cosh t + w_2(t)\sinh t & w_3(t)\cosh t + w_4(t)\sinh t \\ w_5(t)\cosh t + w_6(t)\sinh t & w_7(t)\cosh t + w_8(t)\sinh t \end{bmatrix},$$

$$\operatorname{Vec} Y(t) = e^{tW} \operatorname{Vec} \begin{bmatrix} w_2(t)\cosh t + w_1(t)\sinh t & w_4(t)\cosh t + w_3(t)\sinh t \\ w_6(t)\cosh t + w_5(t)\sinh t & w_8(t)\cosh t + w_7(t)\sinh t \end{bmatrix}.$$

Here, $W = \begin{bmatrix} 0 & 0 & 1 & 2 \\ 0 & 0 & 3 & 4 \\ -1 & -2 & 1 & 2 \\ -3 & -4 & 3 & 4 \end{bmatrix}$,

$w_1(t) = \frac{1}{2}(5 - e^{2t}),\ w_2(t) = 3 + t,\ w_3(t) = -1 + t,\ w_4(t) = \frac{1}{2}(1 + e^{2t}),$ $w_5(t) = 1 + t,\ w_6(t) = 1 + \sin t,\ w_7(t) = 1 - \cos t,\ w_8(t) = -\frac{1}{2}(1 + \cos 2t).$

Acknowledgements

The authors would like to thank King Mongkut's Institute of Technology Ladkrabang Research Fund for financial supports.

References

[1] Z. Al-Zhour, Efficient solutions of coupled matrix and matrix differential equations, *Intell. Cont. Autom.*, 3(2), 176-184 (2012).

[2] G. N. Boshnakov, The asymptotic covariance matrix of the multivariate serial correlations, *Stoch. Proc. Appl.*, 65, 251-258 (1996).

[3] T. Chen, B. A. Francis, *Optimal Sampled-Data Control Systems*, Springer, London, 1995.

[4] J. B. Cruz, C. I. Chen Jr., Series Nash solution of two-person nonzero sum linear differential games, *J. Optimal. Theory*, 7(4), 240-257 (1971).

[5] A. Kilicman, Z. Al-Zhour, The general common exact solutions of coupled linear matrix and matrix differential equations, *J. Anal. Comput.*, 1(1), 15-30 (2005).

[6] J. R. Magnus, H. Neudecker, *Matrix Differential Calculus with Applications in Statistics and Econometrics*, John Wiley & Sons, 1975.

[7] S. G. Mouroutsos, P. D. Sparis, Taylor series approach to system identification, analysis and optimal control, *J. Frankin Inst.*, 319(3), 359-371 (1985).

[8] C. R. Rao, M. B. Rao, *Matrix Algebra and Its Applications to Statistics and Econometrics*, World Scientific, Singapore, 1998.

[9] W. H. Steeb, Y. Hardy, *Matrix Calculus and Kronecker Product: A Practical Approach to Linear and Multilinear Algebra*, World Scientific, Singapore, 2011.

[10] Z. Al-Zhour, The general (vector) solutions of such linear (coupled) matrix fractional differential equations by using Kronecker structures, *Appl. Math. Comput.*, 232, 498-510 (2014).

[11] S. L. Campbell, *Singular systems of differential equations II.*, Pitman, San Francisco, 1982.

[12] R. Ben Taher, M. Rachidi, Linear recurrence relations in the algebra of matrices and applications, *Linear Algebra Appl.*, 330, 15-24 (2001).

[13] H-W. Cheng, SS-T. Yau, More explicit formulas for the matrix exponential, *Linear Algebra Appl.*, 262, 131-163 (1997).

[14] B. Ross, Fractional Calculus and Its Applications, Springer-Verlag, Berlin, 1975.

[15] Z. Al-Zhour, New techniques for solving some matrix and matrix differential equations, *Ain Shams Engineering Journal*, 6, 347-354 (2015).

Expressions of the solutions of some systems of difference equations

M. M. El-Dessoky[1,2], E. M. Elsayed[1,2], E. M. Elabbasy[2] and Asim Asiri[1]
[1]King Abdulaziz University, Faculty of Science,Mathematics Department,
P. O. Box 80203, Jeddah 21589, Saudi Arabia.
[2]Department of Mathematics, Faculty of Science,
Mansoura University, Mansoura 35516, Egypt.
E-mail: dessokym@mans.edu.eg; emmelsayed@yahoo.com;
emelabbasy@mans.edu.e; amkasiri@kau.edu.sa

ABSTRACT

In this paper, we deal with the form of the solutions and the periodicity character of the following systems of nonlinear difference equations of order two

$$z_{n+1}=\frac{z_n t_{n-1}}{\pm t_n \pm t_{n-1}},\ t_{n+1}=\frac{t_n z_{n-1}}{\pm z_n \pm z_{n-1}},$$

where the initial conditions $z_{-1},\ z_0,\ t_{-1}$ and t_0 are nonzero real numbers.

Keywords: recursive sequences, difference equations, periodic solution, solution of difference equation, system of difference equations.

Mathematics Subject Classification: 39A10.

1. INTRODUCTION

Through this paper, we will obtain the form of the solutions of some nonlinear difference equations systems of order two of the following form

$$z_{n+1}=\frac{t_{n-1}z_n}{\pm t_n \pm t_{n-1}},\ t_{n+1}=\frac{z_{n-1}t_n}{\pm z_n \pm z_{n-1}},$$

where the initial conditions $z_{-1},\ z_0,\ t_{-1}$ and t_0 are nonzero real numbers. We will then investigate the periodicity character of the solutions of the systems under study. Finally we will present some numerical examples and some figures will be given to explain the behavior of the obtained solutions.

The study of difference equations is a very rich research field, and difference equations have been applied in several mathematical models in biology, population dynamics, genetics, economics, medicine, and so forth. Solving difference equations and studying the asymptotic behavior of their solutions has attracted the attention of many authors, see for example [1-39].

El-Dessoky et al. [6] studied the periodic nature and the form of the solutions of nonlinear difference equations systems of order four

$$x_{n+1}=\frac{x_n y_{n-3}}{y_{n-2}(\pm 1 \pm x_n y_{n-3})},\ y_{n+1}=\frac{y_n x_{n-3}}{x_{n-2}(\pm 1 \pm y_n x_{n-3})},$$

Grove et al. [7] obtained the existence and behavior of solutions of the rational system

$$x_{n+1}=\frac{a}{x_n}+\frac{b}{y_n},\ \ y_{n+1}=\frac{c}{x_n}+\frac{d}{y_n}.$$

Mansour et al. [8] investigated the periodic nature and get the form of the solutions of the following systems of rational difference equations

$$x_{n+1}=\frac{x_{n-1}}{\pm x_{n-1}y_n - g},\ y_{n+1}=\frac{y_{n-1}}{\pm y_{n-1}x_n - f}.$$

El-Dessoky [9] studied the solutions of the rational equation systems

$$x_{n+1} = \frac{y_{n-1}y_{n-2}}{x_n(\pm 1 \pm y_{n-1}y_{n-2})},\ y_{n+1} = \frac{x_{n-1}x_{n-2}}{y_n(\pm 1 \pm x_{n-1}x_{n-2})}.$$

Touafek et al. [10] investigated the periodic nature and gave the form of the solutions of the following systems of rational second order difference equations

$$x_{n+1} = \frac{y_n}{x_{n-1}(\pm 1 \pm y_n)},\ y_{n+1} = \frac{x_n}{y_{n-1}(\pm 1 \pm x_n)}.$$

Yang et al. [11] studied global behavior of the system of the two nonlinear difference equations

$$x_{n+1} = \frac{Ax_n}{1+y_n^p},\quad y_{n+1} = \frac{By_n}{1+x_n^p}.$$

Din et al. [6] studied the behavior of the solutions of the following system of difference equations

$$x_{n+1} = \frac{\alpha x_{n-3}}{\beta + \gamma y_n y_{n-1} y_{n-2} y_{n-3}},\ y_{n+1} = \frac{\alpha_1 y_{n-3}}{\beta_1 + \gamma_1 x_n x_{n-1} x_{n-2} x_{n-3}}.$$

Definition 1. (Periodicity)

A sequence $\{x_n\}_{n=-k}^{\infty}$ is said to be periodic with period p if $x_{n+p} = x_n$ for all $n \geq -k$.

Definition 2. (Fibonacci Sequence)

The sequence $\{f_m\}_{m=1}^{\infty} = \{1,\ 2,\ 3,\ 5,\ 8,\ 13,\ 21,\ ...\}$ i.e. $f_{m+1} = f_m + f_{m-1},\ m \geq 0,\ f_{-1} = 0,\ f_0 = 1$ is called Fibonacci Sequence.

2. THE FIRST SYSTEM: $Z_{N+1} = \frac{Z_N T_{N-1}}{T_N - T_{N-1}},\ T_{N+1} = \frac{T_N Z_{N-1}}{Z_N - Z_{N-1}}$

In this section, we investigate the solutions of the two difference equations system

$$z_{n+1} = \frac{z_n t_{n-1}}{t_n - t_{n-1}},\ t_{n+1} = \frac{t_n z_{n-1}}{z_n - z_{n-1}}, \tag{1}$$

where $n \in \mathbb{N}_0$ and the initial conditions $z_{-1},\ z_0,\ t_{-1}$ and t_0 are arbitrary nonzero real numbers

Theorem 2.1. *Assume that $\{z_n,\ t_n\}$ are solutions of system (1). Then for $n = 0,\ 1,\ 2,\ ...$, we see that all solutions of system (1) are given by the following formulae*

$$z_{2n-1} = z_{-1}\prod_{i=0}^{n-1}\frac{(f_{2i-2}z_0 - f_{2i-1}z_{-1})(f_{2i-1}t_0 - f_{2i}t_{-1})}{(f_{2i-1}z_0 - f_{2i}z_{-1})(f_{2i}t_0 - f_{2i+1}t_{-1})},\ z_{2n} = z_0\prod_{i=0}^{n-1}\frac{(f_{2i}z_0 - f_{2i+1}z_{-1})(f_{2i-1}t_0 - f_{2i}t_{-1})}{(f_{2i+1}z_0 - f_{2i+2}z_{-1})(f_{2i}t_0 - f_{2i+1}t_{-1})},$$

and

$$t_{2n-1} = t_{-1}\prod_{i=0}^{n-1}\frac{(f_{2i-1}z_0 - f_{2i}z_{-1})(f_{2i-2}t_0 - f_{2i-1}t_{-1})}{(f_{2i}z_0 - f_{2i+1}z_{-1})(f_{2i-1}t_0 - f_{2i}t_{-1})},\ t_{2n} = t_0\prod_{i=0}^{n-1}\frac{(f_{2i-1}z_0 - f_{2i}z_{-1})(f_{2i}t_0 - f_{2i+1}t_{-1})}{(f_{2i}z_0 - f_{2i+1}z_{-1})(f_{2i+1}t_0 - f_{2i+2}t_{-1})},$$

where $\{f_m\}_{m=-2}^{\infty} = \{1,\ 0,\ 1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13,\ ...\}$.

Proof: For $n = 0$ the result holds. Now suppose that $n > 0$ and that our assumption holds for $n-1$. that is,

$$\begin{aligned} z_{2n-3} &= z_{-1}\prod_{i=0}^{n-2}\frac{(f_{2i-2}z_0 - f_{2i-1}z_{-1})(f_{2i-1}t_0 - f_{2i}t_{-1})}{(f_{2i-1}z_0 - f_{2i}z_{-1})(f_{2i}t_0 - f_{2i+1}t_{-1})},\ z_{2n-2} = z_0\prod_{i=0}^{n-2}\frac{(f_{2i}z_0 - f_{2i+1}z_{-1})(f_{2i-1}t_0 - f_{2i}t_{-1})}{(f_{2i+1}z_0 - f_{2i+2}z_{-1})(f_{2i}t_0 - f_{2i+1}t_{-1})}, \\ t_{2n-3} &= t_{-1}\prod_{i=0}^{n-2}\frac{(f_{2i-1}z_0 - f_{2i}z_{-1})(f_{2i-2}t_0 - f_{2i-1}t_{-1})}{(f_{2i}z_0 - f_{2i+1}z_{-1})(f_{2i-1}t_0 - f_{2i}t_{-1})},\ t_{2n-2} = t_0\prod_{i=0}^{n-2}\frac{(f_{2i-1}z_0 - f_{2i}z_{-1})(f_{2i}t_0 - f_{2i+1}t_{-1})}{(f_{2i}z_0 - f_{2i+1}z_{-1})(f_{2i+1}t_0 - f_{2i+2}t_{-1})}, \end{aligned}$$

Now we find from system (1) that

$$\begin{aligned}
z_{2n-1} &= \frac{z_{2n-2}t_{2n-3}}{t_{2n-2}-t_{2n-3}} \\
&= \frac{\left(z_0\prod_{i=0}^{n-2}\frac{(f_{2i}z_0-f_{2i+1}z_{-1})(f_{2i-1}t_0-f_{2i}t_{-1})}{(f_{2i+1}z_0-f_{2i+2}z_{-1})(f_{2i}t_0-f_{2i+1}t_{-1})}\right)\left(t_{-1}\prod_{i=0}^{n-2}\frac{(f_{2i-1}z_0-f_{2i}z_{-1})(f_{2i-2}t_0-f_{2i-1}t_{-1})}{(f_{2i}z_0-f_{2i+1}z_{-1})(f_{2i-1}t_0-f_{2i}t_{-1})}\right)}{\left(t_0\prod_{i=0}^{n-2}\frac{(f_{2i-1}z_0-f_{2i}z_{-1})(f_{2i}t_0-f_{2i+1}t_{-1})}{(f_{2i}z_0-f_{2i+1}z_{-1})(f_{2i+1}t_0-f_{2i+2}t_{-1})}\right)-\left(t_{-1}\prod_{i=0}^{n-2}\frac{(f_{2i-1}z_0-f_{2i}z_{-1})(f_{2i-2}t_0-f_{2i-1}t_{-1})}{(f_{2i}z_0-f_{2i+1}z_{-1})(f_{2i-1}t_0-f_{2i}t_{-1})}\right)} \\
&= \frac{z_0t_{-1}\prod_{i=0}^{n-2}\frac{(f_{2i}z_0-f_{2i+1}z_{-1})(f_{2i-1}t_0-f_{2i}t_{-1})}{(f_{2i+1}z_0-f_{2i+2}z_{-1})(f_{2i}t_0-f_{2i+1}t_{-1})}}{\left(t_0\prod_{i=0}^{n-2}\frac{(f_{2i-1}t_0-f_{2i}t_{-1})(f_{2i}t_0-f_{2i+1}t_{-1})}{(f_{2i+1}t_0-f_{2i+2}t_{-1})(f_{2i-2}t_0-f_{2i-1}t_{-1})}\right)-t_{-1}} = \frac{z_0t_{-1}\prod_{i=0}^{n-2}\frac{(f_{2i}z_0-f_{2i+1}z_{-1})(f_{2i-1}t_0-f_{2i}t_{-1})}{(f_{2i+1}z_0-f_{2i+2}z_{-1})(f_{2i}t_0-f_{2i+1}t_{-1})}}{\left(t_0\frac{(f_{-1}t_0-f_0t_{-1})(f_{2n-4}t_0-f_{2n-3}t_{-1})}{(f_{2n-3}t_0-f_{2n-2}t_{-1})(f_{-2}t_0-f_{-1}t_{-1})}\right)-t_{-1}} \\
&= \frac{z_0\prod_{i=0}^{n-2}\frac{(f_{2i}z_0-f_{2i+1}z_{-1})(f_{2i-1}t_0-f_{2i}t_{-1})}{(f_{2i+1}z_0-f_{2i+2}z_{-1})(f_{2i}t_0-f_{2i+1}t_{-1})}}{-\frac{(f_{2n-4}t_0-f_{2n-3}t_{-1})}{(f_{2n-3}t_0-f_{2n-2}t_{-1})}-1}\left(\frac{f_{2n-3}t_0-f_{2n-2}t_{-1}}{f_{2n-3}t_0-f_{2n-2}t_{-1}}\right) \\
&= \frac{z_0\prod_{i=0}^{n-2}\frac{(f_{2i}z_0-f_{2i+1}z_{-1})(f_{2i-1}t_0-f_{2i}t_{-1})}{(f_{2i+1}z_0-f_{2i+2}z_{-1})(f_{2i}t_0-f_{2i+1}t_{-1})}(f_{2n-3}t_0-f_{2n-2}t_{-1})}{-f_{2n-4}t_0+f_{2n-3}t_{-1}-f_{2n-3}t_0+f_{2n-2}t_{-1}} \\
&= z_0\prod_{i=0}^{n-2}\frac{(f_{2i}z_0-f_{2i+1}z_{-1})(f_{2i-1}t_0-f_{2i}t_{-1})}{(f_{2i+1}z_0-f_{2i+2}z_{-1})(f_{2i}t_0-f_{2i+1}t_{-1})}\frac{(f_{2n-3}t_0-f_{2n-2}t_{-1})}{(-f_{2n-2}t_0+f_{2n-1}t_{-1})} \\
&= z_{-1}\prod_{i=0}^{n-1}\frac{(f_{2i-2}z_0-f_{2i-1}z_{-1})(f_{2i-1}t_0-f_{2i}t_{-1})}{(f_{2i-1}z_0-f_{2i}z_{-1})(f_{2i}t_0-f_{2i+1}t_{-1})},
\end{aligned}$$

$$\begin{aligned}
t_{2n-1} &= \frac{t_{2n-2}z_{2n-3}}{z_{2n-2}-z_{2n-3}} \\
&= \frac{\left(t_0\prod_{i=0}^{n-2}\frac{(f_{2i-1}z_0-f_{2i}z_{-1})(f_{2i}t_0-f_{2i+1}t_{-1})}{(f_{2i}z_0-f_{2i+1}z_{-1})(f_{2i+1}t_0-f_{2i+2}t_{-1})}\right)\left(z_{-1}\prod_{i=0}^{n-2}\frac{(f_{2i-2}z_0-f_{2i-1}z_{-1})(f_{2i-1}t_0-f_{2i}t_{-1})}{(f_{2i-1}z_0-f_{2i}z_{-1})(f_{2i}t_0-f_{2i+1}t_{-1})}\right)}{\left(z_0\prod_{i=0}^{n-2}\frac{(f_{2i}z_0-f_{2i+1}z_{-1})(f_{2i-1}t_0-f_{2i}t_{-1})}{(f_{2i+1}z_0-f_{2i+2}z_{-1})(f_{2i}t_0-f_{2i+1}t_{-1})}\right)-\left(z_{-1}\prod_{i=0}^{n-2}\frac{(f_{2i-2}z_0-f_{2i-1}z_{-1})(f_{2i-1}t_0-f_{2i}t_{-1})}{(f_{2i-1}z_0-f_{2i}z_{-1})(f_{2i}t_0-f_{2i+1}t_{-1})}\right)} \\
&= \frac{z_{-1}t_0\prod_{i=0}^{n-2}\frac{(f_{2i-1}z_0-f_{2i}z_{-1})(f_{2i}t_0-f_{2i+1}t_{-1})}{(f_{2i}z_0-f_{2i+1}z_{-1})(f_{2i+1}t_0-f_{2i+2}t_{-1})}}{z_0\prod_{i=0}^{n-2}\frac{(f_{2i-1}z_0-f_{2i}z_{-1})(f_{2i}z_0-f_{2i+1}z_{-1})}{(f_{2i+1}z_0-f_{2i+2}z_{-1})(f_{2i-2}z_0-f_{2i-1}z_{-1})}-z_{-1}} \\
&= \frac{z_{-1}t_0\prod_{i=0}^{n-2}\frac{(f_{2i-1}z_0-f_{2i}z_{-1})(f_{2i}t_0-f_{2i+1}t_{-1})}{(f_{2i}z_0-f_{2i+1}z_{-1})(f_{2i+1}t_0-f_{2i+2}t_{-1})}}{z_0\frac{(f_{-1}z_0-f_0z_{-1})(f_{2n-4}z_0-f_{2n-3}z_{-1})}{(f_{2n-3}z_0-f_{2n-2}z_{-1})(f_{-2}z_0-f_{-1}z_{-1})}-z_{-1}} \\
&= \frac{t_0\prod_{i=0}^{n-2}\frac{(f_{2i-1}z_0-f_{2i}z_{-1})(f_{2i}t_0-f_{2i+1}t_{-1})}{(f_{2i}z_0-f_{2i+1}z_{-1})(f_{2i+1}t_0-f_{2i+2}t_{-1})}}{-\frac{(f_{2n-4}z_0-f_{2n-3}z_{-1})}{(f_{2n-3}z_0-f_{2n-2}z_{-1})}-1}\left(\frac{f_{2n-3}z_0-f_{2n-2}z_{-1}}{f_{2n-3}z_0-f_{2n-2}z_{-1}}\right) \\
&= \frac{t_0\prod_{i=0}^{n-2}\frac{(f_{2i-1}z_0-f_{2i}z_{-1})(f_{2i}t_0-f_{2i+1}t_{-1})}{(f_{2i}z_0-f_{2i+1}z_{-1})(f_{2i+1}t_0-f_{2i+2}t_{-1})}(f_{2n-3}z_0-f_{2n-2}z_{-1})}{-f_{2n-4}z_0+f_{2n-3}z_{-1}-f_{2n-3}z_0+f_{2n-2}z_{-1}} \\
&= t_0\prod_{i=0}^{n-2}\frac{(f_{2i-1}z_0-f_{2i}z_{-1})(f_{2i}t_0-f_{2i+1}t_{-1})}{(f_{2i}z_0-f_{2i+1}z_{-1})(f_{2i+1}t_0-f_{2i+2}t_{-1})}\frac{(f_{2n-3}z_0-f_{2n-2}z_{-1})}{(-f_{2n-2}z_0+f_{2n-1}z_{-1})} \\
&= t_{-1}\prod_{i=0}^{n-1}\frac{(f_{2i-1}z_0-f_{2i}z_{-1})(f_{2i-2}t_0-f_{2i-1}t_{-1})}{(f_{2i}z_0-f_{2i+1}z_{-1})(f_{2i-1}t_0-f_{2i}t_{-1})}.
\end{aligned}$$

Also, we infer from system (1) that

$$
\begin{aligned}
z_{2n} &= \frac{z_{2n-1}t_{2n-2}}{t_{2n-1}-t_{2n-2}} \\
&= \frac{\left(z_{-1}\prod_{i=0}^{n-1}\frac{(f_{2i-2}z_0-f_{2i-1}z_{-1})(f_{2i-1}t_0-f_{2i}t_{-1})}{(f_{2i-1}z_0-f_{2i}z_{-1})(f_{2i}t_0-f_{2i+1}t_{-1})}\right)\left(t_0\prod_{i=0}^{n-2}\frac{(f_{2i-1}z_0-f_{2i}z_{-1})(f_{2i}t_0-f_{2i+1}t_{-1})}{(f_{2i}z_0-f_{2i+1}z_{-1})(f_{2i+1}t_0-f_{2i+2}t_{-1})}\right)}{\left(t_{-1}\prod_{i=0}^{n-1}\frac{(f_{2i-1}z_0-f_{2i}z_{-1})(f_{2i-2}t_0-f_{2i-1}t_{-1})}{(f_{2i}z_0-f_{2i+1}z_{-1})(f_{2i-1}t_0-f_{2i}t_{-1})}\right)-\left(t_0\prod_{i=0}^{n-2}\frac{(f_{2i-1}z_0-f_{2i}z_{-1})(f_{2i}t_0-f_{2i+1}t_{-1})}{(f_{2i}z_0-f_{2i+1}z_{-1})(f_{2i+1}t_0-f_{2i+2}t_{-1})}\right)} \\
&= \frac{z_{-1}\prod_{i=0}^{n-1}\frac{(f_{2i-2}z_0-f_{2i-1}z_{-1})(f_{2i-1}t_0-f_{2i}t_{-1})}{(f_{2i-1}z_0-f_{2i}z_{-1})(f_{2i}t_0-f_{2i+1}t_{-1})}}{\left(-\prod_{i=0}^{n-1}\frac{(f_{2i-1}z_0-f_{2i}z_{-1})}{(f_{2i}z_0-f_{2i+1}z_{-1})}\prod_{i=0}^{n-2}\frac{(f_{2i}z_0-f_{2i+1}z_{-1})}{(f_{2i-1}z_0-f_{2i}z_{-1})}\right)-1} \\
&= \frac{z_{-1}\prod_{i=0}^{n-1}\frac{(f_{2i-2}z_0-f_{2i-1}z_{-1})(f_{2i-1}t_0-f_{2i}t_{-1})}{(f_{2i-1}z_0-f_{2i}z_{-1})(f_{2i}t_0-f_{2i+1}t_{-1})}}{-\frac{(f_{2n-3}z_0-f_{2n-2}z_{-1})}{(f_{2n-2}z_0-f_{2n-1}z_{-1})}-1}\left(\frac{(f_{2n-2}z_0-f_{2n-1}z_{-1})}{(f_{2n-2}z_0-f_{2n-1}z_{-1})}\right) \\
&= \frac{z_{-1}\prod_{i=0}^{n-1}\frac{(f_{2i-2}z_0-f_{2i-1}z_{-1})(f_{2i-1}t_0-f_{2i}t_{-1})}{(f_{2i-1}z_0-f_{2i}z_{-1})(f_{2i}t_0-f_{2i+1}t_{-1})}(f_{2n-2}z_0-f_{2n-1}z_{-1})}{-f_{2n-3}z_0+f_{2n-2}z_{-1}-f_{2n-2}z_0+f_{2n-1}z_{-1}} \\
&= z_{-1}\prod_{i=0}^{n-1}\frac{(f_{2i-2}z_0-f_{2i-1}z_{-1})(f_{2i-1}t_0-f_{2i}t_{-1})}{(f_{2i-1}z_0-f_{2i}z_{-1})(f_{2i}t_0-f_{2i+1}t_{-1})}\frac{(f_{2n-2}z_0-f_{2n-1}z_{-1})}{(-f_{2n-1}z_0+f_{2n}z_{-1})} \\
&= z_0\prod_{i=0}^{n-1}\frac{(f_{2i}z_0-f_{2i+1}z_{-1})(f_{2i-1}t_0-f_{2i}t_{-1})}{(f_{2i+1}z_0-f_{2i+2}z_{-1})(f_{2i}t_0-f_{2i+1}t_{-1})},
\end{aligned}
$$

and so,

$$
\begin{aligned}
t_{2n} &= \frac{t_{2n-1}z_{2n-2}}{z_{2n-1}-z_{2n-2}} \\
&= \frac{\left(t_{-1}\prod_{i=0}^{n-1}\frac{(f_{2i-1}z_0-f_{2i}z_{-1})(f_{2i-2}t_0-f_{2i-1}t_{-1})}{(f_{2i}z_0-f_{2i+1}z_{-1})(f_{2i-1}t_0-f_{2i}t_{-1})}\right)\left(z_0\prod_{i=0}^{n-2}\frac{(f_{2i}z_0-f_{2i+1}z_{-1})(f_{2i-1}t_0-f_{2i}t_{-1})}{(f_{2i+1}z_0-f_{2i+2}z_{-1})(f_{2i}t_0-f_{2i+1}t_{-1})}\right)}{\left(z_{-1}\prod_{i=0}^{n-1}\frac{(f_{2i-2}z_0-f_{2i-1}z_{-1})(f_{2i-1}t_0-f_{2i}t_{-1})}{(f_{2i-1}z_0-f_{2i}z_{-1})(f_{2i}t_0-f_{2i+1}t_{-1})}\right)-\left(z_0\prod_{i=0}^{n-2}\frac{(f_{2i}z_0-f_{2i+1}z_{-1})(f_{2i-1}t_0-f_{2i}t_{-1})}{(f_{2i+1}z_0-f_{2i+2}z_{-1})(f_{2i}t_0-f_{2i+1}t_{-1})}\right)} \\
&= \frac{\left(t_{-1}\prod_{i=0}^{n-1}\frac{(f_{2i-1}z_0-f_{2i}z_{-1})(f_{2i-2}t_0-f_{2i-1}t_{-1})}{(f_{2i}z_0-f_{2i+1}z_{-1})(f_{2i-1}t_0-f_{2i}t_{-1})}\right)}{\left(-\prod_{i=0}^{n-1}\frac{(f_{2i-1}t_0-f_{2i}t_{-1})}{(f_{2i}t_0-f_{2i+1}t_{-1})}\prod_{i=0}^{n-2}\frac{(f_{2i}t_0-f_{2i+1}t_{-1})}{(f_{2i-1}t_0-f_{2i}t_{-1})}\right)-1} \\
&= \frac{\left(t_{-1}\prod_{i=0}^{n-1}\frac{(f_{2i-1}z_0-f_{2i}z_{-1})(f_{2i-2}t_0-f_{2i-1}t_{-1})}{(f_{2i}z_0-f_{2i+1}z_{-1})(f_{2i-1}t_0-f_{2i}t_{-1})}\right)}{-\frac{(f_{2n-3}t_0-f_{2n-2}t_{-1})}{(f_{2n-2}t_0-f_{2n-1}t_{-1})}-1}\left(\frac{(f_{2n-2}t_0-f_{2n-1}t_{-1})}{(f_{2n-2}t_0-f_{2n-1}t_{-1})}\right) \\
&= \frac{\left(t_{-1}\prod_{i=0}^{n-1}\frac{(f_{2i-1}z_0-f_{2i}z_{-1})(f_{2i-2}t_0-f_{2i-1}t_{-1})}{(f_{2i}z_0-f_{2i+1}z_{-1})(f_{2i-1}t_0-f_{2i}t_{-1})}\right)(f_{2n-2}t_0-f_{2n-1}t_{-1})}{-f_{2n-3}t_0+f_{2n-2}t_{-1}-f_{2n-2}t_0+f_{2n-1}t_{-1}} \\
&= t_{-1}\prod_{i=0}^{n-1}\frac{(f_{2i-1}z_0-f_{2i}z_{-1})(f_{2i-2}t_0-f_{2i-1}t_{-1})}{(f_{2i}z_0-f_{2i+1}z_{-1})(f_{2i-1}t_0-f_{2i}t_{-1})}\frac{(f_{2n-2}t_0-f_{2n-1}t_{-1})}{(-f_{2n-1}t_0+f_{2n}t_{-1})} \\
&= t_{-1}\prod_{i=0}^{n-1}\frac{(f_{2i-2}t_0-f_{2i-1}t_{-1})}{f_{2i-1}t_0-f_{2i}t_{-1})}\frac{(f_{2n-2}t_0-f_{2n-1}t_{-1})}{(-f_{2n-1}t_0+f_{2n}t_{-1})} \\
&= t_0\prod_{i=0}^{n-1}\frac{(f_{2i-1}z_0-f_{2i}z_{-1})(f_{2i}t_0-f_{2i+1}t_{-1})}{(f_{2i}z_0-f_{2i+1}z_{-1})(f_{2i+1}t_0-f_{2i+2}t_{-1})}.
\end{aligned}
$$

The proof is complete.

Example 1. For confirming the results of this section, we consider numerical example for the difference system (1) with the initial conditions $z_{-1} = 0.3$, $z_0 = 0.4$, $t_{-1} = 0.15$ and $t_0 = -0.1$. (See Fig. 1).

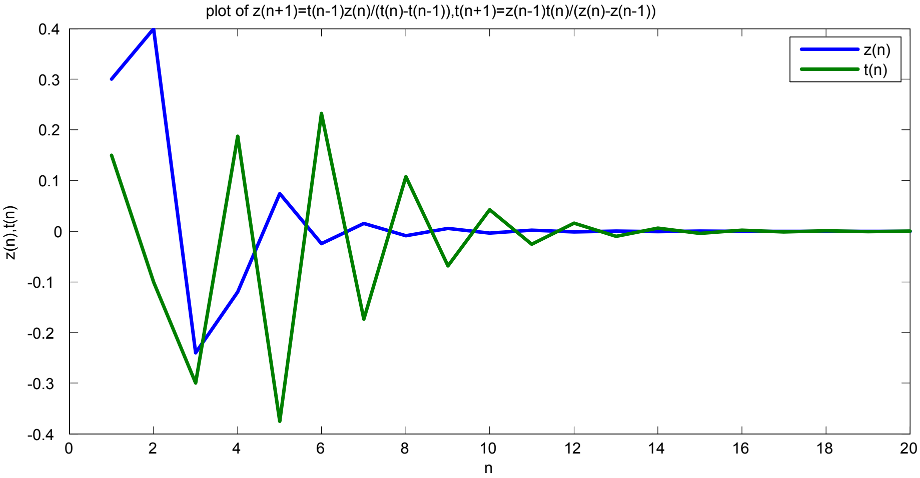


Figure 1. Plot the behavior of the solution of the system (1).

3. THE SECOND SYSTEM: $Z_{N+1} = \frac{Z_N T_{N-1}}{T_N - T_{N-1}}$, $T_{N+1} = \frac{T_N Z_{N-1}}{-Z_N - Z_{N-1}}$

We obtain, in this section, the form of the solutions of the difference equations system

$$z_{n+1} = \frac{z_n t_{n-1}}{t_n - t_{n-1}}, \; t_{n+1} = \frac{t_n z_{n-1}}{-z_n - z_{n-1}}, \tag{2}$$

where $n \in \mathbb{N}_0$ and the initial conditions z_{-1}, z_0, t_{-1} *and* t_0 are arbitrary non zero real numbers with $z_{-1} \neq -z_0$.

THEOREM 3.1. *Let* $\{z_n, t_n\}_{n=-1}^{+\infty}$ *be solutions of system (2).Then* $\{z_n\}_{n=-1}^{+\infty}$ *and* $\{t_n\}_{n=-1}^{+\infty}$ *are given by the formulae for* $n = 0, 1, 2, ...,$

$$\begin{aligned}
z_{4n} &= \frac{-z_{-1}z_0t_{-1}t_0(z_0+z_{-1})}{(f_{2n-2}z_0+f_{2n}z_{-1})(f_{2n-1}z_0+f_{2n+1}z_{-1})(f_{2n-1}t_0-f_{2n-2}t_{-1})(f_{2n}t_0-f_{2n-1}t_{-1})},\\
z_{4n+1} &= \frac{z_{-1}z_0t_{-1}t_0(z_0+z_{-1})}{(f_{2n-2}z_0+f_{2n}z_{-1})(f_{2n-1}z_0+f_{2n+1}z_{-1})(f_{2n}t_0-f_{2n-1}t_{-1})(f_{2n+1}t_0-f_{2n}t_{-1})},\\
z_{4n+2} &= \frac{-z_{-1}z_0t_{-1}t_0(z_0+z_{-1})}{(f_{2n-1}z_0+f_{2n+1}z_{-1})(f_{2n}z_0+f_{2n+2}z_{-1})(f_{2n}t_0-f_{2n-1}t_{-1})(f_{2n+1}t_0-f_{2n}t_{-1})},\\
z_{4n+3} &= \frac{z_{-1}z_0t_{-1}t_0(z_0+z_{-1})}{(f_{2n-1}z_0+f_{2n+1}z_{-1})(f_{2n}z_0+f_{2n+2}z_{-1})(f_{2n+1}t_0-f_{2n}t_{-1})(f_{2n+2}t_0-f_{2n+1}t_{-1})},
\end{aligned}$$

and

$$\begin{aligned}
t_{4n} &= \frac{(f_{2n-2}z_0+f_{2n}z_{-1})(f_{2n}t_0-f_{2n-1}t_{-1})}{(z_0+z_{-1})}, \; t_{4n+1} = \frac{-(f_{2n-1}z_0+f_{2n+1}z_{-1})(f_{2n}t_0-f_{2n-1}t_{-1})}{(z_0+z_{-1})},\\
t_{4n+2} &= \frac{(f_{2n-1}z_0+f_{2n+1}z_{-1})(f_{2n+1}t_0-f_{2n}t_{-1})}{(z_0+z_{-1})}, \; t_{4n+3} = \frac{-(f_{2n}z_0+f_{2n+2}z_{-1})(f_{2n+1}t_0-f_{2n}t_{-1})}{(z_0+z_{-1})}.
\end{aligned}$$

Proof: For $n = 0$ the result holds. Now suppose that $n > 0$ and that our assumption holds for $n - 1$. that is,

$$\begin{aligned}
z_{4n-4} &= \frac{-z_{-1}z_0t_{-1}t_0(z_0+z_{-1})}{(f_{2n-4}z_0+f_{2n-2}z_{-1})(f_{2n-3}z_0+f_{2n-1}z_{-1})(f_{2n-3}t_0-f_{2n-4}t_{-1})(f_{2n-2}t_0-f_{2n-3}t_{-1})},\\
z_{4n-3} &= \frac{z_{-1}z_0t_{-1}t_0(z_0+z_{-1})}{(f_{2n-4}z_0+f_{2n-2}z_{-1})(f_{2n-3}z_0+f_{2n-1}z_{-1})(f_{2n-2}t_0-f_{2n-3}t_{-1})(f_{2n-1}t_0-f_{2n-2}t_{-1})},\\
z_{4n-2} &= \frac{-z_{-1}z_0t_{-1}t_0(z_0+z_{-1})}{(f_{2n-3}z_0+f_{2n-1}z_{-1})(f_{2n-2}z_0+f_{2n}z_{-1})(f_{2n-2}t_0-f_{2n-3}t_{-1})(f_{2n-1}t_0-f_{2n-2}t_{-1})},\\
z_{4n-1} &= \frac{z_{-1}z_0t_{-1}t_0(z_0+z_{-1})}{(f_{2n-3}z_0+f_{2n-1}z_{-1})(f_{2n-2}z_0+f_{2n}z_{-1})(f_{2n-1}t_0-f_{2n-2}t_{-1})(f_{2n}t_0-f_{2n-1}t_{-1})},
\end{aligned}$$

$$\begin{aligned}
t_{4n-4} &= \frac{(f_{2n-4}z_0+f_{2n-2}z_{-1})(f_{2n-2}t_0-f_{2n-3}t_{-1})}{(z_0+z_{-1})}, \ t_{4n-3} = \frac{-(f_{2n-3}z_0+f_{2n-1}z_{-1})(f_{2n-2}t_0-f_{2n-3}t_{-1})}{(z_0+z_{-1})}, \\
t_{4n-2} &= \frac{(f_{2n-3}z_0+f_{2n-1}z_{-1})(f_{2n-1}t_0-f_{2n-2}t_{-1})}{(z_0+z_{-1})}, \ t_{4n-1} = \frac{-(f_{2n-2}z_0+f_{2n}z_{-1})(f_{2n-1}t_0-f_{2n-2}t_{-1})}{(z_0+z_{-1})}.
\end{aligned}$$

Now, we obtain from system (2) that

$$\begin{aligned}
z_{4n+1} &= \frac{z_{4n}t_{4n-1}}{t_{4n}-t_{4n-1}} = \frac{\left(\frac{-z_{-1}z_0t_{-1}t_0(z_0+z_{-1})}{(f_{2n-2}z_0+f_{2n}z_{-1})(f_{2n-1}z_0+f_{2n+1}z_{-1})(f_{2n-1}t_0-f_{2n-2}t_{-1})(f_{2n}t_0-f_{2n-1}t_{-1})}\right)\left(\frac{-(f_{2n-2}z_0+f_{2n}z_{-1})(f_{2n-1}t_0-f_{2n-2}t_{-1})}{(z_0+z_{-1})}\right)}{\left(\frac{(f_{2n-2}z_0+f_{2n}z_{-1})(f_{2n}t_0-f_{2n-1}t_{-1})}{(z_0+z_{-1})}\right)-\left(\frac{-(f_{2n-2}z_0+f_{2n}z_{-1})(f_{2n-1}t_0-f_{2n-2}t_{-1})}{(z_0+z_{-1})}\right)} \\
&= \frac{\left(\frac{z_{-1}z_0t_{-1}t_0(z_0+z_{-1})}{(f_{2n-2}z_0+f_{2n}z_{-1})(f_{2n-1}z_0+f_{2n+1}z_{-1})(f_{2n}t_0-f_{2n-1}t_{-1})}\right)}{(f_{2n-1}t_0-f_{2n-2}t_{-1})+(f_{2n}t_0-f_{2n-1}t_{-1})} \\
&= \frac{z_{-1}z_0t_{-1}t_0(z_0+z_{-1})}{(f_{2n-2}z_0+f_{2n}z_{-1})(f_{2n-1}z_0+f_{2n+1}z_{-1})(f_{2n}t_0-f_{2n-1}t_{-1})(f_{2n+11}t_0-f_{2n}t_{-1})},
\end{aligned}$$

$$\begin{aligned}
t_{4n+1} &= \frac{t_{4n}z_{4n-1}}{-z_{4n}-z_{4n-1}} = \frac{\left(\frac{(f_{2n-2}z_0+f_{2n}z_{-1})(f_{2n}t_0-f_{2n-1}t_{-1})}{(z_0+z_{-1})}\right)\left(\frac{z_{-1}z_0t_{-1}t_0(z_0+z_{-1})}{(f_{2n-3}z_0+f_{2n-1}z_{-1})(f_{2n-2}z_0+f_{2n}z_{-1})(f_{2n-1}t_0-f_{2n-2}t_{-1})(f_{2n}t_0-f_{2n-1}t_{-1})}\right)}{\left[-\left(\frac{-z_{-1}z_0t_{-1}t_0(z_0+z_{-1})}{(f_{2n-2}z_0+f_{2n}z_{-1})(f_{2n-1}z_0+f_{2n+1}z_{-1})(f_{2n-1}t_0-f_{2n-2}t_{-1})(f_{2n}t_0-f_{2n-1}t_{-1})}\right)-\left(\frac{z_{-1}z_0t_{-1}t_0(z_0+z_{-1})}{(f_{2n-3}z_0+f_{2n-1}z_{-1})(f_{2n-2}z_0+f_{2n}z_{-1})(f_{2n-1}t_0-f_{2n-2}t_{-1})(f_{2n}t_0-f_{2n-1}t_{-1})}\right)\right]} \\
&= \frac{\left(\frac{(f_{2n-2}z_0+f_{2n}z_{-1})(f_{2n}t_0-f_{2n-1}t_{-1})}{(z_0+z_{-1})}\right)}{-1+\frac{(f_{2n-3}z_0+f_{2n-1}z_{-1})}{(f_{2n-1}z_0+f_{2n+1}z_{-1})}} = -\frac{\left(\frac{(f_{2n-2}z_0+f_{2n}z_{-1})(f_{2n}t_0-f_{2n-1}t_{-1})}{(z_0+z_{-1})}\right)}{1-\frac{(f_{2n-3}z_0+f_{2n-1}z_{-1})}{(f_{2n-1}z_0+f_{2n+1}z_{-1})}} \\
&= -\frac{\left(\frac{(f_{2n-2}z_0+f_{2n}z_{-1})(f_{2n}t_0-f_{2n-1}t_{-1})}{(z_0+z_{-1})}\right)}{1-\frac{(f_{2n-3}z_0+f_{2n-1}z_{-1})}{(f_{2n-1}z_0+f_{2n+1}z_{-1})}} = -\frac{\left(\frac{(f_{2n-2}z_0+f_{2n}z_{-1})(f_{2n}t_0-f_{2n-1}t_{-1})}{(z_0+z_{-1})}\right)}{\frac{(f_{2n-2}z_0+f_{2n}z_{-1})}{(f_{2n-1}z_0+f_{2n+1}z_{-1})}} \\
&= \frac{-(f_{2n-1}z_0+f_{2n+1}z_{-1})(f_{2n}t_0-f_{2n-1}t_{-1})}{(z_0+z_{-1})}.
\end{aligned}$$

Also, we can prove the other relations. This completes the proof.

Example 2. We assume that the initial conditions for the difference system (2) are $z_{-1} = 0.38$, $z_0 = -17$, $t_{-1} = 0.85$ and $t_0 = 1.26$. (See Fig. 2).

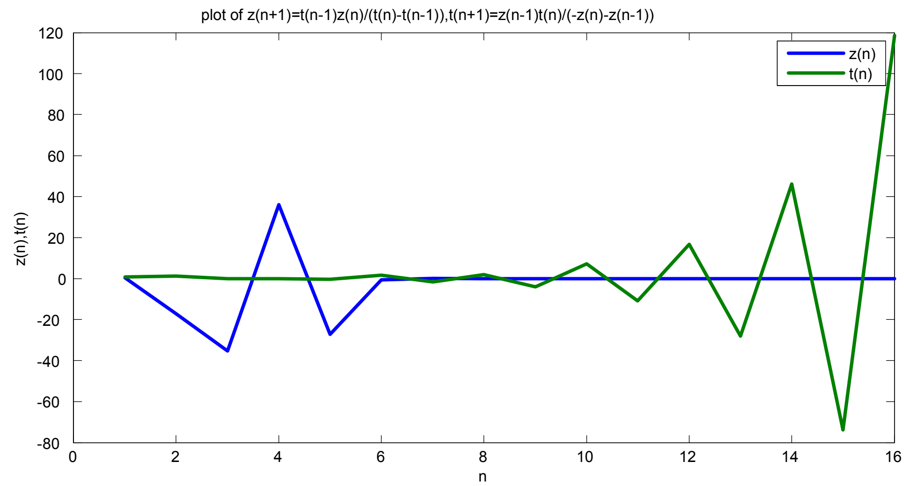


Figure 2. Sketch the behavior of the solution of the system (2).

4. PERIODICITY OF THE SYSTEMS

In this section, we study the periodicity nature of the solutions of the following systems of the difference equations

$$z_{n+1}=\frac{z_n t_{n-1}}{t_n-t_{n-1}},\ t_{n+1}=\frac{t_n z_{n-1}}{z_n+z_{n-1}}. \tag{3}$$

$$z_{n+1}=\frac{z_n t_{n-1}}{t_n+t_{n-1}},\ t_{n+1}=\frac{t_n z_{n-1}}{z_n-z_{n-1}}. \tag{4}$$

$$z_{n+1}=\frac{z_n t_{n-1}}{-t_n-t_{n-1}},\ t_{n+1}=\frac{t_n z_{n-1}}{-z_n-z_{n-1}}. \tag{5}$$

Where $n=0,\ 1,\ 2,\ ...$ and the initial conditions $z_{-1},\ z_0,\ t_{-1}$ and t_0 are arbitrary nonzero real numbers.

Theorem 4.1. *Suppose that $\{z_n,\ t_n\}$ are solutions of difference equation system (3) with $z_0\neq -z_{-1},\ t_0\neq t_{-1}$. Then all solutions of system (3) are periodic with period six and for $n=0,\ 1,\ 2,\ ...$,*

$$z_{6n-1}=z_{-1},\ z_{6n}=z_0,\ z_{6n+1}=\frac{z_0t_{-1}}{t_0-t_{-1}},\ z_{6n+2}=\frac{t_{-1}(z_0+z_{-1})}{(t_{-1}-t_0)},\ z_{6n+3}=\frac{t_0(z_0+z_{-1})}{(t_0-t_{-1})},\ z_{6n+4}=\frac{z_{-1}t_0}{(t_{-1}-t_0)},$$

and

$$t_{6n-1}=t_{-1},\ t_{6n}=t_0,\ t_{6n+1}=\frac{z_{-1}t_0}{z_0+z_{-1}},\ t_{6n+2}=\frac{z_{-1}(t_0-t_{-1})}{(z_0+z_{-1})},\ t_{6n+3}=\frac{z_0(t_{-1}-t_0)}{(z_0+z_{-1})},\ t_{6n+4}=\frac{z_0t_{-1}}{(z_0+z_{-1})}.$$

Proof: For $n=0$ the result holds. Now suppose that $n>0$ and that our assumption holds for $n-1$. that is,

$$z_{6n-7}=z_{-1},\ z_{6n-6}=z_0,\ z_{6n-5}=\frac{z_0t_{-1}}{t_0-t_{-1}},\ z_{6n-4}=\frac{t_{-1}(z_0+z_{-1})}{(t_{-1}-t_0)},\ z_{6n-3}=\frac{t_0(z_0+z_{-1})}{(t_0-t_{-1})},\ z_{6n-2}=\frac{z_{-1}t_0}{(t_{-1}-t_0)},$$

and

$$t_{6n-7}=t_{-1},\ t_{6n-6}=t_0,\ t_{6n-5}=\frac{z_{-1}t_0}{z_0+z_{-1}},\ t_{6n-4}=\frac{z_{-1}(t_0-t_{-1})}{(z_0+z_{-1})},\ t_{6n-3}=\frac{z_0(t_{-1}-t_0)}{(z_0+z_{-1})},\ t_{6n-2}=\frac{z_0t_{-1}}{(z_0+z_{-1})}.$$

Now, we obtain from system (3) that

$$\begin{aligned}
z_{6n-1} &= \frac{z_{6n-2}t_{6n-3}}{t_{6n-2}-t_{6n-3}}=\frac{\left(\frac{z_{-1}t_0}{(t_{-1}-t_0)}\right)\left(\frac{z_0(t_{-1}-t_0)}{(z_0+z_{-1})}\right)}{\left(\frac{z_0t_{-1}}{(z_0+z_{-1})}\right)-\left(\frac{z_0(t_{-1}-t_0)}{(z_0+z_{-1})}\right)}=\frac{z_{-1}t_0z_0}{z_0t_{-1}-z_0(t_{-1}-t_0)}=\frac{z_{-1}t_0z_0}{z_0t_0}=z_{-1},\\
t_{6n-1} &= \frac{t_{6n-2}z_{6n-3}}{z_{6n-2}+z_{6n-3}}=\frac{\left(\frac{z_0t_{-1}}{(z_0+z_{-1})}\right)\left(\frac{t_0(z_0+z_{-1})}{(t_0-t_{-1})}\right)}{\left(\frac{z_{-1}t_0}{(t_{-1}-t_0)}\right)+\left(\frac{t_0(z_0+z_{-1})}{(t_0-t_{-1})}\right)}=\frac{z_0t_{-1}t_0}{-z_{-1}t_0+t_0(z_0+z_{-1})}=\frac{z_0t_{-1}t_0}{t_0z_0}=t_{-1},\\
z_{6n} &= \frac{z_{6n-1}t_{6n-2}}{t_{6n-1}-t_{6n-2}}=\frac{z_{-1}\frac{z_0t_{-1}}{(z_0+z_{-1})}}{t_{-1}-\frac{z_0t_{-1}}{(z_0+z_{-1})}}=\frac{z_{-1}z_0t_{-1}}{t_{-1}(z_0+z_{-1})-z_0t_{-1}}=z_0,\\
t_{6n} &= \frac{t_{6n-1}z_{6n-2}}{z_{6n-1}+z_{6n-2}}=\frac{t_{-1}\frac{z_{-1}t_0}{(t_{-1}-t_0)}}{z_{-1}+\frac{z_{-1}t_0}{(t_{-1}-t_0)}}=\frac{t_{-1}z_{-1}t_0}{z_{-1}(t_{-1}-t_0)+z_{-1}t_0}=t_0.
\end{aligned}$$

We can prove the other relations similarly. The proof is completed.

Theorem 4.2. *If $\{z_n,\ t_n\}$ are solutions of system (4) with $z_0\neq z_{-1},\ t_0\neq -t_{-1}$. Then all solutions of system (4) are periodic with period six and given by the formulae*

$$\begin{aligned}
z_{6n-1} &= z_{-1},\ z_{6n}=z_0,\ z_{6n+1}=\frac{z_0t_{-1}}{t_0+t_{-1}},\ z_{6n+2}=\frac{t_{-1}(z_0-z_{-1})}{(t_0+t_{-1})},\ z_{6n+3}=\frac{t_0(z_{-1}-z_0)}{(t_0+t_{-1})},\ z_{6n+4}=\frac{z_{-1}t_0}{t_0+t_{-1}},\\
t_{6n-1} &= t_{-1},\ t_{6n}=t_0,\ t_{6n+1}=\frac{z_{-1}t_0}{z_0-z_{-1}},\ t_{6n+2}=\frac{z_{-1}(t_0+t_{-1})}{(z_{-1}-z_0)},\ t_{6n+3}=\frac{z_0(t_0+t_{-1})}{(z_0-z_{-1})},\ t_{6n+4}=\frac{z_0t_{-1}}{z_{-1}-z_0}.
\end{aligned}$$

THEOREM 4.3. *Assume that* $\{z_n,\ t_n\}$ *are solutions of difference equation system (5) with* $z_0 \neq -z_{-1},\ t_0 \neq -t_{-1}$. *Then all solutions of system (5) are periodic with period six and for* $n = 0,\ 1,\ 2,\ ...,$

$$\begin{aligned} z_{6n-1} &= z_{-1},\ z_{6n} = z_0,\ z_{6n+1} = -\frac{z_0 t_{-1}}{t_0+t_{-1}},\ z_{6n+2} = \frac{t_{-1}(z_0+z_{-1})}{(t_0+t_{-1})},\ z_{6n+3} = \frac{t_0(z_0+z_{-1})}{(t_0+t_{-1})},\ z_{6n+4} = -\frac{z_{-1}t_0}{t_0+t_{-1}}, \\ t_{6n-1} &= t_{-1},\ t_{6n} = t_0,\ t_{6n+1} = -\frac{z_{-1}t_0}{z_0+z_{-1}},\ t_{6n+2} = \frac{z_{-1}(t_0+t_{-1})}{(z_0+z_{-1})},\ t_{6n+3} = \frac{z_0(t_{-1}+t_0)}{(z_0+z_{-1})},\ \ t_{6n+4} = -\frac{z_0 t_{-1}}{z_0+z_{-1}}. \end{aligned}$$

Example 3. See Figure (3) where we take system (3) with the initial conditions $z_{-1} = 0.18$, $z_0 = 0.17$, $t_{-1} = 0.5$ and $t_0 = 0.86$.

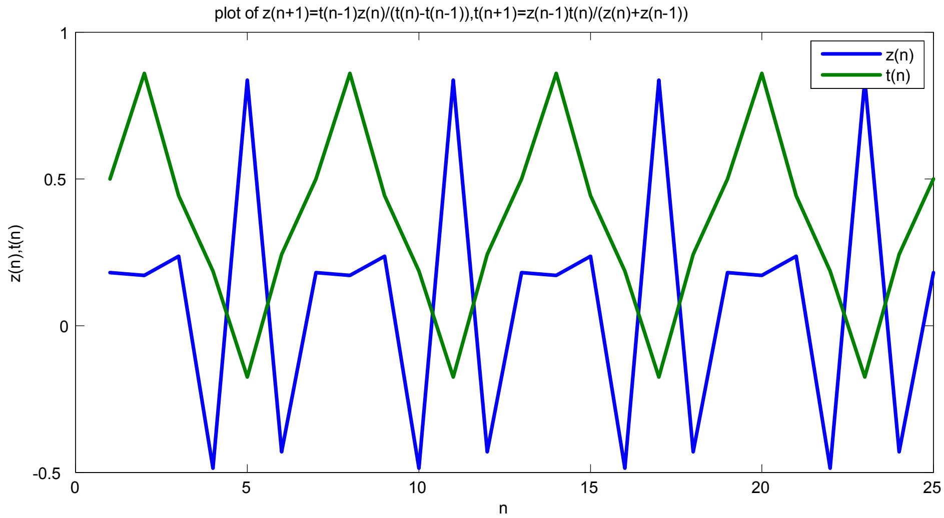


Figure 3. Draw the behavior of the solution of the system (3).

5. OTHER SYSTEMS

In this section, we obtain the form of the solutions of the follwing systems of the difference equations.

THEOREM 5.1. *If* $\{z_n,\ t_n\}$ *are solutions of system*

$$z_{n+1} = \frac{z_n t_{n-1}}{-t_n+t_{n-1}},\ t_{n+1} = \frac{t_n z_{n-1}}{-z_n-z_{n-1}}, \tag{6}$$

where $n \in \mathbb{N}_0$ *and the initial conditions* $z_{-1},\ z_0,\ t_{-1}$ *and* t_0 *are arbitrary non zero real numbers, then for* $n = 0, 1, 2, ...,$

$$\begin{aligned} z_{2n-1} &= z_{-1}\prod_{i=0}^{n-1}\frac{(f_{2i-2}z_0+f_{2i-1}z_{-1})(f_{2i-1}t_0-f_{2i}t_{-1})}{(f_{2i-1}z_0+f_{2i}z_{-1})(f_{2i}t_0-f_{2i+1}t_{-1})},\ z_{2n} = z_0\prod_{i=0}^{n-1}\frac{(f_{2i}z_0+f_{2i+1}z_{-1})(f_{2i-1}t_0-f_{2i}t_{-1})}{(f_{2i+1}z_0+f_{2i+2}z_{-1})(f_{2i}t_0-f_{2i+1}t_{-1})}, \\ t_{2n-1} &= t_{-1}\prod_{i=0}^{n-1}\frac{(f_{2i-1}z_0+f_{2i}z_{-1})(f_{2i-1}t_{-1}-f_{2i-2}t_0)}{(f_{2i}z_0+f_{2i+1}z_{-1})(f_{2i}t_{-1}-f_{2i-1}t_0)},\ t_{2n} = t_0\prod_{i=0}^{n-1}\frac{(f_{2i-1}z_0+f_{2i}z_{-1})(f_{2i}t_0-f_{2i+1}t_{-1})}{(f_{2i}z_0+f_{2i+1}z_{-1})(f_{2i+1}t_0-f_{2i+2}t_{-1})}. \end{aligned}$$

such that $\prod_{i=0}^{-1} A_i = 1.$

THEOREM 5.2. *The solutions of system*

$$z_{n+1}=\frac{z_nt_{n-1}}{-t_n-t_{n-1}},\ t_{n+1}=\frac{t_nz_{n-1}}{-z_n+z_{n-1}}, \tag{7}$$

are given by the relations

$$\begin{aligned} z_{2n-1} &= z_{-1}\prod_{i=0}^{n-1}\frac{(f_{2i-1}z_{-1}-f_{2i-2}z_0)(f_{2i-1}t_0+f_{2i}t_{-1})}{(f_{2i}z_{-1}-f_{2i-1}z_0)(f_{2i}t_0+f_{2i+1}t_{-1})},\ z_{2n}=z_0\prod_{i=0}^{n-1}\frac{(f_{2i}z_0-f_{2i+1}z_{-1})(f_{2i-1}t_0+f_{2i}t_{-1})}{(f_{2i+1}z_0-f_{2i+2}z_{-1})(f_{2i}t_0+f_{2i+1}t_{-1})},\\ t_{2n-1} &= t_{-1}\prod_{i=0}^{n-1}\frac{(f_{2i-1}z_0-f_{2i}z_{-1})(f_{2i-2}t_0+f_{2i-1}t_{-1})}{(f_{2i}z_0-f_{2i+1}z_{-1})(f_{2i-1}t_0+f_{2i}t_{-1})},\ t_{2n}=t_0\prod_{i=0}^{n-1}\frac{(f_{2i-1}z_0-f_{2i}z_{-1})(f_{2i}t_0+f_{2i+1}t_{-1})}{(f_{2i}z_0-f_{2i+1}z_{-1})(f_{2i+1}t_0+f_{2i+2}t_{-1})}, \end{aligned}$$

where $n\in\mathbb{N}_0$ *and the initial conditions* $z_{-1},\ z_0,\ t_{-1}$ *and* t_0 *are arbitrary non zero real numbers and* $\prod_{i=0}^{-1}A_i=1$.

THEOREM 5.3. *Suppose that* $\{z_n,\ t_n\}_{n=-1}^{+\infty}$ *are solutions of system*

$$z_{n+1}=\frac{z_nt_{n-1}}{-t_n-t_{n-1}},\ t_{n+1}=\frac{t_nz_{n-1}}{z_n+z_{n-1}}. \tag{8}$$

where $n\in\mathbb{N}_0$ *and the initial conditions* $z_{-1},\ z_0,\ t_{-1}$ *and* t_0 *are arbitrary non zero real numbers with* $z_{-1}\neq -z_0$. *Then* $\{z_n\}_{n=-1}^{+\infty}$ *and* $\{t_n\}_{n=-1}^{+\infty}$ *are given by the formula for* $n=0,\ 1,\ 2,\ ...,$

$$\begin{aligned} z_{4n} &= \frac{z_{-1}z_0t_{-1}t_0(z_0+z_{-1})}{(f_{2n-2}z_0+f_{2n}z_{-1})(f_{2n-1}z_0+f_{2n+1}z_{-1})(f_{2n-1}t_{-1}+f_{2n-2}t_0)(f_{2n}t_{-1}+f_{2n-1}t_0)},\\ z_{4n+1} &= \frac{-z_{-1}z_0t_{-1}t_0(z_0+z_{-1})}{(f_{2n-2}z_0+f_{2n}z_{-1})(f_{2n-1}z_0+f_{2n+1}z_{-1})(f_{2n}t_{-1}+f_{2n-1}t_0)(f_{2n+1}t_{-1}+f_{2n}t_0)},\\ z_{4n+2} &= \frac{z_{-1}z_0t_{-1}t_0(z_0+z_{-1})}{(f_{2n-1}z_0+f_{2n+1}z_{-1})(f_{2n}z_0+f_{2n+2}z_{-1})(f_{2n}t_{-1}+f_{2n-1}t_0)(f_{2n+1}t_{-1}+f_{2n}t_0)},\\ z_{4n+3} &= \frac{-z_{-1}z_0t_{-1}t_0(z_0+z_{-1})}{(f_{2n-1}z_0+f_{2n+1}z_{-1})(f_{2n}z_0+f_{2n+2}z_{-1})(f_{2n+1}t_{-1}+f_{2n}t_0)(f_{2n+2}t_{-1}+f_{2n+1}t_0)}, \end{aligned}$$

and

$$\begin{aligned} t_{4n} &= \frac{(f_{2n-2}z_0+f_{2n}z_{-1})(f_{2n}t_0+f_{2n-1}t_{-1})}{(z_0+z_{-1})},\ t_{4n+1}=\frac{(f_{2n-1}z_0+f_{2n+1}z_{-1})(f_{2n}t_0+f_{2n-1}t_{-1})}{(z_0+z_{-1})},\\ t_{4n+2} &= \frac{(f_{2n-1}z_0+f_{2n+1}z_{-1})(f_{2n+1}t_0+f_{2n}t_{-1})}{(z_0+z_{-1})},\ t_{4n+3}=\frac{(f_{2n}z_0+f_{2n+2}b)(f_{2n+1}t_0+f_{2n}t_{-1})}{(z_0+z_{-1})}. \end{aligned}$$

THEOREM 5.4. *Let* $\{z_n,t_n\}_{n=-1}^{+\infty}$ *be solutions of system*

$$z_{n+1}=\frac{z_nt_{n-1}}{-t_n+t_{n-1}},\ t_{n+1}=\frac{t_nz_{n-1}}{z_n-z_{n-1}}. \tag{9}$$

Then $\{z_n\}_{n=-1}^{+\infty}$ *and* $\{t_n\}_{n=-1}^{+\infty}$ *are given by the following expressions for* $n=0,1,2,...,$

$$\begin{aligned} z_{4n} &= \frac{z_{-1}z_0t_{-1}t_0(z_0-z_{-1})}{(f_{2n-2}z_0-f_{2n}z_{-1})(f_{2n-1}z_0-f_{2n+1}z_{-1})(f_{2n-1}t_0-f_{2n-2}t_{-1})(f_{2n}t_0-f_{2n-1}t_{-1})},\\ z_{4n+1} &= \frac{z_{-1}z_0t_{-1}t_0(z_0-z_{-1})}{(f_{2n-2}z_0-f_{2n}z_{-1})(f_{2n-1}z_0-f_{2n+1}z_{-1})(f_{2n}t_0-f_{2n-1}t_{-1})(f_{2n+1}t_0-f_{2n}t_{-1})},\\ z_{4n+2} &= \frac{-z_{-1}z_0t_{-1}t_0(z_0-z_{-1})}{(f_{2n-1}z_0-f_{2n+1}z_{-1})(f_{2n}z_0-f_{2n+2}z_{-1})(f_{2n}t_0-f_{2n-1}t_{-1})(f_{2n+1}t_0-f_{2n}t_{-1})},\\ z_{4n+3} &= \frac{-z_{-1}z_0t_{-1}t_0(z_0-z_{-1})}{(f_{2n-1}z_0-f_{2n+1}z_{-1})(f_{2n}z_0-f_{2n+2}z_{-1})(f_{2n+1}t_0-f_{2n}t_{-1})(f_{2n+2}t_0-f_{2n+1}t_{-1})}, \end{aligned}$$

and

$$\begin{aligned} t_{4n} &= \frac{(f_{2n-2}z_0-f_{2n}z_{-1})(f_{2n}t_0-f_{2n-1}t_{-1})}{(z_0-z_{-1})},\ t_{4n+1}=\frac{-(f_{2n-1}z_0-f_{2n+1}z_{-1})(f_{2n}t_0-f_{2n-1}t_{-1})}{(z_0-z_{-1})},\\ t_{4n+2} &= \frac{(f_{2n-1}z_0-f_{2n+1}z_{-1})(f_{2n+1}t_0-f_{2n}t_{-1})}{(z_0-z_{-1})},\ t_{4n+3}=\frac{-(f_{2n}z_0-f_{2n+2}b)(f_{2n+1}t_0-f_{2n}t_{-1})}{(z_0-z_{-1})}. \end{aligned}$$

where $n \in \mathbb{N}_0$ and the initial conditions $z_{-1},\ z_0,\ t_{-1}$ and t_0 are arbitrary non zero real numbers with $z_{-1} \neq z_0$.

THEOREM 5.5. *Let $\{z_n,\ t_n\}_{n=-1}^{+\infty}$ be solutions of system*

$$z_{n+1} = \frac{z_n t_{n-1}}{-t_n - t_{n-1}},\ t_{n+1} = \frac{t_n z_{n-1}}{z_n - z_{n-1}}, \tag{10}$$

where $n \in \mathbb{N}_0$ and the initial conditions $z_{-1},\ z_0,\ t_{-1}$ and t_0 are arbitrary non zero real numbers with $t_{-1} \neq -t_0$. Then $\{z_n\}_{n=-1}^{+\infty}$ and $\{t_n\}_{n=-1}^{+\infty}$ are given by the following relations for $n = 0,\ 1,\ 2,\ ...,$

$$\begin{aligned} z_{4n} &= \frac{(f_{2n}z_0 - f_{2n-1}z_{-1})(f_{2n-2}t_0 + f_{2n}t_{-1})}{t_0 + t_{-1}},\ z_{4n+1} = \frac{(f_{2n}z_0 - f_{2n-1}z_{-1})(f_{2n-1}t_0 + f_{2n+1}t_{-1})}{t_0 + t_{-1}}, \\ z_{4n+2} &= \frac{(f_{2n+1}z_0 - f_{2n}z_{-1})(f_{2n-1}t_0 + f_{2n+1}t_{-1})}{t_0 + t_{-1}},\ z_{4n+3} = \frac{(f_{2n+1}z_0 - f_{2n}z_{-1})(f_{2n}t_0 + f_{2n+2}t_{-1})}{t_0 + t_{-1}}, \end{aligned}$$

and

$$\begin{aligned} t_{4n} &= \frac{-z_0 z_{-1} t_0 t_{-1}(t_0 + t_{-1})}{(f_{2n-1}z_0 - f_{2n-2}z_{-1})(f_{2n}z_0 - f_{2n-1}z_{-1})(f_{2n-2}t_0 + f_{2n}t_{-1})(f_{2n-1}t_0 + f_{2n+1}t_{-1})}, \\ t_{4n+1} &= \frac{z_0 z_{-1} t_0 t_{-1}(t_0 + t_{-1})}{(f_{2n}z_0 - f_{2n-1}z_{-1})(f_{2n+1}z_0 - f_{2n}z_{-1})(f_{2n-2}t_0 + f_{2n}t_{-1})(f_{2n-1}t_0 + f_{2n+1}t_{-1})}, \\ t_{4n+2} &= \frac{-z_0 z_{-1} t_0 t_{-1}(t_0 + t_{-1})}{(f_{2n}z_0 - f_{2n-1}z_{-1})(f_{2n+1}z_0 - f_{2n}z_{-1})(f_{2n-1}t_0 + f_{2n+1}t_{-1})(f_{2n}t_0 + f_{2n+2}t_{-1})}, \\ t_{4n+3} &= \frac{z_0 z_{-1} t_0 t_{-1}(t_0 + t_{-1})}{(f_{2n+1}z_0 - f_{2n}z_{-1})(f_{2n+2}z_0 - f_{2n+1}z_{-1})(f_{2n-1}t_0 + f_{2n+1}t_{-1})(f_{2n}t_0 + f_{2n+2}t_{-1})}. \end{aligned}$$

THEOREM 5.6. *Suppose that $\{z_n,\ t_n\}_{n=-1}^{+\infty}$ be solutions of system*

$$z_{n+1} = \frac{z_n t_{n-1}}{t_n - t_{n-1}},\ t_{n+1} = \frac{t_n z_{n-1}}{-z_n + z_{n-1}}. \tag{11}$$

Then $\{z_n\}_{n=-1}^{+\infty}$ and $\{t_n\}_{n=-1}^{+\infty}$ are given by the following relations for $n = 0,\ 1,\ 2,\ ...,$

$$\begin{aligned} z_{4n} &= \frac{(f_{2n}z_0 - f_{2n-1}z_{-1})(f_{2n-2}t_0 - f_{2n}t_{-1})}{t_0 - t_{-1}},\ z_{4n+1} = \frac{-(f_{2n}z_0 - f_{2n-1}z_{-1})(f_{2n-1}t_0 - f_{2n+1}t_{-1})}{t_0 - t_{-1}}, \\ z_{4n+2} &= \frac{(f_{2n+1}z_0 - f_{2n}z_{-1})(f_{2n-1}t_0 - f_{2n+1}t_{-1})}{t_0 - t_{-1}},\ z_{4n+3} = \frac{-(f_{2n+1}z_0 - f_{2n}z_{-1})(f_{2n}t_0 - f_{2n+2}t_{-1})}{t_0 - t_{-1}}, \end{aligned}$$

and

$$\begin{aligned} t_{4n} &= \frac{-z_0 z_{-1} t_0 t_{-1}(t_0 - t_{-1})}{(f_{2n-1}z_0 - f_{2n-2}z_{-1})(f_{2n}z_0 - f_{2n-1}z_{-1})(f_{2n-2}t_0 - f_{2n}t_{-1})(f_{2n-1}t_0 - f_{2n+1}t_{-1})}, \\ t_{4n+1} &= \frac{z_0 z_{-1} t_0 t_{-1}(t_0 - t_{-1})}{(f_{2n}z_0 - f_{2n-1}z_{-1})(f_{2n+1}z_0 - f_{2n}z_{-1})(f_{2n-2}t_0 - f_{2n}t_{-1})(f_{2n-1}t_0 - f_{2n+1}t_{-1})}, \\ t_{4n+2} &= \frac{-z_0 z_{-1} t_0 t_{-1}(t_0 - t_{-1})}{(f_{2n}z_0 - f_{2n-1}z_{-1})(f_{2n+1}z_0 - f_{2n}z_{-1})(f_{2n-1}t_0 - f_{2n+1}t_{-1})(f_{2n}t_0 - f_{2n+2}t_{-1})}, \\ t_{4n+3} &= \frac{z_0 z_{-1} t_0 t_{-1}(t_0 - t_{-1})}{(f_{2n+1}z_0 - f_{2n}z_{-1})(f_{2n+2}z_0 - f_{2n+1}z_{-1})(f_{2n-1}t_0 - f_{2n+1}t_{-1})(f_{2n}t_0 - f_{2n+2}t_{-1})}. \end{aligned}$$

where $n \in \mathbb{N}_0$ and the initial conditions $z_{-1},\ z_0,\ t_{-1}$ and t_0 are arbitrary non zero real numbers with $t_{-1} \neq t_0$.

THEOREM 5.7. *Let $\{z_n, t_n\}_{n=-1}^{+\infty}$ be solutions of system*

$$z_{n+1} = \frac{z_n t_{n-1}}{t_n + t_{n-1}},\ t_{n+1} = \frac{t_n z_{n-1}}{-z_n - z_{n-1}}. \tag{12}$$

Then $\{z_n\}_{n=-1}^{+\infty}$ and $\{t_n\}_{n=-1}^{+\infty}$ are given by the following relations for $n = 0,\ 1,\ 2,\ ...,$

$$\begin{aligned} z_{4n} &= \frac{(f_{2n}z_0 + f_{2n-1}z_{-1})(f_{2n-2}t_0 + f_{2n}t_{-1})}{t_0 + t_{-1}},\ z_{4n+1} = \frac{(f_{2n}z_0 + f_{2n-1}z_{-1})(f_{2n-1}t_0 + f_{2n+1}t_{-1})}{t_0 + t_{-1}}, \\ z_{4n+2} &= \frac{(f_{2n+1}z_0 + f_{2n}z_{-1})(f_{2n-1}t_0 + f_{2n+1}t_{-1})}{t_0 + t_{-1}},\ z_{4n+3} = \frac{(f_{2n+1}z_0 + f_{2n}z_{-1})(f_{2n}t_0 + f_{2n+2}t_{-1})}{t_0 + t_{-1}}, \end{aligned}$$

and

$$\begin{aligned}
t_{4n} &= \frac{-z_0 z_{-1} t_0 t_{-1}(t_0+t_{-1})}{(f_{2n-1}z_0+f_{2n-2}z_{-1})(f_{2n}z_0+f_{2n-1}z_{-1})(f_{2n-2}t_0+f_{2n}t_{-1})(f_{2n-1}t_0+f_{2n+1}t_{-1})}, \\
t_{4n+1} &= \frac{z_0 z_{-1} t_0 t_{-1}(t_0+t_{-1})}{(f_{2n}z_0+f_{2n-1}z_{-1})(f_{2n+1}z_0+f_{2n}z_{-1})(f_{2n-2}t_0+f_{2n}t_{-1})(f_{2n-1}t_0+f_{2n+1}t_{-1})}, \\
t_{4n+2} &= \frac{-z_0 z_{-1} t_0 t_{-1}(t_0+t_{-1})}{(f_{2n}z_0+f_{2n-1}z_{-1})(f_{2n+1}z_0+f_{2n}z_{-1})(f_{2n-1}t_0+f_{2n+1}t_{-1})(f_{2n}t_0+f_{2n+2}t_{-1})}, \\
t_{4n+3} &= \frac{z_0 z_{-1} t_0 t_{-1}(t_0+t_{-1})}{(f_{2n+1}z_0+f_{2n}z_{-1})(f_{2n+2}z_0+f_{2n+1}z_{-1})(f_{2n-1}t_0+f_{2n+1}t_{-1})(f_{2n}t_0+f_{2n+2}t_{-1})}.
\end{aligned}$$

where $n \in \mathbb{N}_0$ *and the initial conditions* $z_{-1},\ z_0,\ t_{-1}$ *and* t_0 *are arbitrary non zero real numbers with* $t_{-1} \neq -t_0$.

Acknowledgements

This article was funded by the Deanship of Scientific Research (DSR), King Abdulaziz University, Jeddah. The authors, therefore, acknowledge with thanks DSR technical and financial support.

REFERENCES

1. P. Cull, M. Flahive, and R. Robson, Difference Equations: From Rabbits to Chaos, Undergraduate Texts in Mathematics, Springer, New York, NY, USA, 2005.
2. R. J. H. Beverton and S. J. Holt, On the Dynamics of Exploited Fish Populations, Fishery Investigations Series II, Volume 19, Blackburn Press, Caldwell, NJ, USA, 2004.
3. M. R. S. Kulenovic and G. Ladas, Dynamics of Second Order Rational Difference Equations with Open Problems and Conjectures, Chapman & Hall / CRC Press, 2001.
4. V. L. Kocic and G. Ladas, Global Behavior of Nonlinear Difference Equations of Higher Order with Applications, Kluwer Academic Publishers, Dordrecht, 1993.
5. S. Elaydi, An Introduction to Difference Equations, Undergraduate Texts in Mathematics, Springer, New York, NY, USA, 3^{rd} edition, (2005).
6. M. M. El-Dessoky, E. M. Elsayed and M. Alghamdi, Solutions and periodicity for some systems of fourth order rational difference equations, J. Comput. Anal. Appl., Vol. 18(1), (2015), 179-194.
7. E. A. Grove, G. Ladas, L. C. McGrath, and C.T. Teixeira, Existence and behavior of solutions of a rational system, Commun. Appl. Nonlinear Anal., 8 (2001), 1-25.
8. M. Mansour, M. M. El-Dessoky and E. M. Elsayed, On the solution of rational systems of difference equations, J. Comput. Anal. Appl., 15 (5) (2013), 967-976.
9. M. M. El-Dessoky, The form of solutions and periodicity for some systems of third order rational difference equations, Math. Methods Appl. Sci., 39, (2016), 1076-1092.
10. N. Touafek and E. M. Elsayed, On the periodicity of some systems of nonlinear difference equations, Bull. Math. Soc. Sci. Math. Roumanie, Tome 55 (103), No. 2, (2012), 217–224.
11. L. Yang and J. Yang, Dynamics of a system of two nonlinear difference equations, Int. J. Contemp. Math. Sciences, 6 (5) (2011), 209 - 214
12. Q. Din, M. N. Qureshi and A. Qadeer Khan, Dynamics of a fourth-order system of rational difference equations, Adv. Difference Equ., 2012, (2012): 215 doi: 10.1186/1687-1847-2012-215.
13. Q. Din, Asymptotic behavior of an anti-competitive system of second-order difference equations, J. Egyptian Math. Soc., 24, (2016), 37-43.
14. M. M. El-Dessoky, E. M. Elsayed, On a solution of system of three fractional difference equations, J. Comput. Anal. Appl., 19, (2015), 760-769.
15. N. Battaloglu, C. Cinar and I. Yalçınkaya, The dynamics of the difference equation, ARS Combinatoria, 97 (2010), 281-288.
16. M. Aloqeili, Dynamics of a rational difference equation, Appl. Math. Comp., 176(2), (2006), 768-774.
17. C. Cinar, I. Yalçinkaya and R. Karatas, On the positive solutions of the difference equation system $x_{n+1} = m/y_n,\ y_{n+1} = py_n/z_{n-1}y_{n-1}$, J. Inst. Math. Comp. Sci., 18 (2005), 135-136.

18. S. E. Das and M. Bayram, On a system of rational difference equations, World Applied Sciences Journal, 10(11) (2010), 1306–1312.
19. Q. Din, Dynamics of a discrete Lotka-Volterra model, Adv. Difference Equ., 2013, (2013): 95.
20. E. O. Alzahrani, M. M. El-Dessoky, E. M. Elsayed and Y. Kuang, Solutions and Properties of Some Degenerate Systems of Difference Equations, J. Comput. Anal. Appl., Vol. 18(2), (2015), 321-333.
21. A. Q. Khan, M. N. Qureshi, Global dynamics of some systems of rational difference equations, J. Egyptian Math. Soc., 24, (2016), 30-36.
22. E. M. Elabbasy, H. El-Metwally and E. M. Elsayed, Global behavior of the solutions of difference equation, Adv. Differ. Equ., 2011, 2011:28.
23. M. M. El-Dessoky, M. Mansour, E. M. Elsayed, Solutions of some rational systems of difference equations, Utilitas Mathematica, 92, (2013), 329-336.
24. M. M. El-Dessoky, On the solutions and periodicity of some nonlinear systems of difference equations, J. Nonlinear Sci. Appl., 9(5), (2016), 2190-2207.
25. E. M. Elsayed, A. M. Ahmed, Dynamics of a three-dimensional systems of rational difference equations, Math. Methods Appl. Sci., 39, (2016), 1026-1038.
26. N. Touafek and E. M. Elsayed, On a second order rational systems of difference equations, Hokkaido Math. J., 44, (2015), 29–45.
27. Y. Yazlik, D. T. Tollu, N. Taskara, On the Behaviour of Solutions for Some Systems of Difference Equations, J. Comput. Anal. Appl., 18 (1), (2015), 166-178.
28. Qianhong Zhang, Jingzhong Liu, and Zhenguo Luo, Dynamical Behavior of a System of Third-Order Rational Difference Equation, Discrete Dyn. Nat. Soc., 2015, (2015), Article ID 530453, 6 pages.
29. M. M. El-Dessoky, On a solvable for some systems of rational difference equations, J. Nonlinear Sci. Appl., Vol. 9(6), (2016), 3744-3759.
30. Wenqiang Ji, Decun Zhang and Liying Wang, Dynamics and behaviors of a third-order system of difference equation, Mathematical Sciences, 2013, (2013),:34.
31. Q. Zhang and W. Zhang, On a system of two high-order nonlinear difference equations, Adv. Math. Phys., 2014, (2014), Article ID 729273, 8 pages.
32. Mehmet Gümüş and Yüksel Soykan, Global Character of a Six-Dimensional Nonlinear System of Difference Equations, Discrete Dyn. Nat. Soc., 2016, (2016), Article ID 6842521, 7 pages.
33. M. M. El-Dessoky, Solution of a rational systems of difference equations of order three, Mathematics,4(3), (2016), 1-12.
34. A. Gelisken, On A System of Rational Difference Equations, J. Comput. Anal. Appl., Vol. 23(4), (2017), 593-606.
35. N. Haddad, N. Touafek, Julius Fergy T. Rabago, Solution form of a higher-order system of difference equations and dynamical behavior of its special case, Math. Methods Appl. Sci.,40(10), (2017), 3599-3607.
36. Chang-you Wang, Xiao-jing Fang, Rui Li, On the dynamics of a certain four-order fractional difference equations , J. Comput. Anal. Appl., Vol. 22(5), (2017), 968-976.
37. M. M. El-Dessoky, E. M. Elsayed and E. O. Alzahrani, The form of solutions and periodic nature for some rational difference equations systems, J. Nonlinear Sci. Appl., Vol., 9(10), (2016), 5629–5647.
38. M. M. El-Dessoky, Abdul Khaliq and Asim Asiri, On some rational systems of difference equations, J. Nonlinear Sci. Appl., Vol. 11(1), (2018), 49-72.
39. Asim Asiri, M. M. El-Dessoky and E. M. Elsayed, Solution of a third order fractional system of difference equations , J. Comput. Anal. Appl., Vol., 24(3), (2018), 444-453.

Hardy type inequalities for Choquet integrals

George A. Anastassiou
Department of Mathematical Sciences
University of Memphis
Memphis, TN 38152, U.S.A.
ganastss@memphis.edu

Abstract

Here we present Hardy type integral inequalities for Choquet integrals. These are very general inequalities involving convex and increasing functions. Initially we collect a rich machinery of results about Choquet integrals needed next, and we prove also results of their own merit such as, Choquet-Hölder's inequalities for more than two functions and a multivariate Choquet-Fubini's theorem. The main proving tool here is the property of comonotonicity of functions. We finish with independent estimates on left and right Riemann-Liouville-Choquet fractional integrals.

2010 AMS Mathematics Subject Classification: 26A33, 26D10 26D15, 26E50, 28E10.

Keywords and Phrases: Choquet integral, Hardy inequality, comonotonicity, fractional integral, convexity.

1 Introduction

To motivate the work in this article we mention the Riemann-Liouville fractional integrals, see [9]. Let $[a,b]$, $(-\infty < a < b < \infty)$ be a finite interval on the real axis $\mathbb{R}$. The left and right Riemann-Liouville fractional integrals $I_{a+}^{\alpha} f$ and $I_{b-}^{\alpha} f$ (respectively) of order $\alpha > 0$ are defined by

$$\left(I_{a+}^{\alpha} f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x f(t)(x-t)^{\alpha-1} dt, \quad (x > a),$$
$$\left(I_{b-}^{\alpha} f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_x^b f(t)(t-x)^{\alpha-1} dt, \quad (x < b),$$

where Γ is the Gamma function.

We mention a basic property of the operators $I_{a+}^{\alpha} f$ and $I_{b-}^{\alpha} f$ of order $\alpha > 0$, see also [11]: It holds that the fractional integral operators $I_{a+}^{\alpha} f$ and $I_{b-}^{\alpha} f$ are

1

bounded in $L_p(a,b)$, $1 \leq p \leq \infty$, that is

$$\left\| I_{a+}^{\alpha} f \right\|_p \leq K \|f\|_p, \quad \left\| I_{b-}^{\alpha} f \right\|_p \leq K \|f\|_p,$$

where

$$K = \frac{(b-a)^{\alpha}}{\alpha \Gamma(\alpha)}.$$

The first inequality that is the result involving the left-sided fractional integral, was proved by H.G. Hardy in one of his first papers, see [7]. He did not write down the constant, but the calculation of the constant was hidden inside his proof.

General Hardy inequalities of the above type were derived also in [8] and [1]. We continue this kind of research for Choquet integrals based on the comonotonicity property of functions and convexity. We derive a wide range of Choquet integral inequalities of Hardy type.

2 Background

In this section we give some definitions and basic properties of Choquet integral essential for this work.

Definition 1 *([15]) Let X be a non-empty set, $\mathcal{F}$ be a σ-algebra of subsets of X and $\mu : \mathcal{F} \to [0,\infty]$ be a nonnegative real-valued set function, μ is said to be a fuzzy measure iff:*

(1) $\mu(\varnothing) = 0$,

(2) for any $A, B \in \mathcal{F}$, $A \subseteq B$ implies $\mu(A) \leq \mu(B)$ (monotonicity),

(3) for $\{A_n\} \subseteq \mathcal{F}$, $A_1 \subseteq A_2 \subseteq ... \subseteq A_n \subseteq ...$, implies $\lim_{n\to\infty} \mu(A_n) = \mu(\cup_{n=1}^{\infty} A_n)$ (continuity from below)

(4) for $\{A_n\} \subseteq \mathcal{F}$, $A_1 \supseteq A_2 \supseteq ... \supseteq A_n \supseteq ...$, $\mu(A_1) < \infty$, implies $\lim_{n\to\infty} \mu(A_n) = \mu(\cap_{n=1}^{\infty} A_n)$ (continuity from above).

If μ is a fuzzy measure from $\mathcal{F}$ to $[0,1]$ with $\mu(X) = 1$, μ is called a regular fuzzy measure. If μ is a fuzzy measure, $(X, \mathcal{F}, \mu)$ is called a fuzzy measure space and $(X, \mathcal{F})$ is a fuzzy measurable space. Clearly μ is not necessarily an additive measure. Let F be the set of all real-valued nonnegative measurable functions defined on X.

Definition 2 *([10]) Let $(X, \mathcal{F}, \mu)$ be a fuzzy measure space, μ is said to be submodular (supermodular) if*

$$\mu(A \cap B) + \mu(A \cup B) \leq (\geq) \mu(A) + \mu(B), \quad \forall\ A, B \subseteq \mathcal{F}. \tag{1}$$

Definition 3 *([4]) Let $f, g \in F$, f and g are said to be comonotonic iff $f(x) < f(x')$ implies $g(x) \leq g(x')$, $\forall\ x, x' \in X$.*

Definition 4 *([5], [16]) Let $(X, \mathcal{F}, \mu)$ be a fuzzy measure space, $f \in F$ and $A \in \mathcal{F}$. The Choquet integral of f with respect to μ on A is defined by*

$$(C) \int_A f d\mu = \int_0^\infty \mu \left(A \cap \{x | f(x) \geq \alpha\} \right) d\alpha. \tag{2}$$

If $(C) \int_X f d\mu < \infty$, we call f (C)-integrable, $L_1(\mu)$ is the set of all (C)-integrable function.

Clearly $(C) \int_X f d\mu < \infty$, implies $(C) \int_A f d\mu < \infty$.

Theorem 5 *([14]) Let $(X, \mathcal{F}, \mu)$ be a fuzzy measurable space, $\{f_1, f_2, f\} \subset F$, $A, B \in \mathcal{F}$ and $c \geq 0$ constant. Then,*

(1) if $\mu(A) = 0$, then $(C) \int_A f d\mu = 0$,

(2) $(C) \int_A c d\mu = c\mu(A)$,

(3) if $f_1 \leq f_2$, then

$$(C) \int_A f_1 d\mu \leq (C) \int_A f_2 d\mu, \tag{3}$$

(4) if $A \subset B$, then $(C) \int_A f d\mu \leq (C) \int_B f d\mu$,

(5) $(C) \int_A (f + c) d\mu = (C) \int_A f d\mu + c\mu(A)$,

(6) $(C) \int_A c f d\mu = c \left((C) \int_A f d\mu \right)$.

Theorem 6 *([5]) Let $(X, \mathcal{F}, \mu)$ be a fuzzy measure space and $f, g \in F$. Then*

(1) if f, g are comonotonic, then for any $A \in \mathcal{F}$,

$$(C) \int_A (f + g) d\mu = (C) \int_A f d\mu + (C) \int_A g d\mu, \tag{4}$$

(2) if μ is submodular, then for any $A \in \mathcal{F}$,

$$(C) \int_A (f + g) d\mu \leq (C) \int_A f d\mu + (C) \int_A g d\mu. \tag{5}$$

The Jensen's inequality for Choquet integrals follows:

Theorem 7 *([13]) Let $(X, \mathcal{F}, \mu)$ be a fuzzy measure space and $f \in L_1(\mu)$. If μ is a regular fuzzy measure and $\Phi : [0, \infty) \to [0, \infty)$ is a convex function, then*

$$\Phi \left((C) \int_X f d\mu \right) \leq (C) \int_X \Phi(f) d\mu. \tag{6}$$

Corollary 8 *([13]) Let $(X, \mathcal{F}, \mu)$ be a fuzzy measure space and $f \in L_1(\mu)$. If μ is a regular fuzzy measure, then*

$$\left((C) \int_X f d\mu \right)^p \leq (C) \int_X f^p d\mu, \tag{7}$$

for any $1 < p < \infty$.

Theorem 9 *([13]) (Hölder's inequality) Let $(X, \mathcal{F}, \mu)$ be a fuzzy measure space and $f, g \in F$. If μ is a submodular fuzzy measure and $1 < p, q < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$, then*

$$(C)\int_X fg d\mu \leq \left((C)\int_X f^p d\mu\right)^{\frac{1}{p}} \left((C)\int_X g^q d\mu\right)^{\frac{1}{q}}. \tag{8}$$

Theorem 10 *([13]) (Minkowski inequality) Let $(X, \mathcal{F}, \mu)$ be a fuzzy measure space and $f, g \in F$. If μ is a submodular fuzzy measure and $1 \leq p < \infty$, then*

$$\left((C)\int_X (f+g)^p d\mu\right)^{\frac{1}{p}} \leq \left((C)\int_X f^p d\mu\right)^{\frac{1}{p}} + \left((C)\int_X g^p d\mu\right)^{\frac{1}{p}}. \tag{9}$$

We give

Theorem 11 *(Hölder's inequality for three functions) Let $(X, \mathcal{F}, \mu)$ be a fuzzy measure space and $f_1, f_2, f_3 \in F$. If μ is a submodular fuzzy measure and $1 < p_1 \leq p_2 \leq p_3 < \infty$ with $\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = 1$, then*

$$(C)\int_X f_1 f_2 f_3 d\mu \leq \left((C)\int_X f_1^{p_1} d\mu\right)^{\frac{1}{p_1}} \left((C)\int_X f_2^{p_2} d\mu\right)^{\frac{1}{p_2}} \left((C)\int_X f_3^{p_3} d\mu\right)^{\frac{1}{p_3}}. \tag{10}$$

Proof. Let $p = \frac{p_3}{p_3 - 1} > 1$ and $q = p_3$. Notice that $\frac{1}{p} + \frac{1}{q} = 1$.

We apply (8) as follows

$$(C)\int_X f_1 f_2 f_3 d\mu \leq \left((C)\int_X (f_1 f_2)^p d\mu\right)^{\frac{1}{p}} \left((C)\int_X f_3^{p_3} d\mu\right)^{\frac{1}{p_3}}. \tag{11}$$

We see that

$$\frac{p}{p_1} + \frac{p}{p_2} = p\left(\frac{1}{p_1} + \frac{1}{p_2}\right) = p\left(1 - \frac{1}{p_3}\right) = p\left(\frac{p_3 - 1}{p_3}\right) = 1. \tag{12}$$

Clearly it holds $\frac{p_1}{p}, \frac{p_2}{p} > 1$.

Therefore we get

$$(C)\int_X f_1^p f_2^p d\mu \overset{(8)}{\leq} \left((C)\int_X f_1^{p\frac{p_1}{p}} d\mu\right)^{\frac{p}{p_1}} \left((C)\int_X f_2^{p\frac{p_2}{p}} d\mu\right)^{\frac{p}{p_2}} = \tag{13}$$

$$\left((C)\int_X f_1^{p_1} d\mu\right)^{\frac{p}{p_1}} \left((C)\int_X f_2^{p_2} d\mu\right)^{\frac{p}{p_2}}.$$

That is

$$\left((C)\int_X (f_1 f_2)^p d\mu\right)^{\frac{1}{p}} \leq \left((C)\int_X f_1^{p_1} d\mu\right)^{\frac{1}{p_1}} \left((C)\int_X f_2^{p_2} d\mu\right)^{\frac{1}{p_2}}. \tag{14}$$

Combining (11) and (14), we produce (10). ■

In general we have

4

Theorem 12 *(Hölder's inequality for n functions) Let $(X, \mathcal{F}, \mu)$ be a fuzzy measure space and $f_i \in F$, $i = 1, ..., n \in \mathbb{N}$. If μ is a submodular fuzzy measure and $1 < p_1 \leq p_2 \leq ... \leq p_n < \infty$ with $\sum_{i=1}^{n} \frac{1}{p_i} = 1$, then*

$$(C) \int_X \prod_{i=1}^{n} f_i d\mu \leq \prod_{i=1}^{n} \left((C) \int_X f_i^{p_i} d\mu \right)^{\frac{1}{p_i}} . \tag{15}$$

Proof. By induction. ■

Remark 13 *Let $\mathcal{A}$ be a σ-algebra, and let $\{A_k\}_{k \in \mathbb{N}} \subseteq \mathcal{A}$ be a family of pairwise disjoint sets. Here $\mathcal{P}$ is a probability measure on $(X, \mathcal{A})$ with only the finite additivity property valid: i.e.,*

$$P\left(\cup_{k=1}^{n} A_k\right) = \sum_{k=1}^{n} P\left(A_k\right), \quad \forall\ n \in \mathbb{N}.$$

We observe that

$$P\left(\cup_{k=1}^{\infty} A_k\right) = \lim_{n \to \infty} P\left(\cup_{k=1}^{n} A_k\right) = \lim_{n \to \infty} \sum_{k=1}^{n} P\left(A_k\right) = \sum_{k=1}^{\infty} P\left(A_k\right). \tag{16}$$

That is, the countable additivity property holds, hence P is a usual probability measure.

Notice that a σ-algebra on X is also an algebra of subsets of X.

Definition 14 *([3], [6]) For every space Ω and algebra $\mathcal{A}$ of subsets of Ω a set-function $\sigma : \mathcal{A} \to \mathbb{R}$ is called a (normalized) capacity if it satisfies the following:*

(i)

$$\sigma(\varnothing) = 0, \quad \sigma(\Omega) = 1, \tag{17}$$

(ii) $\forall\ A, B \in \mathcal{A} : A \subseteq B \Rightarrow \sigma(A) \leq \sigma(B)$.

From (i) and (ii) we get that the range of σ is contained in $[0, 1]$.

In general the Choquet integral is defined as follows:

Definition 15 *([3], [12]) Let $(\Omega, \mathcal{A})$ be an algebra and $f : \Omega \to \mathbb{R}$ is a bounded $\mathcal{A}$-measurable function and σ is any (normalized) capacity on Ω we define the Choquet integral of f with respect to σ to be the number*

$$(C) \int_\Omega f(\omega) \, d\sigma(\omega) = \int_0^\infty \sigma\left(\{\omega \in \Omega : f(\omega) \geq \alpha\}\right) d\alpha + \tag{18}$$

$$\int_{-\infty}^{0} \left[\sigma\left(\omega \in \Omega : f(\omega) \geq \alpha\}\right) - 1\right] d\alpha,$$

5

where the integrals are taken in the sense of Riemann.

A (normalized) capacity σ is called probability ([6]) iff

$$\forall\ A, B \in \mathcal{A} : \sigma(A \cup B) + \sigma(A \cap B) = \sigma(A) + \sigma(B). \tag{19}$$

Notice that since the integrands are monotone, the Choquet integral always exists, and if σ is a probability it collapses to a usual Lebesgue integral.

Definition 16 *([6]) Let $f, g : \Omega \to \mathbb{R}$ be two bounded $\mathcal{A}$-measurable functions. We say that f and g are comonotonic, if for every $\omega, \omega' \in \Omega$,*

$$(f(\omega) - f(\omega'))(g(\omega) - g(\omega')) \geq 0. \tag{20}$$

A class of functions $\mathcal{F}^*$ is said to be comonotonic if for every $f, g \in \mathcal{F}^*$, f and g are comonotonic.

Proposition 17 *([6]) If σ and λ are (normalized) capacities on the algebra $(\Omega, \mathcal{A})$, and $f, g : \Omega \to \mathbb{R}$ are bounded $\mathcal{A}$-measurable functions then:*

(i)

$$(C)\int_{\Omega} 1_A d\sigma = \sigma(A),\ \ \forall\ A \in \mathcal{A}, \tag{21}$$

where 1_A is the characteristic function on A,

(ii) (positive homogeneity)

$$(C)\int_{\Omega} pf d\sigma = p\left((C)\int_{\Omega} f d\sigma\right),\ \ \text{for every } p \geq 0, \tag{22}$$

(iii) (monotonicity) $f \geq g$ implies

$$(C)\int_{\Omega} f d\sigma \geq (C)\int_{\Omega} g d\sigma, \tag{23}$$

(iv)

$$(C)\int_{\Omega} (f + p) d\sigma = (C)\int_{\Omega} f d\sigma + p,\ \ \forall\ p \in \mathbb{R}, \tag{24}$$

(v) (comonotonic additivity) If f, g are comonotonic then

$$(C)\int_{\Omega} (f + g) d\sigma = (C)\int_{\Omega} f d\sigma + (C)\int_{\Omega} g d\sigma. \tag{25}$$

We need the very important

Lemma 18 *([6]) Let $(\Omega, \mathcal{A})$ be an algebra. Suppose that $\mathcal{F}^*$ is a comonotonic class of bounded and $\mathcal{A}$-measurable functions from Ω into $\mathbb{R}$ and σ is a (normalized) capacity on $(\Omega, \mathcal{A})$. Then there exists a probability measure P on $(\Omega, \mathcal{A})$ such that for every $f \in \mathcal{F}^*$*

$$\int_{\Omega} f d\sigma = \int_{\Omega} f dP. \tag{26}$$

Here $\int_{\Omega} f dP$ is a standard integral of Lebesgue type.

Based on Remark 13, Lemma 18 is still valid in case that $(\Omega, \mathcal{A})$ is a σ-algebra.

Definition 19 *([6]) Let X, Y be two sets and $Z = X \times Y$. Let $f : Z \to \mathbb{R}$. We say that f has comonotonic x-sections if for every $x, x' \in X$, $f(x, \cdot) : Y \to \mathbb{R}$, and $f(x', \cdot) : Y \to \mathbb{R}$ are comonotonic functions. Comonotonicity of y-sections is similarly defined. We call f slice-comonotonic if it has both comonotonic x-sections and y-sections.*

Remark 20 *Notice that Definitions 14-16 and Proposition 17, are still valid when $(\Omega, \mathcal{A})$ is a σ-algebra.*

Next we mention Fubini's theorem for Choquet integrals.

Theorem 21 *([2]) Let (Ω_1, Σ_1), (Ω_2, Σ_2) be σ-algebras. Let u_i, $i = 1, 2$ be submodular (or supermodular) regular fuzzy measures on Ω_i, respectively. Let $\Omega = \Omega_1 \times \Omega_2$ be endowed with the product σ-algebra $\Sigma = \Sigma_1 \otimes \Sigma_2$. Let $f : \Omega_1 \times \Omega_2 \to \mathbb{R}$ be a slice-comonotonic bounded Σ-measurable mapping, then:*

1) $f(\cdot, \omega_2)$ is Σ_1-measurable and $\omega_2 \in \Omega_2 \to (C)\int_{\Omega_1} f(s, \omega_2)\, du_1(s)$ is bounded and Σ_2-measurable,

$f(\omega_1, \cdot)$ is Σ_2-measurable and $\omega_1 \in \Omega_1 \to (C)\int_{\Omega_2} f(\omega_1, t)\, du_2(t)$ is bounded and Σ_1-measurable,

2) the iterated integrals $(C)\int_{\Omega_2}\int_{\Omega_1} f du_1 du_2$, $(C)\int_{\Omega_1}\int_{\Omega_2} f du_2 du_1$ exist and are equal:

$$(C)\int_{\Omega_2}\left((C)\int_{\Omega_1} f(\omega_1, \omega_2)\, du_1\right) du_2 = (C)\int_{\Omega_1}\left((C)\int_{\Omega_2} f(\omega_1, \omega_2)\, du_2\right) du_1. \tag{27}$$

We give

Definition 22 *Let $f : \prod_{i=1}^{n} \Omega_i \to \mathbb{R}$, $n \in \mathbb{N}$. If the i-sections $f(x_1, ..., x_{i-1}, \cdot, x_{i+1}, ..., x_n)$ and $f(x_1', ..., x_{i-1}', \cdot, x_{i+1}', ..., x_n')$ are comonotonic functions, for all $i = 1, ..., n$; where the vectors $(x_1, ..., x_{i-1}, x_{i+1}, ..., x_n)$, $(x_1', ..., x_{i-1}', x_{i+1}', ..., x_n') \in \prod_{\substack{j=1 \\ j \neq i}}^{n-1} \Omega_j$ are different, for all $i = 1, 2, ..., n$, we call f slice-n-comonotonic function.*

We denote by θ a permutation of the set $\{1, 2, ..., n\}$ into itself, $n \in \mathbb{N}$. There are $n!$ permutations.

In [2] is mentioned that Theorem 21 can be generalized for n spaces. Next we state in brief Fubini's theorem for n Choquet iterated integrals.

7

Theorem 23 *Let (Ω_i, Σ_i) be σ-algebras, $i = 1, 2, ..., n \in \mathbb{N}$. Let u_i, $i = 1, 2, ..., n$ be submodular (or supermodular) regular fuzzy measures on Ω_i, respectively. Let $\Omega = \prod_{i=1}^{n} \Omega_i$ be endowed with the product σ-algebra $\Sigma = \otimes_{i=1}^{n} \Sigma_i$. Let $f : \prod_{i=1}^{n} \Omega_i \to \mathbb{R}$ be a slice-comonotonic bounded Σ-measurable mapping, then*

$$(C) \int_{\Omega_n} \int_{\Omega_{n-1}} ... \int_{\Omega_1} f du_1 du_2 ... du_n =$$

$$(C) \int_{\Omega_{\theta(n)}} \int_{\Omega_{\theta(n-1)}} ... \int_{\Omega_{\theta(1)}} f du_{\theta(1)} du_{\theta(2)} ... du_{\theta(n)}, \tag{28}$$

for any permutation θ on the set $\{1, ..., n\}$. All the iterated Choquet integrals in (28) exist and are equal.

Proof. By induction, (23) and using Theorem 21. ■

Remark 24 *If μ is a countably additive bounded measure, then the Choquet integral $(C) \int_A f d\mu$ reduces to the usual Lebesgue type integral (see, e.g. [5], p. 62, or [17], p. 226), above it is $A \subseteq \Omega$.*

3 Main Results

This section is motivated by [8].

Let the fuzzy measure spaces $(\Omega_1, \Sigma_1, \mu_1)$ and $(\Omega_2, \Sigma_2, \mu_2)$, where μ_1, μ_2 are regular fuzzy measures, furthermore μ_1, μ_2 are assumed to be submodular.

Let $k : \Omega_1 \times \Omega_2 \to \mathbb{R}_+$ which is a bounded measurable function and $k(x, y)$ is $slice-$comonotonic and belongs to a comonotonic class F_1^* $as\ a\ function\ of\ y$.

Consider the function

$$K(x) = (C) \int_{\Omega_2} k(x, y) d\mu_2(y), \quad x \in \Omega_1, \tag{29}$$

and assume that $K(x) > 0$.

Notice that K is bounded.

Denote by $W(k)$ the class of functions $g : \Omega_1 \to \mathbb{R}_+$, such that

$$g(x) = (C) \int_{\Omega_2} k(x, y) f(y) d\mu_2(y), \tag{30}$$

where $f : \Omega_2 \to \mathbb{R}_+$ is a bounded measurable function, such that $k(x, y) f(y)$ is $slice-$comonotonic and belongs to a comonotonic class F_2^* $as\ a\ function\ of\ y$.

Notice that g is also bounded.

We give

Theorem 25 *Let u be a nonnegative measurable function on Ω_1. Assume that $\frac{u(x)}{K(x)}$ is bounded on Ω_1. Define υ on Ω_2 by*

$$\upsilon(y) = (C)\int_{\Omega_1} \frac{u(x)}{K(x)} k(x,y)\, d\mu_1(x), \tag{31}$$

which is bounded. Let $\Phi : \mathbb{R}_+ \to \mathbb{R}_+$ be a convex and increasing function, such that $k(x,y)\Phi(f(y))$ is x-section comonotonic with comonotonic class F_3^*. Assume here that $(F_1^* \cup F_2^* \cup F_3^*) \subseteq F^*$, where F^* is one comonotonic class of functions on Ω_2. Assume further that $u(x)(K(x))^{-1} k(x,y)\Phi(f(y))$ is slice-comonotonic. Then

$$(C)\int_{\Omega_1} u(x)\Phi\left(\frac{g(x)}{K(x)}\right) d\mu_1(x) \leq (C)\int_{\Omega_2} \upsilon(y)\Phi(f(y))\, d\mu_2(y), \tag{32}$$

holds for all $g \in W(k)$, with f as in (30).

Proof. We observe that

$$(C)\int_{\Omega_1} u(x)\Phi\left(\frac{g(x)}{K(x)}\right) d\mu_1(x) =$$

$$(C)\int_{\Omega_1} u(x)\Phi\left(\frac{1}{K(x)}(C)\int_{\Omega_2} k(x,y) f(y)\, d\mu_2(y)\right) d\mu_1(x) = \tag{33}$$

(next we use Lemma 18, where P is a probability measure on Ω_2)

$$(C)\int_{\Omega_1} u(x)\Phi\left(\frac{1}{K(x)}(C)\int_{\Omega_2} k(x,y) f(y)\, dP(y)\right) d\mu_1(x) \leq$$

(we can also write $K(x) = \int_{\Omega_2} k(x,y)\, dP(y)$, hence by classic Jensen's inequality)

$$(C)\int_{\Omega_1} u(x)(K(x))^{-1}\left((C)\int_{\Omega_2} k(x,y)\Phi(f(y))\, dP(y)\right) d\mu_1(x) = \tag{34}$$

(again by Lemma 18)

$$(C)\int_{\Omega_1} u(x)(K(x))^{-1}\left((C)\int_{\Omega_2} k(x,y)\Phi(f(y))\, d\mu_2(y)\right) d\mu_1(x) =$$

$$(C)\int_{\Omega_1}\left((C)\int_{\Omega_2} u(x)(K(x))^{-1} k(x,y)\Phi(f(y))\, d\mu_2(y)\right) d\mu_1(x) =$$

(since the functions $\Phi(f(y))$ and $u(x)(K(x))^{-1} k(x,y)\Phi(f(y))$ are bounded and the second one is slice-comonotonic, we can apply Fubini's Theorem 21)

$$(C)\int_{\Omega_2}\left((C)\int_{\Omega_1} u(x)(K(x))^{-1} k(x,y)\Phi(f(y))\, d\mu_1(x)\right) d\mu_2(y) =$$

$$(C)\int_{\Omega_2}\Phi(f(y))\left((C)\int_{\Omega_1}u(x)(K(x))^{-1}k(x,y)\,d\mu_1(x)\right)d\mu_2(y)\overset{(31)}{=} \tag{35}$$
$$(C)\int_{\Omega_2}\Phi(f(y))\,\upsilon(y)\,d\mu_2(y),$$

proving the claim. ■

We also give

Corollary 26 *All as in Theorem 25, with $\Phi =$ identity mapping. Then*

$$(C)\int_{\Omega_1}\frac{u(x)}{K(x)}g(x)\,d\mu_1(x)\leq(C)\int_{\Omega_2}\upsilon(y)\,f(y)\,d\mu_2(y), \tag{36}$$

holds for all $g\in W(k)$, with f as in (30).

Corollary 27 *All as in Theorem 25, with $\Phi(x)=x^p$, $\forall\ x\in\mathbb{R}_+$, $p>1$. Then*

$$(C)\int_{\Omega_1}\frac{u(x)}{K^p(x)}g^p(x)\,d\mu_1(x)\leq(C)\int_{\Omega_2}\upsilon(y)\,f^p(y)\,d\mu_2(y), \tag{37}$$

holds for all $g\in W(k)$, with f as in (30).

Corollary 28 *All as in Theorem 25, with $\Phi(x)=e^x$, $\forall\ x\in\mathbb{R}_+$. Then*

$$(C)\int_{\Omega_1}u(x)\,e^{\frac{g(x)}{K(x)}}d\mu_1(x)\leq(C)\int_{\Omega_2}\upsilon(y)\,e^{f(y)}d\mu_2(y), \tag{38}$$

holds for all $g\in W(k)$, with f as in (30).

Corollary 29 *All as in Theorem 25, with $\Phi =$ identity mapping and $u(x)=K(x)$. Then*

$$(C)\int_{\Omega_1}g(x)\,d\mu_1(x)\leq(C)\int_{\Omega_2}\upsilon(y)\,f(y)\,d\mu_2(y), \tag{39}$$

holds for all $g\in W(k)$, with f as in (30). Here $\upsilon(y)=(C)\int_{\Omega_1}k(x,y)\,d\mu_1(x)$ is bounded.

Corollary 30 *All as in Theorem 25, with $\Phi(x)=x^p$, $\forall\ x\in\mathbb{R}_+$, $p>1$, and $u(x)=K^p(x)$. Then*

$$(C)\int_{\Omega_1}g^p(x)\,d\mu_1(x)\leq(C)\int_{\Omega_2}\upsilon(y)\,f^p(y)\,d\mu_2(y), \tag{40}$$

holds for all $g\in W(k)$, with f as in (30). Here

$$\upsilon(y)=(C)\int_{\Omega_1}K^{p-1}(x)\,k(x,y)\,d\mu_1(x)\ \text{ is bounded.} \tag{41}$$

10

Remark 31 *(on Corollary 30) Let us assume that* $k(x,y) \leq M$, $M > 0$, $\forall$ $(x,y) \in \Omega_1 \times \Omega_2$, *then* $K(x) \leq M$. *And from (41),* $\upsilon(y) \leq M^p$.

Consequently, from (40), it holds

$$(C)\int_{\Omega_1} g^p(x)\,d\mu_1(x) \leq M^p \left((C)\int_{\Omega_2} f^p(y)\,d\mu_2(y)\right), \tag{42}$$

and even better written

$$\left((C)\int_{\Omega_1} g^p(x)\,d\mu_1(x)\right)^{\frac{1}{p}} \leq M\left((C)\int_{\Omega_2} f^p(y)\,d\mu_2(y)\right)^{\frac{1}{p}}. \tag{43}$$

Next we rewrite the result of (43) in detail.

Theorem 32 *Assume that* $k(x,y) \leq M$, $M > 0$, $\forall$ $(x,y) \in \Omega_1 \times \Omega_2$, *and let* $p > 1$. *Define*

$$\upsilon(y) = (C)\int_{\Omega_1} K^{p-1}(x)\,k(x,y)\,d\mu_1(x), \tag{44}$$

which is bounded. Here $k(x,y)(f(y))^p$ *is* x*-section comonotonic with comonotonic class* F_3^*. *Assume that* $(F_1^* \cup F_2^* \cup F_3^*) \subseteq F^*$, *where* F^* *one comonotonic class on* Ω_2. *Assume further that* $(K(x))^{p-1}k(x,y)(f(y))^p$ *is slice-comonotonic. Then*

$$\left((C)\int_{\Omega_1} g^p(x)\,d\mu_1(x)\right)^{\frac{1}{p}} \leq M\left((C)\int_{\Omega_2} f^p(y)\,d\mu_2(y)\right)^{\frac{1}{p}}, \tag{45}$$

holds for all $g \in W(k)$, *with* f *as in (30).*

Remark 33 *Assume that* $k(x,y) \leq M$, $M > 0$, $\forall$ $(x,y) \in \Omega_1 \times \Omega_2$. *Hence directly by (30) we get*

$$g(x) \leq M\left((C)\int_{\Omega_2} f(y)\,d\mu_2(y)\right), \quad \forall\ x \in \Omega_1.$$

Therefore

$$\int_{\Omega_1} g(x)\,d\mu_1(x) \leq M\left((C)\int_{\Omega_2} f(y)\,d\mu_2(y)\right), \tag{46}$$

holds for all $g \in W(k)$, *with* f *as in (30).*

Theorem 34 *Define* υ *on* Ω_2 *by* $\upsilon(y) = (C)\int_{\Omega_1} k(x,y)\,d\mu_1(x)$, *which is bounded. Let* $p > 1$. *Here* $k(x,y)(f(y))^p$ *is slice−comonotonic and belongs to a comonotonic class* F_3^* *as a function of* y. *Assume that* $(F_1^* \cup F_2^* \cup F_3^*) \subseteq F^*$, *where* F^* *one comonotonic class on* Ω_2. *Then*

$$(C)\int_{\Omega_1} (K(x))^{1-p}\,g^p(x)\,d\mu_1(x) \leq (C)\int_{\Omega_2} \upsilon(y)\,f^p(y)\,d\mu_2(y), \tag{47}$$

holds for all $g \in W(k)$, *with* f *as in (30).*

Proof. By Theorem 25, take $f(x) = x^p$, $x \geq 0$, $p > 1$, and $u(x) = K(x)$. ■

Corollary 35 *All as in Theorem 34. Then*

$$\left((C)\int_{\Omega_1} g^p(x)\,d\mu_1(x)\right)^{\frac{1}{p}} \leq M\left((C)\int_{\Omega_2} f^p(y)\,d\mu_2(y)\right)^{\frac{1}{p}}. \tag{48}$$

holds for all $g \in W(k)$, with f as in (30). Here $k(x,y) \leq M$, $M > 0$, $\forall$ $(x,y) \in \Omega_1 \times \Omega_2$.

Proof. Since $p > 1$, $1 - p < 0$. Hence the left hand side of (47) is greater equal to $M^{1-p}\left((C)\int_{\Omega_1} g^p(x)\,d\mu_1(x)\right)$, by $K(x) \leq M$ and $(K(x))^{1-p} \geq M^{1-p}$. And the right hand side of (47) is less equal to $M\left((C)\int_{\Omega_2} f^p(y)\,d\mu_2(y)\right)$, by $v(y) \leq M$. Therefore

$$M^{1-p}\left((C)\int_{\Omega_1} g^p(x)\,d\mu_1(x)\right) \leq M\left((C)\int_{\Omega_2} f^p(y)\,d\mu_2(y)\right), \tag{49}$$

proving the claim. ■

4 Appendix

Here $\mathcal{B}$ stands for the Borel σ-algebra on $[a,b]$.

Let the fuzzy measure spaces $([a,b], \mathcal{B}, \mu_1)$ and $([a,b], \mathcal{B}, \mu_2)$, where $[a,b] \subset \mathbb{R}$ and μ_1, μ_2 are bounded fuzzy measures with μ_2 submodular. Let $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Let $f : [a,b] \to \mathbb{R}_+$ which is bounded and $\mathcal{B}$-measurable.

We define the left and right Riemann-Liouville-Choquet fractional integrals of order $\alpha > 1$ (respectively):

$$\left(I_{a+}^{\alpha} f\right)(x) = \frac{1}{\Gamma(\alpha)}(C)\int_a^x (x-t)^{\alpha-1} f(t)\,d\mu_2(t), \tag{50}$$

and

$$\left(I_{b-}^{\alpha} f\right)(x) = \frac{1}{\Gamma(\alpha)}(C)\int_x^b (t-x)^{\alpha-1} f(t)\,d\mu_2(t), \tag{51}$$

$\forall$ $x \in [a,b]$, where Γ is the gamma function.

We assume that $\left(I_{a+}^{\alpha} f\right)$ and $\left(I_{b-}^{\alpha} f\right)$ are $\mathcal{B}$-measurable functions. Clearly $I_{a+}^{\alpha} f, I_{b-}^{\alpha} f$ are nonnegative and bounded over $[a,b]$.

Remark 36 *By Theorem 9 we obtain*

$$\left(I_{a+}^{\alpha} f\right)(x) \leq \frac{1}{\Gamma(\alpha)}\left((C)\int_a^x (x-t)^{p(\alpha-1)}\,d\mu_2(t)\right)^{\frac{1}{p}}\left((C)\int_a^x f^q(t)\,d\mu_2(t)\right)^{\frac{1}{q}} \leq \tag{52}$$

$$\frac{1}{\Gamma(\alpha)}\left((b-a)^{p(\alpha-1)}\mu_2([a,b])\right)^{\frac{1}{p}}\left((C)\int_a^b f^q(t)\,d\mu_2(t)\right)^{\frac{1}{q}}.$$

Hence it holds

$$\left(\left(I_{a+}^{\alpha}f\right)(x)\right)^p \leq \frac{1}{(\Gamma(\alpha))^p}(b-a)^{p(\alpha-1)}\mu_2([a,b])\left((C)\int_a^b f^q(t)\,d\mu_2(t)\right)^{\frac{p}{q}}, \tag{53}$$

$\forall\ x \in [a,b]$.

Therefore

$$(C)\int_a^b \left(\left(I_{a+}^{\alpha}f\right)(x)\right)^p d\mu_1(x) \leq$$

$$\frac{\mu_1([a,b])}{(\Gamma(\alpha))^p}(b-a)^{p(\alpha-1)}\mu_2([a,b])\left((C)\int_a^b f^q(t)\,d\mu_2(t)\right)^{\frac{p}{q}}. \tag{54}$$

We have proved that

$$\left((C)\int_a^b \left(\left(I_{a+}^{\alpha}f\right)(x)\right)^p d\mu_1(x)\right)^{\frac{1}{p}} \leq$$

$$\frac{\left(\mu_1([a,b])\,\mu_2([a,b])\right)^{\frac{1}{p}}(b-a)^{(\alpha-1)}}{\Gamma(\alpha)}\left((C)\int_a^b f^q(t)\,d\mu_2(t)\right)^{\frac{1}{q}}. \tag{55}$$

Similarly, we have

$$\left(I_{b-}^{\alpha}f\right)(x) \overset{(8)}{\leq} \frac{1}{\Gamma(\alpha)}\left((C)\int_x^b (t-x)^{p(\alpha-1)}d\mu_2(t)\right)^{\frac{1}{p}}\left((C)\int_x^b f^q(t)\,d\mu_2(t)\right)^{\frac{1}{q}} \tag{56}$$

$$\leq \frac{1}{\Gamma(\alpha)}\left((b-a)^{p(\alpha-1)}\mu_2([a,b])\right)^{\frac{1}{p}}\left((C)\int_a^b f^q(t)\,d\mu_2(t)\right)^{\frac{1}{q}}.$$

As before we obtain

$$\left((C)\int_a^b \left(\left(I_{b-}^{\alpha}f\right)(x)\right)^p d\mu_1(x)\right)^{\frac{1}{p}} \leq$$

$$\frac{\left(\mu_1([a,b])\,\mu_2([a,b])\right)^{\frac{1}{p}}(b-a)^{(\alpha-1)}}{\Gamma(\alpha)}\left((C)\int_a^b f^q(t)\,d\mu_2(t)\right)^{\frac{1}{q}}. \tag{57}$$

We have proved

13

Theorem 37 *Here $\alpha > 1$ and the rest are as in this section. It holds*

$$\max\left\{\left((C)\int_a^b \left(\left(I_{a+}^{\alpha}f\right)(x)\right)^p d\mu_1(x)\right)^{\frac{1}{p}}, \left((C)\int_a^b \left(\left(I_{b-}^{\alpha}f\right)(x)\right)^p d\mu_1(x)\right)^{\frac{1}{p}}\right\}$$

$$\leq \frac{\left(\mu_1([a,b])\,\mu_2([a,b])\right)^{\frac{1}{p}}(b-a)^{(\alpha-1)}}{\Gamma(\alpha)}\left((C)\int_a^b f^q(t)\,d\mu_2(t)\right)^{\frac{1}{q}}. \tag{58}$$

Remark 38 *From (52) we get*

$$\left(I_{a+}^{\alpha}f\right)(x) \leq \frac{1}{\Gamma(\alpha)}\left((x-a)^{p(\alpha-1)}\mu_2([a,x])\right)^{\frac{1}{p}}\left((C)\int_a^b f^q(t)\,d\mu_2(t)\right)^{\frac{1}{q}}, \tag{59}$$

and from (56) we derive (by exchanging the roles of p and q)

$$\left(I_{b-}^{\alpha}f\right)(x) \leq \frac{1}{\Gamma(\alpha)}\left((b-x)^{q(\alpha-1)}\mu_2([x,b])\right)^{\frac{1}{q}}\left((C)\int_a^b f^p(t)\,d\mu_2(t)\right)^{\frac{1}{p}}. \tag{60}$$

Therefore by multiplying (59), (60) we get

$$\left(I_{a+}^{\alpha}f\right)(x)\left(I_{b-}^{\alpha}f\right)(x) \leq \frac{1}{(\Gamma(\alpha))^2}\left((x-a)^{p(\alpha-1)}\mu_2([a,x])\right)^{\frac{1}{p}}\cdot \tag{61}$$

$$\left((b-x)^{q(\alpha-1)}\mu_2([x,b])\right)^{\frac{1}{q}}\left((C)\int_a^b f^q(t)\,d\mu_2(t)\right)^{\frac{1}{q}}\left((C)\int_a^b f^p(t)\,d\mu_2(t)\right)^{\frac{1}{p}}$$

(using Young's inequality for $a,b \geq 0$, $a^{\frac{1}{p}}b^{\frac{1}{q}} \leq \frac{a}{p}+\frac{b}{q}$)

$$\leq \frac{1}{(\Gamma(\alpha))^2}\left(\frac{(x-a)^{p(\alpha-1)}\mu_2([a,x])}{p}+\frac{(b-x)^{q(\alpha-1)}\mu_2([x,b])}{q}\right)\cdot$$

$$\left((C)\int_a^b f^q(t)\,d\mu_2(t)\right)^{\frac{1}{q}}\left((C)\int_a^b f^p(t)\,d\mu_2(t)\right)^{\frac{1}{p}}. \tag{62}$$

We have that

$$\frac{\left(I_{a+}^{\alpha}f\right)(x)\left(I_{b-}^{\alpha}f\right)(x)}{\left[\frac{(x-a)^{p(\alpha-1)}\mu_2([a,x])}{p}+\frac{(b-x)^{q(\alpha-1)}\mu_2([x,b])}{q}\right]} \leq$$

$$\frac{1}{(\Gamma(\alpha))^2}\left((C)\int_a^b f^q(t)\,d\mu_2(t)\right)^{\frac{1}{q}}\left((C)\int_a^b f^p(t)\,d\mu_2(t)\right)^{\frac{1}{p}}. \tag{63}$$

Notice that the denominator of left hand side of (63) is never zero.

14

Integrating (63) with respect to x we obtain:

Theorem 39 *Here $\alpha > 1$ and the rest are as in this section. It holds*

$$(C)\int_a^b \frac{\left(I_{a+}^{\alpha} f\right)(x)\left(I_{b-}^{\alpha} f\right)(x)\, d\mu_1(x)}{\left[\frac{(x-a)^{p(\alpha-1)}\mu_2([a,x])}{p} + \frac{(b-x)^{q(\alpha-1)}\mu_2([x,b])}{q}\right]} \leq$$

$$\frac{\mu_1([a,b])}{(\Gamma(\alpha))^2}\left((C)\int_a^b f^p(t)\, d\mu_2(t)\right)^{\frac{1}{p}}\left((C)\int_a^b f^q(t)\, d\mu_2(t)\right)^{\frac{1}{q}}. \tag{64}$$

Inequality (64) is a Hilbert-Pachpatte type inequality for Choquet fractional integrals.

References

[1] G. Anastassiou, *Intelligent Comparisons: Analytic Inequalities*, Springer, Heidelberg, New York, 2016.

[2] A. Chateauneuf, J.P. Lefort, *Some Fubini theorems on product sigma-algebras for non-additive measures*, Internat. J. Approx. Reason, 48 (2008), no. 3, 686-696.

[3] G. Choquet, *Theory of capacities*, Ann. Inst. Fourier, 5 (1953), 131-295.

[4] L.M. de Campos, M.J. Bolanos, *Characterization and comparison of Sugeno and Choquet integrals*, Fuzzy Sets Syst., 52 (1992), 61-67.

[5] D. Denneberg, *Nonadditive Measure and Integral*, Kluwer Academic, Dordrecht, 1994.

[6] P. Ghirardato, *On independence for non-additive measures, with a Fubini theorem*, J. Economic Theory, 73 (1997), 261-291.

[7] H.G. Hardy, *Notes on some points in the integral calculus*, Messenger of Mathematics, vol. 47, no. 10, 1918, 145-150.

[8] S. Iqbal, K. Krulic, J. Pecaric, *On an inequality of G. Hardy*, J. of Inequalities and Applications, Vol. 2010, Article ID 264347, 23 pages.

[9] A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, *Theory and Applications of Fractional Differential Equations*, vol. 204 of North-Holland Mathematics Studies, Elsevier, New York, NY, USA, 2006.

[10] E. Pap, *Null-Additive Set Functions*, Kluwer Academic, Dordrecht, 1995.

[11] S.G. Samko, A.A. Kilbas, O.I. Marichev, *Fractional Integral and Derivatives: Theory and Applications*, Gordon and Breach Science Publishers, Yverdon, Switzerland, 1993.

[12] D. Schmeidler, *Subjective probability and expected utility without additivity*, Econometrica, 57 (1989), 571-587.

[13] Rui-Sheng Wang, *Some inequalities and convergence theorems for Choquet integrals*, J. Appl. Math. Comput. 35 (2011), 305-321.

[14] Z. Wang, *Convergence theorems for sequences of Choquet integrals*, Int. J. Gen. Syst. 26 (1997), 133-143.

[15] Z. Wang, G. Klir, *Fuzzy Measure Theory*, Plenum, New York, 1992.

[16] Z. Wang, G.J. Klir, W. Wang, *Monotone set functions defined by Choquet integrals*, Fuzzy Sets Syst. 81 (1996), 241-250.

[17] Z. Wang, G.J. Klir, *Generalized Measure Theory*, Springer, New York, 2009.

TABLE OF CONTENTS, JOURNAL OF COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 27, NO. 7, 2019